The Painlevé III Equation and the Iwasawa Decomposition

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Abstract

For the third Painlevé equation an explicit isomorphism between the monodromy data and the data of the approach of Dorfmeister-Pedit-Wu, based on the Iwasawa decomposition of the loop groups, is established. As an application, this provides a simple algebraic way to calculate the monodromy data in terms of the Cauchy data at zero.

1 Introduction

Recent progress in differential geometry of surfaces obtained by the methods of the theory of the integrable equations attracts attention to this area and supplies new problems for the soliton theory, which induce a development of new methods. One of the most interesting problems in this area is a construction of the constant mean curvature (CMC) surfaces. The corresponding Gauss equation describing these surfaces, written in invariant form, reads as follows (for details see, for example, [2]):

$$u_{z\bar{z}} + 2e^{u} - 2Q\bar{Q}e^{-u} = 0. \tag{1}$$

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Here z is a local complex variable on a Riemann surface, $e^u dz d\bar{z}$ is a conformal metric of the surface in Euclidean space and $Q(z)dz^2$ is the holomorphic Hopf differential. By the transformation

$$\frac{dw}{dz} = \sqrt{Q}, \;\; v = u - \log|Q| \tag{2}$$

this equation for a simply connected domain can be reduced to the elliptic sinh-Gordon equation

$$v_{w\bar{w}} + 4\sinh v = 0. \tag{3}$$

The umbilic points are the zeroes of Q. Obviously, the solution $v(w,\bar{w})$ is singular at these points if the metric $u(z,\bar{z})$ is smooth.

To integrate these equations Dorfmeister, Pedit and Wu recently suggested a new method [4], , generating all the solutions and based on the Iwasawa decomposition of the loop groups. In fact, this approach can be made an effective tool [11, 9] to computer construction of simply connected CMC surfaces with arbitrary disposition of umbilics. The Dorfmeister-Pedit-Wu (DPW) method parametrizes the solutions by holomorphic data (see Sect.2). The question of understanding this parametrization in the conventional terms of the soliton theory arises.

In the present paper we treat the rotational symmetric case of (1), which leads to the Painlevé III equation, and identify this holomorhpic parametrization with the monodromy data standard in the theory of the Painlevé equations [6]. Unexpectedly, this identification provides us an elegant way to connect the asymptotics of the Painlevé III function at zero with the monodromy data. The usual schemes for deriving the corresponding connection formulas are based (see, for example, [7],[6]) on the nontrivial asymptotic analysis of the auxiliary linear system. The Iwasawa decomposition approach, which we present in Sect.5, reduces the needs of the asymptotic analysis to trivial references to the known asymptotic properties of the Bessel functions.

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2 The method of Dorfmeister-Pedit-Wu

Let $\mathcal{D}$ be a simply connected domain, $z = 0 \in \bar{\mathcal{D}}$ and $\xi(\lambda, z)$ be an element

$$\xi = \sum_{k=-1}^{\infty} \xi_k \lambda^k$$

of the twisted loop algebra

$$\lambda_\tau^{\mathbf{C}} = \{\eta = \sum_{k=-\infty}^{\infty} \eta_k \lambda^k, \eta_k \in sl(2, \mathbf{C}), \sigma_3 \eta(-\lambda) \sigma_3 = \eta(\lambda)\},$$

which is a holomorphic function of z. We use the standard notations

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for the Pauli matrices.

The solution of the initial problem

$$C_z = \xi C, \quad C(z = 0) = I \tag{4}$$

on $\mathcal{D}$ belongs to the twisted loop group

$$\Lambda_\tau^{\mathbf{C}} = \{C = \sum_{k=-\infty}^{\infty} C_k \lambda^k, C_k \in Mat(2, \mathbf{C}), \det C = 1, \sigma_3 C(-\lambda) \sigma_3 = C(\lambda)\}.$$

Let us define the subgroups $\Lambda_{\tau,+}^{\mathbf{C}}, \Lambda_\tau$ of $\Lambda_\tau^{\mathbf{C}}$

$$\Lambda_{\tau,+}^{\mathbf{C}} = \{B \in \Lambda_\tau^{\mathbf{C}}, B = \sum_{k=0}^{\infty} B_k \lambda^k\},$$
$$\Lambda_\tau = \{\Psi \in \Lambda_\tau^{\mathbf{C}}, \bar{\Psi}(1/\bar{\lambda}) = \sigma_2 \Psi(\lambda) \sigma_2\}.$$

Theorem 1 *Any element $C \in \Lambda_\tau^{\mathbf{C}}$ can be uniquely factorized into a product*

$$C = B\Psi, \tag{5}$$

where $\Psi \in \Lambda_\tau, B \in \Lambda_{\tau,+}^{\mathbf{C}}$,

$$B(\lambda = 0) = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}, \rho > 0.$$

The map $\Lambda_\tau^{\mathbf{C}} \to \Lambda_{\tau,+}^{\mathbf{C}} \times \Lambda_\tau$ is a diffeomorphism.

For the proof of this theorem see [4], [12].

To show that so defined Ψ is the Ψ-function of the corresponding linear problem for the equation (1) let us consider the logarithmic derivatives of Ψ, which belong to λ_τ

$$U(\lambda) = \Psi_z \Psi^{-1} = B^{-1}\xi B - B^{-1}B_z = \textstyle\sum_{k=-1}^{\infty} U_k \lambda^k,$$
$$V(\lambda) = \Psi_{\bar z} \Psi^{-1} = -B^{-1}B_{\bar z} = \textstyle\sum_{k=0}^{\infty} V_k \lambda^k$$

with

$$V_0 = -\frac{\rho_{\bar z}}{\rho}\sigma_3, \quad U_{-1} = \begin{pmatrix} 0 & \rho^{-2} f \\ \rho^2 g & 0 \end{pmatrix},$$

where we denote the coefficient ξ_{-1} by

$$\xi_{-1} = \begin{pmatrix} 0 & f \\ g & 0 \end{pmatrix}.$$

The symmetry $\Psi \in \Lambda_\tau$ implies

$$\bar U(1/\bar\lambda) = \sigma_2 V(\lambda)\sigma_2$$

and the following structure of $U(\lambda), V(\lambda)$:

$$U(\lambda) = U_{-1}\lambda^{-1} + U_0, \;\; V(\lambda) = \lambda V_1 + V_0,$$
$$U_0 = \frac{\rho_z}{\rho}\sigma_3, \;\; V_1 = \begin{pmatrix} 0 & -\rho^2 \bar g \\ -\rho^{-2}\bar f & 0 \end{pmatrix}.$$

Taking

$$\rho = e^{u/4}, \;\; f = Q, \;\; g = 1$$

one gets the following theorem.

Theorem 2 *Let $C(z,\lambda)$ be a solution of (4) with ξ_{-1} of the form*

$$\xi_{-1} = \begin{pmatrix} 0 & Q \\ 1 & 0 \end{pmatrix}.$$

The function $\Psi(\lambda)$ determined by the Iwasawa decomposition (5) of $C(z,\lambda)$ satisfies

$$\Psi_z = U\Psi, \;\; \Psi_{\bar z} = V\Psi \tag{6}$$

with U, V of the form

$$U = \begin{pmatrix} \dfrac{u_z}{4} & \dfrac{1}{\lambda} Q e^{-u/2} \\ \dfrac{1}{\lambda} e^{u/2} & -\dfrac{u_z}{4} \end{pmatrix}, V = \begin{pmatrix} -\dfrac{u_{\bar z}}{4} & -\lambda e^{u/2} \\ -\lambda \bar Q e^{-u/2} & \dfrac{u_{\bar z}}{4} \end{pmatrix}. \tag{7}$$

The system (6,7) is nothing else but the Lax representation for the Gauss equation (1) of the CMC surfaces [2].

Remark. Function $\Psi(\lambda)$ can be thought of as a solution of the Riemann-Hilbert problem for the two component contour consisting of a small circle about $\lambda = \infty$ and a small circle about $\lambda = 0$ with the matrix functions $\sigma_2 \bar{C}(1/\bar{\lambda})\sigma_2$ and $C(\lambda)$ as the corresponding jump-matrices. This Riemann-Hilbert problem is an elliptic reduction of the general complex construction $(z \to x, \bar{z} \to t)$ proposed by Krichever [8]. As an important consequence of Theorem 1 we get the global solvability of this Riemann-Hilbert problem under the symmetry in question.

3 Reduction to the third Painlevé equation

Let us take ξ of the special form

$$\xi = \frac{1}{\lambda}\begin{pmatrix} 0 & z^{\alpha} \\ 1 & 0 \end{pmatrix} \tag{8}$$

and C be a solution of

$$C_z = \xi C, \; C(z=0) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \; a > 0. \tag{9}$$

This problem is clearly gauge equivalent to (4).

Theorem 3 *The solution $u(z,\bar{z})$ of the equation (1) generated by the data (8, 9) is rotationaly symmetric and depends on $|z|$ only.*

To prove this theorem we remark that the problem (8,9) possesses the symmetry

$$C(z,\lambda) \to \tilde{C}(\tilde{z},\tilde{\lambda}) = \exp(i\frac{\alpha\phi}{4}\sigma_3)C(ze^{i\phi}, \lambda e^{i\phi}e^{i\alpha\phi/4}).$$

This transformation clearly does not effect $B(\lambda = 0)$ in the Iwasawa decomposition (5)

$$\rho(ze^{i\phi}) = \rho(z).$$

The variables w, v in (2) in the case $Q = z^{\alpha}$ under consideration are equal to

$$w = \frac{z^{\beta}}{\beta}, \; e^{v} = \rho^4 |z|^{-\alpha}, \beta = 1 + \frac{\alpha}{2}. \tag{10}$$

Function v is a function of $|z|$ only. It satisfies an ordinary differential equation

$$v_{xx} + \frac{1}{x}v_x + \sinh v = 0, \tag{11}$$

where x is defined by

$$x = 4|w| = 4\frac{|z|^\beta}{\beta}. \tag{12}$$

Equation (11) is a particular case of the third Painlevé equation.

The DPW method selects smooth for $x \in (0, \infty]$ solutions to equation (11) parametrized by the exponent α in (8) and initial data a in (9),

$$v = v(x|\alpha, a).$$

This poses a question about the relationship between the Iwasawa - decomposition parameters (α, a) and the standard parametrization of the third Painlevé transcendents by the monodromy data of the auxiliary linear system, which corresponds equation (11) according to the isomonodromy method [6]. This question is treated in the next section. We conclude this section by the following theorem.

Theorem 4 *The behavior of the solution $v(x|\alpha, a)$ as $x \to 0$ is described as follows:*

$$v = r\log x + s + o(1), \tag{13}$$

where

$$r = -\frac{2\alpha}{2+\alpha},\ s = 4\log a - \frac{2\alpha}{2+\alpha}\log\frac{2+\alpha}{8}. \tag{14}$$

According to Theorem 1, ρ is a smooth function of $x \in [0, \infty)$. Moreover, $C(z=0) = D$ implies

$$\rho(x=0) = a.$$

Now to conclude the proof one needs only to take into account formulas (10, 12).

4 Connection formulas between the Iwasawa-decomposition data and the monodromy data

In the framework of the isomonodromy method the third Painlevé equation (11) is associated [5] with the linear system

$$\Phi_\zeta = (-\frac{ix^2}{16}\sigma_3 + \frac{xv_x}{4\zeta}\sigma_1 + \frac{i}{\zeta^2}\cosh v\sigma_3 + \frac{1}{\zeta^2}\sinh v\sigma_2)\Phi, \tag{15}$$

which monodromy data are preserved by equation (11).

The monodromy data for system (15) can be defined (for the details see [6]) through the connection matrix

$$Q = \Phi_0^{-1}(\zeta)\Phi_\infty(\zeta). \tag{16}$$

In equation (16) $\Phi_0,\ \Phi_\infty$ denote the canonical solutions of (15), which correspond to the irregular singular points $\zeta = 0$ and $\zeta = \infty$ respectively:

$$\begin{gathered}
\Phi_0(\zeta) = (\cosh \tfrac{v}{2} I + \sinh \tfrac{v}{2}\sigma_1)(I + O(\zeta))\exp(-\tfrac{i}{\zeta}\sigma_3),\\
\zeta \to 0,\ -\pi < \arg\zeta < \pi,\\
\Phi_\infty(\zeta) = (I + O(\zeta^{-1}))\exp(-\tfrac{ix^2}{16}\zeta\sigma_3),\\
\zeta \to \infty,\ -\pi < \arg\zeta < \pi.
\end{gathered}$$

Matrix Q has the form

$$Q = (1 - |p|^2)^{-1/2}\begin{pmatrix} 1 & p \\ \bar{p} & 1 \end{pmatrix}. \tag{17}$$

Theorem 5 *[5] p is the first integral of the Painlevé equation (11).*

The complex number p provides an alternative parametrization for the third Painlevé transcendent $v(x|\alpha, a)$,

$$v(x|\alpha, a) \equiv v(x|p).$$

Theorem 6 *The connection formulas between the parameters p and (α, a) are given by relations*

$$\begin{aligned}
p &= \frac{Ae^{-i\frac{\pi}{2\beta}+i\frac{\pi}{2}} - Be^{i\frac{\pi}{2\beta}-i\frac{\pi}{2}}}{A+B},\\
A &= (2\beta)^{1-1/\beta}\Gamma^2(1-\frac{1}{2\beta})a^{-2},\\
B &= (2\beta)^{\frac{1}{\beta}-1}\Gamma^2(\frac{1}{2\beta})a^2,\ \beta = 1 + \frac{\alpha}{2},
\end{aligned} \tag{18}$$

where $\Gamma(t)$ is the Euler gamma-function.

To prove the theorem let us consider the basic equation (9), which defines element $C \in \Lambda_\tau^{\mathbf{C}}$. This equation can be easily solved in terms of the Bessel functions, which gives the following representations for the entries $C_{kl}, k,l=1,2$:

$$
\begin{aligned}
C_{11} &= iz^{\beta-\frac{1}{2}}\lambda^{\frac{1}{2\beta}-1}a\Gamma(1+\frac{1}{2\beta})(2\beta)^{\frac{1}{2\beta}}e^{-\frac{i\pi}{4\beta}}J_{\frac{1}{2\beta}-1}(\frac{iz^\beta}{\beta\lambda}),\\
C_{12} &= -iz^{\beta-\frac{1}{2}}\lambda^{-\frac{1}{2\beta}}\frac{1}{a}\Gamma(1-\frac{1}{2\beta})(2\beta)^{-\frac{1}{2\beta}}e^{\frac{i\pi}{4\beta}}J_{-\frac{1}{2\beta}+1}(\frac{iz^\beta}{\beta\lambda}),\\
C_{21} &= z^{\frac{1}{2}}\lambda^{\frac{1}{2\beta}-1}a\Gamma(1+\frac{1}{2\beta})(2\beta)^{\frac{1}{2\beta}}e^{-\frac{i\pi}{4\beta}}J_{\frac{1}{2\beta}}(\frac{iz^\beta}{\beta\lambda}),\\
C_{22} &= z^{\frac{1}{2}}\lambda^{-\frac{1}{2\beta}}\frac{1}{a}\Gamma(1-\frac{1}{2\beta})(2\beta)^{-\frac{1}{2\beta}}e^{\frac{i\pi}{4\beta}}J_{-\frac{1}{2\beta}}(\frac{iz^\beta}{\beta\lambda}).
\end{aligned}
\tag{19}
$$

The nontrivial dependence on z is due to the combination z^β/λ. Therefore, one can expect a differential λ-equation for C. In fact, setting

$$C_0 = C\lambda^{\frac{\beta-1}{2\beta}\sigma_3},$$

we obtain that C satisfies the equation

$$C_{0\ \lambda} = \left(-\frac{1}{\beta\lambda^2}\begin{pmatrix} 0 & z^{2\beta-1} \\ z & 0 \end{pmatrix} + \frac{1}{2\beta\lambda}\begin{pmatrix} \beta-1 & 0 \\ 0 & 1-\beta \end{pmatrix}\right) C_0.$$

Performing the same transformation with the element Ψ,

$$\Psi \to \Psi_0 = \Psi\lambda^{\frac{\beta-1}{2\beta}\sigma_3},$$

and repeating the same arguments as in Sect.2 we come to the equation

$$\Psi_{0\ \lambda}\Psi^{-1} = -\frac{1}{\beta}\begin{pmatrix} 0 & \rho^2\bar{z} \\ \rho^{-2}\bar{z}^{2\beta-1} & 0 \end{pmatrix} - \frac{1}{4\lambda}xv_x\sigma_3 - \frac{1}{\beta\lambda^2}\begin{pmatrix} 0 & \rho^{-2}z^{2\beta-1} \\ \rho^2 z & 0 \end{pmatrix}.$$

Recall that $\rho = e^{v/4}|z|^{\frac{\beta-1}{2}}$, where $v \equiv v(x|\alpha,a), x = 4|z|^\beta/\beta$. To recover the ζ- equation (15) it remains to make an obvious gauge transformation

$$
\begin{gathered}
\Psi_0 \to \Phi = T^{-1}\Psi_0,\\
T = \begin{pmatrix} \rho\bar{z}^{\frac{1-\beta}{2}} & -\rho\bar{z}^{\frac{1-\beta}{2}} \\ \rho^{-1}\bar{z}^{\frac{-1+\beta}{2}} & \rho^{-1}\bar{z}^{\frac{-1+\beta}{2}} \end{pmatrix}
\end{gathered}
$$

and to rescale variable λ

$$\lambda = iz^\beta\zeta\frac{1}{\beta}.$$

Taking into account equations

$$\Phi = T^{-1}\Psi\lambda^{\frac{\beta-1}{2\beta}\sigma_3},\ \bar{\Psi}(1/\bar{\lambda}) = \sigma_2\Psi(\lambda)\sigma_2\ \ \Psi = B^{-1}C,$$

explicit formula (19) for C and known asymptotics of the Bessel functions [1], we conclude that

$$\begin{gathered}\Phi(x,\zeta) = \sqrt{\tfrac{\beta}{2\pi}}(\cosh\tfrac{v}{2}I + \sinh\tfrac{v}{2}\sigma_1)(I+O(\zeta))\exp(-\tfrac{i}{\zeta}\sigma_3)T_0 D,\\ \zeta\to 0,\ -\pi < \arg\zeta < \pi,\\ \Phi(x,\zeta) = \sqrt{\tfrac{\beta}{2\pi}}(I+O(\zeta^{-1}))\exp(-\tfrac{ix^2}{16}\zeta\sigma_3)T_\infty\sigma_2\bar{D}\sigma_2,\\ \zeta\to\infty,\ -\pi < \arg\zeta < \pi,\end{gathered}$$

where

$$T_0 = \begin{pmatrix} \Gamma(1+\frac{1}{2\beta})(2\beta)^{\frac{1}{2\beta}} & \Gamma(1-\frac{1}{2\beta})(2\beta)^{-\frac{1}{2\beta}} \\ -ie^{-\frac{i\pi}{2\beta}}\Gamma(1+\frac{1}{2\beta})(2\beta)^{\frac{1}{2\beta}} & -ie^{\frac{i\pi}{2\beta}}\Gamma(1-\frac{1}{2\beta})(2\beta)^{-\frac{1}{2\beta}} \end{pmatrix},$$

$$T_\infty = \begin{pmatrix} ie^{\frac{-i\pi}{2\beta}}\Gamma(1-\frac{1}{2\beta})(2\beta)^{-\frac{1}{2\beta}} & -ie^{\frac{i\pi}{2\beta}}\Gamma(1+\frac{1}{2\beta})(2\beta)^{\frac{1}{2\beta}} \\ -\Gamma(1-\frac{1}{2\beta})(2\beta)^{-\frac{1}{2\beta}} & \Gamma(1+\frac{1}{2\beta})(2\beta)^{\frac{1}{2\beta}} \end{pmatrix}.$$

This leads to the following representations for the canonical solutions $\Phi_{0,\infty}$ of the system (15):

$$\Phi_0 = \sqrt{\frac{2\pi}{\beta}}\Phi D^{-1}T_0^{-1},$$

$$\Phi_\infty = \sqrt{\frac{2\pi}{\beta}}\Phi\sigma_2\bar{D}^{-1}\sigma_2 T_\infty^{-1},$$

that yields

$$Q = T_0 D\sigma_2\bar{D}^{-1}\sigma_2 T_\infty^{-1} \tag{20}$$

for the corresponding connection matrix. Substituting explicit expressions for $T_{0,\infty}$ into (20), we obtain (17) with the monodromy parameter p defined by the equations (18). This completes the proof of the theorem.

Remark. If $\beta = 1$ ($\alpha = 0$ - regular at $x = 0$ solution u), then equation (18) is simplified up to the formula $p = (a^{-2} - a^2)/2$.

5 The connection formulas between the monodromy data and the Cauchy data for $v(x|p)$.

As an unexpected consequence of the analysis presented above we obtain an *elementary* and *rigorous* derivation of the connection formula between the monodromy parameter p and the Cauchy data (r,s) from (13).

Theorem 7 *The monodromy parameter p of the solution $v(x|p)$ can be expressed in terms of the Cauchy data (r,s) according to the equations*

$$\begin{aligned} p &= \frac{Ae^{-\frac{ir}{4}\pi} - Be^{\frac{ir}{4}\pi}}{A+B}, \\ A &= 2^{-\frac{3}{2}r}e^{-\frac{s}{2}}\Gamma^2(\frac{1}{2}-\frac{r}{4}), \\ B &= 2^{\frac{3}{2}r}e^{\frac{s}{2}}\Gamma^2(\frac{1}{2}+\frac{r}{4}). \end{aligned} \tag{21}$$

Solving equations (14) for $\beta = 1 + \frac{\alpha}{2}$ and a we get

$$\beta = \frac{2}{r+2}, \quad a = e^{s/4}(2r+4)^{r/4}.$$

Now, the theorem is an immediate consequence of the formulae (18).

Connection formulae (21) were first obtained by Novokshenov [10] by using different and more complicated approach based on rather delicate asymptotic analysis of the ζ -system (15) and taking formula (13) as an prior information about the behavior of the solution of the Painlevé equation (11) at $x = 0$. Here we have shown that the connection formula (21) can be derived purely algebraically and rigorously from the Iwasawa decomposition only without any prior information about solution u.

It was shown in [10] (see also [6]) that for $|p| < 1$ solution $v(x|p)$ has the following asymptotics as $x \to \infty$:

$$v(x|p) = \gamma x^{-1/2}\sin\{x + \frac{\alpha^2}{16}\log x + \phi\} + o(x^{-1/2}), \tag{22}$$

where the asymptotic parameters (γ, ϕ) are

$$\gamma = -\frac{8}{\pi}\log(1-|p|^2), \quad \gamma > 0, \tag{23}$$

$$\phi = \frac{\alpha^2}{8}\log 2 + \frac{3\pi}{4} - \arg\Gamma(\frac{i\alpha^2}{16}) - \arg p. \tag{24}$$

Theorem 6 provides us with an explicit expression for p in terms of the Iwasawa decomposition data (α, a). Therefore, given $\alpha \geq 0, a > 0$ we can evaluate the asymptotic behavior of the function $v(x|\alpha, a)$ at $x \to \infty$ as well as at $x \to 0$. In other words, we possess a complete *global* information about third Painlevé transcendent $v(x|\alpha, a)$.

Remark. Apparently, a deductive derivation of the asymptotics (22) and the connection formulas (23,24) can not be obtained purely algebraically without an appropriate asymptotic analysis of the monodromy problem (16) in the spirit of the nonlinear steepest-descent method proposed recently by Deift and Zhou [3].

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