On Selberg Zeta Functions, Topological Zeroes and Determinant Formulas

Ulrich Bröcker

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Abstract

Let X be a compact locally symmetric space form of a noncompact symmetric space G/K of rank one. So X is of the form $X = \Gamma\backslash G/K$, where G is a real connected noncompact semisimple Lie group of real rank one, K is a maximal compact and Γ a cocompact and torsionfree discrete subgroup of G. We additionally denote by A the noncompact abelian subgroup of G arising in the Iwasawa decomposition $G = KAN$ of G and by M the centralizer of A in K.

Then we can assocciate to every irreducible unitary representation $\sigma \in Rep(M)$ a *Selberg zeta function* Z_σ *of* X. It is known from [JU 93], that the multiplicities of the so-called topological zeroes of Z_σ admit a description as indices of certain differential complexes. Applying results from [SCHM 71] we give them an explicit form. As a consequence of it in *Proposition 1* we prove a functional equation of the *Gamma factor of* Z_σ, which is the zeta regularized determinant of the sequence of topological zeroes of Z_σ.

Furthermore if $\sigma = 1$ we get an explicit *factorisation of* Z_1 *into zeta regularized determinants* by carefully integrating a special Selberg trace formula and using the proven reflection formula for the Gamma factor. The resulting formula is stated in *Proposition 3*. It is a generalisation of a result of A.Voros ([VO 87]) for the case of a hyperbolic surface.

1 Spectral functions of special sequences and their properties

We examine spectral functions in the sence of A. Voros associated to special sequences. Therefore we first collect all the definitions and statements from [VO 87] that will be applied in the sequel. Then we introduce certain sequences and consider reflection properties and asymptotic behaviour of the associated so-called functional determinants. A resulting formula is the basis of the proof of a reflection formula strongly connected with Selberg zeta functions in the next section. Nevertheless this theory is relatively independent of that application.

1.1 Definitions

We start with a sequence of positive real numbers $\{\lambda_k\}_{k=0}^{\infty}$ with

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \ldots \quad , \quad \lambda_k \uparrow \infty$$

The two basic assumptions are that the *partition function of the sequence*,

$$\theta(t) \stackrel{df}{=} \sum_{k=0}^{\infty} e^{-t\lambda_k},$$

converges for all t with $\mathrm{Re}(t) > 0$ and that it admits for $t \to 0$ a full asymptotic expansion (i.e. to all orders in t)

$$\theta(t) \sim \sum_{m=0}^{\infty} c_{i_m}\, t^{i_m} \tag{1}$$

for a certain increasing sequence of real exponents $\{i_m\}_{m=0}^{\infty}$ with $i_m \uparrow \infty$ and $i_0 < 0$. The positive number $\mu \stackrel{df}{=} -i_0$ is called the *order of the sequence*. Because μ is an integer in all of our cases we make it to an additional assumption and replace $[\mu]$ by μ whenever it occurs in [VO 87].
We consider four types of spectral functions and their connection between each other : Zeta functions, functional determinants, Weierstrass products and resolvents.
The *Zeta function* of the sequence $\{\lambda_k\}_{k=0}^{\infty}$ is defined as

$$Z(s) \stackrel{df}{=} \sum_{k=0}^{\infty} \lambda_k^{-s} .$$

It is a holomorphic function in the half-plane $\mathrm{Re}(t) > \mu$ and admits a meromorphic continuation to the whole complex plane with poles only in

$$s = -i_k \quad , \quad \text{residue:} \quad c_{i_k} \Big/ \Gamma(-i_k) .$$

(This is how the expansion 1 is encoded in Z. We will use this later.) The negative or zero integers are always regular values for Z with

$$Z(-m) = (-1)^m \cdot m! \cdot c_m \quad , \quad m = 0, 1, 2, \ldots .$$

The *generalized* (or two parameter) *Zeta function* of the sequence,

$$Z(s,a) \stackrel{df}{=} \sum_{k=0}^{\infty} (\lambda_k + a)^{-s} \qquad , \qquad a \geq -\lambda_0 \ , \ \mathrm{Re}(s) > \mu \ ,$$

(and analytic continuation elsewhere) can be viewed as the Zeta function of the shifted sequence $\{\lambda_k + a\}_{k=0}^{\infty}$.
The (regularized) *Weierstrass product* of the sequence is given by

$$\Delta(\lambda) \stackrel{df}{=} \prod_{k=0}^{\infty} \left[\left(1 - \frac{\lambda}{\lambda_k}\right) \exp\left(\sum_{r=1}^{\mu} \frac{\lambda^r}{r \cdot \lambda_k^r} \right) \right] \qquad .$$

It is the unique entire function of order μ having the zeros $\{\lambda_k\}$ and subjected to the normalisation conditions

$$\left(\frac{d}{d\lambda}\right)^j \left(\log \Delta(\lambda) \right)\Big|_{\lambda=0} = 0 \qquad , \qquad j = 0, \ldots, \mu \qquad .$$

Our interest is finally directed to the *functional determinant* of the sequence, which is the zeta regularized determinant

$$D(\lambda) \stackrel{df}{=} \exp\left\{ -\frac{d}{ds} \left(Z(s, -\lambda) \right) \Big|_{s=0} \right\} \qquad . \tag{2}$$

We will need the connection between the two regularisations of the infinite product $\prod_{k=0}^{\infty} (\lambda_k - \lambda)$

$$D(\lambda) \quad = \quad \exp\left\{ -Z'(0) - \sum_{m=1}^{\mu} \mathrm{F.P.} Z(m) \frac{\lambda^m}{m} - \sum_{m=2}^{\mu} c_{-m} \left(\sum_{r=1}^{m-1} \frac{1}{r} \right) \frac{\lambda^m}{m!} \right\} \cdot \Delta(\lambda) \tag{3}$$

There we have used the *finite part* prescription as usual

$$\mathrm{F.P.} f(s) \stackrel{df}{=} \begin{cases} f(s) & \text{if s is not a pole} \\ \lim_{\epsilon \to 0} \left(f(s+\epsilon) - \frac{\text{residue}}{\epsilon} \right) & \text{if s is a simple pole} \end{cases} \qquad .$$

We consider also the μ-th resolvent R_μ , defined as the convergent series

$$R_\mu(\lambda) \stackrel{df}{=} \mu! \cdot \sum_{k=0}^{\infty} \frac{1}{(\lambda_k - \lambda)^{\mu+1}} \qquad .$$

It follows

$$R_\mu(\lambda) = \left(\frac{d}{d\lambda}\right)^{\mu+1} \left(-\log \Delta(\lambda) \right) = \left(\frac{d}{d\lambda}\right)^{\mu+1} \left(-\log D(\lambda) \right) \tag{4}$$

and backwards

$$-\log \Delta(\lambda) = \left(\int_0^{\lambda} dt \right)^{\mu+1} \left(R_\mu(t) \right) \qquad .$$

To achieve a similar result getting back $D(\lambda)$ from $R_\mu(\lambda)$ integrating it $(\mu+1)$-times A. Voros exhibitted a special integral operator $\int_{-\infty}^{\lambda} dt$ (defined via meromorphic continuation of the integration rule

$$\int_{-\infty}^{\lambda} (-t)^{-s}\, dt = \frac{(-\lambda)^{1-s}}{s-1}$$

viewed as a function of s defined on $s > 1$). To calculate $\left(\int_{-\infty}^{\lambda} dt\right)(f(t))$ one has to look at the asymptotic expansion of $f(t)$ for $t \to -\infty$ and extract enough leading terms such that the rest gives a finite contribution. For the extracted terms one has to use the integration rules

$$\begin{aligned} \int_{-\infty}^{\lambda} 0\, dt = 0 \quad , \quad \int_{-\infty}^{\lambda} (-t)^{-s}\, dt &= \frac{(-\lambda)^{1-s}}{s-1} \qquad (s \neq 1) \\ \int_{-\infty}^{\lambda} (-t)^{-1}\, dt &= -\log(-\lambda) \\ \left(\int_{-\infty}^{\lambda} dt\right)^{m} (\log(-t)) &= \frac{(\lambda)^m}{m!}\left(\log(-\lambda) - \sum_{r=1}^{m}\frac{1}{r}\right) \quad . \end{aligned} \tag{5}$$

Because of the special choise of a primitive only *linear* changes of variables are allowed if the ordinar integral does not converge (If it does, of course, $\int_{-\infty}^{\lambda}$ has the usual meaning.). With the help of this procedure one gets

$$-\log D(\lambda) = \left(\int_{-\infty}^{\lambda} dt\right)^{\mu+1} (R_\mu(t)) \qquad , \tag{6}$$

The fact that $Z(s,a)$ is joined with $\mathrm{e}^{-t\,a}\theta(t)$ via Mellin transform (cf. [VO 87]), the observation $R_\mu(\lambda) = \mu!\; Z(\mu+1,\lambda)$ and definition (2) give us the two asymptotic expansions for $\lambda \to -\infty$

$$R_\mu(\lambda) \sim \sum_{m=0}^{\infty} c_{i_m} \cdot \Gamma(i_m+\mu+1) \cdot (-\lambda)^{-i_m-\mu-1} \tag{7}$$

$$-\log D(\lambda) \sim \sum_{\substack{m=0 \\ i_m \notin -\mathbb{N}}}^{\infty} \frac{c_{i_m}\Gamma(i_m)}{(-\lambda)^{i_m}} - \sum_{k=0}^{\mu} c_{-k}\left[\log(-\lambda) - \sum_{r=1}^{k}\frac{1}{r}\right]\frac{\lambda^k}{k!} \tag{8}$$

where the c_{i_k} are the coefficients arising from (1) .

Finally we give

Lemma 1 *1. The functional determinants D and $D^{(a)}$ of the sequences $\{\lambda_k\}$ and $\{\lambda_k + a\}$ are related by*

$$D^{(a)}(\lambda) = D(\lambda - a) \quad .$$

2. The functional determinants D_1 , D_2 and D_{all} of the sequences $\{\lambda_k^{(1)}\}$, $\{\lambda_k^{(2)}\}$ and $\{\lambda_k^{(1)}\} \cup \{\lambda_k^{(2)}\}$ satisfying the above conditions are related by

$$D_{\text{all}}(\lambda) = D_1(\lambda) \cdot D_2(\lambda) \quad .$$

3. *The relation between the functional determinants D and $\tilde{D}$ of the sequences $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\lambda_k\}_{k=0}^{\infty}$ resp. , where λ_0 is an arbitrary complex number, is*
$$\tilde{D}(\lambda) = (\lambda_0 - \lambda) \cdot D(\lambda) \quad .$$

Proof. The first two relations are simple consequences of the given definitions. For the third one one conciders the associated μ-th resolvents to find
$$\tilde{R}_\mu(\lambda) = R_\mu(\lambda) + \frac{\mu!}{(\lambda_0 - \lambda)^{\mu+1}} \quad .$$
Having in mind the definition of the integral operator $\int_{-\infty}^{\lambda} dt$ one gets with relation (6)
$$-\log \tilde{D}(\lambda) = -\log D(\lambda) - \log(\lambda_0 - \lambda)$$
hence the stated equation. □

1.2 Elementary sequences

Lets consider the (so-called elementary) sequences
$$\Big\{ k \quad \text{with multiplicity} \quad k^{j-1} \ \Big\}_{k=1}^{\infty} \quad , \; j = 1, 2, \dots \, , \tag{9}$$
and denote the associated spectral functions introduced in the previous section by $\theta_j \, , D_j \, , \dots$.
Standard calculations give us
$$\begin{aligned} \theta_j(t) &= \left(-\frac{d}{dt}\right)^{j-1} \left(\frac{1}{\mathrm{e}^t - 1}\right) \\ &= \frac{\Gamma(j)}{t^j} + (-1)^{j-1} \sum_{k=0}^{\infty} \frac{B_{k+j}}{k+j} \cdot \frac{t^k}{k!} \end{aligned}$$
where the B_k are the Bernoulli numbers defined via
$$\begin{aligned} B_0 &\overset{df}{=} 1 \\ 0 &= \sum_{j=0}^{k} \binom{k+1}{j} B_j \qquad \text{for} \quad k \geq 1 \, . \end{aligned}$$

Hence the order of the sequences (9) is j and thats why that index of the spectral functions. For the coefficients $c_{i_k}^{(j)}$ of this representation of $\theta_j(t)$ are often used in this theory (cf. section 1.1) we give them explicitely.
$$c_{-j}^{(j)} = \Gamma(j) \quad , \qquad c_k^{(j)} = (-1)^{j-1} \frac{B_{k+j}}{(k+j)k!} \qquad (k \geq 0) \quad .$$
Using $B_{2k+1} = 0$ for $k = 1, 2, \dots$ we have for all $j \geq 1$ and $k \geq 0$
$$\begin{aligned} c_{2k+1}^{(2j)} &= c_{2k}^{(2j-1)} = 0 \qquad (\text{only } c_0^{(1)} = B_1 = -\frac{1}{2}) \quad , \\ c_{2k}^{(2j)} &= -\frac{B_{2k+2j}}{(2k+2j)(2k)!} \quad , \quad c_{2k+1}^{(2j-1)} = \frac{B_{2k+2j}}{(2k+2j)(2k+1)!} \quad . \end{aligned}$$

Also one has

$$\begin{aligned} Z_j(s) &= \zeta(s-j+1) \quad , \\ \Delta_j(z) &= \prod_{k=1}^{\infty}\left[\left(1-\frac{z}{k}\right)\cdot\exp\left(\sum_{r=1}^{j}\frac{z^r}{r\cdot k^r}\right)\right]^{k^{j-1}} \quad , \\ D_j(z) &= \exp\left\{-\zeta'(1-j)-\sum_{k=1}^{j-1}\zeta(k-j+1)\frac{z^k}{k}-\left(\gamma+\sum_{r=1}^{j-1}\frac{1}{r}\right)\frac{z^j}{j}\right\}\cdot\Delta_j(z) \end{aligned}$$

where we have used, that the Riemann Zeta function $\zeta(s)=\sum_{k=1}^{\infty}k^{-s}$ $(\mathrm{Re}(s)>1)$ admits a meromorphic continuation to the whole complex plane with only one pole ($s=1$, simple pole) with residue 1 and F.P.$\zeta(1)=\gamma$, ($\gamma\ldots$Eulers constant , cf. [PA 88]).

Using the special values

$$\begin{aligned} \zeta(0) &= -\frac{1}{2} \quad , \\ \zeta(-2m) &= 0 \quad , \ m=1,2,\ldots \quad , \\ \zeta(1-2m) &= -\frac{B_{2m}}{2m} \quad , \ m=1,2,\ldots \quad , \end{aligned}$$

one gets

$$\begin{aligned} D_{2j+1}(z) = \exp\Bigg\{&-\zeta'(-2j)+\sum_{k=0}^{j-1}\frac{B_{2j-2k}\,z^{2k+1}}{(2j-2k)(2k+1)} \\ &+\frac{z^{2j}}{4j}-\left(\gamma+\sum_{r=1}^{2j}\frac{1}{r}\right)\frac{z^{2j+1}}{2j+1}\Bigg\}\cdot\Delta_{2j+1}(z) \end{aligned} \tag{10}$$

and

$$\begin{aligned} D_{2j}(z) = \exp\Bigg\{&-\zeta'(1-2j)+\sum_{k=1}^{j-1}\frac{B_{2j-2k}\,z^{2k}}{(2j-2k)(2k)} \\ &+\frac{z^{2j-1}}{4j-2}-\left(\gamma+\sum_{r=1}^{2j-1}\frac{1}{r}\right)\frac{z^{2j}}{2j}\Bigg\}\cdot\Delta_{2j}(z) \quad . \end{aligned} \tag{11}$$

We continue with reflection formulas of the determinants D_j . It turns out that the behaviour of D_1 is a little different to that of the other D_j . So we use the explicit formula of [VO 87]

$$D_1(z)=\frac{\sqrt{2\pi}}{\Gamma(1-z)}$$

to derive

$$D_1(z)\cdot D_1(1-z)=2\sin\pi z \quad .$$

Now we rewrite results of N. Kurokawa stated in [KU 91]. There he introduced the meromorphic function

$$F_j(z) \stackrel{df}{=} \exp\left(\frac{z^{j-1}}{j-1}\right) \cdot \Delta_j(z) \cdot \Big(\Delta_j(-z)\Big)^{(-1)^{j-1}} \quad , \quad j \geq 2 \quad .$$

It satisfies the functional equation

$$F_j'(z) = \pi z^{j-1} \cot(\pi z) F_j(z) \quad , \tag{12}$$

which is obviously equivalent to

$$F_j(z) = \exp\left\{ \int_0^z \pi t^{j-1} \cot(\pi t)\, dt \right\} \quad .$$

Thereby the integral is an contour integral in the complex λ-plane (the meromorphic integrand has only simple poles with *integer* residues, hence the integral is determined up to a multiple of $2\pi i$, and the term on the right side is well-defined). For a proof N. Kurokawa referred to [KU Pre]. Unfortunately this paper seems to be out of reach here, so we prove (12) without knowing its contents.

First observe that it suffices to prove the equality

$$\pi z^{j-1} \cos(\pi z) = \sin(\pi z) \cdot \frac{d}{dz} \log(F_j(z))$$

for $z \in \mathbb{C}$ with $|z| < 1$ because both sides are well-defined holomorphic functions on the complex plane. Inserting the power series representation of sin and cos and the definition of F_j (and Δ_j therein) we have to prove

$$\begin{aligned}\sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k+1} z^{2k+j-1}}{(2k)!} \quad &= \quad \left[\sum_{k=0}^{\infty} (-1)^k \frac{\pi^{2k+1} z^{2k+1}}{(2k+1)!} \right] \Bigg[z^{j-2} \\ &+ \sum_{l=1}^{\infty} l^{j-1} \frac{d}{dz} \left(\log\left((1-\frac{z}{l})(1+\frac{z}{l})^{(-1)^{j-1}} \right) + \sum_{m=1}^{j} (1+(-1)^{j-1+m}) \frac{z^m}{m\, l^m} \right) \Bigg]\end{aligned}$$

what is equivalent to

$$\begin{aligned}\sum_{k=1}^{\infty} (-1)^k \frac{2k\pi^{2k+1} z^{2k+j-1}}{(2k+1)!} \quad &= \quad \left[\sum_{i=0}^{\infty} (-1)^i \frac{\pi^{2i+1} z^{2i+1}}{(2i+1)!} \right] \\ &\left[\sum_{l=1}^{\infty} l^{j-1} \frac{d}{dz} \left(\log\left((1-\frac{z}{l})(1+\frac{z}{l})^{(-1)^{j-1}} \right) + \sum_{m=1}^{j} (1+(-1)^{j-1+m}) \frac{z^m}{m\, l^m} \right) \right] \quad .\end{aligned}$$

For $|z/l| \leq |z| < 1$ we can use the power series of $\log(1 \pm z/l)$ so we have to show

$$\begin{aligned}\sum_{k=1}^{\infty} (-1)^k \frac{2k\pi^{2k+1} z^{2k+j-1}}{(2k+1)!} \quad &= \quad \left[\sum_{i=0}^{\infty} (-1)^i \frac{\pi^{2i+1} z^{2i+1}}{(2i+1)!} \right] \\ &\left[\sum_{l=1}^{\infty} l^{j-1} \frac{d}{dz} \left(- \sum_{m=j+1}^{\infty} (1+(-1)^{j-1+m}) \frac{z^m}{m\, l^m} \right) \right]\end{aligned}$$

We transform the right side of this equation

$$\begin{aligned}
r.s. &= \sum_{i=0}^{\infty}\sum_{l=1}^{\infty}(-1)^{i+1}\frac{\pi^{2i+1}z^{2i+1}}{(2i+1)!}\sum_{m=1}^{\infty}\left((1+(-1)^{m-1})\frac{z^{m+j-1}}{l^{m+1}}\right)\\
&= \sum_{i=0}^{\infty}\sum_{l=1}^{\infty}(-1)^{i+1}\frac{\pi^{2i+1}z^{2i+1}}{(2i+1)!}\sum_{m=1}^{\infty}\left(2\frac{z^{2m+j-2}}{l^{2m}}\right)\\
&= \sum_{i=0}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}(-1)^{i+1}\frac{2\pi^{2i+1}z^{2m+2i+j-1}}{(2i+1)!l^{2m}}\\
&= \sum_{k=1}^{\infty}\sum_{m=1}^{k}(-1)^{k-m+1}\frac{2\pi^{2k-2m+1}z^{2k+j-1}}{(2k-2m+1)!}\sum_{l=1}^{\infty}l^{-2m}\\
&= \sum_{k=1}^{\infty}\sum_{m=1}^{k}(-1)^{k-m+1}\frac{2\pi^{2k-2m+1}\zeta(2m)}{(2k-2m+1)!}z^{2k+j-1} \quad .
\end{aligned}$$

Using the well-known relation for the Riemann zeta function

$$\zeta(1-s)\Gamma(\frac{1-s}{2})\pi^{(s-1)/2} = \zeta(s)\Gamma(\frac{s}{2})\pi^{-s/2}$$

and again

$$\zeta(1-2m) = -\frac{B_{2m}}{2m} \qquad (m \geq 1)$$

this changes into

$$\begin{aligned}
r.s. &= \sum_{k=1}^{\infty}\sum_{m=1}^{k}(-1)^{k}\frac{\pi^{2k+1}2^{2m}B_{2m}}{(2k-2m+1)!(2m)!}z^{2k+j-1}\\
&= \sum_{k=1}^{\infty}(-1)^{k}\frac{\pi^{2k+1}}{(2k+1)!}\left(\sum_{m=1}^{k}\binom{2k+1}{2m}2^{2m}B_{2m}\right)z^{2k+j-1} \quad .
\end{aligned}$$

To conclude the proof we rewrite a relation between the Bernoulli numbers which we claim in the next subsection using $B_{2i+1} = 0\,(i \geq 1)$.

$$\begin{aligned}
0 &= \sum_{i=0}^{2k+1}\binom{2k+1}{i}2^{i}B_{i}\\
&= 1+(2k+1)2(-\frac{1}{2})+\sum_{i=1}^{k}\binom{2k+1}{2i}2^{2i}B_{2i}\\
2k &= \sum_{i=1}^{k}\binom{2k+1}{2i}2^{2i}B_{2i}
\end{aligned}$$

(Surely there will be a direct proof of it using only its inductive definition.)
□

From (10) and (11) resp. we derive for $j \geq 1$

$$\begin{aligned}
F_{2j+1}(z) &= \frac{D_{2j+1}(z)\,D_{2j+1}(-z)}{D_{2j+1}(0)^2} \quad ,\\
F_{2j}(z) &= \frac{D_{2j}(z)}{D_{2j}(-z)} \quad .
\end{aligned}$$

So it holds for every $j \geq 2$

$$\frac{D_j(z)}{D_j(0)} \cdot \left(\frac{D_j(-z)}{D_j(0)}\right)^{(-1)^{j-1}} = \exp\left\{\int_0^z \pi t^{j-1} \cot(\pi t)\, dt\right\}$$

To derive asymptotic expansions recall (8) and insert the given values for $c_k^{(j)}$. We comprehence the proved properties in

Lemma 2 (Properties of the D_j)

1. **Reflection**

   (a) $$D_1(z) \cdot D_1(1-z) = 2\sin \pi z$$

   (b) $$\frac{D_j(z)}{D_j(0)} \cdot \left(\frac{D_j(-z)}{D_j(0)}\right)^{(-1)^{j-1}} = \exp\left\{\int_0^z \pi t^{j-1} \cot(\pi t)\, dt\right\} \quad (j \geq 2)$$

2. **Asymptotic behaviour** $(j \geq 1)$

$$-\log D_1(z) \overset{z\to-\infty}{\sim} \sum_{k=1}^{\infty} \frac{B_{2k}(-z)^{1-2k}}{2k(2k-1)} + \frac{\log(-z)}{2} - (\log(-z)-1)z$$

$$-\log D_{2j}(z) \overset{z\to-\infty}{\sim} -\sum_{k=1}^{\infty} \frac{B_{2k+2j}(-z)^{-2k}}{(2k+2j)2k} + \frac{B_{2j}}{2j}(\log(-z)) - \left(\log(-z) - \sum_{r=1}^{2j} \frac{1}{r}\right) \frac{z^{2j}}{2j}$$

$$-\log D_{2j+1}(z) \overset{z\to-\infty}{\sim} \sum_{k=1}^{\infty} \frac{B_{2k+2j}(-z)^{1-2k}}{(2k+2j)(2k-1)} - \left(\log(-z) - \sum_{r=1}^{2j+1} \frac{1}{r}\right) \frac{z^{2j+1}}{2j+1}$$

□

1.3 Shifted elementary sequences

Lets consider the (so-called shifted elementary) sequences

$$\left\{k + \frac{1}{2} \quad \text{with multiplicity} \quad (2k+1)^{j-1} \right\}_{k=0}^{\infty} \quad , \; j = 1, 2, \ldots \; , \tag{13}$$

and denote the associated spectral functions by $\theta_j^+, D_j^+, \ldots$ resp. .
First we look at the Zeta function

$$\begin{aligned} Z_j^+(s) &= \sum_{k=0}^{\infty} (2k+1)^{j-1}(k+\tfrac{1}{2})^{-s} \\ &= \left(2^s - 2^{j-1}\right) \zeta(s-j+1) \end{aligned}$$

This gives us the coefficients $c_{i_k}^{j,+}$ because they are also the residues at the points $s=-i_k$ of the product $Z_j^+(s)\,\Gamma(s)$ (this fact is due to the already mentioned relation between this product and $\theta_j^+(t)$ via Mellin transform (cf. [VO 87])). The product only has poles at $s=-m$ and $s=j$ with residues $Z_j^+(-m)\cdot\dfrac{(-1)^m}{m!}$ and $2^{j-1}\,\Gamma(j)$ resp. . So we find the coefficients $c_k^{j,+}$ as

$$\begin{aligned}
c_{-2j-1}^{2j-1,+} &= 2^{2j-2}\Gamma(2j-1) \qquad , \qquad c_{2k}^{2j-1,+}=0 \quad (k\geq 0)\\
c_{2k+1}^{2j-1,+} &= -\frac{B_{2k+2j}}{(2k+2j)}\cdot\frac{2^{2k+2j-1}-1}{2^{2k+1}(2k+1)!} \quad (k\geq 0)\\
c_{-2j}^{2j,+} &= 2^{2j-1}\Gamma(2j) \qquad , \qquad c_{2k+1}^{2j,+}=0 \quad (k\geq 0)\\
c_{2k}^{2j,+} &= \frac{B_{2k+2j}}{(2k+2j)}\cdot\frac{2^{2k+2j-1}-1}{2^{2k}(2k)!} \quad (k\geq 0)
\end{aligned}$$

with B_k the Bernoulli numbers. Hence the order of the sequences (13) is also j. Now we look at $\theta_j^+(t)$.

$$\begin{aligned}
\theta_j^+(t) &= \sum_{k=1}^{\infty}(2k-1)^{j-1}\mathrm{e}^{-t\left(k-\frac{1}{2}\right)}\\
&= 2^{j-1}\left(-\frac{d}{dt}\right)^{j-1}\left(\mathrm{e}^{\frac{t}{2}}\cdot\frac{1}{\mathrm{e}^t-1}\right)\\
&= 2^{j-1}\left(-\frac{d}{dt}\right)^{j-1}\left[\frac{1}{t}+\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k+1}2^iB_i\binom{k+1}{i}\right)\frac{t^k}{2^{k+1}(k+1)!}\right]\\
&= 2^{j-1}\left[\frac{\Gamma(j)}{t^j}+(-1)^{j-1}\sum_{k=0}^{\infty}\left(\sum_{i=0}^{k+j}2^iB_i\binom{k+j}{i}\right)\frac{t^k}{2^{k+j}(k+j)k!}\right]
\end{aligned}$$

Comparing with the already calculated c_{i_k} we have got a relation between the B_k using $B_{2k+1}=0\ (k\geq 1)$.

$$\sum_{i=0}^{k}\binom{k}{i}2^iB_i = \begin{cases} -B_k\left(2^k-2\right) & k \text{ even}\\ 0 & k \text{ odd}\end{cases} .$$

We used this to prove (12) in the previous subsection.
Our next task is to express the determinants D_j^+ in terms of the D_i to derive a reflection formula. Therefore we look at the defining sequence

$$\begin{aligned}
&\left\{\ k+\frac{1}{2}\text{ with multiplicity }\quad(2k+1)^{j-1}\right\}_{k=0}^{\infty}\\
=&\left\{\ k-\frac{1}{2}\text{ with multiplicity }\quad(2k-1)^{j-1}\right\}_{k=1}^{\infty}\\
=&\left\{\frac{1}{2}\right\}\cup\left\{\ k+\frac{1}{2}\text{ with multiplicity }\quad(2k+1)^{j-1}\right\}_{k=1}^{\infty}
\end{aligned} \tag{14}$$

With Lemma 1 we get from the second line immediately

$$D_j^+(z)=\prod_{i=1}^{j}\left(D_i\left(z+\frac{1}{2}\right)\right)^{\binom{j-1}{i-1}2^{i-1}(-1)^{j-i}} \tag{15}$$

and from the third line of (14) we get the factorisation

$$D_j^+(z) = \left(\frac{1}{2} - z\right) \cdot \prod_{i=1}^{j} \left(D_i\left(z - \frac{1}{2}\right)\right)^{\binom{j-1}{i-1} 2^{i-1}} . \tag{16}$$

With these two factorisations we derive

$$\begin{aligned} D_j^+(z) \cdot \left(D_j^+(-z)\right)^{(-1)^{j-1}} &= \left(\left(\frac{1}{2} - z\right) \prod_{i=1}^{j} \left(D_i\left(z - \frac{1}{2}\right)\right)^{\binom{j-1}{i-1}2^{i-1}}\right) \\ &\quad \cdot \left(\prod_{i=1}^{j} \left(D_i\left(\frac{1}{2} - z\right)\right)^{\binom{j-1}{i-1}2^{i-1}(-1)^{j-i}}\right)^{(-1)^{j-1}} \end{aligned}$$

$$\begin{aligned} &= D_1\left(z + \frac{1}{2}\right) D_1\left(\frac{1}{2} - z\right) \prod_{i=2}^{j} \left[D_i\left(z - \frac{1}{2}\right)\left[D_i\left(\frac{1}{2} - z\right)\right]^{(-1)^{i-1}}\right]^{\binom{j-1}{i-1}2^{i-1}} \\ &= \cos(\pi z) \cdot \prod_{i=2}^{j} \left[D_i\left(z - \frac{1}{2}\right) \cdot \left(D_i\left(\frac{1}{2} - z\right)\right)^{(-1)^{i-1}}\right]^{\binom{j-1}{i-1}2^{i-1}} \end{aligned}$$

By division by the same equation at $z = 0$ we get with Lemma 2

$$\begin{aligned} &\frac{D_j^+(z)}{D_j^+(0)} \cdot \left(\frac{D_j^+(-z)}{D_j^+(0)}\right)^{(-1)^{j-1}} \\ &= \cos(\pi z) \cdot \prod_{i=2}^{j} \left[\frac{D_i\left(z - \frac{1}{2}\right)}{D_i\left(-\frac{1}{2}\right)} \cdot \left(\frac{D_i\left(\frac{1}{2} - z\right)}{D_i\left(\frac{1}{2}\right)}\right)^{(-1)^{i-1}}\right]^{\binom{j-1}{i-1}2^{i-1}} \\ &= \cos(\pi z) \cdot \exp\left\{\int_{-\frac{1}{2}}^{z-\frac{1}{2}} \pi \left(\sum_{i=2}^{j} \binom{j-1}{i-1} (2t)^{i-1}\right) \cot(\pi t)\, dt\right\} \\ &= \cos(\pi z) \cdot \exp\left\{-\int_0^z \pi \left(\sum_{i=1}^{j-1} \binom{j-1}{i} (2t-1)^i\right) \tan(\pi t)\, dt\right\} \\ &= \exp\left\{-\int_0^z \pi (2t)^{j-1} \tan(\pi t)\, dt\right\} \end{aligned}$$

Inserting also the explicitely known quantities in (3) and (8) we have proven

Lemma 3 (Properties of the D_j^+ , $j \geq 1$)

1. **Relation with Δ_j^+**

$$D_{2j}^+(z) = \exp\left\{-\left(Z_{2j}^+\right)'(0) + \sum_{k=1}^{j-1} \frac{2^{2k} - 2^{2j-1}}{2k} \cdot \frac{B_{2j-2k}}{2j-2k} \cdot z^{2k}\right.$$

$$
\begin{aligned}
&\qquad\qquad -2^{2j-1}\left(\gamma+\sum_{r=1}^{2j-1}\frac{1}{r}\right)\frac{z^{2j}}{2j}\Bigg\}\cdot\Delta^{+}_{2j}(z) \qquad (17)\\
D^{+}_{2j-1}(z) &= \exp\left\{-\left(Z^{+}_{2j-1}\right)'(0)+\sum_{k=0}^{j-1}\frac{2^{2k+1}-2^{2j-2}}{2k+1}\cdot\frac{B_{2j-2k}}{2j-2k}\cdot z^{2k+1}\right.\\
&\qquad\qquad \left.-2^{2j-2}\left(\gamma+\sum_{r=1}^{2j-2}\frac{1}{r}\right)\frac{z^{2j-1}}{2j-1}\right\}\cdot\Delta^{+}_{2j-1}(z)
\end{aligned}
$$

2. **Reflection**

$$
\frac{D^{+}_{j}(z)}{D^{+}_{j}(0)}\cdot\left(\frac{D^{+}_{j}(-z)}{D^{+}_{j}(0)}\right)^{(-1)^{j-1}} = \exp\left\{-\int_{0}^{z}\pi(2t)^{j-1}\tan(\pi t)\,dt\right\}
$$

3. **Asymptotic behaviour** $(j \geq 1)$

$$
\begin{aligned}
-\log D^{+}_{2j-1}(z) &\overset{z\to-\infty}{\sim} \sum_{k=0}^{\infty}\frac{B_{2k+2j}}{2k+2j}\cdot\frac{2^{2j-2}-2^{-2k-1}}{(2k+1)(-z)^{2k+1}}\\
&\qquad -2^{2j-2}\left(\log(-z)-\sum_{r=1}^{2j-1}\frac{1}{r}\right)\frac{z^{2j-1}}{2j-1}\\
-\log D^{+}_{2j}(z) &\overset{z\to-\infty}{\sim} -\sum_{k=1}^{\infty}\frac{B_{2k+2j}}{2k+2j}\cdot\frac{2^{2j-1}-2^{-2k}}{2k(-z)^{2k}}\\
&\quad +\left(2^{2j-1}-1\right)\frac{B_{2j}}{2j}(\log(-z))-2^{2j-1}\left(\log(-z)-\sum_{r=1}^{2j}\frac{1}{r}\right)\frac{z^{2j}}{2j}
\end{aligned}
$$

□

1.4 Sequences with polynomial multiplicities, Reflection formula

Now we apply the results for the two types of elementary sequences to the functional determinant D associated to a sequence

$$
\begin{cases}
\{\ k \text{ with multiplicity } Q(k)\ \}_{k=1}^{\infty} & \text{(case of integers, } \mathbb{Z}\text{)}\\
\left\{\ k+\frac{1}{2} \text{ with multiplicity } Q\left(k+\frac{1}{2}\right)\ \right\}_{k=0}^{\infty} & \text{(case of half integers, } \mathbb{Z}+\frac{1}{2}\text{)}
\end{cases}
$$

where Q is an odd polynomial of degree 2m-1, hence of the form

$$
Q(z)=\sum_{j=1}^{m} b_{j-1}\, z^{2j-1}
$$

and with *integer values* at $\mathbb{Z}$ and $\mathbb{Z}+\frac{1}{2}$ resp. . If we want to comprehence both cases we denote the sequence by

$$
\{\ \lambda_k \text{ with multiplicity } m_k\ \}_{k=0}^{\infty}
$$

with $m_k \in \mathbb{Z}$. From this definition we get the factorisation

$$D(z) = \begin{cases} \prod_{j=1}^{m} \left(D_{2j}(z)\right)^{b_{j-1}} & (\mathbb{Z}) \\ \prod_{j=1}^{m} \left(D_{2j}^{+}(z)\right)^{b_{j-1}2^{1-2j}} & (\mathbb{Z}+\frac{1}{2}) \end{cases} .$$

Applying Lemma 2 ($\mathbb{Z}$) and Lemma 3 ($\mathbb{Z}+\frac{1}{2}$) resp. we find

$$\begin{aligned} \frac{D(z)}{D(-z)} &= \begin{cases} \exp\left\{\int_0^z \pi \left(\sum_{j=1}^{m} b_{j-1} t^{2j-1}\right) \cot(\pi t)\, dt\right\} & (\mathbb{Z}) \\ \exp\left\{-\int_0^z \pi \left(\sum_{j=1}^{m} \frac{b_{j-1}}{2^{2j-1}} (2t)^{2j-1}\right) \tan(\pi t)\, dt\right\} & (\mathbb{Z}+\frac{1}{2}) \end{cases} \\ &= \begin{cases} \exp\left\{\int_0^z \pi Q(t) \cot(\pi t)\, dt\right\} & (\mathbb{Z}) \\ \exp\left\{-\int_0^z \pi Q(t) \tan(\pi t)\, dt\right\} & (\mathbb{Z}+\frac{1}{2}) \end{cases} \end{aligned}$$

Thereby one has to look carefully at the integrality of the residues of the integrands in every term of the form

$$\exp\left\{\int_a^b f(t)\, dt\right\} .$$

So first one can only take an integer power of the determinants D_j and D_j^{+}, so we got the upper equations only taken to a certain integer power. But both sides are defined also without this exponent and equal at $z = 0$.

We state the final result of this section.

Lemma 4 (Properties of D)

1. **Reflection formula**

$$\frac{D(z)}{D(-z)} = \exp\left\{\int_0^z \pi\, Q(t) \begin{Bmatrix} \cot(\pi t) \\ -\tan(\pi t) \end{Bmatrix} dt\right\} \quad \begin{cases} (\mathbb{Z}) \\ (\mathbb{Z}+\frac{1}{2}) \end{cases}$$

$$\frac{d}{dz} \log \frac{D(z)}{D(-z)} = \begin{cases} \pi Q(z) \cot(\pi z) & (\mathbb{Z}) \\ -\pi Q(z) \tan(\pi z) & (\mathbb{Z}+\frac{1}{2}) \end{cases}$$

2. **Symmetrisation**

$$\left(-\frac{d}{dz^2}\right)^{m+1} \left(-\log D(z) D(-z)\right) = \sum_{k=0}^{\infty} \frac{m_k\, m!}{\left(z^2 - \lambda_k^2\right)^{m+1}}$$

3. **Asymptotic expansion for** $z \to -\infty$

$$-\log D(z) \quad \sim \quad \sum_{k=1}^{m} b_{k-1}\left(\log(-z) - \sum_{r=1}^{2k}\frac{1}{r}\right)\frac{z^{2k}}{2k}$$

$$\begin{cases} +\displaystyle\sum_{j=1}^{m} b_{j-1}\left(\frac{B_{2j}}{2j}\log(-z) - \sum_{k=1}^{\infty}\frac{B_{2k+2j}(-z)^{-2k}}{(2k+2j)2k}\right) & (\mathbb{Z}) \\ \\ -\displaystyle\sum_{j=1}^{m} b_{j-1}\left(1-2^{1-2j}\right)\left(\frac{B_{2j}}{2j}\log(-z) + \sum_{k=1}^{\infty}\frac{B_{2k+2j}(-z)^{-2k}}{(2k+2j)2k}\right) & \\ & (\mathbb{Z}+1/2) \end{cases}$$

Proof. The reflection formula is already proved. For the asymptotic expansions one uses

$$-\log D(z) = \begin{cases} \displaystyle\sum_{j=1}^{m} b_{j-1}\left(-\log D_{2j}(z)\right) & (\mathbb{Z}) \\ \displaystyle\sum_{j=1}^{m} \frac{b_{j-1}}{2^{2j-1}}\left(-\log D^{+}_{2j}(z)\right) & (\mathbb{Z}+\frac{1}{2}) \end{cases}$$

and the results for the determinants occuring on the right side.
Also one has (cf. (11) and (17)) for any m with $m \geq j \geq 1$

$$\left(-\frac{d}{dz^2}\right)^{m+1}\left(-\log D_{2j}(z)\right) = \left(-\frac{d}{dz^2}\right)^{m+1}\left(-\log \Delta_{2j}(z)\right)$$

and

$$\left(-\frac{d}{dz^2}\right)^{m+1}\left(-\log D^{+}_{2j}(z)\right) = \left(-\frac{d}{dz^2}\right)^{m+1}\left(-\log \Delta^{+}_{2j}(z)\right)$$

and by an easy calculation on

$$\Delta_{2j}(z) = \prod_{k=1}^{\infty}\left[\left(1-\frac{z}{k}\right)\cdot\exp\left\{\sum_{r=1}^{2j}\frac{z^r}{r\cdot k^r}\right\}\right]^{k^{2j-1}}$$

and

$$\Delta^{+}_{2j}(z) = \prod_{k=0}^{\infty}\left[\left(1-\frac{z}{k+\frac{1}{2}}\right)\cdot\exp\left\{\sum_{r=1}^{2j}\frac{z^r}{r\cdot(k+\frac{1}{2})^r}\right\}\right]^{(2k+1)^{2j-1}}$$

one gets

$$\left(-\frac{d}{dz^2}\right)^{m+1}\left(-\log\left[\Delta_{2j}(z)\,\Delta_{2j}(-z)\right]\right) \quad = \quad \sum_{k=1}^{\infty}\left(-\frac{d}{dz^2}\right)^{m}\frac{k^{2j-1}}{\left(z^2-k^2\right)}$$

and

$$\left(-\frac{d}{dz^2}\right)^{m+1}\left(-\log\left[\Delta^{+}_{2j}(z)\,\Delta^{+}_{2j}(-z)\right]\right) \quad = \quad \sum_{k=0}^{\infty}\left(-\frac{d}{dz^2}\right)^{m}\frac{(2k+1)^{2j-1}}{\left(z^2-(k+\frac{1}{2})^2\right)}$$

Combining these relations one gets

$$\left(-\frac{d}{dz^2}\right)^{m+1}(-\log D(z)D(-z)) = \begin{cases} \displaystyle\sum_{k=1}^{\infty} \frac{Q(k)\; m!}{\left(z^2-k^2\right)^{m+1}} & (\mathbb{Z}) \\ \displaystyle\sum_{k=0}^{\infty} \frac{Q\left(k+\frac{1}{2}\right)\; m!}{\left(z^2-\left(k+\frac{1}{2}\right)^2\right)^{m+1}} & (\mathbb{Z}+\frac{1}{2}) \end{cases}$$

and as a comprehension of both cases equality 2. $\square$

2 Application to Selberg zeta functions

2.1 Standard results concerning rank one Lie groups

Let G be a noncompact real, connected and semisimple Lie group with real Lie algebra g_0 and its complexification g. We denote by B the Killing form on g

$$B(X,Y) = \text{trace}(\text{ad}X \circ \text{ad}Y)\ , \quad X,Y \in g \quad .$$

Let K be a maximal compact subgroup (k_0 its real Lie algebra, k its complexification). We have the Cartan Killing decomposition $g = k \oplus p$, which is the eigenspace decomposition with respect to the Cartan involution $\theta : g \to g$. With the help of B and θ we define a scalar product on g_0

$$\langle X,Y\rangle_\theta = -B(X,\theta Y)\ , \quad X,Y \in g_0 \quad .$$

We assume once and for all that the real rank of G is one. Hence a maximal abelian subalgebra a_0 of p_0 has dimension one.
So there are exactly the cases

1. $G = SO(1,2n)^\circ$ (case $\mathbb{R}$)
2. $G = SU(1,n)$ (case $\mathbb{C}$)
3. $G = Sp(1,n)$ (case $\mathbb{H}$)
4. $G = F_{4(-20)}$ (case $\mathbb{C}$a) .

Up to conjugacy in G we have exactly two Cartan subgroups of G. The Lie subgroup $A \cong \mathbb{R}$ corresponding to a_0 and a maximal torus T of the centralizer M of A in K give us a noncompact Cartan subgroup H of G

$$H = TA \quad .$$

A maximal abelian subgroup $H_c \subset K$ is a compact Cartan subgroup of G. H and H_c are joined via the Cayley transform $H \to H_c$. It induces an isomorphism of the corresponding Lie algebras $h \cong h_c$. Now we decompose g into rootspaces w.r.t. the root systems $\Delta_c = \Delta(g,h_c)$ and $\Delta = \Delta(g,h)$. That is for

$$g_\lambda \stackrel{df}{=} \left\{X \in g \;\middle|\; [H,X] = \lambda(H)\,X \quad \forall H \in h\right\}\ ,\ \lambda \in h^*\ ,$$

we have $\Delta \overset{df}{=} \left\{ \lambda \in h^* \;\middle|\; g_\lambda \neq 0 \;\right\}$ and this decomposition into root spaces reads

$$g = h \oplus \left(\bigoplus_{\lambda \in \Delta} g_\lambda \right) \quad .$$

In our cases the dimension of the g_λ are at most one (cf. [DI 77]). We choose H_c as $H_c = TA_c$ with $A_c \cong S^1$ 1-dimensional and introduce compatible linear orderings for Δ and Δ_c. The positive root systems w.r.t. that orderings are denoted by Δ^+ and Δ_c^+ resp. . For the half sum of the positive roots we use the symbols

$$\begin{aligned}
\rho_G &\overset{df}{=} \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha \quad = \quad \rho_A + \rho_M \\
\rho_M &\overset{df}{=} \frac{1}{2} \sum_{\alpha \in \Delta_M^+} \alpha \quad , \quad \Delta_M^+ = \left\{ \alpha \in \Delta^+ \,|\, \alpha\big|_a = 0 \right\} \quad , \\
\rho_A &\overset{df}{=} \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha \quad , \quad \Delta_1^+ = \Delta^+ \backslash \Delta_M^+ \quad .
\end{aligned}$$

Let δ be the real root, that is the unique root in Δ^+ giving real values on h_0. In our cases we have $\Delta_1^+|_a = \{\beta\} = \{\lambda\}$ ($\mathbb{R}$) or $\Delta_1^+|_a = \{\beta, 2\beta\} = \{\lambda/2, \lambda\}$ ($\mathbb{C}, \mathbb{H}, \mathbb{C}\mathrm{a}$). β is called the short restricted root and λ the long restricted root. It holds $\delta|_a = \lambda$ and $\rho_A = d \cdot \delta$ for a certain number $d \in \mathbb{Z}/2$. Now we introduce the spaces that are of interest here

$$X \overset{df}{=} \Gamma \backslash G/K \quad , \quad Y \overset{df}{=} \Gamma \backslash G/H_c$$

Constants used very often are

$$\begin{aligned}
m &\overset{df}{=} \frac{1}{2} \dim_{\mathbb{R}}(X) \\
m_2 &\overset{df}{=} \frac{1}{2} \dim_{\mathbb{R}}(Y) = \# \Delta^+ \\
p &\overset{df}{=} \#\{\alpha \in \Delta_1^+ : \quad \alpha|_a = \beta\} \\
q &\overset{df}{=} \#\{\alpha \in \Delta_1^+ : \quad \alpha|_a = 2\beta\}
\end{aligned}$$

In order to get concrete results we realize the introduced Lie groups and Lie algebras as matrix groups as usual with

$$E_k^l \overset{df}{=} \begin{pmatrix} I_l & 0 \\ 0 & -I_k \end{pmatrix} \in GL(k+l, \mathbb{R})$$

$$\begin{aligned}
SO(1,2n)^\circ &= C.C.I. \left\{ A \in GL(2n+1, \mathbb{R}) \;\middle|\; A^T E_{2n}^1 A = E_{2n}^1 \;\right\} \\
SU(1,n) &= \left\{ A \in GL(n+1, \mathbb{C}) \;\middle|\; A^T E_n^1 \overline{A} = E_n^1 \quad , \quad \det A = 1 \;\right\} \\
Sp(1,n) &= \left\{ A \in GL(2n+2, \mathbb{C}) \;\middle|\; A^T \begin{pmatrix} E_n^1 & 0 \\ 0 & E_n^1 \end{pmatrix} \overline{A} = \begin{pmatrix} E_n^1 & 0 \\ 0 & E_n^1 \end{pmatrix}, \right. \\
&\qquad \left. A^T \begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix} A = \begin{pmatrix} 0 & I_{n+1} \\ -I_{n+1} & 0 \end{pmatrix} \right\}
\end{aligned}$$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{C}$a
G	$SO(1,2n)^\circ$	$SU(1,n)$	$Sp(1,n)$	$F_{4(-20)}$
K	$SO(2n)$	$U(n)$	$Sp(1)\times Sp(n)$	$Spin(9)$
H	$SO(2)^n$	$U(1)^n$	$U(1)^{n+1}$	$SO(2)^4$
M	$SO(2n-1)$	2fold $[U(n-1)]$	$Sp(1)\times Sp(n-1)$	$Spin(7)$
T	$SO(2)^{n-1}$	$U(1)^{n-1}$	$U(1)^n$	$SO(2)^3$
$dim_{\mathbb{R}} G$	$n(2n+1)$	n^2+2n	$(n+1)(2n+3)$	52
$dim_{\mathbb{R}} K$	$n(2n-1)$	n^2	$3+n(2n+1)$	36
$dim_{\mathbb{R}} X$	$2n$	$2n$	$4n$	16
m	n	n	$2n$	8
$m_2=\#\Delta^+$	n^2	$n(n+1)/2$	$(n+1)^2$	24
$\#\Delta_M^+$	$(n-1)^2$	$(n-2)(n-1)/2$	$(n-1)^2+1$	9
$\#\Delta_1^+$	$2n-1$	$2n-1$	$4n-1$	15
p	$2n-1$	$2n-2$	$4n-4$	8
q	0	1	3	7
$2p+8q$	$2(2n-1)$	$4(n+1)$	$8(n+2)$	72
$p/2+q$	$n-1/2$	n	$2n+1$	11
d	$n-1/2$	$n/2$	$n+1/2$	$11/2$

Table I

F_4 has to be handled abstractly (see e.g. [CA 14]).

Let $E_{i,j}$ denote the elementary quadratic matrix with 1 in the ij th place and zeroes elsewhere (the dimension will be clear in all cases and is omitted). We summarize the results concerning bases of Lie algebras, Killing form, root decompositions and some needed constants (which are all standard of course) in table I and II.

In table II we also show the explicit form of the Plancherel density

$$\mu_\sigma(it\lambda) = C_m \cdot d_\sigma \cdot Q_\sigma(t) \cdot \phi_\sigma(t) \quad ,$$

associated to the principal series representation with coordinates $\sigma \in \hat{M}$ and $it\lambda \in a^*$ (a^* parametrized by the long restricted root λ) which was calculated for our cases by R. Miatello in [MI 79] w.r.t. a normalisation of the Haar measure of G given there. Here $Q_\sigma(t)$ is a monic polynomial, d_σ is the dimension of the representation σ

$$d_\sigma = \prod_{\alpha\in\Delta_M^+} \frac{(\alpha, \Lambda_\sigma+\rho_M)}{(\alpha,\rho_M)}$$

and in all cases we have $\quad C_m = \dfrac{(-1)^m}{\pi^m\Gamma(m)} \quad$ and $\quad \phi_\sigma(t) \in \begin{cases} \tan(\pi t) \\ -\cot(\pi t)\end{cases}$.

Volumes.

We complete this introduction with the relation between the normalisations of the Haar measure of G from [GA 77], [SCHM 71] and [MI 79] and our normalisation called *Standard* due to the used standard matrix realisation. Therefore we compare the forms ω on the Lie algebras which define by G-invariant continuation (and projection in the case of $\Gamma\backslash G$) the volume forms.

1. $\underline{\mathrm{vol}_{\mathrm{Standard}}}$ We consider a basis $\{b_j\}$ of the Lie algebra g_0 with

$$\langle b_j, b_k\rangle_\theta = \delta_{jk}(2p+8q) \quad .$$

		$\mathbb{R}$	$\mathbb{C}$
1	Basis of k_0 $2 \leq j \leq n+1$ $j < k \leq n+1$	$K_{kj} \stackrel{df}{=} E_{kj} - E_{jk}$ $2 \leq j < k \leq 2n+1$	$K_{kj}^{(1)} \stackrel{df}{=} E_{kj} - E_{jk}$ $K_{kj}^{(2)} \stackrel{df}{=} iE_{kj} + iE_{jk}$ $K_j \stackrel{df}{=} iE_{jj} - iE_{11}$
2	Basis of p_0 $j = 2, \ldots, n+1$	$P_j^{(1)} \stackrel{df}{=} E_{j1} + E_{1j}$ $P_j^{(2)} \stackrel{df}{=} E_{j+n,1} + E_{1,j+n}$	$P_j^{(1)} \stackrel{df}{=} E_{j1} + E_{1j}$ $P_j^{(2)} \stackrel{df}{=} iE_{j1} - iE_{1j}$
3	Basis of a_0	$H_1 = P_2^{(1)}$	$H_1 = P_2^{(1)}$
4	Basis of t_0	$H_j = K_{2j+1,2j}$	$H_j = (2K_{j+1} - K_2)/\sqrt{3}$
5	Basis of $(a_c)_0$	$H_1^c = K_{3,2}$	$H_1^c = K_2$
6	Δ	$\{\pm\epsilon_j, \pm\epsilon_j \pm \epsilon_k\}_{j=1,k=j+1}^{n\quad n}$ $\epsilon_j \stackrel{df}{=} iH_j^*,\ j\neq 1$ $\epsilon_1 \stackrel{df}{=} H_1^*$ $\epsilon_1^c \stackrel{df}{=} i(H_1^c)^*$	$\{\pm(\epsilon_j - \epsilon_k)\}_{j=1,k=j+1}^{n\quad n+1}$ $\epsilon_j \stackrel{df}{=} (2i/\sqrt{3})H_j^*,\ j\neq 1, n+1$ $\epsilon_1 \stackrel{df}{=} H_1^* - \frac{1}{2}\sum_{j=2}^{n} \epsilon_j$ $\epsilon_{n+1} \stackrel{df}{=} -H_1^* - \frac{1}{2}\sum_{j=2}^{n} \epsilon_j$ $\epsilon_1^c \stackrel{df}{=} i(H_1^c)^* - \frac{1}{2}\sum_{j=2}^{n} \epsilon_j$ $\epsilon_{n+1}^c \stackrel{df}{=} -i(H_1^c)^* - \frac{1}{2}\sum_{j=2}^{n} \epsilon_j$
7	Δ^+	$\{\epsilon_j, \epsilon_j \pm \epsilon_k\}_{j=1,k=j+1}^{n\quad n}$	$\{\epsilon_j - \epsilon_k\}_{j=1,k=j+1}^{n\quad n+1}$
8	Δ_1^+	$\{\epsilon_1, \epsilon_1 \pm \epsilon_k\}_{k=2}^{n}$	$\{\epsilon_1 - \epsilon_j, \epsilon_j - \epsilon_{n+1}\}_{j=2}^{n}$ $\cup\{\epsilon_1 - \epsilon_{n+1}\}$
9	general highest weight Λ_σ	$\sum_{j=2}^{n} s_j\epsilon_j \quad , s_j \in \mathbb{Z}$ $s_2 \geq \ldots \geq s_n \geq 0$	$\sum_{j=2}^{n}(s_j + \frac{s}{2})\epsilon_j, s, s_j \in \mathbb{Z}$ $s_2 \geq \ldots \geq s_n = 0$
10	ρ_M	$\sum_{j=2}^{n}(n - j + \frac{1}{2})\,\epsilon_j$	$\sum_{j=2}^{n}(\frac{n}{2} - j + 1)\,\epsilon_j$
11	δ	$\epsilon_1 = H_1^*$	$\epsilon_1 - \epsilon_{n+1} = 2H_1^*$
12	$\phi_\sigma(t)$	$\tan(\pi t)$	$\tan(\pi t) \iff s + n$ odd
13	$Q_\sigma(t)$	$t\prod_{j=2}^{n}\left(t^2 - (s_j + n - j + \frac{1}{2})^2\right)$	$t\prod_{j=1}^{n-1}\left(t^2 - (s_j + \frac{s+n}{2} - j)^2\right)$

TABLE II.1

	$\mathbb{H}$
1	$K_{kj}^{(1)} \stackrel{df}{=} E_{kj} - E_{jk} + E_{n+1+k,n+1+j} - E_{n+1+j,n+1+k}$ $K_{kj}^{(2)} \stackrel{df}{=} i(E_{kj} + E_{kj} - E_{n+1+k,n+1+j} - E_{n+1+j,n+1+k})$ $K_{kj}^{(3)} \stackrel{df}{=} E_{n+1+k,j} + E_{n+1+j,k} - E_{k,n+1+j} - E_{j,n+1+k}$ $K_{kj}^{(4)} \stackrel{df}{=} i(E_{n+1+k,j} + E_{n+1+j,k} + E_{k,n+1+j} + E_{j,n+1+k})$ $\left.\begin{array}{l} K_j^{(1)} \stackrel{df}{=} i\sqrt{2}(E_{jj} - E_{n+1+j,n+1+j}) \\ K_j^{(2)} \stackrel{df}{=} \sqrt{2}(E_{n+1+j,j} - E_{j,n+1+j}) \\ K_j^{(3)} \stackrel{df}{=} i\sqrt{2}(E_{n+1+j,j} + E_{j,n+1+j}) \end{array}\right\} \quad 1 \leq j \leq n+1$
2	$P_j^{(1)} \stackrel{df}{=} E_{j1} + E_{1j} - E_{n+1+j,n+2} - E_{n+2,n+1+j}$ $P_j^{(2)} \stackrel{df}{=} i(E_{j1} - E_{1j} + E_{n+1+j,n+2} - E_{n+2,n+1+j})$ $P_j^{(3)} \stackrel{df}{=} E_{n+1+j,1} + E_{n+2,j} + E_{j,n+2} + E_{1,n+1+j}$ $P_j^{(4)} \stackrel{df}{=} i(E_{n+1+j,1} + E_{n+2,j} - E_{j,n+2} - E_{1,n+1+j})$
3	$H_1 = P_2^{(1)}$
4	$H_2 = (K_1^{(1)} + K_2^{(1)})/\sqrt{2}\,, \quad H_j = K_j^{(1)} \quad (j \geq 3)$
5	$H_1^c = (K_1^{(1)} - K_2^{(1)})/\sqrt{2}$
6	$\{\pm 2\epsilon_j, \pm\epsilon_j \pm \epsilon_k\}_{j=1\,,\,k=j+1}^{n+1\,,\,n+1}$, $\epsilon_j \stackrel{df}{=} i\sqrt{2}H_j^* \quad (j \neq 1,2)$ $\epsilon_1 \stackrel{df}{=} H_1^* + iH_2^*, \epsilon_2 \stackrel{df}{=} H_1^* - iH_2^*$ $\epsilon_1^c \stackrel{df}{=} i(H_1^c)^* + iH_2^*, \epsilon_2^c \stackrel{df}{=} i(H_1^c)^* - iH_2^*$
7	$\{2\epsilon_j, \epsilon_j \pm \epsilon_k\}_{j=1\,,\,k=j+1}^{n+1\,,\,n+1}$
8	$\{2\epsilon_1, 2\epsilon_2, \epsilon_1 + \epsilon_2\} \cup \{\epsilon_1 \pm \epsilon_k, \epsilon_2 \pm \epsilon_k\}_{k=3}^{n+1}$
9	$\sum_{j=3}^{n+1} s_j\epsilon_j + s\cdot(\epsilon_1 - \epsilon_2) \quad , \quad s \in \frac{\mathbb{Z}}{2}, s \geq 0, s_j \in \mathbf{Z}, s_3 \geq \ldots \geq s_{n+1} \geq 0$
10	$\frac{1}{2}(\epsilon_1 - \epsilon_2) + \sum_{j=3}^{n+1}(n+2-j)\epsilon_j$
11	$\epsilon_1 + \epsilon_2 = 2H_1^*$
12	$\tan(\pi t) \iff s \in \mathbf{Z}$
13	$t\left(t^2 - (s+\frac{1}{2})^2\right)\prod_{j=3}^{n+1}\left[\left(t^2 - (s_j - s + n - j + \frac{3}{2})^2\right)\left(t^2 - (s_j + s + n - j + \frac{5}{2})^2\right)\right]$

TABLE II.2

	$\mathbb{C}$a
6	$\{\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\} \cup \{\pm\epsilon_j, \pm\epsilon_j \pm \epsilon_k\}_{j=1\,,\,k=j+1}^{4\,,\,4}$
7	$\{\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\} \cup \{\epsilon_j, \epsilon_j \pm \epsilon_k\}_{j=1\,,\,k=j+1}^{4\,,\,4}$
8	$\{\epsilon_1, \epsilon_1 \pm \epsilon_k\}_{k=2}^{4} \cup \{\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\}$
9	$\sum_{j=2}^{4} s_j\epsilon_j \quad , \quad s_j \in \frac{\mathbb{Z}}{2}, s_j - s_k \in \mathbf{Z}, s_2 \geq s_3 \geq s_4 \geq 0$
10	$\frac{1}{2}(5\epsilon_2 + 3\epsilon_3 + \epsilon_4)$
11	ϵ_1
12	$\tan(\pi t) \iff s_j \in \mathbf{Z}$
13	$t\left(t^2 - (\frac{s_4+1}{2})^2\right)\left(t^2 - (\frac{s_3+3}{2})^2\right)\left(t^2 - (\frac{s_2+5}{2})^2\right)\left(t^2 - (s_2 - s_3 - s_4 + \frac{1}{2})^2\right)$ $\left(t^2 - (s_2 - s_3 + s_4 + \frac{3}{2})^2\right)\left(t^2 - (s_2 + s_3 - s_4 + \frac{7}{2})^2\right)\left(t^2 - (s_2 + s_3 + s_4 + \frac{9}{2})^2\right)$

TABLE II.3

For SO, SU and Sp the fixed matrix bases have this property (except the basis of t_0 for $\mathbb{C}$ what gives a factor for $\mathrm{vol}(H)$ and for $\mathrm{vol}(K)$), for F_4 one can take it as a definition. Taking for a Lie subalgebra b of g_0 a positive ordered subbasis of $\{b_j\}$ we consider the wedge product of all elements of the dual basis, a volume form, and the associated positive volume of the corresponding Lie group B is called $\mathrm{vol}_{\mathrm{Standard}}(B)$.

To calculate the needed volumes of G^d/K, K and H in our first three cases we use the isomorphisms $SO(j)/SO(j-1) \cong S^{j-1}$, $U(j)/U(j-1) \cong S^{2j-1}$ and $Sp(j)/Sp(j-1) \cong S^{4j-1}$ and look at the corresponding bases of the tangent spaces (Only for the diagonal matrices in the quaternionic case we get a factor) . For the volume of the compact Cayley plane see [HE 84]. One gets

$$\mathrm{vol}_{\mathrm{Standard}}(F_4/Spin(9)) = \frac{6\pi^8}{\Gamma(12)} \quad .$$

2. $\underline{\mathrm{vol}_{\mathrm{Miatello}}(\Gamma\backslash G)}$ In p_0 we have to choose a basis $\{\tilde{P}_j\}$ with

$$\langle \tilde{P}_j, \tilde{P}_k \rangle = \frac{B(\tilde{P}_j, \tilde{P}_k)}{B(H,H)} = \delta_{jk} \quad ,$$

where H is the element of a_0 with $\lambda(H) = 2$ and on k_0 we have to take the form which gives $\mathrm{vol}(K) = 1$. So if $\lambda = 2(P_2^{(1)})^*$ this is our standard basis of p_0 and if $\lambda = (P_2^{(1)})^*$ we get $\tilde{P}_j = 2P_j$ hence with

$$\mathrm{vol}_{\mathrm{Standard}}(X) = \frac{\mathrm{vol}_{\mathrm{Standard}}(\Gamma\backslash G)}{\mathrm{vol}_{\mathrm{Standard}}(K)}$$

$$\mathrm{vol}_{\mathrm{Miatello}}(\Gamma\backslash G) = \begin{cases} \dfrac{\mathrm{vol}_{\mathrm{Standard}}(X)}{2^{2n}} & G = SO(1,2n) \\ \mathrm{vol}_{\mathrm{Standard}}(X) & \text{sonst} \end{cases} \tag{18}$$

3. $\underline{\mathrm{vol}_{\mathrm{Gangolli}}(\Gamma\backslash G)}$ We omit the detailled definition of this volume because in [GA 77] was already calculated the corresponding form on g_0 as

$$\omega_{\mathrm{Gangolli}} = \frac{\Gamma(m) 2^{p+2q-1}}{\pi^m (2p+8q)^m} \cdot \frac{\omega_\theta}{\omega_\theta(K)} \quad ,$$

where the form ω_θ comes from a orthonormal basis of g_0 w.r.t. the scalar product $\langle\ ,\ \rangle_\theta$. So we find immediately

$$\mathrm{vol}_{\mathrm{Gangolli}}(\Gamma\backslash G) = \frac{\Gamma(m)}{\pi^m} \cdot 2^{p+2q-1} \cdot \mathrm{vol}_{\mathrm{Standard}}(X) \tag{19}$$

4. $\underline{\mathrm{vol}_{\mathrm{Schmid}}(\Gamma\backslash G)}$ We have to take on h_0 the form which gives $\mathrm{vol}(H) = 1$ and on g_0/h_0 the form

$$\omega_{\mathrm{Schmid}}^{(G/H)} = i^{m_2} \bigwedge_{j=1}^{m_2} w^j \wedge \bar{w}^j \quad ,$$

	G^d/K	K	H
$\mathbb{R}$	$2\pi^{n+\frac{1}{2}}/\Gamma\left(n+\frac{1}{2}\right)$	$(2\pi)^{n^2}\left(\Gamma(n)\prod_{j=0}^{n-2}(2j+1)!\right)^{-1}$	$(2\pi)^n$
$\mathbb{C}$	$\pi^n/\Gamma(n+1)$	$\sqrt{(n+1)/2}\cdot 2^n\pi^{n(n+1)/2}\left(\prod_{j=2}^{n}\Gamma(j)!\right)^{-1}$	$\sqrt{(n+1)/2}\,(2\pi)^n$
$\mathbb{H}$	$\pi^{2n}/\Gamma(2n+2)$	$2^{n+1}\pi^{n^2+n+2}\left(\sqrt{2}^{3n+3}\prod_{j=1}^{n}\Gamma(2j)\right)^{-1}$	$(2\pi)^{n+1}/\sqrt{2}^{n+1}$

Special values of $\mathrm{vol}_{\mathrm{Standard}}$

where $\{w^j, \bar{w}^j\}$ is the dual basis of a basis $\{\bar{v}_j, v_j\}$ of $g_0/h_0 \cong n^+ \oplus n^-$ (with $n^\pm \overset{df}{=} \bigoplus_{\pm\lambda\in\Delta_c^+} g_\lambda$ as usual) and $\{v_j\}$ is chosen as an orthonormal basis of n^- w.r.t the inner product

$$(X,Y) = -B(X,\theta(\bar{Y})) = \langle X,\bar{Y}\rangle_\theta \quad .$$

But n^+ and n^- are nilpotent Lie algebras so the Killing form degenerates there. Hence if we define

$$a_j \overset{df}{=} \frac{\bar{v}_j + v_j}{\sqrt{2}} \quad , \quad b_j \overset{df}{=} i\cdot\frac{\bar{v}_j - v_j}{\sqrt{2}} ...$$

we get with $\{a_j, b_j\}$ an orthonormal basis w.r.t. $\langle\ ,\ \rangle_\theta$ and it holds

$$\bigwedge_{j=1}^{m_2} a_j^* \wedge b_j^* = i^{m_2}\bigwedge_{j=1}^{m_2} w^j\wedge\bar{w}^j \quad .$$

Hence we find

$$\omega_{\mathrm{Schmid}} = \frac{\omega_\theta}{\omega_\theta(H)} \quad ,$$

and so

$$\mathrm{vol}_{\mathrm{Schmid}}(\Gamma\setminus G) = (2p+8q)^{m_2}\cdot\frac{\mathrm{vol}_{\mathrm{Standard}}(K)}{\mathrm{vol}_{\mathrm{Standard}}(H)}\cdot\mathrm{vol}_{\mathrm{Standard}}(X)$$

$$\begin{aligned}
&= \mathrm{vol}_{\mathrm{Standard}}(X)\cdot\frac{(2(2n-1))^{n^2}(2\pi)^{n^2}}{(2\pi)^n\Gamma(n)\prod_{j=0}^{n-2}(2j+1)!} && \text{for } \mathbb{R}\\
&= \mathrm{vol}_{\mathrm{Standard}}(X)\cdot\frac{(4(n+1))^{n(n+1)/2}\cdot\pi^{n(n-1)/2}}{\prod_{j=2}^{n}\Gamma(j)} && \text{for } \mathbb{C} \qquad (20)\\
&= \mathrm{vol}_{\mathrm{Standard}}(X)\cdot\frac{(8(n+2))^{(n+1)^2}\cdot\pi^{n^2+1}}{2^{n+1}\prod_{j=1}^{n}\Gamma(2j)} && \text{for } \mathbb{H}
\end{aligned}$$

2.2 The Selberg zeta function and its topological zeroes

The Selberg zeta function Z_σ associated to $\sigma \in \hat{M}$ is a meromorphic function on a^* and is for $s \in a^*$ with $\mathrm{Re}(s) > 2\rho_A$ defined as

$$Z_\sigma(s) \overset{df}{=} \prod_{c \in C_{\mathrm{geo}}} \prod_{\mathrm{N}=0}^{\infty} \det\left(I - \mathrm{S}^{\mathrm{N}}\, P_c^s\, \sigma(m_c)\, \mathrm{e}^{-s\, l_c}\right) \quad . \tag{21}$$

The notation we have used will be described in the sequel.
C_{geo} is the set of all primitive oriented closed geodesics of the locally homogeneous space $X = \Gamma \backslash G/K$.
Let Φ denote the geodesic flow on SX (note that orbits of Φ can be viewed as geodesics in X as well). P_c denotes the linear Poincaré map of a closed orbit c having the period l_c

$$P_c : T_p(SX) \to T_p(SX)\ ,\ p \in c\ ,$$

defined pointwise as

$$P_{c,p} \overset{df}{=} (d\,\Phi_{\,l_c}\,)_p\ .$$

Since for any points $p, q \in c$ these maps are conjugated P_c is well defined (we look only at the eigenvalues of this linear map).
P_c^s denotes the restriction to the stable part of the tangent space $T_p(SX)$ (the linear hull of all eigenvectors for eigenvalues with $|\cdot| < 1$) and $\mathrm{S}^{\mathrm{N}}\, P_c^s$ is the induced map on the N-th symmetric tensor product $\mathrm{S}^{\mathrm{N}}\, T_p(SX)$ of the space $T_p(SX)$. Finally $\sigma(m_c)$, $m_c \in M$, is defined to be the monodromy of the associated closed curve $\tilde{c}$ in the twisted geodesic flow in the induced vector bundle $\Gamma\backslash G \times_M V_\sigma$ over $\Gamma \backslash G/M = S(X)$ (see e.g. [2]).
For the spherical case $\sigma = 1$ we give R. Gangollis definition of Z_1 in [GA 77] from the group theoretical point of view. We want to use his results as well as show that the mysterious exponent κ in the functional equation for Z_1 (introduced to give certain residues an integer value) is not necessary and can be omitted. He always used β to identify a^* with the complex s-plane.

$$Z_1(s\beta) = \prod_{\gamma \in \mathrm{Prim}_{\,\Gamma}} \prod_{\nu \in L} \left(\det\left(I - \left[\xi_\nu(h(\gamma))\right]^{-1}\, \mathrm{e}^{-s\, u_\gamma}\right)\right)^{m_\nu} \quad . \tag{22}$$

Here $\mathrm{Prim}_{\,\Gamma}$ is a representation system of primitive G-conjugacy classes in Γ , L is the set of linear functions on h which are of the form $\sum_{i=1}^t m_i\alpha_i$ with m_i nonnegative and integral (we have used $\Delta^+ = \{\alpha_1, \ldots, \alpha_t\}$). Furthermore for $\nu \in L \quad m_\nu$ denotes the number of distinct ordered t-tuples $(m_1, \ldots, m_t)$ such that $\nu = \sum_{i=1}^t m_i\alpha_i$, ξ_ν is defined to be the character of the Cartan subgroup H which corresponds to ν. To define $h(\gamma) \in H$, $\gamma \in \Gamma$, we first introduce the Lie subgroup $A^+ = \exp(a^+)$ corresponding to the positive Weyl chamber $a^+ \subset a$. Then $h(\gamma) = h_p(\gamma)h_k(\gamma)$ is an arbitrary element of H which is conjugated to γ and for which $h_p(\gamma)$ lies in A^+. Finally he has used $u_\gamma = \beta(\log h_p(\gamma))$.
In [GA 77] Gangolli considered as first the logarithmic derivative of Z_1, proved the integrality of the residues of all its (simple) poles up to an integer multiple κ and normalized Z_1 by stipulating $\lim_{s\to\rho}(s-\rho)^{-g}Z_1(s) = 1$,

g being the order of the zero at $s = \rho$. Then a constant C occurs as a factor in the product representation (22). We have chosen a normalisation of Z_1 to achieve $C = 1$. Note that the logarithmic derivative does not change. As we will see from the equivalence of the two given definitions (and also explicitely from the equality of some calculated indices and the residues calculated in [GA 77]) Z_1 is defined not only up to an integer power κ and therefore we have already chosen $\kappa = 1$ whenever it occurs in [GA 77].

Since we want to use the *long* restricted root $\lambda = \delta|_a$ for parametrizing a^* we have to modify the left side of (22) and consider $Z_1(s\lambda)$. Redefining $u_\gamma = \lambda(\log h_p(\gamma))$ we have preserved Z_1 as a function on a^*. Clearly the consequence of this modification in the case $\lambda = 2\beta$ is that the complex coordinates of all elements of a^* (e.g. the singularities of Z_1) are now half of their former value. Now the equivalence of the two given definitions follows by comparing the corresponding objects under the equivalence

$$\begin{array}{ccc} C_{\text{geo}} & \cong & \operatorname{Prim}_\Gamma \\ c & \longmapsto & \gamma_c \end{array}$$

As a consequence we get for the logarithmic derivative of Z_1

$$\begin{aligned} \frac{d}{ds}\log Z_1(s) &= \sum_{\gamma\in\operatorname{Prim}_\Gamma}\sum_{\nu\in L}\frac{d}{ds}\log\left(\det\left(I - [\xi_\nu(h(\gamma))]^{-1}\,\mathrm{e}^{-s\,u_\gamma}\right)\right)^{m_\nu} \\ &= \sum_{\gamma\in\operatorname{Prim}_\Gamma}\sum_{\nu\in L}\sum_{j=1}^{\infty}\left(u_\gamma\, m_\nu\,\xi_\nu(h(\gamma))^{-j}\;\mathrm{e}^{-j\,s\,u_\gamma}\right) \\ &= \sum_{\gamma\in C_\Gamma\setminus\{e\}} |u_\gamma|\, j(\gamma)^{-1} C(h(\gamma))\mathrm{e}^{(d-s)\,|u_\gamma|} \end{aligned} \tag{23}$$

with $C_\gamma = \bigcup_{\delta\in\operatorname{Prim}_\Gamma}\{\delta^j\,;\, j \geq 0\}$ a set of representatives in Γ for the Γ-Conjugacy class of elements of Γ (note $u_\gamma \geq 0$ here), $C(h(\gamma))$ a certain positive function on A, which actual value plays no role here, and $j(\gamma)$ the exponent in the representation of γ by an element of $\operatorname{Prim}_\Gamma$. For a detailled derivation of the last line and the definitions thereby see [GA 77]. We will need the last expression in section 3 to reformulate a trace formula.

Return now to the general situation. In [JU 93] A. Juhl proved a functional equation of the form

$$\frac{Z_\sigma(s+\rho_A)}{Z_\sigma(-s+\rho_A)} \cong \left(\frac{Z_\sigma^+(s+\rho_A)}{Z_\sigma^+(-s+\rho_A)}\right)^{-2}. \tag{24}$$

There Z_σ^+ is the functional determinant (a definition of this you find in section 1.1) associated to the sequence

$$\left\{\chi \in a_0^* \quad \text{with multiplicity} \quad a_{-\chi,\sigma} \quad \right\}_{\substack{\chi>\rho_A \\ \chi^d\in\hat{A}^d \quad \sigma\text{-admissible}}} \tag{25}$$

with $a_{k,\sigma}$ certain integers defined in the sequel, $A^d = \exp ia_0$, $\chi^d \in \hat{A}^d$ is defined as

$$\chi^d(\exp(iX)) = \exp i\chi(X)$$

and $\chi^d \in \hat{A}^d$ is called *admissible for* σ iff it holds for all $m \in A^d \cap M$ $\chi^d(m) \cdot \sigma(m) = 1$ .The relation $\cong$ denotes equality up to a factor of the form exp{holomorphic polynomial} .
First we modify the defining sequence and introduce the functional determinant $D^{\mathbf{t}}_\sigma$ ($\mathbf{t}$ means topological, see below) associated to the sequence

$$\Big\{\chi \in a_0^* \quad \text{with multiplicity} \quad a_{\chi-\rho_A,\sigma} \quad \Big\}_{\substack{\chi>0 \\ (\chi-\rho_A)^d \in \hat{A}^d \;\; \sigma\text{-admissible}}} \tag{26}$$

As we will see the numbers $a_{\chi,\sigma}$ satisfy

$$a_{-\chi,\sigma} = -a_{\chi-2\rho_A,\sigma}$$

and so we derive

$$D^{\mathbf{t}}_\sigma(s) = (Z^+_\sigma(s+\rho_A))^{-1}$$

hence

$$\frac{Z_\sigma(s+\rho_A)}{Z_\sigma(-s+\rho_A)} \cong \left(\frac{\mathrm{D}^{\mathbf{t}}_\sigma(s)}{\mathrm{D}^{\mathbf{t}}_\sigma(-s)}\right)^2 \quad . \tag{27}$$

$D^{\mathbf{t}}_\sigma$ (or better $D^{\mathbf{t}}_\sigma(-s-\rho_A)^2$) is sometimes called the Gamma factor of Z_σ , because there is a close analogy to the relation between the Riemann Zeta function and the classical Gamma function. There it is the function Γ, that makes the meromorphic function ζ into a function

$$\xi(s) = \zeta(s)\Gamma(s/2)\pi^{-s/2}$$

with no singularities outside a critical strip ($0 \leq \mathrm{Re}(s) \leq 1$) and satisfying a reflection formula

$$\xi(s) = \xi(1-s) \quad .$$

Here the product

$$\Xi(s) = Z_\sigma(s)\ \left(\mathrm{D}^{\mathbf{t}}_\sigma(-s-\rho_A)\right)^2$$

has the same properties. There are no singularities outside the strip $0 \leq \mathrm{Re}(s) \leq 2\rho_A$ and it holds a reflection formula

$$\Xi(s) \cong \Xi(2\rho_A - s) \quad .$$

Return now to the quantities $a_{\chi,\sigma}$.
From the root decomposition

$$g = h_c \oplus \left(\bigoplus_{\lambda\in\Delta_c^+} g_\lambda\right) \oplus \left(\bigoplus_{-\lambda\in\Delta_c^+} g_\lambda\right)$$

we get a decomposition of the tangent space $T^{\mathbb{C}}_{eH_c}(G/H_c)$

$$\begin{aligned} T^{\mathbb{C}}_{eH_c}(G/H_c) &= T^{\mathbb{C}\,(1,0)}_{eH_c}(G/H_c) \oplus T^{\mathbb{C}\,(0,1)}_{eH_c}(G/H_c) \\ &= \bigoplus_{\lambda\in\Delta_c^+} g_\lambda \quad \oplus \quad \bigoplus_{-\lambda\in\Delta_c^+} g_\lambda \end{aligned}$$

and by G-invariant continuation an almost complex structure of G/H_c (of course we use multiplication with the imaginary unit i in the occuring complex Lie algebras) which is G-invariant, so $Y = \Gamma\backslash G\,/\,H_c$ becomes a complex manifold. Also we have the natural isomorphy

$$\hat{H}_c \cong \Lambda$$

of $\hat{H}_c$, the group of characters of H_c, and a lattice $\Lambda \subseteq h_\Delta \overset{df}{=} (i \cdot (h_c)_0)^*$. So we can consider for each $\mu \in \Lambda$ the line bundle L_μ

$$L_\mu \overset{df}{=} G \times \mathbb{C} \big/ \sim \qquad \Big[(g,z) \sim (gh^{-1}, \mathrm{e}^{\mu}(h)z) \quad \forall g \in G, \forall z \in \mathbb{C}, \forall h \in H_c\Big]$$

associated to the corresponding character. As it was shown in [GR/SCHM 69] L_μ admits a special holomorphic structure, which is used here (its definition we omit because we don't need it).
Let finally be $H^j(Y, \Gamma\backslash L_\mu)$ be the $\bar\partial-$cohomology groups of $\Gamma\backslash L_\mu$.

Then $a_{\chi,\sigma}$ is defined to be the index of this complex

$$a_\mu \overset{df}{=} \sum_{j=0}^{\mathrm{top}} (-1)^j \dim_{\mathbb{C}} \left(H^j(Y, \Gamma\backslash L_\mu) \right)$$

for the special μ

$$\mu = \chi^c + \Lambda_\sigma$$

Here we used the Cayley transform $H \to H_c$ to assign to each $\chi^c \in h_c^*$ an element $\chi \in a^* \subset h^*$ (this is nothing else than $\epsilon_j^c \to \epsilon_j$) together with the fact that we consider only χ in the image of this assignment.
Now we concretize what it means that $\chi^d \in \hat{A}_d$ is $\sigma-$admissible. First note $\hat{A}_d = (\mathbb{Z} \cdot \beta)^d$. Next calculate M and A^d in our matrix realisation to find $M \cap A^d = \{e\}$ ($\mathbb{R}$) and $M \cap A^d = \{e, s_0\}$ ($\mathbb{C}, \mathbb{H}, \mathbb{C}$a) resp. with $e \in G$ the identity element and $s_0 = diag(-1,-1,1,\ldots,1)$ ($\mathbb{C}$),
$s_0 = diag(-1,-1,1,\ldots,1,-1,-1,1,\ldots,1)$ ($\mathbb{H}$). Let Λ_σ be the highest weight of the representation $\sigma \in \hat{M}$. We have to decide between $\sigma(s_0) = \pm I$ so we look at Weyl's character formula

$$\mathrm{tr}\;\sigma(s) = \frac{\sum_{w \in W(m,t)} \det(w) \exp w(\Lambda_\sigma + \rho_M)(s)}{\sum_{w \in W(m,t)} \det(w) \exp w(\rho_M)(s)} \quad , s \in T \quad \text{regular,}$$

where tr denotes the trace of the representation σ and $W(m,t)$ the Weyl group of (m,t) (that is the group generated by the reflections $s_\alpha(\alpha) = -\alpha$, $\alpha \in \Delta(m,t) \cong \Delta_M$) . We have to calculate $\lim_{s \to s_0} \mathrm{tr}\;\sigma(s)$ to decide whether $\mathrm{tr}\sigma(s_0) > 0$ or not. Therefore we verify $\exp(\epsilon_j)(s_0) = 1$ except $\exp(\epsilon_1 - \epsilon_2)(s_0) = -1$, ($\mathbb{H}$) and $\exp(\epsilon_j)(s_0) = -1, j = 2,3,4,$ ($\mathbb{C}$a). The Weyl group $W(m,t)$ is nothig else than

$$W(m,t) = \begin{cases} Perm(\{2,\ldots,n\}) & \mathbb{C} \\ Perm(\{3,\ldots,n+1\})\langle s_{\epsilon_1-\epsilon_2}, s_{\epsilon_3}, \ldots, s_{\epsilon_{n+1}} \rangle & \mathbb{H} \\ Perm(\{2,3,4\})\langle s_{\epsilon_2}, s_{\epsilon_3}, s_{\epsilon_4} \rangle & \mathbb{C}\mathrm{a} \end{cases} .$$

So we find

$$\sigma(s_0) = \begin{cases} (-1)^s & \mathbb{C} \\ (-1)^{2s} & \mathbb{H} \\ (-1)^{2(s_2+s_3+s_4)} & \mathbb{C}a \end{cases}$$

where the numbers s, s_j on the right side come from Λ_σ (see table II). Hence the following $\chi \in a_0^*$ give σ-admissible $\chi^d \in \hat{A}^d$

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{C}a$
$(\mathbb{Z})\lambda$	always	s even	$s \in \mathbb{Z}$	$s_j \in \mathbb{Z}$
$(\mathbb{Z}+\frac{1}{2})\lambda$	$-$	s odd	$s \in \mathbb{Z}+\frac{1}{2}$	$s_j \in \mathbb{Z}+\frac{1}{2}$

.

Now we use once and for all the long root λ to identify a^* with $\mathbb{C}$. Then the functions Z_σ and $D_\sigma^{\mathbf{t}}$ become meromorphic functions on the complex plane. If as before we denote by d the number satisfying

$$\rho_A = d \cdot \lambda = \rho \cdot \beta \quad ,$$

then we have found the sequence belonging to $D_\sigma^{\mathbf{t}}$ as

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{C}a$
$\mathbb{N}$	$-$	$s+n$ even	$s \in \mathbb{Z}+\frac{1}{2}$	$s_j \in \mathbb{Z}+\frac{1}{2}$
$\mathbb{N}_0+\frac{1}{2}$	always	$s+n$ odd	$s \in \mathbb{Z}$	$s_j \in \mathbb{Z}$

and that′s the reason why we′ve considered such sequences in the first section.

2.3 Topological zeroes, Reflection formula

Now we apply results of W. Schmid in [SCHM 71] to get explicit values of the introduced indeces $a_{\chi,\sigma}$.

A comprehension of Theorem 1 and 3 from [SCHM 71] gives us

$$a_\mu = \frac{(-1)^m}{(2\pi)^{m_2}} \cdot \mathrm{vol}_{\mathrm{Schmid}}(\Gamma\backslash G) \cdot \prod_{\alpha \in \Delta_c^+} (\mu + \rho_G \, , \, \alpha) \tag{28}$$

with ρ_G as before and $(\ ,\ )$ coming from the Killing form B on g by restriction to h_c and duality. From now on we omit the index c at all the occuring root systems because nothing changes looking at the product $\prod_{\alpha \in \Delta^+} (\mu + \rho_G \, , \, \alpha)$ for the corresponding μ. So this can be rewritten to

$$\begin{aligned} a_\mu &= \frac{(-1)^m}{(2\pi)^{m_2}} \cdot \mathrm{vol}_{\mathrm{Schmid}}(\Gamma\backslash G) \cdot \prod_{\alpha \in \Delta_M^+} (\mu + \rho_G \, , \, \alpha) \cdot \prod_{\alpha \in \Delta_1^+} (\mu + \rho_G \, , \, \alpha) \\ &= C(X) \cdot P(\mu) \end{aligned}$$

where we have used the abbreviations

$$\begin{aligned} C(X) &\overset{df}{=} \frac{(-1)^m}{(2\pi)^{m_2}} \cdot \mathrm{vol}_{\mathrm{Schmid}}(\Gamma\backslash G) \cdot \prod_{\alpha \in \Delta_M^+} (\rho_M \, , \, \alpha) \cdot \prod_{\alpha \in \Delta_1^+} (\delta \, , \, \alpha) \ , \\ P(\mu) &\overset{df}{=} \prod_{\alpha \in \Delta_1^+} \frac{(\mu + \rho_G \, , \, \alpha)}{(\delta \, , \, \alpha)} \cdot \prod_{\alpha \in \Delta_M^+} \frac{(\mu + \rho_G \, , \, \alpha)}{(\rho_M \, , \, \alpha)} \quad . \end{aligned}$$

The constant C(X).
First we specify (,) using the explicite representation of the ϵ_j together with our choise of a basis of g (c.f. the definition of $\mathrm{vol}_{\mathrm{standard}}$) and get

$$(\epsilon_j, \epsilon_k) = \begin{cases} \delta_{jk}/2(2n-1) & \mathbb{R} \\ (\delta_{jk} - 1/(n+1))/2(n+1) & \mathbb{C} \\ \delta_{jk}/4(n+2) & \mathbb{H} \end{cases}$$

and then we transform $\mathrm{vol}_{\mathrm{Schmid}}(\Gamma\backslash G)$ into $\mathrm{vol}_{\mathrm{Standard}}(X)$, use the Hirzebruch proportionality

$$\frac{\chi(X)}{\chi(G^d/K)} = (-1)^m \frac{\mathrm{vol}_{\mathrm{Standard}}(X)}{\mathrm{vol}_{\mathrm{Standard}}(G^d/K)}$$

(see e.g. [HI 56]) and insert all known constants to verify step by step for $\mathbb{R}$

$$\begin{aligned} \prod_{\alpha\in\Delta_M^+}(\rho_M\,,\,\alpha)\prod_{\alpha\in\Delta_1^+}(\delta\,,\,\alpha) &= \frac{1}{(2(2n-1))^{2n-1}}\cdot\frac{\prod_{j=0}^{n-2}(2j+1)!}{(2(2n-1))^{(n-1)^2}2^{n-1}} \\ C(X) &= (-1)^n\frac{\mathrm{vol}_{\mathrm{Standard}}(X)}{\pi^n\Gamma(n)2^{2n-1}} = (-1)^n\frac{\mathrm{vol}_{\mathrm{Gangolli}}(X)}{\Gamma(n)^2\,2^{4n-3}} \\ =\frac{\chi(X)}{\Gamma(2n)} &= (-1)^n\frac{2\mathrm{vol}_{\mathrm{Miatello}}(X)}{\pi^n\Gamma(n)} = \frac{2\chi(X)}{(2n-1)!\chi(G^d/K)} \end{aligned}$$

for $\mathbb{C}$

$$\begin{aligned} \prod_{\alpha\in\Delta_M^+}(\rho_M\,,\,\alpha)\prod_{\alpha\in\Delta_1^+}(\delta\,,\,\alpha) &= \frac{2}{(2(n+1))^{2n-1}}\cdot\frac{\prod_{j=1}^{n-2}j!}{(2(n+1))^{(n-2)(n-1)/2}} \\ C(X) &= (-1)^n\frac{2\mathrm{vol}_{\mathrm{Standard}}(X)}{\pi^n\Gamma(n)} = (-1)^n\frac{\mathrm{vol}_{\mathrm{Gangolli}}(X)}{\Gamma(n)^2\,2^{2n-2}} \\ =\frac{2\chi(X)}{\Gamma(n)\Gamma(n+2)} &= (-1)^n\frac{2\mathrm{vol}_{\mathrm{Miatello}}(X)}{\pi^n\Gamma(n)} = \frac{2\chi(X)}{(n-1)!n!\chi(G^d/K)} \end{aligned}$$

and for $\mathbb{H}$

$$\begin{aligned} \prod_{\alpha\in\Delta_M^+}(\rho_M\,,\,\alpha)\prod_{\alpha\in\Delta_1^+}(\delta\,,\,\alpha) &= \frac{2^3}{(4(n+2))^{4n-1}}\cdot\frac{2^{n-1}\prod_{j=1}^{n-1}\Gamma(2j)}{(4(n+2))^{n^2-2n+2}} \\ C(X) &= \frac{2\mathrm{vol}_{\mathrm{Standard}}(X)}{\pi^{2n}\Gamma(2n)} = \frac{\mathrm{vol}_{\mathrm{Gangolli}}(X)}{\Gamma(2n)^2\,2^{4n}} \\ =\frac{2\chi(X)}{\Gamma(2n)\Gamma(2n+2)} &= \frac{2\mathrm{vol}_{\mathrm{Miatello}}(X)}{\pi^{2n}\Gamma(2n)} = \frac{2\chi(X)}{\Gamma(2n)\Gamma(2n+2)\chi(G^d/K)} \end{aligned}$$

The polynomial part $\mathbf{P(\mu)}$.
First we rewrite it for $\mu = z\delta + \Lambda_\sigma$. Note that δ is orthogonal to Δ_M . So we get

$$P(z\delta+\Lambda_\sigma) = \prod_{\alpha\in\Delta_1^+}\frac{((z+d)\delta+\Lambda_\sigma+\rho_M\,,\,\alpha)}{(\delta\,,\,\alpha)}\cdot\prod_{\alpha\in\Delta_M^+}\frac{(\Lambda_\sigma+\rho_M\,,\,\alpha)}{(\rho_M\,,\,\alpha)}$$

and the second product is nothing else than the dimension d_σ of the representation σ and we have

$$P(z\delta+\Lambda_\sigma) = d_\sigma \cdot \prod_{\alpha \in \Delta_1^+} \left(z + d + \frac{(\Lambda_\sigma + \rho_M\, ,\, \alpha)}{(\delta\, ,\, \alpha)} \right)$$

By a direct computation we verify that the remaining product is exactly $Q_\sigma(z+d)$, $Q_\sigma(t)$ introduced above as (monic) polynomial part of the plancherel density $\mu_\sigma(it\lambda)$. So we have proven

Lemma 5 (Index formula)

$$a_{z\delta,\sigma} = C(X) \cdot d_\sigma \cdot Q_\sigma(z+d) \tag{29}$$

Following Gangolli we denote the singularities of Z_σ that are related to the singularities of the associated plancherel density by *topological zeroes* (as usual a pole will be a zero with negative multiplicity) and all remaining singularities (which are all proper zeroes) as *spectral zeroes*. So the sequence which defines $D_\sigma^{\mathbf{t}}$ is (up to reflection and translation) the sequence of the topological zeroes of Z_1^σ with half multiplicity. In detail $D_\sigma^{\mathbf{t}}$ is associated to the sequence

$$\begin{cases} \left\{ k \quad \text{with multiplicity} \quad C(X) d_\sigma Q_\sigma(k) \right\}_{k=1}^{\infty} & (\text{case} \quad \mathbf{Z}) \\ \left\{ k+\frac{1}{2} \quad \text{with multiplicity} \quad C(X) d_\sigma Q_\sigma(k+\frac{1}{2}) \right\}_{k=1}^{\infty} & (\text{case} \quad \mathbf{Z}+\frac{1}{2}) \end{cases}$$

Here the case $\mathbf{Z}$ means the case that exactly the $(\mathbf{Z}\lambda)^d$ are σ-admissible and for $\mathbf{Z}+1/2$ analogously. Recall that these multiplicities are indeces of certain differential complexes hence *integers*. Also Q_σ is an odd polynomial (c.f. table II) , so all assumptions from subsection 1.4 are valid and we deduce from Lemma 4

$$\frac{D_\sigma^{\mathbf{t}}(s)}{D_\sigma^{\mathbf{t}}(-s)} = \begin{cases} \exp\left[C(X)\, d_\sigma \int_0^s \pi\, Q_\sigma(t)\, \cot(\pi t)\, dt \right] & (\mathbf{Z}) \\ \exp\left[-C(X)\, d_\sigma \int_0^s \pi\, Q_\sigma(t)\, \tan(\pi t)\, dt \right] & (\mathbf{Z}+\frac{1}{2}) \end{cases}$$

Using the associated plancherel density we can comprehence both formulas to the main result in this section

Proposition 1 (Functional equation for the Gamma factor)

$$\frac{D_\sigma^{\mathbf{t}}(s)}{D_\sigma^{\mathbf{t}}(-s)} = \exp\left[-2\pi \mathrm{vol}(\Gamma\backslash G) \int_0^s \mu_\sigma(it\lambda)\, dt \right] \tag{30}$$

where for $\mathrm{vol}(\Gamma\backslash G)$ and $\mu_\sigma(it\lambda)$ the same normalisation of the Haar measure on G is used.

Corollary 1 With the functional equation (27) one proves

$$\frac{Z_\sigma(s+d)}{Z_\sigma(-s+d)} \quad \cong \quad \exp\left[-4\pi \mathrm{vol}(\Gamma\backslash G) \int_0^s \mu_\sigma(it\lambda)\, dt \right]$$

Now we relate the functional equation for the spherical case $\sigma = 1 \in \hat{M}$ (and $d_\sigma = 1$ of course) to the results of R. Gangolli in [GA 77] and N. Kurokawa in [KU 92] . Therefore we use the explicite structure of Harish-Chandra's c-function

$$\frac{1}{c(r)} = \frac{\Gamma((p+q)/2)\Gamma(ir+p/2)\Gamma(ir/2+p/4+q/2)}{\Gamma(p+q)\Gamma(ir)\Gamma(ir/2+p/4)}$$

in our cases to verify

$$\begin{aligned}\frac{1}{c(ir)\,c(-ir)} &= \begin{cases} (-1)^m \dfrac{4\pi Q_1(r)\tan(\pi r)}{\Gamma(m)^2\, 2^{2m+p+2q-1}} & \mathbb{R} \\ (-1)^m \dfrac{2\pi Q_1(r/2)\phi_1(r/2)}{\Gamma(m)^2\, 2^{p+2q-1}} & \text{else} \end{cases} \\ &= \mu_1^{\text{Gangolli}}(it\beta) \cdot \begin{cases} 4\pi & \mathbb{R} \\ 2\pi & \text{else} \end{cases}\end{aligned}$$

so the reflection formula reads here (if we now parametrize a^* by β !)

$$\frac{D^{\mathbf{t}}(s\beta)}{D^{\mathbf{t}}(-s\beta)} = \exp\left[-\frac{\text{vol}_{\text{Gangolli}}(\Gamma\backslash G)}{2} \int_0^s \frac{dt}{c(ir)\,c(-ir)}\right] \quad . \tag{31}$$

Corollary 2 The Selberg Zeta function in [GA 77] is defined if one puts there $\kappa = 1$ (we have shown here explicitely the residues of the logarithmic derivative being integers) and so

$$\frac{Z_1((s+\rho)\beta)}{Z_1((-s+\rho)\beta)} = \left(\frac{D^{\mathbf{t}}(s\beta)}{D^{\mathbf{t}}(-s\beta)}\right)^2 = \exp\left[-\text{vol}_{\text{Gangolli}}(\Gamma\backslash G) \int_0^s \frac{dt}{c(ir)\,c(-ir)}\right]$$

hence exactly the same result as in [GA 77] .

Corollary 3 For the case of $\mathbb{C}$a using the results of [GA 77] we get

$$\begin{aligned} C(X) &= \frac{\text{vol}_{\text{Gangolli}}(\Gamma\backslash G)}{\Gamma(8)^2\, 2^{20}} = \frac{2\text{vol}_{\text{Standard}}(X)}{\pi^8\,\Gamma(8)} \\ &= \frac{2\text{vol}_{\text{Miatello}}(\Gamma\backslash G)}{\pi^8\,\Gamma(8)} = \frac{4\,\chi(X)}{\Gamma(8)\Gamma(12)} = \frac{2}{11!4\cdot 5\cdot 6\cdot 7}\cdot\frac{\chi(X)}{\chi(G^d/K)} \end{aligned}$$

In [KU 92] (only the spherical case) we find the same structure of the functional equation for the Gamma factor of Z_1 defined there as the zeta regularized determinant

$$\Gamma_X(s) = \det{}_\zeta(\sqrt{\Delta_{G^d/K} + d^2} + s + d)$$

but volume normalisations and other constants are not specified. So the formulas contain less information. Nevertheless using the explicitely given multiplicities of the eigenvalues of $\Delta_{G^d/K}$ we can rewrite our Gamma factor as

$$D_1^{\mathbf{t}}(s) = \det{}_\zeta(\sqrt{\Delta_{G^d/K} + d^2} - s)^{\frac{\chi(X)}{\chi(G^d/K)}} \quad .$$

3 Determinant formula for the spherical case

Throughout this section we consider only the case

$$\sigma = 1 \in \hat{M} \qquad .$$

Again we parametrize a^* by the long root λ (not by the short one as in [GA 77]).
Let $s_j^{\pm} = d \pm i r_j^{+}$ (multiplicity n_j) be the spectral zeroes of Z_1 with r_j^{+} chosen nonnegative (or purely imaginary in $(0, id]$ for the finitely many $s_j^{\pm}$ not lying on the axis $\mathrm{Re}(t) = d$ but in the interval $[0, 2d]$) and define a new sequence

$$\lambda_j \overset{df}{=} s_j^{+} \cdot s_j^{-} = d^2 + (r_j^{+})^2 \qquad .$$

The sequence $\{\lambda_j$, multiplicity $n_j\}$ will be our sequence containing the information about the spectral zeroes of Z_1 . Let r_k be the poles of

$$\begin{aligned} f(r) &= \pi \,\mathrm{vol}(\Gamma\backslash G)\mu_1(r\lambda) \\ &= \mathrm{vol}_{\mathrm{Gangolli}}(\Gamma\backslash G) \cdot \begin{cases} (4c(r)c(-r))^{-1} & \mathbb{R} \\ (2c(2r)c(-2r))^{-1} & \text{else} \end{cases} \end{aligned}$$

($c(r)$ the spherical Harish-Chandra c-function as above) in the upper half plane with corresponding residue d_k . What we have changed comparing with the notation in [GA 77] ? For $\mathbb{R}$ nothing has changed but a factor $\mathrm{vol}_{\mathrm{Gangolli}}(\Gamma\backslash G)/4$ for d_k and in the other cases (where we've used a really different parametrisation) the $s_j^{\pm}, r_j^{+}$ and r_k being half of their former value and again d_k gets only the factor $\mathrm{vol}_{\mathrm{Gangolli}}(\Gamma\backslash G)/4$. Here $\{ir_k - d\}$ is the set of topological zeroes of Z_1 with multiplicity $-4id_k$ and the sequence which defines the Gamma factor $D^{\mathbf{t}}$ of Z_1 is given in this notation by

$$\Big\{ - ir_k \quad \text{with multiplicity} \quad 2id_k \ \Big\}_{k=0}^{\infty} \qquad .$$

3.1 A Selberg trace formula

From [GA 77] we take the trace formula

$$\sum_{j=0}^{\infty} n_j F^*(r_j) = \int_{-\infty}^{\infty} \frac{F^*(r)\, f(r)}{\pi}\, dr + \sum_{\gamma \in C_\Gamma \backslash \{e\}} |u_\gamma|\, j(\gamma)^{-1} C(h(\gamma)) F(u_\gamma) \qquad . \tag{32}$$

There we have on the right side used the notations from section 2.2. The pair (F, F^*) has to be a pair of "admissible" functions. There F^* is the Fourier transform of F

$$F(u) = \frac{1}{2\pi} \int F^*(r) \exp(-iru)\, dr \qquad .$$

Let us consider the test function

$$F_s^*(r) = \frac{m!}{(r^2 + k^2)^{m+1}} \qquad , \qquad \mathrm{Re}(k) > d \,,\ s = d^2 - k^2 \tag{33}$$

Because of $F_s^*(r) = \left(-\frac{d}{dk^2}\right)^m \left(\frac{1}{r^2+k^2}\right)$ we deduce by inversion formula of the Fourier transformation

$$\begin{aligned} F_s(u) &= \left(-\frac{d}{dk^2}\right)^m \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(-iru)}{k^2+r^2}\,dr\right) \\ &= \left(-\frac{d}{dk^2}\right)^m \left(\frac{\exp(-k\,|u|)}{2k}\right) \end{aligned} \tag{34}$$

Lemma 6 For $\mathrm{Re}(k) > d$ the pair $(F_s\,, F_s^*)$ is a pair of admissible functions for (32).

Proof First one gets by induction over m

$$\begin{aligned} \left(-\frac{d}{dk^2}\right)^n \left(\frac{\exp(-k\,|u|)}{2k}\right) &= \frac{\exp(-k\,|u|)\left(\sum_{i=0}^{m} \tilde{C}_i (k\,|u|)^i\right)}{(2k)^{2m+1}} \\ &= \exp(-k\,|u|)\sum_{i=0}^{m} C_i \frac{|u|^i}{k^{2m+1-i}} \end{aligned}$$

for some constants C_i and $\tilde{C}_i$ resp. . Therefore it holds if we choose an $\epsilon > 0$ with $\mathrm{Re}(k) > d + 2\epsilon$

$$\begin{aligned} &\sup_u \left\{ \exp((d+\epsilon)\,|u|)\ |F_s(u)| \right\} \\ &\leq \ \mathrm{Const}\cdot \exp((d+\epsilon-\mathrm{Re}(k))\,|u|)\cdot \sum_{i=0}^{m} \frac{|u|^i}{|k|^{2m+1-i}} \\ &< \ \mathrm{Const}\cdot \exp(-\epsilon\,|u|)\cdot \sum_{i=0}^{m} \frac{|u|^i}{|k|^{2m+1-i}} \quad < \quad \infty \end{aligned}$$

The other conditions in [GA 77] are obviously valid. □

So we look at the trace formula (32) for these test functions. We apply formula (23) from section 2.2 and the trace formula reads

$$\sum_{j=0}^{\infty} \frac{n_j\ m!}{(r_j^2+k^2)^{m+1}} = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{m!\ f(r)}{(k^2+r^2)^{m+1}}\,dr + \left(-\frac{d}{dk^2}\right)^{m+1} (-\log Z_1(k+d))$$

Considering the sequence

$$\left\{\lambda_j \quad \text{with multiplicity} \quad n_j\ \right\}_{j=0}^{\infty}$$

defined above we find that it is a sequence of nonnegative numbers having order $m = \dim(G/K)/2$ in the sence of A. Voros (cf. [GA 77] Proposition 1.2). We introduce the associated spectral functions

$$R_m^{\mathbf{s}}\,,\ D^{\mathbf{s}}\,,\ \ldots,$$

where the index s means *spectral* due to the connection between the $\{\lambda_j\}$ and the so-called spectral zeroes of Z_1 , which are indeed related to the spectrum of the Laplacian on X. The zeroes $s_j^{\pm}$ are eigenvalues of $\pm\sqrt{\Delta_X}$ and so

$$D^{\mathbf{s}}(s) = \det{}_{\zeta}(\Delta_X - s)\,.$$

In this notation we have

$$\begin{aligned}\left(\frac{d}{ds}\right)^{m+1}\Big(-\log D^{\mathbf{s}}(s)\Big) &= R^{\mathbf{s}}_m(s)\\ &= \sum_{j=0}^{\infty}\frac{n_j\,m!}{(\lambda_j - s)^{m+1}} = \sum_{j=0}^{\infty}\frac{n_j\,m!}{(r_j^2 + k^2)^{m+1}}\quad ,\end{aligned}$$

what is exactly the left side of our trace formula. Finally we denote by I the contribution of the identity class,

$$I \overset{df}{=} \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{m!\,f(r)}{(k^2+r^2)^{m+1}}\,dr \quad ,$$

and the trace formula becomes

$$\left(\frac{d}{ds}\right)^{m+1}\Big(-\log D^{\mathbf{s}}(s)\Big) = I + \left(-\frac{d}{dk^2}\right)^{m+1}(-\log Z_1(k+d)) \quad . \tag{35}$$

3.2 Contribution of the Identity

Main purpose of this subsection is to prove a special form of the contribution

$$I = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{m!\,f(r)}{(k^2+r^2)^{m+1}}\,dr$$

of the identity class to the trace formula (35).

Proposition 2 *For* Re*(k)¿0 we have*

$$I = \left(-\frac{d}{dk^2}\right)^{m+1}\Big(-2\log D^{\mathbf{t}}(-k)\Big) \tag{36}$$

There $D^{\mathbf{t}}$ is the Gamma factor of Z_1 (introduced in section 2.2).
The trace formula (35) reads now

$$\left(\frac{d}{ds}\right)^{m+1}\Big(-\log D^{\mathbf{s}}(s)\Big) = \left(-\frac{d}{dk^2}\right)^{m+1}\Big(-2\log D^{\mathbf{t}}(-k) - \log Z_1(k+d)\Big) \tag{37}$$

Proof. The proof of (36) is in 4 steps. We use the abbreviation

$$int(r) \overset{df}{=} \frac{m!\,f(r)}{(k^2+r^2)^{m+1}} \quad .$$

and the fact that $f(r)$ is a meromorphic function having only simple poles ($\{r_j\}$ in the upper half plane with corresponding residues $\{d_j\}$).
For convenience we additionally assume that $ik \notin \{r_j\}$. $\{r_j\}$ is a discrete

subset of $\mathbf{C}$, so it suffices to prove the upper equality in this case.
The 1st step is to rewrite the right side of the stated equality with Lemma 4 from section 1.4 and with the reflection formula (30)

$$\begin{aligned}
&\left(-\frac{d}{dk^2}\right)^{m+1}\Big(-2\log D^{\mathbf{t}}(-k)\Big)\\
= &\left(-\frac{d}{dk^2}\right)^{m+1}\left(-\log(D^{\mathbf{t}}(-k)D^{\mathbf{t}}(k))+\log\left(\frac{D^{\mathbf{t}}(k)}{D^{\mathbf{t}}(-k)}\right)\right)\\
= &\left(\sum_{j=0}^{\infty}\frac{2i\,d_j\,m!}{\left(k^2+r_j^2\right)^{m+1}}+\left(-\frac{d}{dk^2}\right)^m\left[\frac{2f(ik)}{2k}\right]\right)\\
= &\ 2i\cdot\left(\sum_{j=0}^{\infty}\frac{d_j\,m!}{\left(k^2+r_j^2\right)^{m+1}}+\frac{1}{i}\left(-\frac{d}{dk^2}\right)^m\left[\frac{f(ik)}{2k}\right]\right)\quad.
\end{aligned}$$

The 2nd step is to prove (under our assumption on k)

$$\begin{aligned}
\operatorname{Res}_{r=r_j}(int(r)) &= \frac{d_j\,m!}{\left(k^2+r_j^2\right)^{m+1}}\\
\operatorname{Res}_{r=ik}(int(r)) &= \frac{1}{i}\left(-\frac{d}{dk^2}\right)^m\left[\frac{f(ik)}{2k}\right]\quad.
\end{aligned}$$

This is easy to obtain because

$$int(r)=f(r)\cdot\frac{m!}{(k^2+r^2)^{m+1}}=f(r)\cdot\frac{m!}{((r+ik)(r-ik))^{m+1}}$$

and if $ik\notin\{r_j\}$ $int(r)$ has at $r=r_j$ a simple pole with residue we wanted to prove and at $r=ik$ a pole of order $m+1$, so the residue is

$$\begin{aligned}
\operatorname{Res}_{r=ik}(int(r)) &= \left(\frac{d}{dr}\right)^m\bigg|_{r=ik}\left(\frac{f(r)}{(r+ik)^{m+1}}\right)\\
&= \frac{1}{i}\cdot\left(-\frac{d}{dr}\right)^m\bigg|_{r=k}\left(\frac{f(ir)}{(r+k)^{m+1}}\right)\\
&= \frac{1}{i}\cdot\left(-\frac{d}{dr}\right)^m\bigg|_{r=k}\left(\frac{2r}{(r+k)^{m+1}}\cdot\frac{f(ir)}{2r}\right)\\
&= \frac{1}{i}\cdot\left(-\frac{1}{2r}\frac{d}{dr}\right)^m\bigg|_{r=k}\left(\frac{f(ir)}{2r}\right)
\end{aligned}$$

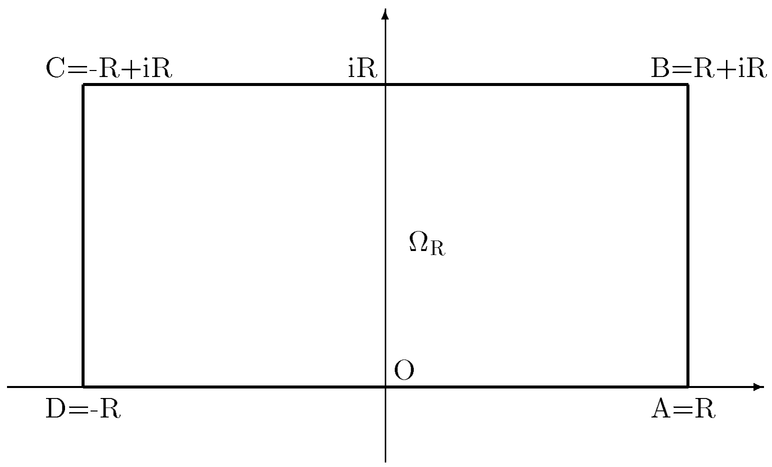
Thereby the equality of the last two lines one gets verifying by product differentiation rule and induction over m that both equal the term

$$\frac{1}{i}\cdot\sum_{j=1}^{m}(-1)^{m-j}\frac{(2m-j-1)!}{(m-j)!\,(j-1)!\,(2k)^{2m-j}}\left[\left(\frac{d}{dr}\right)^j\left(\frac{f(ir)}{2r}\right)\bigg|_{r=k}\right]\quad.$$

So it remains to check

$$\int_{-\infty}^{\infty} int(r)\,dr = 2\pi i\Big(\sum_{j=0}^{\infty} \mathrm{Res}_{\,r=r_j}\,(int(r)) + \mathrm{Res}_{\,r=ik}\,(int(r))\Big) \quad .$$

Therefore let Ω_{R} be the rectangular contour in the complex r-plane with vertices at $A = -\mathrm{R}$, $B = \mathrm{R}$, $C = \mathrm{R} + i\mathrm{R}$ and $D = -\mathrm{R} + i\mathrm{R}$ and consider only $\mathrm{R} \in \mathbb{R}$ with $\mathrm{R} > |k|$ and $i\mathrm{R} \notin \{r_k\}_{k=0}^{\infty}$.



So on the one hand we have

$$\int_{\Omega_{\mathrm{R}}} int(r)\,dr = \int_{-\mathrm{R}}^{\mathrm{R}} int(r)\,dr + I_+ + I_- + J$$

with $I_+ = \int_A^B int(r)\,dr$,$I_- = \int_C^D int(r)\,dr$ and $J = \int_B^C int(r)\,dr$,

and on the other hand by Cauchy$'$s Residue theorem

$$\int_{\Omega_{\mathrm{R}}} int(r)\,dr = 2\pi i\Big(\mathrm{Res}_{\,r=ik}\,(int(r)) + \sum_{j=0}^{N_{\mathrm{R}}} \mathrm{Res}_{\,r=r_j}\,(int(r))\Big) \quad ,$$

where $r_0, \ldots, r_{N_{\mathrm{R}}}$ are that r_j inside of Ω_{R} (following Gangolli we assumed a suitable ordering). Of course $N_{\mathrm{R}} \to \infty$ for $\mathrm{R} \to \infty$, hence we have to prove

$$\lim_{\mathrm{R}\to\infty} I_+ = \lim_{\mathrm{R}\to\infty} I_- = \lim_{\mathrm{R}\to\infty} J = 0 \quad .$$

Therefore we use that (as mentioned above) in our cases $r \to c(r)^{-1}$ is of the special form

$$\frac{1}{c(r)} = \frac{\Gamma((p+q)/2)\Gamma(ir+p/2)\Gamma(ir/2+p/4+q/2)}{\Gamma(p+q)\Gamma(ir)\Gamma(ir/2+p/4)}$$

and hence with $2m = p + q + 1$ =dim(G/K) a tempered function satisfying the inequality

$$\left|\frac{1}{c(r)\,c(-r)}\right| \leq \mathrm{Const} \cdot |r|^{2m}$$

So

$$|J| \quad \leq \quad \mathrm{Const}\cdot \int\limits_{\mathrm{R}+i\mathrm{R}}^{\mathrm{R}-i\mathrm{R}} \frac{dr}{|r|^2} \quad < \quad \mathrm{Const}\cdot \frac{\mathrm{R}}{2\,\mathrm{R}^2} \quad \overset{\mathrm{R}\to\infty}{\longrightarrow} \quad 0$$

and in the same way for I_+ and I_-. This concludes the proof of the above proposition.

3.3 Integration of the trace formula, Factorisation formula

We use the integral operator $\int_{-\infty}^{s}$ in the sence of A. Voros introduced in the first section. This operator is particularly well-suited to integrate our trace formula (37) because of the following facts:

1. $\left(\int_{-\infty}^{s}\right)^{m+1}\left(\frac{d}{ds}\right)^{m+1}\Big(-\log D^{\mathbf{s}}(s)\Big) = -\log D^{\mathbf{s}}(s)$ (cf. (6))

2. If the standard integral $\int_{-\infty}^{s} g(s')\,ds'$ converges then applying the operator $\int_{-\infty}^{s}$ to $g(s)$ gives the same result, hence one can change variables in the usual way (in that case!). So due to the exponential decrease of $\log Z_1(k)$ for $k\to\infty$ we have also (remember $s = d^2 - k^2$)

$$\begin{aligned}
&\left(\int_{-\infty}^{s}\right)^{m+1}\left(-\frac{d}{dk^2}\right)^{m+1}\Big(-\log Z_1(k+d)\Big)\\
&= \left(\int_{k}^{\infty} 2k'\,dk'\right)^{m+1}\left(-\frac{1}{2k'}\frac{d}{dk'}\right)^{m+1}\Big(-\log Z_1(k'+d)\Big)\\
&= -\log Z_1(k+d)
\end{aligned}$$

Applied to (37) we get the integrated trace formula in its first version

$$\log Z_1(k+d) = \log D^{\mathbf{s}}(s) - 2\left(\int\limits_{-\infty}^{s}\right)^{m+1}\left(-\frac{d}{dk^2}\right)^{m+1}(\log D^{\mathbf{t}}(-k)) \qquad (38)$$

To get back $-2\log D^{\mathbf{t}}(-k)$ from the second term on the right side we have to be more careful because the standard integral does not converge. Hence we have to look at its asymptotic behaviour for $k\to\infty$ (resp. $s\to-\infty$) and to calculate the contribution of the divergent leading terms.

Remember the asymptotic expansion from Lemma 4 which is of the form

$$-\log D^{\mathbf{t}}(-k) \overset{k\to\infty}{\sim} \sum_{j=1}^{\infty}\frac{c_{2j}^{\mathbf{t}}\Gamma(2j)}{k^{2j}} - \sum_{j=0}^{m} c_{-2j}^{\mathbf{t}}\left[\log k - \sum_{r=1}^{2j}\frac{1}{r}\right]\frac{k^{2j}}{(2j)!}$$

After a termwise differentiation this gives

$$\begin{aligned}
&\left(-\frac{d}{dk^2}\right)^{m+1}\Big(-\log D^{\mathbf{t}}(-k)\Big)\\
&\underset{s\to-\infty}{\overset{k\to\infty}{\sim}} \sum_{j=1}^{\infty}\frac{c_{2j}^{\mathbf{t}}\Gamma(2j)(j+m)!}{k^{2j+2m+2}(j-1)!} - \sum_{j=0}^{m}\frac{(-1)^{j+1}j!(m-j)!c_{-2j}^{\mathbf{t}}}{2(2j)!k^{2m+2-2j}}\\
&= \sum_{j=1}^{\infty}\frac{c_{2j}^{\mathbf{t}}\Gamma(2j)(j+m)!}{(d^2-s)^{j+m+1}(j-1)!} - \sum_{j=0}^{m}\frac{(-1)^{j+1}j!(m-j)!c_{-2j}^{\mathbf{t}}}{2(2j)!(d^2-s)^{m+1-j}} \quad .
\end{aligned}$$

Integrating it carefully (having in mind the integration rules (5) for $s = d^2 - k^2$) we find

$$\left(\int\limits_{-\infty}^{s}\right)^{m+1}\left(-\frac{d}{dk^2}\right)^{m+1}\Big(-\log D^{\mathbf{t}}(-k)\Big)$$

$$\overset{\substack{s\to-\infty\\ k\to\infty}}{\sim} \sum_{j=1}^{\infty}\frac{c^{\mathbf{t}}_{2j}\Gamma(2j)}{(d^2-s)^j}+\sum_{j=0}^{m}\frac{c^{\mathbf{t}}_{-2j}}{2}\left[\log(d^2-s)-\sum_{r=1}^{j}\frac{1}{r}\right]\frac{(d^2-s)^j}{(2j)!}$$

$$= \sum_{j=1}^{\infty}\frac{c^{\mathbf{t}}_{2j}\Gamma(2j)}{k^{2j}}-\sum_{j=0}^{m}c^{\mathbf{t}}_{-2j}\left[\log k-\sum_{r=1}^{j}\frac{1}{2r}\right]\frac{k^{2j}}{(2j)!}\quad .$$

Now we compare this result with the asymptotics of $-\log D^{\mathbf{t}}(-k)$ and get

$$\left(\int\limits_{-\infty}^{s}\right)^{m+1}\left(-\frac{d}{dk^2}\right)^{m+1}\Big(-\log D^{\mathbf{t}}(-k)\Big)$$

$$\overset{k\to\infty}{\sim} -\log D^{\mathbf{t}}(-k) \quad - \quad \sum_{j=1}^{m}c^{\mathbf{t}}_{-2j}\left[\sum_{r=1}^{j}\frac{1}{2r-1}\right]\frac{k^{2j}}{(2j)!}$$

We have the same singularities on both sides ($-d/dk^2 = d/ds$) and $D^{\mathbf{t}}$ is an entire function of order $2m$. So this is an equality not only asymptotically. Hence remembering (38) we find

$$\log Z_1(k+d) = \log D^{\mathbf{s}}(s) - 2\log D^{\mathbf{t}}(-k) - 2\sum_{j=1}^{m}c^{\mathbf{t}}_{-2j}\left(\sum_{r=1}^{j}\frac{1}{2r-1}\right)\frac{k^{2j}}{(2j)!}$$

Exponentiating we have the factorisation formula

$$Z_1(k+d) = \exp[-2\sum_{j=1}^{m}c^{t}_{-2j}\left(\sum_{r=1}^{j}\frac{1}{2r-1}\right)\frac{k^{2j}}{(2j)!}]\cdot\Big(D^{\mathbf{t}}(-k)\Big)^{-2}\cdot D^{\mathbf{s}}(d^2-k^2)$$

We want to find out where the exponential factor comes from. From the trace formula (37) we have the asymptotic equality

$$\left(\frac{d}{ds}\right)^{m+1}\Big(-\log D^{\mathbf{s}}(s)\Big) \overset{\substack{k\to\infty\\ s\to-\infty}}{\sim} \left(-\frac{d}{dk^2}\right)^{m+1}\Big(-2\log D^{\mathbf{t}}(-k)\Big)$$

Introducing the coefficients $c^{\mathbf{s}}_{-d^2,j}$ associated to the functional determinant $D^{\mathbf{s}\,(-d^2)}$ of the shifted sequence $\{\lambda_j - d^2\}$ and remembering the relation $D^{\mathbf{s}\,(-d^2)}(s-d^2) = D^{\mathbf{s}}(s)$ from Lemma 1 we find asymptotic expansions of both sides.

$$\text{r.s.} \quad \overset{k\to\infty}{\sim} \quad 2\sum_{j=1}^{\infty}\frac{c^{\mathbf{t}}_{2j}\Gamma(2j)(j+m)!}{k^{2j+2m+2}(j-1)!}-\sum_{j=0}^{m}\frac{(-1)^{j+1}j!(m-j)!c^{\mathbf{t}}_{-2j}}{(2j)!k^{2m+2-2j}}$$

$$\text{l.s.} \quad \overset{s-d^2\to-\infty}{\sim} \quad \sum_{j=-m}^{\infty}\frac{c^{\mathbf{s}}_{-d^2,j}\Gamma(j+m+1)}{(d^2-s)^{j+m+1}}$$

$$= \quad \sum_{j=1}^{\infty}\frac{c^{\mathbf{s}}_{-d^2,j}(j+m)!}{k^{2j+2m+2}}+\sum_{j=0}^{m}\frac{(m-j)!c^{\mathbf{s}}_{-d^2,-j}}{k^{2m+2-2j}}\quad .$$

Combining these asymptotics we find the relations

$$\begin{aligned}(-1)^j \frac{c^{\mathbf{s}}_{-d^2,-j}}{j!} &= \frac{c^{\mathbf{t}}_{-2j}}{(2j)!} \quad (0 \leq j \leq m) \\ c^{\mathbf{s}}_{-d^2,j} &= \frac{2\Gamma(2j)}{\Gamma(j)} c^{\mathbf{t}}_{2j} \quad (j \geq 1)\end{aligned}$$

so we can express the exponential factor $\exp\left[\sum_{j=1}^m T_j(k^2)\right]$ in the factorisation formula in terms of s and the $c^{\mathbf{s}}_{-d^2,-j}$ as well,

$$\begin{aligned}T_j(k^2) &\overset{df}{=} -2c^{\mathbf{t}}_{-2j}\left(\sum_{r=1}^{j}\frac{1}{2r-1}\right)\frac{(k^2)^j}{(2j)!} &(39)\\ &= -2c^{\mathbf{s}}_{-d^2,-j}\left(\sum_{r=1}^{j}\frac{1}{2r-1}\right)\frac{(-k^2)^j}{j!} \\ &= -2c^{\mathbf{s}}_{-d^2,-j}\left(\sum_{r=1}^{j}\frac{1}{2r-1}\right)\frac{(s-d^2)^j}{j!} \quad , &(40)\end{aligned}$$

or better in a mixed form (the difference of the divergent leading terms in the asymptotics of both functional determinants)

$$\begin{aligned}T_j(k^2) &= -2c^{\mathbf{t}}_{-2j}\left(\sum_{r=1}^{2j}\frac{1}{r} - \sum_{r=1}^{j}\frac{1}{2r}\right)\frac{(k^2)^j}{(2j)!} \\ &= \left[(-1)^j\frac{c^{\mathbf{s}}_{-d^2,-j}}{j!}\left(\sum_{r=1}^{j}\frac{1}{r}\right) - \frac{2c^{\mathbf{t}}_{-2j}}{(2j)!}\left(\sum_{r=1}^{2j}\frac{1}{r}\right)\right]\cdot k^{2j} \\ &= c^{\mathbf{s}}_{-d^2,-j}\left(\sum_{r=1}^{j}\frac{1}{r}\right)\frac{(s-d^2)^j}{j!} - 2c^{\mathbf{t}}_{-2j}\left(\sum_{r=1}^{2j}\frac{1}{r}\right)\frac{(k^2)^j}{(2j)!} &(41)\end{aligned}$$

We comprehence our result in

Proposition 3 (Factorisation formula) *Let Z_1 be the spherical Selberg zeta function of X with topological zeroes $ir_k - d$, $k \geq 0$, (multiplicity $-4id_k$) and spectral zeroes $s_j^{\pm}$, $j \geq 0$, (mult. n_j). Then we can express it as the product of two zeta regularized determinants as*

$$\begin{aligned}Z_1(k+d) &= \exp\left[\sum_{j=1}^m T_j(k^2)\right] \cdot \left(D^{\mathbf{t}}(-k)\right)^{-2} \cdot D^{\mathbf{s}}(d^2-k^2) \\ &= \exp\left[\sum_{j=1}^m T_j(k^2)\right] \det\nolimits_\zeta\left(\sqrt{-\Delta_{G^d/K}+d^2}-k\right)^{-\frac{2\chi(X)}{\chi(G^d/K)}} \det\nolimits_\zeta(\Delta_X - s)\end{aligned}$$

with $D^{\mathbf{t}}$ the functional determinant of $\{-ir_k\,,\ \text{mult. } 2id_k\}$, $D^{\mathbf{s}}$ the functional determinant of $\{s_j^+ \cdot s_j^-\,,\ \text{mult. } n_j\}$ and $T_j(k)$ as in one of the equivalent expressions (39), (40), (41).

The last expression for T_j explains why it occurs in the factorisation formula. For an explicite calculation however one has to use the expression (39) because only the $c^{\mathbf{t}}_{-2j}$ are known. Looking at Lemma 4 one has to determine the coefficient b_j of z^{2j+1} in the odd polynomial $Q(z)$ that gives the multiplicities of the defining sequence and gets

$$c^{\mathbf{t}}_{-2j} = \Gamma(2j)\, b_{j-1}$$

In the previous section we have calculated the multiplicity of the topological zero z_k as $C(X)\ Q_1(z_k)$ with $C(X)$ a constant and Q_1 the polynomial in the spherical plancherel density (see table 2). So the $c^{\mathbf{t}}_{-2j}$ are calculable in every case.
For the case of a real hyperbolic surface we calculated the multiplicity as $z \cdot \chi(X) = z \cdot (2-2g)$ with g the genus of the surface. So there is only $b_0 = 2-2g$ nonvanishing and the exponential factor in the factorisation formula is

$$\exp(T_1(k^2)) = \exp(2g-2)k^2$$

which is after comparing the notation precisely the same result as in [VO 87].

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