Harmonic Spinors on Conformally flat Manifolds with S^1-Symmetry

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1 Introduction

The n-dimensional sphere S^n is the conformal compactification of the product $H^{n-1}\times S^1$ of the hyperbolic space H^{n-1} by the circle S^1. Indeed, consider $H^{n-1}=\{(x,t)\in R^{n-2}\times R^1 : t>0\}$ with the hyperbolic metric $g_o=\dfrac{dx^2+dt^2}{t^2}$ and $S^1=\{z\in\mathbb{C}: |z|=1\}$. The map

$$j: H^{n-1}\times S^1 \longrightarrow R^n\cup\{\infty\}=S^n, \quad j(x,t,z)=(x,t\cdot z)$$

is a conformal embedding and the complement $S^n\backslash j(H^{n-1}\times S^1)=S^{n-2}$ is the totally geodesic sphere S^{n-2}. Let $\Gamma\subset Iso_o(H^{n-1})\subset \mathrm{Conf}(S^n)$ be a geometrically finite Kleinian group of compact type (see [1]) and denote by $\Lambda(\Gamma)\subset S^{n-2}$ its limit set. Γ acts on the sphere S^n as a group of conformal transformations and $j: H^{n-1}\times S^1\longrightarrow S^n$ is a Γ-equivariant map. Therefore, the manifold

$$X^n(\Gamma)=(S^n\backslash\Lambda(\Gamma))/\Gamma$$

is the conformal compactification of $H^{n-1}/\Gamma\times S^1$ and j defines a conformal, S^1-equivariant embedding $H^{n-1}/\Gamma\times S^1\longrightarrow X^n(\Gamma)$. There is a conformal S^1-action on $X^n(\Gamma)$ and it is free away from the fixed point set $\Omega(\Gamma)/\Gamma$ being the complement of $H^{n-1}/\Gamma\times S^1$ in $X^n(\Gamma)$. $\Omega(\Gamma)\subset S^{n-2}$ denotes the region of discontinuity of the Kleinian group Γ. (see [4]). In this paper we prove the following

Theorem: Let $\Gamma\subset Iso_o(H^{n-1})$ be a geometrically finite Kleinian group of compact type and denote by $X^n(\Gamma)=(S^n\backslash\Lambda(\Gamma))/\Gamma$ the conformal compactification of the space $H^{n-1}/\Gamma\times S^1$ Then the kernel of the Dirac operator on $X^n(\Gamma)$ is trivial.

Remark: If the Hausdorff dimension $\dim_H\Lambda(\Gamma)\leq\frac{n-2}{2}$ of $\Lambda(\Gamma)$ is not greater then $\frac{n-2}{2}$ the manifold $X^n(\Gamma)$ admits in its conformal structure a Riemannian metric of constant scalar curvature $R\geq 0$ (see [6]). In this case the theorem follows directly from the well-known Lichnerowicz formula for the square of the Dirac operator.

In particular, in dimension $n=4$ the twistor space Z of $X^4(\Gamma)$ is a complex manifold and under the Penrose correspondence the kernel of the Dirac operator on $X^4(\Gamma)$ coincides with the cohomology (see [5])

$$H^1(Z;\mathcal{O}\ (-1))\oplus H^1(Z;\mathcal{O}\ (-3)).$$

Consequently, we obtain

Corollary: Let Z be the twistor space of a compact, conformally flat manifold $X^4(\Gamma)$ with a geometrically finite Kleinian group $\Gamma \subset PSL(2;\mathbb{C})$ of compact type. Then the cohomology groups

$$H^1(Z;\mathcal{O}\ (-1)) = 0 = H^1(Z;\mathcal{O}\ (-3)) \quad \text{vanish.}$$

2 Proof of the Theorem

We fix an S^1-invariant metric g in the conformal structure of $X^n(\Gamma)$ given by

$$g = v^2(x,t)\{g_o \oplus d\theta^2\}$$

on $H^{n-1}/\Gamma \times S^1 \longrightarrow X^n(\Gamma)$. The kernel of the Dirac operator is a finite-dimensional S^1-representation and therefore we may assum that the harmonic
spinor ψ on $X^n(\Gamma)$ satisfies the conditions

$$D\psi = 0, \quad (e^{i\theta}\psi)(m) = e^{ik\theta}\psi(m).$$

ψ is a harmonic spinor with respect to the metric g if and only if $\varphi := v^{\frac{n-1}{2}}\psi$ is a harmonic spinor for the metric $g_o \oplus d\theta^2$ (see [2]). Now φ is a S^1-parameter family of tuples of spinors on H^{n-1} $\quad \varphi = \varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta))$ with the transformation rule $\varphi(2\theta) = \varphi(0)e^{ik\theta}$ since a double cover of S^1 lifts into the spin structure of $X^n(\Gamma)$. We write

$$\varphi_i(x,t,\theta) = \varphi_i^\star(x,t)e^{i\frac{k}{2}\theta}$$

and then the n-dimensional Dirac equation in the metric $g_o \oplus d\theta^2$ is equivalent to the $(n-1)$-dimensional Dirac equations

$$D\varphi_1(\theta) + (-1)^m i\frac{\partial\varphi_2}{\partial\theta}(\theta) = 0$$

$$D\varphi_2(\theta) - (-1)^m i\frac{\partial\varphi_1}{\partial\theta}(\theta) = 0$$

on $(H^{n-1}/\Gamma, g_o)$, where $m = \left[\frac{n-1}{2}\right]$ (see [3], page 13-14). Finally we obtain

$$D\varphi_1^\star + (-1)^m \tfrac{k}{2}\varphi_2^\star = 0$$

$$D\varphi_2^\star - (-1)^m \tfrac{k}{2}\varphi_1^\star = 0$$

and the spinor fields $\varphi^\star = \varphi_1^\star \pm i\varphi_2^\star$ are solutions of the equation

$$D\varphi^\star \pm (-1)^m i\tfrac{k}{2}\varphi^\star = 0$$

on H^{n-1}/Γ. Fix a point $x \in \Omega(\Gamma)$ of the region of discontinuity. The limit $\lim_{t\to 0} |\psi(x,t)|$ exists since ψ is a smooth spinor on $X^n(\Gamma)$. With respect to

$$g = \left(\frac{v(x,t)^2}{t}\right)\{dx^2 + dt^2 + t^2 d\theta^2\}$$

the limit $\lim_{t\to 0} \frac{v(x,t)}{t} > 0$ exists too. Consequently ($\varphi = v^{\frac{n-1}{2}}\psi$), the length $|\varphi^\star(x,t)| \sim t^{\frac{n-1}{2}}$ behaves like $t^{\frac{n-1}{2}}$ in case $x \in \Omega(\Gamma)$. Using the fact that ψ is a solution of an elliptic differential operator with analytic coefficients on can see that $|\varphi^\star(x,t)|$ is a power series in $\sqrt{t}$ starting at $t^{\frac{n-1}{2}}$. In any case we see that in the finite number of ends of the hyperbolic space H^{n-1}/Γ the length $|\varphi^\star(x,t)|^2$ behaves like t^{n-1}. The volume from of H^{n-1}/Γ is given by $\frac{dx\wedge dt}{t^{n-1}}$ and $\Omega(\Gamma)/\Gamma$ is compact. Finally we conclude that $\varphi^\star$ is a square-integrable spinor field on H^{n-1}/Γ satisfying the equation

$$D\varphi^\star = \pm\tfrac{k}{2} i\varphi^\star$$

.

H^{n-1}/Γ is a complete Riemannian manifold and the Dirac operator is essentially self-adjoint. In particular, its defect indizees

$$\dim\ker\left(D \pm \frac{k}{2}i\right) = 0 \quad (k \neq 0)$$

with the respect to the Hilbert space L^2 of square integrable spinors vanish.
Now we investigate the case $k = 0$, i.e. S^1-invariant harmonic spinors on $X^n(\Gamma)$. We restrict ourself - for technical reasons - to the case of dimension $n = 4$. On the hyperbolic space H^3 we use the orthonormal frame

$$e_1 = t\frac{\partial}{\partial x} \quad e_2 = t\frac{\partial}{\partial y} \quad e_3 = t\frac{\partial}{\partial t}$$

as well as the representation of the Clifford algebra given by

$$e_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad e_3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

A spinor field $\varphi^\star$ (on H^3) is given by its components f and g and the Dirac equation $D\varphi^\star = 0$ is equivalent to

$$t\{f_t - 2g_{\bar z}\} = f$$

$$t\{2f_z + g_t\} = g,$$

$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$. We derive now - as already mentioned before - that near a point $(x,y) \in \Omega(\Gamma)$ the functions f and g have power series expansions of type

$$f(x,y,t) = \sum_{i\geq 3}^{\infty} (\sqrt{t})^i f_i(x,y)$$
$$g(x,y,t) = \sum_{i\geq 3}^{\infty} (\sqrt{t})^i g_i(x,y)$$

The spinor φ has the components $f(x,y,t), g(x,y,t)$ with respect to the frame $e_1 = t\frac{\partial}{\partial x}$ $e_2 = t\frac{\partial}{\partial y}$ $e_3 = t\frac{\partial}{\partial t}$ $e_4 = \frac{\partial}{\partial \theta}$ on $H^3/\Gamma \times S^1$. Consequently, the spinor φ has the same components with respect to the frame

$$e_1^\star = \frac{t}{v}\frac{\partial}{\partial x} \quad e_2^\star = \frac{t}{v}\frac{\partial}{\partial y} \quad e_3^\star = \frac{t}{v}\frac{\partial}{\partial t} \quad e_4^\star = \frac{1}{v}\frac{\partial}{\partial \theta}$$

being an orthonormal frame for the metric $g = v^2\{g_o + d\theta^2\}$ (see [2], page 136-138). Locally we introduce the coordinates $\alpha = t\cos\theta \quad \beta = t\sin\theta$. Then

$$g = \left(\frac{v}{t}\right)^2 \{dx^2 + dy^2 + d\alpha^2 + d\beta^2\}$$

and the components of φ with respect to the frame $e_1^{\star\star} = e_1^\star, \quad e_2^{\star\star} = e_2^\star$

$$e_3^\star = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} e_3^{\star\star} + \frac{\beta}{\sqrt{\alpha^2+\beta^2}} e_4^{\star\star},$$
$$e_4^\star = -\frac{\beta}{\sqrt{\alpha^2+\beta^2}} e_3^{\star\star} + \frac{\alpha}{\sqrt{\alpha^2+\beta^2}} e_4^{\star\star}$$

are the functions

$$f(x,y,t)\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}} + i\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right) = f(x,y,t)e^{i\theta}$$
$$g(x,y,t)\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2}} - i\frac{\beta}{\sqrt{\alpha^2+\beta^2}}\right) = g(x,y,t)e^{-i\theta}$$

Let F and G be the components of the spinor $\psi = \frac{1}{v^{3/2}}\varphi$ with respect to the frame $e_1^{\star\star}, \cdots, e_4^{\star\star}$. Then

$$v^{3/2}F(x,y,\alpha,\beta) = f(x,y,t)e^{i\theta}$$

Since ψ is reall-analytic it follows that F is reall-analytic and consequently $(\theta = 0)$ $f(x,y,t) = v^{3/2}F(x,y,t,0)$ admits a power series expansions in $\sqrt{t}$. Finally we use the power series expansions for f and g as well as the differential equation $(\star)$. Then we obtain $(i \geq 5)$

$$f_3 = 0 \quad f_4 = 0 \quad (i/2-1)f_i - 2(g_{i-2})_{\bar{z}} = 0$$

$$g_3 = 0 \quad g_4 = 0 \quad 2(f_{i-2})_z + (i/2-1)g_i = 0.$$

The relations immedatly imply that all coefficients f_i and g_i vanish. This proves that the kernel of the Dirac operator on $X^4(\Gamma)$ is trivial.

Remark: The same argument applies to the eigenspinors of the Dirac operator on $X^n(\Gamma)$.

Consider $D\psi = \lambda\psi$ and $(e^{i\theta}\psi) = e^{ik\theta}\cdot\psi$. Then $\varphi := v^{\frac{n-1}{2}}\psi$ satisfies the equation

$$D\varphi = \lambda v \varphi$$

(see [2]) and for the pair $(\varphi_1^\star, \varphi_2^\star)$ we obtain the following equations on H^{n-1}/Γ depending on the dimension n:

a.) in case $n = 2m+2$ is even

$$D\varphi_1^\star + (-1)^{m+1}\tfrac{k}{2}\varphi_2^\star = \lambda v \varphi_1^\star \qquad (\star\star)$$

$$D\varphi_2^\star - (-1)^{m+1}\tfrac{k}{2}\varphi_1^\star = -\lambda v \varphi_2^\star$$

b.) in case $n = 2m+1$ is odd

$$D\varphi_1^\star - (-1)^{m}\tfrac{k}{2}\varphi_2^\star = \lambda v \varphi_2^\star \qquad (\star\star\star)$$

$$D\varphi_2^\star + (-1)^{m}\tfrac{k}{2}\varphi_1^\star = \lambda v \varphi_1^\star .$$

The first case provides to the equation

$$D^2\varphi_1^\star - \lambda\cdot grad(v)\cdot\varphi_1^\star - \lambda^2 v^2 \varphi_1^\star = -(\tfrac{k}{2})^2\varphi_1^\star$$

and finally - since all spinor fields belong to the Hilbert space L^2 -

$$\int\limits_{H^{n-1}/\Gamma} (D\varphi_1^\star, D\varphi_1^\star) = \int\limits_{H^{n-1}/\Gamma} (\lambda^2 v^2 - (\tfrac{k}{2})^2)|\varphi_1^\star|^2 .$$

The last equation yields

$$\left(\frac{k}{2}\right)^2 \le \lambda^2 \max\{v^2(x) : x \in H^{n-1}/\Gamma\},$$

i.e. a bound for the S^1-weight of the representation $E_\lambda(D)$ depending on λ ($n = 2m+2$). Conversely, every solution of the system $(\star\star)$ or $(\star\star\star)$ satisfying the condition $|\varphi^\star(x,t)| \sim t^{\frac{n-1}{2}}$ defines a spinor field ψ on $X^n(\Gamma)\setminus\Omega(\Gamma)/\Gamma = H^{n-1}/\Gamma\times S^1$ with finite L^2-norm. Then ψ is a smooth solution of the Dirac operator $D\psi = \lambda\psi$ on $X^n(\Gamma)$ since $\Omega(\Gamma)/\Gamma$ has codimension two (see for example [4], page 287-288).

3 Further Remarks

A similar method can be used for the investigation of the behaviour of harmonic spinors near a fixed point set of codimension two on a Riemannian manifold with S^1-symmetry. Consider $M^n (n = 2m+1, 2m+2)$ with a S^1-action and denote by Σ a compact fixed point set of codimension 2 in M^n. We suppose moreover that the S^1-action lifts into the spin structure of M^n. Furthermore, we assume that the normal bundle of Σ is topological trivial and in a neighbourhood $\Sigma \subset U \approx \Sigma \times R^2(\alpha,\beta) \subset M^n$ of Σ the metric g has the form

$$g = v^2(g_\Sigma + d\alpha^2 + d\beta^2)$$

for some Riemannian metric g_Σ on Σ. S^1 acts by rotations in the $R^2(\alpha,\beta)$ plane and ℓ denotes the S^1-weight of the normal bundle.
Consider a harmonic spinor and suppose that

$$D\psi = 0, \quad (e^{i\theta}\psi) = e^{ik\theta}\psi$$

.

We restrict ψ to U and consider there the metric

$$g = v^2(g_\Sigma + d\alpha^2 + d\beta^2) = (vt)^2\{\frac{g_\Sigma + dt^2}{t^2} + d\theta^2\}$$

where t and θ are polar coordinates in the plane $R^2(\alpha,\beta)$. ψ is harmonic if and only if $\varphi := (vt)^{\frac{n-1}{2}}\psi$ is a harmonic spinor with respect to the metric $\dfrac{g_\Sigma + dt^2}{t^2} \oplus d\theta^2$. The spinor φ is an S^1-parameter family of pairs $\varphi(\theta) = (\varphi_1(\theta), \varphi_2(\theta))$ where $\varphi_i(\theta)$ are spinors on $\Sigma \times (0,\varepsilon)$ and

$$\varphi_i(\ell\theta) = \varphi_i(0)e^{ik\theta}$$

Similarly as in the proof of the theorem we reduce the n-dimensional Dirac equation $D\varphi = 0$ to the $(n-1)$-dimensional Dirac equation

$$D\varphi^\star \pm (-1)^m \tfrac{k}{l}\, i\varphi^\star = 0$$

on $\Sigma \times (0,\varepsilon)$ and with respect to the metric $(g_\Sigma + d\alpha^2 + d\beta^2) = \dfrac{g_\Sigma + dt^2}{t^2}$.

Now we introduce $\varphi^{\star\star} = \dfrac{1}{t^{\frac{n-2}{2}}}\varphi^\star = v^{\frac{n-1}{2}}\sqrt{t}\psi$. The latter equation is equivalent to

$$tD(\varphi^{\star\star}) \pm (-1)^m \tfrac{k}{l} i\varphi^{\star\star} = 0$$

for the Dirac operator D of the metric $g_\Sigma \oplus dt^2$ (see [2], page 136-138). Consider now the case $n = 2m+2$. Then $\varphi^{\star\star}$ is a t-parameter family of spinors on Σ^{2m} and therefore it splits into $\varphi^{\star\star} = \varphi^+ \oplus \varphi^-$. The Dirac equation under consideration splits also into equations

$$t\{D_\Sigma\varphi^+ - i\dot{\varphi}^-\} = \pm(-1)^m \tfrac{k}{l}\, i\varphi^-$$

$$t\{D_\Sigma \varphi^- + i\dot{\varphi}^+\} = \pm(-1)^m \tfrac{k}{l}\, i\varphi^+$$

where D_Σ is the Dirac operator on the closed manifold Σ^{2m}. We multiply this equation by φ^-:

$$t\{(D_\Sigma \varphi^+, \varphi^-) - i(\dot{\varphi}^-, \varphi^-)\} = \pm(-1)^m \tfrac{k}{l}\, i|\varphi^-|^2$$

$$t\{(\varphi^-, D_\Sigma \varphi^+) + i(\varphi^-, \dot{\varphi}^-)\} = \mp(-1)^m \tfrac{k}{l}\, i|\varphi^-|^2$$

and now we integrate over Σ^{2m}. Then we obtain

$$t \cdot \frac{d}{dt}\|\varphi^-\|^2 = \mp(-1)^m 2\frac{k}{l}\,\|\varphi^-\|^2$$

for the L^2-norm $\|\ \|$ of $\varphi^-(t)$ over Σ. Finally we have

$$\|\varphi^-\|^2 = \mathcal{C}_1 t^{\mp(-1)^m 2k/l} \quad \text{and} \quad \|\varphi^+\|^2 = \mathcal{C}_2 t^{\pm(-1)^m 2k/l}.$$

The spinor field $\varphi^{\star\star} = v^{\frac{n-1}{2}}\sqrt{t}\psi$ tends to zero as $t \to 0$. Therefore, in case $k = 0$ we conclude $\mathcal{C}_1 = 0 = \mathcal{C}_2$ and consequently $\psi \equiv 0$. In case $k \neq 0$ only one of the exponents $\pm(-1)^m 2k/l$ is positive. This observation yields $C_1 = 0$ or $C_2 = 0$ and consequently $\varphi^+ \equiv 0$ or $\varphi^- \equiv 0$. The Dirac equation reduces now to ($\varphi^- \equiv 0$)

$$D_\Sigma \varphi^+ = 0$$

$$t\dot{\varphi}^+ = |k/l|\varphi^+$$

and we obtain the solution $\varphi^+(x,t) = \varphi_0(x)t^{|k/l|}$ for a harmonic spinor φ_0 on Σ^{2m}. We can handle the case $n = 2m+1$ in a similary way. We split $\varphi^{\star\star}$ over $\Sigma^{2m+1} \times (0,\varepsilon)$ into $\varphi^{\star\star} = \varphi^{\star\star}_+ + \varphi^{\star\star}_-$. Then the equation $tD(\varphi^{\star\star}) \pm (-1)^m \frac{k}{l}\, i\varphi^{\star\star} = 0$ yields

$$tD(\varphi^{\star\star}_+) \pm (-1)^m \tfrac{k}{l}\, i\varphi^{\star\star}_- = 0$$

$$tD(\varphi^{\star\star}_-) \pm (-1)^m \tfrac{k}{l}\, i\varphi^{\star\star}_+ = 0$$

and now we interprete $\varphi^{\star\star}_+$ and $\varphi^{\star\star}_-$ as t-parameter families of spinors on Σ^{2m+1}. Finally we obtain the following

Theorem: Let M^n be a Riemannian spin manifold and suppose that S^1 acts as a group of isometries with a compact fixed point set Σ^{n-2} of codimension 2. Suppose that the

normal bundle of Σ^{n-2} is trivial and has S^1-weight ℓ. In a normal neighbourhood $\Sigma \subset U$ the metric g let be given by
$g = v^2(g_\Sigma + d\alpha^2 + d\beta^2)$. If $\psi \neq 0$ is a harmonic spinor on M^n such that $(e^{i\theta}\psi) = e^{ik\theta} \cdot \psi$, then

1.) $k \neq 0$

2.) There exists a harmonic spinor φ_o on Σ^{n-2} such that

$$|\psi(x,t,\theta)|^2 = |\varphi_o(x)|^2 \frac{t^{2|\frac{k}{l}|-1}}{v^{n-1}}$$

.

Corollary: If Σ^{n-2} has no non-trivial harmonic spinors then M^n too.

Corollary: Let Σ be a Riemannian spin manifold and suppose $\ker(D_\Sigma) = 0$. Then any harmonic spinor on $\Sigma \times S^2$ is zero.

Literatur

[1] B.N. Apanasov, Discrete groups in space and uniformization problems, Kluwer Academic Publishers 1991

[2] H.Baum, Spin-Strukturen und Dirac-Operatoren über pseudo-Riemannschen Mannigfaltigkeiten, Teubner Verlag / Leipzig 1981

[3] H.Baum, Th.Friedrich, R.Grunewald, I.Kath, Twistors and Killing Spinors on Riemannian Manifolds, Teubner Verlag Stuttgart/ Leipzig 1991

[4] P.J.Braam, A Kaluza-Klein approach to hyperbolic three manifolds, L Enseignement Mathematiques t. 34 (1988), 275-311

[5] N.J.Hitchin, Linear field equations on self-dual spaces, Proc. R. Soc. London A 370 (1980), 173-191

[6] R.Schoen, S.T.Yau, Conformally flat manifolds, Kleinian groups and scalar curvature, Invent. Math. 92 (1988), 47-71.

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