ON THE REAL POLARIZATION OF CHERN–SIMONS THEORY

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Abstract

A construction of the Hilbert space of a $2+1$–dimensional topological quantum field theory due to Lisa Jeffrey and Jonathan Weitsman is studied in detail. We complete the argument for isotropy of the leaves of a certain foliation on the moduli space of flat $SU(2)$–connections. This defines a real polarization of the prequantum system. Then the Hilbert space is defined to be generated by the set of 'Bohr–Sommerfeld–leaves', i.e. of those leaves admitting a parallel section in the prequantum line bundle. The characterization of these leaves is done alternatively using arguments familiar in general constructions of TQFT.

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1 Introduction

Since the celebrated work of Witten [W] there have been many successful attempts to construct 2+1–dimensional topological field theories ([APW, Hi], for a definition see [A] et al). One of the main tools therein (suggested by Witten) is the quantization via Kähler polarization of the following prequantum system:

Denote by g the genus of the closed Riemann surface Σ^g and by $\bar{\mathcal{S}}_g(\kappa,\lambda)$ the moduli space of projectively flat connections with fixed determinant on the vector bundle of rank κ and degree λ. On the latter a symplectic stratification is given, i.e. one the strata of which are open symplectic manifolds whose symplectic structures obey certain gluing conditions (see [H] for details). Denote by ω the symplectic form on each stratum. This ω defines an integral class in the second cohomology. According to Narasimhan and Seshadri [NS] $\bar{\mathcal{S}}_g(\kappa,\lambda)$ is a $(\kappa^2(g-1)+1-g)$–dimensional algebraic variety and the open and dense set $\mathcal{S}_g$ of irreducible connections is a symplectic manifold; moreover, in the spirit of the first statement, an open algebraic manifold. The Riemannian structure on Σ^g defines almost complex structures on the strata of $\bar{\mathcal{S}}_g$ which coincide with the algebraic structure described above. But then $\omega \in H^{1,1}$ on each stratum.

First, one constructs the prequantum line bundle, i.e. a line bundle $\mathcal{L}$ over $\bar{\mathcal{S}}_g$ which is holomorphic and has first Chern class $c_1(\mathcal{L}) = [\omega]$. Knowing that our symplectic form ω is a Kähler form for the Narasimhan–Seshadri construction it suffices to construct a Hermitian line bundle $\mathcal{L}$ together with a Hermitian connection ∇ such that the curvature satisfies $R^\nabla = 2\pi i\omega \in H^{1,1}$. Then the corresponding complex structure is integrable and the topological type of $\mathcal{L}$ is of the kind we are looking for. To each element ϕ of the mapping class group of Σ^g we may assign a symplectomorphism Φ on $\bar{\mathcal{S}}_g$. There is a lift of that action to the group of parallel bundle automorphisms of $\mathcal{L}$. Now the Hilbert space of the quantization is defined to be the space of holomorphic sections $\mathcal{H}_k^{Kahler}(\Sigma^g) = H^0(\mathcal{L}^k)$. The (projective unitary) action of the mapping class group and the pairing appear naturally [APW, Hi]. Difficulties arise in constructing a vacuum vector associated to a three–manifold bounded by Σ^g. An approach for the handlebody is given in [CLM] (but again up to a phase factor only).

Jeffrey and Weitsman have given in [JW1] an alternative construction of a 2+1–dimensional topological quantum field theory based on a real polarization of the moduli space of flat $SU(2)$–connections ($\kappa = 2$, $\lambda = 0$, trivial fixed determinant) on a closed, oriented Riemann surface.

For $SU(2)$–bundles ($\kappa = 2, \lambda = 0$) they construct a set of $(3g-3)$ real functions $h_i : \bar{\mathcal{S}}_g \longrightarrow R$ using the trace of the holonomy of the flat connection around each element of a system of $(3g-3)$ simple, non–intersecting loops on Σ^g. The image of $\pi = (h_1, \ldots, h_{3g-3})$ is a polyhedron in R^{3g-3} denoted by B_g. From [G, W] one knows that these functions Poisson commute according to the fact that two arbitrary loops of our system do not intersect.

One can show that the preimages of π are isotropic, i.e. the intersection with each stratum of the symplectic stratification is isotropic. Jeffrey and Weitsman have shown this for the preimages where $h_i \neq 0; 1$, i.e. none of the holonomies around the loops lie in the center of $SU(2)$ (see [W]) and for the case of a handlebody, i.e. where all holonomies around the chosen loops are trivial. We complete the argument using ideas coming from TQFT.

This provides us with a 'Lagrangian foliation', although there may be many 'leaves' not having half the dimension, which, nevertheless, cannot be neglected but have to be taken into account in the computations. So, we have a kind of generalized real polarization here (see [Wo] for a definition).

Restricting $(\mathcal{L}, \nabla)$ to the isotropic leaves we end up with flat Hermitian line bundles. Suppressing the subtleties coming from using half densities the quantum Hilbert space of Σ^g is roughly defined as

$$\mathcal{H}_k(\Sigma^g) = \bigoplus_{b \in B_g^k} H^0(\mathcal{L}^k|_{\pi^{-1}(b)}, \nabla^{\mathcal{L}^k}|_{\pi^{-1}(b)}).$$

Here $B_s^k \subset B_g$ denots the set of those leaves where the flat connection is actually trivial, the so–called Bohr–Sommerfeld leaves. It turns out that the dimension of these Hilbert spaces is exactly the expected dimension of the Hilbert spaces in the Kähler polarization as predicted by Velinde's formulas [V]. The advantage of this polarization is a rather easy construction of the vacuum vectors in favourable cases of three–manifolds (such as handlebodies and 'generalized handlebodies' described in the paper). Difficulties arise in the construction of the action of the mapping class group. An ansatz is given in [JW3]. However, the techniques of the real polarization have been proven to be a powerful tool in investigating the moduli space $\bar{\mathcal{S}}_g$ itself. Jeffrey and Weitsman computed in [JW2] the volume and Donaldson suggested similar techniques in [D] to compute the cohomology ring without making use of Verlinde's dimension formula for the Hilbert spaces of the Kähler polarization. Conversely, supposing that the Riemann–Roch–Hirzebruch formula holds on $\bar{\mathcal{S}}_g$ and that there are vanishing theorems for the higher cohomology groups, one may obtain the Verlinde formula in this rather geometrical way.

The paper is organized as follows. In Chapter 2 we construct the prequantum system for moduli spaces of flat $SU(2)$– and $SO(3)$–connections (the latter corresponds more or less to $\kappa = 2, \lambda = 0$ or 1). Line bundle and connection are constructed via the Chern–Simons cocycle (see also [RSW]). In Chapter 3 we construct the vacuum vector for certain three-manifolds (the so–called 'generalized handlebodies'.) Next we give a sketch of the construction of a real polarization (Chapter 4) and the characterization of the set of Bohr–Sommerfeld–leaves (Chapter 5) which is done in [JW2] in detail.

However, we will give another proof for the non–generic leaves using once more arguments coming from TQFT (used in Theorem 3.1 of [JW2] for the construction of the parallel section $s : L_H \longrightarrow \mathcal{L}^k_{|L_H}$) for the 'generalized handlebody' via the Chern–Simons functional CS.

Now having some experience with Chern–Simons gauge theory we are able to give a rather geometrical proof of the following identity (mentioned in [DS]):
Consider a Heegard splitting of a closed, oriented 3–manifold M^3 into two handlebodies of genus g.

$$M^3 = H_1 \cup_{\Sigma^g} H_2$$

This defines embedded Lagrangian subvarieties $L_{H_1}, L_{H_2} \subset \bar{\mathcal{S}}_g$. A flat connection A on M^3 can then also be viewed as an intersection point of the two Lagrangians. Under certain hypotheses, which are satisfied in our situation, one can define a symplectic action Φ_ω for smooth paths joining one Lagrangian with the other, in particular at the intersection points (see e.g. [F, S]). A flat connection A corresponds to the constant path at the intersection point. Finally, the identity, which will follow easily from all the other constructions, is

$$\Phi_\omega(A) = CS(A).$$

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2 The Prequantum System

2.1 The Symplectic Structure of Moduli Spaces over Surfaces

We list some facts on moduli spaces of flat connections over a closed Riemann surface (which actually hold in the more general frame work of projectively flat connections as considered in [H]). Let G

be a compact, semi–simple Lie group and g its Lie algebra equipped with the Killing form $-tr(. \quad .)$. Moreover, let P be a flat principal G–bundle over Σ. Denote by $\mathcal{A}(P)$ the set of smooth connections on P, by $\mathcal{A}_F(P) \subset \mathcal{A}(P)$ the subset of flat ones, by $\mathcal{G}(P)$ the group of gauge transformations of P and define

$$\bar{\mathcal{S}}(P) = \mathcal{A}_F(P)/\mathcal{G}(P).$$

Sometimes we write $\mathcal{A}(\Sigma), \mathcal{A}_F(\Sigma), \mathcal{G}(\Sigma), \bar{\mathcal{S}}(\Sigma)$ instead if P is the (trivial) $SU(2)$–bundle, and sometimes we write $\bar{\mathcal{S}}_g$ instead of $\bar{\mathcal{S}}(\Sigma)$ if Σ is the connected, closed Riemann surface of genus g.

There is a natural constant symplectic form on $\mathcal{A}(P)$ defined by

$$\omega'(\alpha,\beta) = const. \int_\Sigma tr(\alpha \wedge \beta),$$

with $\alpha, \beta \in \Omega^1(adP)$. The curvature turns out to be the moment map for the action of the gauge group. Intuitively one 'defines' a symplectic structure on $\bar{\mathcal{S}}(P)$ via Marsden–Weinstein reduction (see [AB]).

To make it more precise: $\bar{\mathcal{S}}(P)$ admits a stratification whose strata are symplectic orbifolds (and manifolds in favourable situations). It is given in the following way: Pick a $\rho \in \bar{\mathcal{S}}(P)$. There is an identification of the stabilizer as a Lie subgroup Stab$(\rho) \subset G$ and therefore of its Lie algebra

$$\text{stab}(\rho) \cong H^0(ad\rho) \subset g.$$

G acts on g via conjugation and so it acts on the set of Lie subalgebras. We obtain the set S of conjugacy classes of Lie subalgebras. Denote by $[t]$ the conjugacy class of a Lie subalgebra t. Note that Stab(ρ) acts on stab(ρ). Now we define a partition

$$\bar{\mathcal{S}}(P) = \coprod_{s \in S} \{[\rho] | [\text{stab}(\rho)] = s\}.$$

We write $\mathcal{S}^s(P)$ for the different strata and omit the superscript s if it is the center $s = C(g)$. The claim is, that

Proposition 2.1 *(1) The partition defines a stratification of the topological space $\bar{\mathcal{S}}(P)$.*
(2) $\mathcal{S}^s(P)$ carries the structure of a symplectic orbifold (see [G1]).
(3) There is a subspace $\mathcal{P}$ of the continuous functions on $\bar{\mathcal{S}}(P)$ which are smooth on each stratum on which the Poisson brackets on each stratum define a Poisson structure via restriction (see [H1]).

Sketch of the proof. Note, that the orbifolds are manifolds if all elements in $\mathcal{S}^s(P)$ have the same type of stabilizer, i.e. if the stratum $\mathcal{S}^s(P)$ is exactly a stratum in the stratification given by the stabilizers rather than its Lie algebras. Moreover, let $[\rho] \in \mathcal{S}^s(P)$. Consider the deformation complex which describes its neighbourhood in $\mathcal{S}^s(P)$

$$0 \longrightarrow \Omega^0(ad(\rho)|_{C(\text{stab}(\rho))}) \xrightarrow{d_\rho} \Omega^1(ad(\rho)|_{C(\text{stab}(\rho))}) \xrightarrow{d_\rho} \Omega^2(ad(\rho)|_{C(\text{stab}(\rho))}) \longrightarrow 0.$$

Here $C(\text{stab}(\rho))$ denotes the centralizer of the Lie subalgebra. With

$$\Gamma(\rho) = \pi_0((C(\text{Stab}(\rho)) \cap \text{Stab}(\rho))$$

a neighbourhood of $[\rho]$ in $\mathcal{S}^s(P)$ is given by $\{\alpha \in H^1(ad(\rho)|_{C(\text{stab}(\rho))})|[\alpha,\alpha] = 0\}/\Gamma(\rho)$ (see [G]). Note that the obstruction space is $H^2(ad(\rho)_{C(\text{stab}(\rho))}) \cong (H^0(ad(\rho)|_{C(\text{stab}(\rho))}))^*$. But for $\phi \in H^0(ad(\rho)|_{C(\text{stab}(\rho))}) \cong C(\text{stab}(\rho)) \cap \text{stab}(\rho)$ the obstruction $[\alpha,\alpha]$ with $\alpha \in H^1(ad(\rho)|_{C(\text{stab}(\rho))})$ always satisfies

$$\begin{aligned}\int_\Sigma tr([\alpha,\alpha]\phi) &= \int_\Sigma tr(\alpha \wedge [\alpha,\phi]) \\ &= 0,\end{aligned}$$

on the account of $[\alpha,\phi] = 0$ and vanishes. So, the neighbourhood is described by the Zariski–deformations $H^1(ad(\rho)|_{C(\text{stab}(\rho))})/\Gamma(\rho)$ itself and the strata are clearly of orbifold type. The symplectic structure ω_s is defined via the cup–product on H^1. The stratum is a manifold if $\Gamma(\rho)$ acts trivial on H^1 on the whole stratum which is the case for all strata if $G = SU(N)$.

Now, restricting ω' to the tangent space of the strata in the stratification defined by stab(A) we obtain via Hodge theory for the above complex

$$\omega'|_{\{A|F_A=0;[\text{stab}(A)]=s\}} = \pi^*\omega_s,$$

with the projection $\pi : \mathcal{A}_F(P) \longrightarrow \bar{\mathcal{S}}(P)$. We will denote ω' by ω from now on.

2.2 The $SU(2)$–case

We sketch the details of the very careful discussion of the construction of the line bundle $\mathcal{L}_g \longrightarrow \bar{\mathcal{S}}_g$ in [JW1, JW3, RSW]. Denote by $\mathcal{A}_F(\Sigma)$ the space of all flat connection one–forms of the (trivial) $SU(2)$–bundle over the (not necessarily connected) Riemann surface $\Sigma = \coprod_\iota \Sigma^{g_\iota}$. Then the moduli space of flat $SU(2)$–connections can be described alternatively as

$$\bar{\mathcal{S}}_\Sigma \cong \prod_\iota Hom(\pi_1(\Sigma^{g_\iota}); SU(2))/\text{Ad} \cong \mathcal{A}_F(\Sigma)/\mathcal{G}(\Sigma),$$

where $\mathcal{G}(\Sigma)$ is the gauge group of the bundle. Using the second description we can give a construction of the desired bundle. First we define a $U(1)$–cocycle λ for the action of the gauge group. Let $A \in \mathcal{A}_F$ and $g \in \mathcal{G}$ be two arbitrary elements. Pick any compact oriented three–manifold N bounded by Σ. Now extend A and g to a (not necessarily flat) connection $\tilde{A}$ and a gauge transformation $\tilde{g}$ on the (trivial) $SU(2)$–bundle over N, respectively. Then we define

$$\begin{aligned}\lambda(A,g) &= \exp(i(CS(\tilde{g}\tilde{A}) - CS(\tilde{A}))) \\ CS(B) &= \frac{1}{4\pi}\int_N tr(B \wedge dB + \frac{2}{3}B \wedge B \wedge B),\end{aligned}$$

for a $SU(2)$–connection one form B on N. Note that $CS(gA) - CS(A) \in 2\pi Z$ on closed, oriented three–manifolds. This follows from

$$CS(\tilde{g}\tilde{A}) - CS(\tilde{A}) = \frac{1}{4\pi}\int_N \tilde{g}^* dSU(2) + \frac{1}{4\pi}\int_\Sigma tr(A \wedge g^{-1}dg).$$

From that we deduce that λ does not depend on the extensions and it is indeed a cocycle, i.e.

$$\lambda(A, gh) = \lambda(gA, h)\lambda(A, g),$$

and so defines a Hermitian line bundle $\mathcal{L}_\Sigma \longrightarrow \bar{\mathcal{S}}_\Sigma$ by the equivalence relation

$$(A, z) \sim (gA, \lambda(A, g)z) \in \mathcal{A}_F \times C.$$

We can define continuous trivializations using the slices Ker$(d_A^*)/Stab(A)$ for the gauge action near A. For irreducible connections the trivial section will do the job and one easily sees that $\mathcal{L} \longrightarrow \bar{\mathcal{S}}_g$ is actually a smooth line bundle on each stratum. For reducible A we have to define a section of the trivial line bundle over Ker(d_A^*) which is invariant under the $Stab(A)$–action defined by λ. Note that $\lambda(A, g) = 1$ for $g \in Stab(A)$. For $a \in$ Kerd_A^* define on $[0, 1] \times \Sigma$

$$A(a) = A + ta.$$

Then $\phi(a) = \exp(iCS(A(a)))$ defines a trivialization in a neighbourhood of $[A]$.

One of the basic properties is that

$$\begin{aligned} \mathcal{L}_{\Sigma \sqcup \Sigma'} &= \mathcal{L}_\Sigma \boxtimes \mathcal{L}_{\Sigma'} \\ \mathcal{L}_{-\Sigma} &= \mathcal{L}_\Sigma^*, \end{aligned}$$

where $-\Sigma$ is Σ equipped with the opposite orientation. It is an easy exercise to check that the following connection one–form η on the trivial line bundle over $\mathcal{A}_F$ descends to a connection ∇_Σ on $\mathcal{L}$: For $\alpha \in T_A\mathcal{A}_F \cong \Omega^1(M; \underline{su}(2))$ we define

$$\eta_A(\alpha) = \frac{i}{4\pi} \int_{\Sigma^g} tr(A \wedge \alpha).$$

Again

$$\begin{aligned} \nabla_{\Sigma \sqcup \Sigma'} &= \nabla_\Sigma \boxtimes \nabla_{\Sigma'} \\ \nabla_{-\Sigma} &= \nabla_\Sigma^*. \end{aligned}$$

Remember that the symplectic form ω can be considered as given by the Mardsen–Weinstein reduction of the constant symplectic form ω' on the affine space of all connection one–forms $\mathcal{A}$ over the vector space $\Omega^1(\Sigma^g; \underline{su}(2))$

$$\omega'(\alpha, \beta) = \frac{1}{2\pi} \int_{\Sigma^g} tr(\alpha \wedge \beta).$$

Now, noting that the curvature of η is $d\eta = i\omega'_{|\mathcal{A}_F}$ we see that the curvature of ∇ is $R^\nabla = i\omega$, and the first Chern class and accordingly the topological type of $\mathcal{L}$ are exactly as we wished.

Consider a diffeomorphism ϕ of Σ. This induces a symplectomorphism Φ of $\bar{\mathcal{S}}(\Sigma)$ via

$$\Phi([A]) = [\phi^* A],$$

for a flat connection A which depends on the homotopy type of ϕ only. There exists an oriented three–manifold N^3 bounded by Σ such that Φ extends to a diffeomorphism on N^3. One has to check that this is true for connected surfaces Σ. But there as it has been known since Dehn ϕ is homotopic to a sequence of Dehn twists. Successively pushing this sequence into the handle body we obtain a sequence of Dehn surgeries along the components of a link and, finally, N^3 together with an extension $\tilde{\phi}$ of ϕ. But then for extensions $\tilde{A}, \tilde{g}$ of a connection A and a gauge transformation g on Σ, $\tilde{\phi}^* \tilde{A}$ and $\tilde{\phi}^* \tilde{g}$ are extensions of $\phi^* A$ and $\phi^* g$, respectively. So,

$$\lambda(A, g) = \lambda(\phi^* A, \phi^* g),$$

and with $(\Phi)^* \eta = \eta$ we conclude that the trivial lift to the line bundle $\mathcal{A}_F \times C \longrightarrow \mathcal{A}_F$ descends to a parallel automorphism of $\mathcal{L}$.

2.3 The $SO(3)$–case

There are also bundles $\mathcal{L}_\epsilon \longrightarrow \bar{\mathcal{S}}_g^\epsilon$ on the moduli spaces of flat connections on the $SO(3)$–bundles P_ϵ with $w_2(P_\epsilon)[\Sigma^g] = \epsilon \in Z_2)$ over Σ^g together with Hermitian connections ∇_ϵ.

Consider first the trivial bundle. Given a connection A and a gauge transformation g of the bundle we look for a possibly nontrivial $SO(3)$–bundle P_N over an oriented three–manifold N which is bounded by Σ^g such that A and g extend to $\tilde{A}$ and $\tilde{g}$, respectively, on P_N and which admits flat connections. Such bundle always exists (actually we could construct a three–manifold such that g as a map to $SO(3)$ extends and take then the trivial bundle). In order to define the Chern–Simons functional we fix a flat connection A_0 on P_N. Then for a connection A on P_N we define

$$CS(A) = \frac{1}{4\pi} \int_N tr(\alpha \wedge d_{A_0} \alpha + \frac{2}{3} \alpha \wedge \alpha \wedge \alpha),$$

with $\alpha = A - A_0$ and identifying the Lie algebras of $SU(2)$ and $SO(3)$. This depends of course on the choice of A_0. The pull back $\tilde{g}^* dSO(3)$ is an invariant 3–form on P_N which projects down to N. In this way we write

$$CS(\tilde{g}\tilde{A}) - CS(\tilde{A}) = \frac{1}{4\pi} \int_N \tilde{g}^* dSO(3) + \frac{1}{4\pi} \int_\Sigma tr(\alpha \wedge g^{-1} d_{A_0} g).$$

The volume of $SO(3)$ is $4\pi^2$ and so in the case of a closed three–manifold N the first term on the right–hand side is $4\pi^2 k$ with $k = deg(\tilde{g}) \in Z$ (for a definition see [DS]). There are gauge transformations of odd degree on many closed three-manifolds. So we have to take $\lambda(A, g) = \exp(2i(CS(\tilde{g}\tilde{A}) - CS(\tilde{A})))$ in order to gain gauge invariance modulo $2\pi Z$. Then the connection one–form is $\eta_0 = 2\eta$. So

$$\begin{aligned} p^* \mathcal{L}_0 &= \mathcal{L}^{\otimes 2} \\ p^* \nabla_0 &= \nabla^{\otimes 2}, \end{aligned}$$

under the projection $p : \bar{\mathcal{S}}_g \longrightarrow \bar{\mathcal{S}}_g^0$. On the other hand we may define a symplectic $H^1(\Sigma^g, Z_2)$–action on $\bar{\mathcal{S}}_g$ looking like

$$(A_1, B_1, \ldots, A_g, B_g) \mapsto (\pm A_1, \pm B_1, \ldots, \pm A_g, \pm B_g)$$

in terms of generators of the fundamental group. Clearly, $\bar{\mathcal{S}}_g^0$ is the quotient of $\bar{\mathcal{S}}_g$ under this action. It lifts to an action on the bundle $\mathcal{L}$. To see this we construct the action on $\mathcal{A}_F$. Let $[\phi] \in H^1(\Sigma^g; Z)$ be given by a closed 1-form $\phi = d\psi$ for a smooth function ψ on $\Sigma^g \setminus C$ which is 0 in a negative collar of C and 1 in a positive one. That defines a gauge transformation

$$\begin{aligned} \Phi : \quad & \Sigma^g \setminus C \longrightarrow SU(2) \\ \Phi(x) \quad = \quad & \exp(i\pi\psi) \end{aligned}$$

outside a single closed curve C whose class $[C]$ is the dual to ϕ. This transformation projects down to a smooth $SO(3)$–transformation which cannot be lifted to $SU(2)$, whereas Φ^2 can. Nevertheless, the transformation $A \mapsto \Phi A$ can be defined for $SU(2)$–connections and defines a symplectomorphism on the moduli level. It is obvious that $[\Phi^2 A] = [A]$ on the moduli level. We define

$$\theta(A, \Phi) = \exp\left(\frac{i}{4\pi} \int\limits_{\Sigma^g} tr(A \wedge \Phi^{-1} d\Phi)\right).$$

Then the lift of the action to the line bundle is given by

$$(A, \zeta) \mapsto (\Phi A, \theta(A, \phi)\zeta).$$

With the formula for the cocycle given above one easily deduces, that this gives a well–defined lift. In order to show that it preserves the connection ∇ one has to show that the difference of the one–form η and its pull–back is exactly the differential $(\theta(., \Phi))^{-1} d\theta(., \Phi)$:

$$\begin{aligned} \Phi^*\eta_A(\alpha) \quad - \quad & \eta_A(\alpha) \\ = \quad & \tfrac{i}{4\pi} \int\limits_{\Sigma^g} tr(\Phi A \Phi^{-1} + (d\Phi)\Phi^{-1} \wedge \Phi\alpha\Phi^{-1}) - \tfrac{i}{4\pi} \int\limits_{\Sigma^g} tr(A \wedge \alpha) \\ = \quad & \tfrac{i}{4\pi} \int\limits_{\Sigma^g} tr(d\Phi)\Phi^{-1} \wedge \Phi\alpha\Phi^{-1}) \\ = \quad & -(\theta(., \Phi))^{-1} d\theta(., \Phi). \end{aligned}$$

Careful analysis of the fixed points shows that this action is nontrivial over their fibres. So, the bundle $\mathcal{L}_g$ does not come from the $SO(3)$–moduli space $\bar{\mathcal{S}}_g^0$.

Now consider the nontrivial bundle P_1. In order to construct the cocycle in this situation we cannot use an oriented three–manifold bounded N by Σ^g, because each $SO(3)$–bundle P over N restricts to P_0 on Σ^g as $\omega_2(P|_\Sigma) = \omega_2(P)|_\Sigma = 0$ on account of $\Sigma^g = \partial N$ shows! Instead we look for bundles over oriented three–manifolds bounded by a torus T^2 and Σ^g which restricts to the nontrivial bundles on both boundary components. On the torus there is (up to gauge equivalence) exactly one flat connection on the nontrivial bundle (see e.g. [BD]). Let us fix a connection A_0 in this class. Given a flat connection A and a gauge transformation g on $P_1 \longrightarrow \Sigma^g$ we look for such bundles P_N which allow extensions of A and g to $\tilde{A}$ and $\tilde{g}$ on P_N such that $\tilde{A}|_{T^2} = A_0$ and $\tilde{g}|_{T^2} = Id$ and a flat extension of A_0 (which we also denote by A_0). We may define CS as in the case of a trivial bundle and again the cocycle is given by $\lambda(A, g) = \exp(2i(CS(\tilde{g}\tilde{A}) - CS(\tilde{A})))$.

We push the whole construction further and suppose $P \longrightarrow \Sigma$ is a $SO(3)$–bundle over the (possibly nonconnected) surface $\Sigma = \coprod_\iota \Sigma^{g_\iota}$. Given a flat connection A and a gauge transformation g on P we look for an oriented three–manifold N bounded by Σ and a collection of tori and an $SO(3)$–bundle P_N which restricts to P over Σ together with extensions $\tilde{A}$ and $\tilde{g}$ such that $\tilde{g}|_{T_i^2} = Id$ and $\tilde{A}|T_i^2 = 0$ or A_0 depending on whether $P_N|_{T_i^2}$ is the trivial or nontrivial bundle over the ith torus T_i^2, respectively. The cocycle for $\mathcal{L}_P$ is given by $\lambda(A,g) = \exp(2i(CS(\tilde{g}\tilde{A}) - CS(\tilde{A})))$. Again

$$\begin{aligned} \mathcal{L}_{P \sqcup P'} &= \mathcal{L}_P \boxtimes \mathcal{L}_{P'} \\ \mathcal{L}_{-P} &= \mathcal{L}_P^* \\ \nabla_{P \sqcup P'} &= \nabla_P \boxtimes \nabla_{P'} \\ \nabla_{-P} &= \nabla_P^*, \end{aligned}$$

where $-P$ denotes the bundle P with a flip in the orientation of the base.

3 Parallel Sections in $\mathcal{L}(\mathcal{L}_0)$

We construct parallel constant sections over certain Lagrangians in exterior tensor products of the line bundles given in Chapter 2 (see [JW3] Thm. 3.1). Given $SU(2)$ $(SO(3))$–bundles $P_\iota \longrightarrow \Sigma_\iota$ over a set of Riemann surfaces, suppose N is an oriented three–manifold with $\partial N = \coprod_\iota \Sigma_\iota$, and $P_N \longrightarrow N$ is an $SU(2)(SO(3))$–bundle such that $P_N|_{\Sigma_\iota} = P_\iota$. L_{P_N} denotes the (maybe empty) set of gauge equivalence classes of restrictions of flat connections on P_N to Σ. Denote by $r : \mathcal{A}(P_N) \longrightarrow \prod_\iota \mathcal{A}(P_\iota)$ the restriction map to the boundary (for the gauge group $r : \mathcal{G}(P_N) \longrightarrow \prod_\iota \mathcal{G}(P_\iota)$ likewise). On the one hand $r(\mathcal{A}_F(P_N))$ projects down to L_{P_N}. But the latter can be identified with the projection of $r(\mathcal{A}_F(P_N))$ to $\prod_\iota \mathcal{A}_F(P_\iota)/r\mathcal{G}(P_N)$. (Note that in the $SU(2)$–case $r\mathcal{G}(P_N) = \prod_\iota \mathcal{G}(P_\iota)$, and there is no difference between these point of views.)

Proposition 3.1 *We assume that N satisfies $H^1(N, \partial N; adA) = 0$ for all flat $SU(2)$($SO(3)$)–connections on N and that the induced restriction map on the moduli level,*

$$r : \mathcal{A}_F(P_N)/\mathcal{G}(P_N) \longrightarrow L_{P_N},$$

is one–to–one;
Then L_{P_N} is a Lagrange submanifold.

Proof. The first part will follow from the proof of the second. We restrict ourselves to points in the top stratum of L_{P_N}, i.e. flat connections on N which restrict to irreducible flat connections A on the boundary. We obtain the following long exact sequence of homology groups with values adA:

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(N,\partial N) & \longrightarrow & H^1(N) & & \\ & & & & & \overset{r}{\searrow} & \\ & & \| & & \| & & H^1(\partial N) \\ & & & & & \overset{\partial}{\swarrow} & \\ 0 & \longleftarrow & H^2(N) & \longleftarrow & H^2(N,\partial N) & & \end{array}$$

The vertical isomorphisms are due to Poincaré duality, and together with Stoke's theorem we see that Im(r) is a Lagrangian subspace, even in the case of a general three–manifold. For generalized handlebodies the obstruction space $H^2(N)$ vanishes and $[A]$ is a smooth point in the moduli space of flat connections on the three–manifold, i.e. its Zariski tangential space $H^1(N, \partial N)$ is its tangential space.

Now we define

$$\begin{array}{rcll} \tilde{s} & : & r(\mathcal{A}_F(P_N)) \longrightarrow r(\mathcal{A}_F(P_N)) \times C & \\ \tilde{s}(r(A)) & = & (r(A), \exp(iCS(A))) & \text{in the } \quad SU(2)\text{–case} \\ \tilde{s}(r(A)) & = & (r(A), \exp(2iCS(A))) & \text{in the } \quad SO(3)\text{–case.} \end{array}$$

Theorem 3.2 (Jeffrey/Weitsman,[JW3] Theorem 3.1) *Under the assumptions of Proposition 3.1 $\tilde{s}$ defines a parallel section on $\mathcal{L}|_{L_{P_N}}$ ($\mathcal{L}_0|_{L_{P_N}}$).*

Sketch of the proof. One has to check first that this is a correct definition, i.e. having a gauge transformation $g \in \mathcal{G}_\Sigma(P_N)$ which restricts to the identity on the boundary $\exp(2iCS(gA)) = \exp(2iCS(A))$. Moreover, this section of the trivial bundle over $r(\mathcal{A}_F(P_N))$ is invariant under the $r\mathcal{G}(P_N)$–action given by the Chern–Simons cocycle of the last paragraph. So it defines a section of $\mathcal{L}|_{L_{P_N}}$ ($\mathcal{L}_0|_{L_{P_N}}$). Finally, one sees that this section is parallel using Stoke's theorem (see [JW3] for details).

We will make use of the theorem for a special class of three manifolds:

Definition 3.3 *We call a (connected) oriented three-manifold N with boundary $\partial N = \coprod \iota \Sigma_\iota$ a* generalized handlebodies *iff there is a connected component Σ_0 of ∂N such that N is homotopic to the bucket*

$$\bigvee_{\iota \neq 0} \Sigma_\iota \vee \bigvee_{i=1}^{g(\Sigma_0)} S^1$$

The handlebody itself is of course a generalized handlebody. It retracts to a bucket of circles while the boundary retracts to the same bucket. We will see more general examples later on. For generalized handlebodies the assumption of Proposition 3.1 is satisfied.

4 The Polarization of the Moduli Space

4.1 The case of the $SU(2)$–bundle

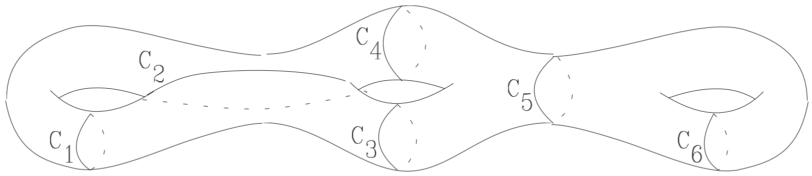
First we sketch the construction of the function π . Given a flat connection $A \in \bar{\mathcal{S}}_g$ we define

$$h_C(A) = \frac{1}{\pi} \arccos(\frac{1}{2} tr(hol_C(A))) \in [0, 1],$$

for a simple closed curve C. This is well defined because different choices of base points on C as well as different representations of A as a connection one–form lead to conjugate holonomies. Goldman in [G] and for the special case of $G = SU(2)$, Weitsman in [W] proved the following

Proposition 4.1 *(1) The s-gradient of this function is defined outside its values $\{0; 1\}$. The gradient flow of s-grad(h_C) is periodic of period 1 if C is a nonseparating loop and $\frac{1}{2}$ if C is a separating loop.*
(2) Let C, C' be two simple, nonintersecting loops. Then $h_C, h_{C'}$ Poisson commute, i.e. the Lie bracket of their symplectic gradients vanishes.

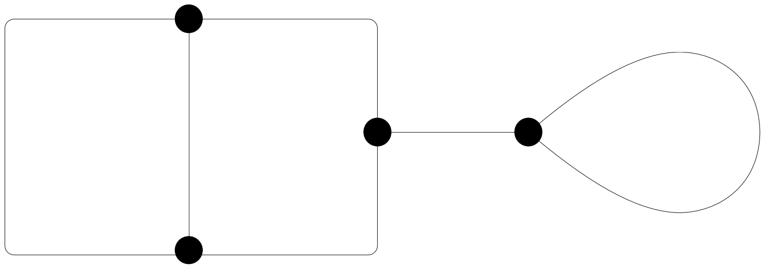
Now pick a system of $3g-3$ simple, nonintersecting loops $\{C_1, \ldots, C_{3g-3}\}$. There may be nonseparating as well as separating loops (like C_5 in the picture below) which correspond to a nontrivial element in $H_1(\Sigma)$ or not.



Define $\pi = (h_{C_1}, \ldots, h_{C_{3g-3}})$.

Corollary 4.2 *Outside the singular sets where the differential of any h_i is not defined the commuting gradient flows define a symplectic torus action. The dimension of the torus is $3g-3$.*

That means we can hope for a completely integrable system. However, it turns out that this is not quite the case. Let us denote by B_g the image of π. The system of loops decomposes Σ^g into $2g-2$ pants (three holed spheres or trinions). Denote this set by Γ. Each pant decomposition corresponds to a trivalent graph whose knots are indexed by elements of Γ. The edges of the graph correspond to the loops of the system and an edge is emerging from a knot γ iff the loop bounds the pant γ as it is depicted for the example above by:



The space $\bar{\mathcal{S}}_g$ is obtained by gluing the spaces of flat connections of the trinions. Given the data $\pi(A)$ on each trinion $\gamma \in \Gamma$ we seek for the space $\mathcal{M}(h_{i(\gamma)}, h_{j(\gamma)}, h_{k(\gamma)})$ of gauge equivalence classes of flat connections on the trinion γ having the corresponding data on the boundaries $C_{i(\gamma)}, C_{j(\gamma)}, C_{k(\gamma)}$. Elementary calculation leads to the following result:

Lemma 4.3 (Prop.3.1. in [JW1]) *$\mathcal{M}(h_{i(\gamma)}, h_{j(\gamma)}, h_{k(\gamma)})$ consists of exactly one point if*

$$|h_{i(\gamma)} - h_{j(\gamma)}| \leq h_{k(\gamma)} \leq min(h_{i(\gamma)} + h_{j(\gamma)}, 2\pi - (h_{i(\gamma)} + h_{j(\gamma)})),$$

where i, j, k correspond to the loops around the three holes of γ, and is empty elsewhere.

An equality is assumed if $A|_\gamma$ is reducible on the pant. To obtain a leaf $\pi^{-1}(\pi(A))$ we reglue the restrictions of A to the pants with elements in

$$Stab(A|_{C_{i(\gamma)}}) \times Stab(A|_{C_{j(\gamma)}}) \times Stab(A|_{C_{k(\gamma)}}).$$

This gives the same elements in $\bar{\mathcal{S}}_g$ iff the gluing data come from restrictions of $Stab(A|_\gamma)$ to the boundary, i.e.

Lemma 4.4 ([JW2] Prop. 3.7)

$$\pi^{-1}(\pi(A)) \cong \prod_i Stab(A|_{C_i}) / \prod_\gamma Stab(A|_\gamma).$$

From this, one deduces

Theorem 4.5 ([JW1, JW2]) *(1) B_g is a convex polyhedron contained in the cube $[0,1]^{3g-3}$.*
(2) The s-gradients of $\pi = (h_1, \dots, h_{3g-3})$ are linearly independent at $[A] \in \mathcal{S}_g$ iff A restricted to each trinion is irreducible. The linear group generated by the s–gradients acts transitively on a preimage $\pi^{-1}(b)$ iff $b \notin \{0; 1\}^{3g-3}$. In that case the number of degeneracies is exactly the number of pants on which the restriction of A is reducible if A itself is irreducible and $2g-3$ if A is reducible.
(3) Another way of stating (2) is that the system is linearly independent iff $\pi([A])$ lies in the interior of B_g and the group action is transitive iff it lies in the interior of the cube.
(4) Besides the degeneracies described in (2) the stabilizer of the time-one action of the s–gradient flow is generated by $\{h_i\}$, $g_\gamma = \{\frac{1}{2}(h_{i(\gamma)} + h_{j(\gamma)} + h_{k(\gamma)})\}$ for $\gamma \in \Gamma$ and $\{\frac{h_j}{2}\}$ if C_j is seperating.

Corollary 4.6 *(2) of the theorem implies that a generic leaf $\pi^{-1}(b)$ admits a free $(3g-3-j)$–torus action of the torus T^{3g-3-j} defined by the lattice given in (4). As an immediate consequence we see that for such b the leaf is the disjoint union of $(3g-3-j)$–dimensional tori which are isotropic because the commutator of two fundamental fields vanishes according to Proposition 4.1. The analysis via pants shows that it actually consists of just one such torus.*

Remark 4.7 *As it stands, this corollary has been deduced by Jeffrey and Weitsman in their papers. However, the conclusion should be true for all leaves. This is summarized in the following Theorem 4.8. Actually, one can show that the intersection of each leaf with any stratum of the symplectic stratification of $\bar{S}_g$ is a Lagrangian submanifold. This is clear for Case (2) of the theorem and calls for some extra arguments in the general case.*

4.2 The case of the $SO(3)$–bundles

Everything constructed and stated up to now can be done for the two $SO(3)$–bundles P_0, P_1. Note that the moduli space $\bar{\mathcal{S}}_g^0$ of flat $SO(3)$–connections on P_0 carries a stratification whose strata are symplectic *orbifolds* while the moduli space $\bar{\mathcal{S}}_g^1$ of flat connections on P_1 is a symplectic orbifold itself: an $SO(3)$–connection with no nontrivial parallel section in the adjoint bundle may have trivial stabilizer or Z_2 or the Kleinian group K_0, a connection with a one–dimensional family of parallel sections may have stabilizer $SO(2)$ or $O(2)$ (see [BD]). You may define functions h_i^0 in a similar way just by

$$h_i^0([A]) = \frac{1}{2\pi} \arccos(\frac{1}{2}(tr(hol_{C_i}(A)) - 1)) \in [0, \frac{1}{2}].$$

This time we obtain a transitive torus action iff $h_i^0 \neq \{0; \frac{1}{2}\}$ for all i. Some sublety arises if some of the h_i^0 are $\frac{1}{2}$. The differential and so the symplectic gradient are not defined for these functions. But nonetheless there is a symplectic S^1–action on $\{h_i^0 = \frac{1}{2}\}$ coming from that loop C_i. The degeneration is reflected in the fact that there are flat connections on both P_0 and P_1 having these data. To shed light on this fact: On $\{h_i^0 = \frac{1}{2}\}$ for a distinguished i we obtain a "twisting operation" in the following way: Suppose the flat connection A is on a tubular neighbouhood of C_i of the form $\frac{i}{2}d\theta$ such that the holonomy around C_i is $diag(1; -1; -1)$. We cut Σ and the bundle along C_i. We act on $A|_{\Sigma \setminus C_i}$ with the gauge transformation $\Phi = \exp(j\pi\frac{\psi}{2})$ with the function ψ on the complement of C_i as in the preceding section. In no case this transformation descends to an $SO(3)$–gauge transformation. The point is that $\Phi^*(A|_{\Sigma \setminus C_i})$ closes to a global connection on Σ^g but on a different bundle! Note that $\Phi^* A$ is now of the form $\frac{i}{2}d\theta$ on the negative side of the collar and $-\frac{i}{2}d\theta$ on the positive. So, we have to reglue the bundle by $\exp(i\theta)$ which changes the topological type of the bundle and preserves the restriction of the symplectic structures (the latter is a rather immediate consequence of the definitions). For a certain connection of that kind all other data h_j remain unchanged. Note that this operation is not natural: we could also use $\Phi = \exp(j'\pi\frac{\psi}{2})$ instead with j' purely imaginary and perpendicular to i. So it suffices to prove the isotropy for the tori in $\bar{\mathcal{S}}_g^0$. The two–fold covering $SU(2) \longrightarrow SO(3)$ induces a symplectic branched covering $p : \bar{\mathcal{S}}_g \longrightarrow \bar{\mathcal{S}}_g^0$. The leaves are mapped to leaves and the value $h_i^0 = \frac{1}{2}$ on $\bar{\mathcal{S}}_g^0$ corresponds to the same value of h_i on $\bar{\mathcal{S}}_g$. But there it is a non–degenerate value and defines a 1–periodic symplectic flow which (after a choice of one of the tori in the preimage) defines a symplectic action on the set $\{h_i^0 = \frac{1}{2}\}$ which commutes with all the other symplectic actions.

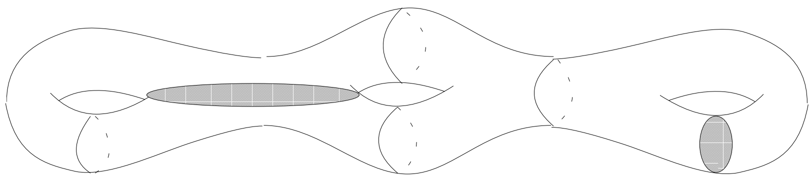
4.3 Isotropy of the leaves

Theorem 4.8 *Let $x \in B_g$.*
(1) Then the preimage $\pi^{-1}(x)$ is an isotropic subvariety of $\bar{\mathcal{S}}_g$, i.e. it is isotropic in each symplectic stratum.
(2) ([W] Prop. 3.7.) Assume $x \in (0,1)^{3g-3} \cap B_g$. Then the preimage is an isotropic submanifold either in $\mathcal{S}_g$ or in its complement, i.e. it entirely consists either of irreducible or reducible connections. The leaf is a torus of dimension $3g-3-j$ where j is the number of degeneracies described above.

Proof of (2). The isotropy in the case of (2) of the theorem is a consequence of the transitive symplectic torus action on such leaves (as pointed out in [W]). One might think that the other assertion follows from the fact that a flat connection A is reducible iff $A|_\gamma$ is reducible for all trinions $\gamma \in \Gamma$. Note that this is not true in general: Assume there is a separating curve C_i in the system. If A is reducible then $h_i = 0$. But there are abundant labellings of the graph of the pant decomposition such that $A|_\gamma$ is reducible for all γ but $h_i \neq 0; 1$. One rather uses the one–dimensionality of all $Stab(A|_{C_i})$ to reglue the nonvanishing element of $H^0(adA)$ along the boundaries of the pants to obtain a nonvanishing element of $H^0(adA')$ for the twisted connection A'.

The subtlety in proving isotropy arises when the holonomy around one of the circles lies in the center of $SU(2)$. There is actually not a big difference between $\pm Id$. In the case of the minus sign one can consider this holonomy to be trivial as well, reducing to the $SO(3)$–setting. A subset of $\bar{\mathcal{S}}_g$ has isotropic intersections with each stratum iff its projection to $\bar{\mathcal{S}}_g^0$ has.

Now for each C of our system around which the holonomy of A lies in the center (that means it is trivial in $SO(3)$) we fill in a disc lying inside the handlebody H as shown for our example if $h_2(A), h_6(A) = 0; 1$.



Then we take a regular neighbourhood of the resulting cell complex in R^3 and finally obtain a cobordism between Σ^g and a not necessarily connected Riemann surface Σ'. This cobordism is a generalized handlebody. We will sometimes refer to Σ' as a *characteristic surface of the connection A*. There is a straightforward pushing down procedure for the system of loops on Σ to a system of loops on Σ'. Again this can be seen by noting that the cobordism deformation retracts to a bucket of circles and the components of Σ'. Clearly, each flat connection A on Σ^g having holonomy in the center around the above loops, can be extended to a flat connection in a bundle P_N over the whole generalized handlebody (this is basically the reason why we chose this notation). Denote by A' the flat connection on Σ' obtained by restriction. Note that this may have nontrivial Stiefel–Whitney class on some components of Σ'. The construction is made in a way that there is no loop in the induced system on the characteristic surface having trivial holonomy with respect to A' .
We are now in the position to prove statement (1) of 4.8.

Proof of 4.8 (1). First we treat the possibly occuring spheres in the characteristic surface Σ'. Note that P_N restricted to such a sphere is always trivial. Now along these spheres we glue copies of B^3 to the cobordism, because there is no flat connection but the trivial one on the trivial bundle over S^2. Now fix $A \in \mathcal{S}_g$ with $\pi(A) = x$ and families $\{A_t\}, \{B_t\} \subset \pi^{-1}(x)$ with $A_0, B_0 = A$ representing tangent vectors α, β of $\pi^{-1}(x)$ in A. Next we consider the characteristic surface Σ' and the corresponding generalized handlebody N. The surface depends of course only on x and not on the connection A chosen in its preimage. So, the two families can be extended to families of flat connections on N which we will again denote by A_t, B_t. On each component of Σ' the holonomy of their restrictions along the induced curve system is nontrivial in the $SO(3)$–setting. But in that case we know isotropy already and it follows that

$$\omega_{\Sigma'}(\alpha, \beta) = \frac{1}{2\pi} \int\limits_{\Sigma'} tr(\alpha \wedge \beta) = 0.$$

Now mimicking the idea of Jeffrey and Weitsman in the case N is the handlebody H^g itself with Stoke's Theorem

$$\omega(\alpha, \beta) = \frac{1}{2\pi} \int\limits_{N} d(tr(\alpha \wedge \beta)) + \omega_{\Sigma'}(\alpha, \beta) = 0,$$

because the integrand vanishes on account of the vanishing of $d_A\alpha = d_A\beta = 0$.

5 The Bohr–Sommerfeld Leaves of the Foliation

In this section we will give a short description of the leaves in the foliation described in Chapter 4 on which the restriction of the Hermitian connection on $\mathcal{L}$ is trivial. Note in passing that for all the discussions using holonomy arguments around loops in the moduli space it is important to know the flatness of *all* the restrictions of ∇ to the leaves. The latter follows immediately from the isotropy proved in the preceding section.

5.1 The $SU(2)$–case

Using the pant decomposition of Σ^g Jeffrey and Weitsman prove in [JW1] the following

Theorem 5.1 (Theorem 8.1 in [JW1]) *The set of Bohr–Sommerfeld points in B_g is given by*

$$\left\{ x \in B_g \,\middle|\, \begin{array}{lll} (a) \quad h_i = l_i/k & l_i = 0, \ldots, k & \textit{for all } i \\ (b) \quad l_{i(\gamma)} + l_{j(\gamma)} + l_{k(\gamma)} \in 2Z & & \textit{for all pants } \gamma \end{array} \right\}.$$

Proof. We follow a slightly different line here. First of all, we compute the holonomy of ∇ around the closed Hamiltonian flow lines explicitely in Lemma 5.2 from which the theorem follows for leaves with transitive torus action. Then we complete the argument using the parallel section of Chapter 3 rather then computing the holonomy around elements in the fundamental group of the leaf not coming from closed flow lines.

The main point we use to prove the statement is again the structure of the moduli space $\bar{\mathcal{S}}_g$ and its toric action coming from the careful analysis of the pant decomposition in [JW1]. The s–gradient of h_C is defined outside the preimages of $\{0, 1\}$, i.e. if A has not holonomy in the center of $SU(2)$ around C. Then the flow of $[A]$ under s-gradh_C is periodic with period 1 or $\frac{1}{2}$ for C nonseperating and seperating, respectively. Denote by ρ_C the image of $[A]$ under the Hamiltonian flow of h_C. Having the above system of curves we consider the space of functions generated by $\{h_i\}$. The time–one flow defines the toric action. The periods define a lattice containing

$$\{kh_i\}_{C_i \quad nonseperating} + \{(k/2)h_i\}_{C_i \quad seperating}.$$

It turns out that there are $2g-2$ additional functions $\{g_\gamma = \frac{1}{2}(h_{i(\gamma)} + h_{j(\gamma)} + h_{k(\gamma)})\}$ for each trinion $\gamma \in \Gamma$ with trivial time–one flow (Prop. 5.4. in [JW1]). If the holonomies of A around the boundary loops $C_{i(\gamma)}, C_{j(\gamma)}, C_{k(\gamma)}$ are not in the center then the s–gradient of g_γ is well–defined and has periodic orbits with period 1. Denote by ρ_γ the image of its flow.
We need to prove the following crucial

Proposition 5.2 *Let C be a simple closed curve and A a flat connection with holonomy around C not in the center of $SU(2)$. Then for the image of $[A]$ under the Hamiltonian flow*

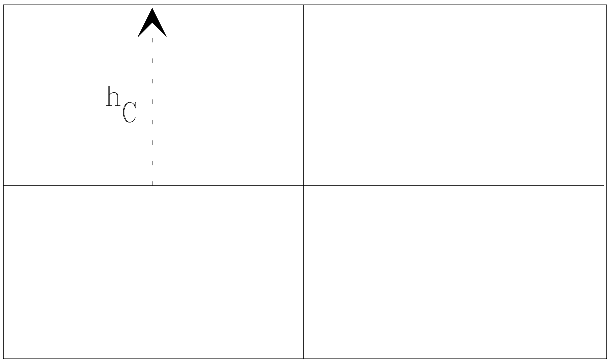
$$\begin{aligned} hol_\nabla(\rho_C) &= \exp(2\pi i h_C) && \textit{for } C \textit{ a nonseparating, and} \\ hol_\nabla(\rho_C) &= \exp(\pi i h_C) && \textit{for } C \textit{ a separating curve.} \end{aligned}$$

Moreover, suppose the holonomies of A along the boundary components of a pant γ are not in the center. Then

$$hol_\nabla(\rho_\gamma) = \exp(2\pi i g_\gamma).$$

Proof of Lemma 5.2. We will give an argument borrowed from well–known TQFT techniques reducing everything to the torus where one can compute the result explicitely.

Note that on the torus there is no pant decomposition but only one curve in the system which cuts the torus into a cylinder. On the other hand every flat connection is reducible and the moduli space is diffeomorphic to the pillowcase $\bar{\mathcal{S}}_1 \cong T^2/Z_2$ where the Z_2–action sends $z \mapsto -z$ in the flat torus C/Γ. In the picture below a fundamental domain of the torus is chosen such that the center coincides with the origin in the plane. Toplogically the pillowcase is obtained by the two upper rectangles glued together along the sides, geometrically by gluing smoothly along the interior of the sides. Then $h = h_C$ defines a real polarization (the one defined by the height function of the rectangles in the picture) with two singular fibres homeomorphic to $[0,1]$.

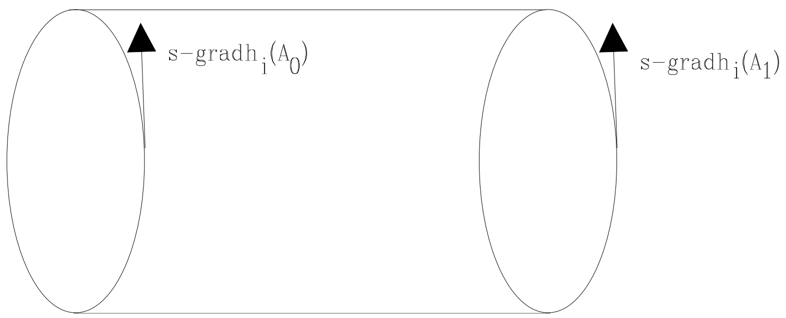


The noncritical levels are smooth curves in the top stratum. As the value of h tends to 0 or 1 the holonomy along the level curve tends to zero as the analysis of the covering torus T^2 shows. But then

$$hol_\nabla(h^{-1}(s)) = \exp(i \int_{h^{-1}([0,s])} \omega) = \exp(2\pi i s),$$

where the first equality is obvious now and the second can be seen from the geometry of the torus again.

Now we suppose $g \geq 2$. First note $h_i^{-1}(s)$ is connected in each stratum (which an analysis via pants will show) and so each regular point (where the differential is defined) can be smoothly connected to any other on the same level within a stratum. With the Hamiltonian flow of h_i we obtain a smoothly embedded cylinder C whose levels are s–gradient flow lines of h_i:



Then we gain (for C_i nonseperating)

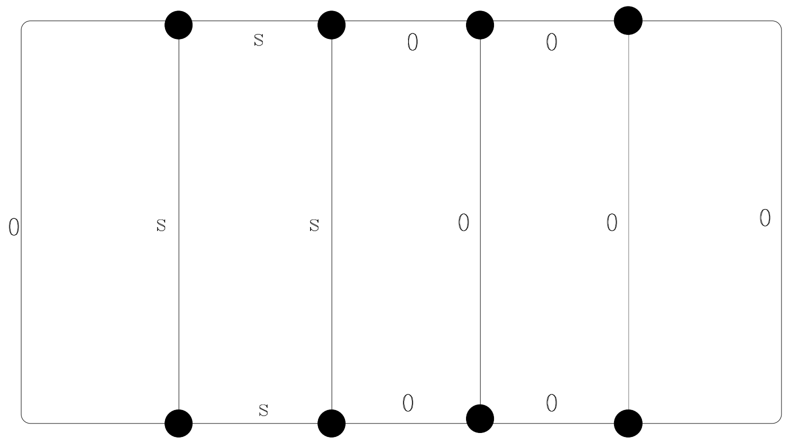
$$\begin{aligned} hol_\nabla(\rho_i(A_1))hol_\nabla(\rho_i(A_0))^{-1} &= \exp(i\int_C \omega) \\ &= \exp(i\int_{[0,1]}\int_{S^1} \omega(\frac{\partial}{\partial t}, \text{s-grad}h_i)dtds) \\ &= \exp(2\pi i\int_0^1 dt\frac{\partial h_i}{\partial t}) \\ &= \exp(2\pi i(h_i(A_1) - h_i(A_0))) \\ &= 0. \end{aligned}$$

This holds true for g_γ and for C_i seperating with the only difference that

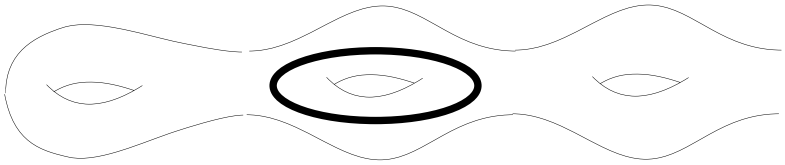
$$hol_\nabla(\rho_i(A_1))(hol_\nabla(\rho_i(A_0)))^{-1} = \exp(\pi i(h_i(A_1) - h_i(A_0))).$$

So, (as pointed out in [JW1]) it suffices to find reference points in all strata (actually in the top and the middle one only), where the differential of the function in question is defined and the assertion holds.

Pick a nonseperating curve C_i first. Seek a closed subgraph in the graph corresponding to the pant decomposition. Label all the edges of the subgraph by $s \neq 0; 1$ all the remaining by 0 (according to $g \geq 2$ there is a remaining edge). The result may e.g. look like the following labelled trivalent graph:



The gluing of the pants along these 0–labelled loops allows for constructing both: reducible and irreducible connections having these data. From the handlebody we drill out a full torus corresponding to the subgraph. The left hand part of the generalized handlebody in the example above is depicted below:

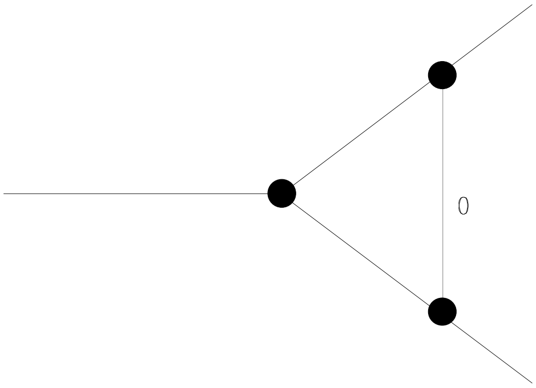


With Theorem 3.2, we obtain a generalized handlebody, i.e. we obtain a parallel section in $\mathcal{L}'^* \boxtimes \mathcal{L}$ (for the line bundle $\mathcal{L}'$ over $\bar{\mathcal{S}}_1$) over the Lagrangian $L \subset \bar{\mathcal{S}}_1 \times \bar{\mathcal{S}}_g$ defined by restriction of the flat connections on the drilled handlebody. This Lagrangian is the graph of the map

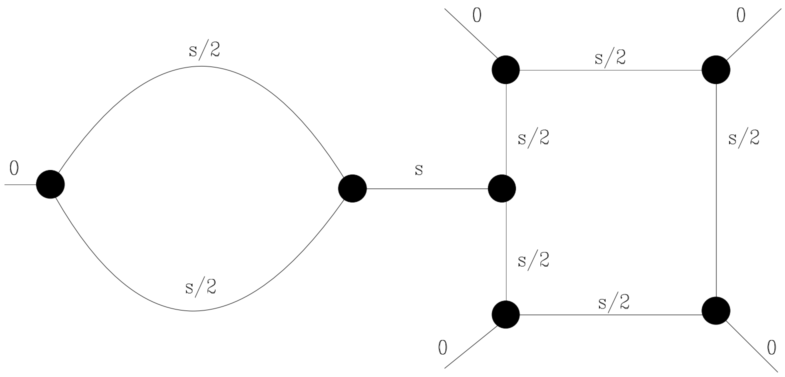
$$\Phi : \{[A] \in \bar{\mathcal{S}}_g | \quad h_i([A]) = 0 \text{ for all } i \text{ with } C_i \text{ labelled } 0\} \longrightarrow \bar{\mathcal{S}}_1$$

induced by the bordism. In the image we have $h = s$ and Hamiltonian flow lines of h_i for the subgraph are carried to Hamiltonian flow lines of h on $\bar{\mathcal{S}}_1$. The parallel section on $\mathcal{L}|_L$ can be considered as a parallel section in $Hom(\Phi^*\mathcal{L}'^*, \mathcal{L})$ with respect to $\Phi^*\nabla'^* \otimes \nabla$. So, the holonomies along the flow lines in the different spaces with respect to the different connections are the same. But we have computed the holonomy along the flow lines of h on $\bar{\mathcal{S}}_1$ already.

Now suppose C_i is separating. Label the corresponding edge by s. W.l.o.g. we assume that the two other boundary loops C_j, C_k of the pant are nonseparating. In the opposite case we glue in a handle with a loop labelled zero on it as indicated in the picture of the corresponding part of the graph below:



That gives an equivalent situation via a bordism which collapses the additional handle. Now we choose a closed subgraph containing the two nonseperating edges. It again corresponds to drilling a full torus out of the handlebody. Label all the edges in the subgraph by $\frac{s}{2}$. Do the same at the other node of C_i. Label the remaining edges by 0. We obtain a graph like



The functions h_j corresponding to these edges descend to $h = \frac{s}{2}$ on the torus. On the other hand the Hamiltonian flows of h_i, h_j, h_k are linearily dependent on the corresponding leaf (after choosing the right orientation in such a way, that s-grad(h_i) = s-grad(h_j) + s-grad(h_k)). Furthermore s-grad$(\frac{h_j}{2})$ = s-grad$(\frac{h_k}{2})$ on the leaf because all other edges going out from the subgraph are labelled by zero. Then the assertion follows from the first case.

The holonomy along ρ_γ is computed by labelling the edges of the node in question by $s, \frac{s}{2}, \frac{s}{2}$, where the latter value is given the separating loop if it occurs (supposing not all three boundary components are separating as above). Then the time–one flow line of g_γ is exactly the time–one flow line ρ_i where i is labelled with s. The formula for the holonomy follows immediately.

Remark 5.3 *(1) One can compute the holonomies using the explicit form of the Hamiltonian flows (given in [W]) and the definition of $\mathcal{L}$ via the Chern–Simons cocycle.*
(2) There should also be a direct geometric approach using the cylinder argument above where at level zero the cylinder is a piecewise smooth curve in $L_H = \pi^{-1}(0,\ldots,0)$, the Lagrangian defined by the handlebody. The holonomy around the piecewise smooth loop is zero and the assertion can be concluded if one can embed the cylinder in such a way that the holonomy around the level loop converges as the level tends to zero. Lacking this cylinder we were looking for regular reference points.

Now we are ready to complete the proof of the theorem. First we pick a leaf of our foliation consisting of connections which do not have holonomy in the center of $SU(2)$. Then we know from the discussion in Chapter 4 that the torus (of perhaps lesser than half the dimension) acts transitively on the leaf (which in turn is a torus itself). It suffices to check that the holonomies around the generators of the (Abelian) fundamental group of these tori are trivial iff the conditions in the theorem are satisfied. This is the same as saying that the holonomies around the closed time–one orbits of elements of the lattice introduced above are trivial in these cases which is exactly proved in Lemma 5.2. So, we are left with the cases in which the torus acts not transitively, i.e. the holonomies of the connections of the leaf L around some loops of the system are in the center. The difficulties here arise from elements in the fundamental group of the leaf which are not representable by an orbit of the action. One has to use a different argument to conclude the vanishing of the holonomy around those loops.

At first we treat the loops in the system which have trivial holonomy. The technique we are going to describe applies in the remaining cases successively. For the loops with trivial holonomy we fill in discs, so the boundary of a regular neighbourhood of this complex gives again our cobordism N of Σ^g to a, possibly nonconnected, characteristic surface $\Sigma' = \coprod_\iota \Sigma_\iota$. Clearly, the flat connections in the leaf L extend to flat connections on this cobordism and so give flat connections in the product of leaves $\prod_\iota L_\iota$ on Σ'. These have no nontrivial holonomy around loops in the corresponding sytem of curves consisting of loops of the system on Σ^g which were not contracted. In terms of the corresponding graph: the new one is obtained by erasing all edges labelled with $h_j = 0$ and identifying edges which meet in a node with two edges.

The h'_j satisfy the conditions of the theorem iff the h_j on Σ^g do. On the other hand $\exp(iCS(A))$ is a nowhere vanishing covariant constant section in

$$\boxtimes_\iota \mathcal{L}_\iota^* \boxtimes \mathcal{L}|_{L_N},$$

where L_N is the Lagrangian in $\prod_\iota \bar{\mathcal{S}}_{g_\iota} \times \bar{\mathcal{S}}_g$ which consists of those flat connections coming from the restriction of flat connections on the generalized handlebody (see Chapter 3). As in the prove of Lemma 5.2 this is the graph of the induced map

$$\Phi : \{[A] \in \bar{\mathcal{S}}_g | \quad h_i([A]) = 0 \text{ for all } i \text{ with } C_i \text{ labelled } 0\} \longrightarrow \prod_\iota \bar{\mathcal{S}}_{g_\iota}.$$

Again $\exp(iCS(A))$ may be interpreted as a covariant constant section in $Hom(\Phi^* \boxtimes_\iota \mathcal{L}_\iota, \mathcal{L})$. So this provides an isomorphism between the covariant constant sections

$$\bigotimes_\iota \Gamma(\mathcal{L}_\iota|_{L_\iota}; \nabla|_{L_\iota}) \longrightarrow \Gamma(\mathcal{L}|_L; \nabla|_L).$$

Here ι runs over the connected components of Σ'. $\mathcal{L}_\iota$ and L_ι are the line bundles and leaves in the moduli space of the connected component ι corresponding to $\mathcal{L}$ and L, respectively. To see this, note that $\exp(iCS(A))$ defines an isomorphism of the line bundles, which is parallel with respect to the induced connection as stated in Chapter 3. The spaces on the left– and right–hand side are zero- or one–dimensional, so this isomorphism of bundles induces indeed an isomorphism of the corresponding section spaces.

Now we need some more intricate arguments to tackle the cases where the holonomies around some loops are $-Id$. First we assume that k is even. Then we go over to the $SO(3)$–setting. Seeking for parallel sections in $\mathcal{L}^k$ is then equivalent to seeking parallel sections in $\mathcal{L}_0^{k/2}$. But holonomy $-Id$ goes over to trivial holonomy and we may apply the procedure above.

Suppose that k is odd. W.l.o.g. assume that there is no trivial holonomy along a curve in the system. We start with the case where the holonomy is $-Id$ around a nonseparating loop. Pick a closed subgraph of the graph of the pant decomposition. This corresponds one–to–one to an element of $Z_2^g \cong H_1(H^3, Z_2) \subset H_1(\Sigma, Z_2)$ for the handlebody H^3.

The action of the corresponding dual as described in Chapter 2 changes only the data on the subgraph from h_i to $1 - h_i$. So, the holonomy on our loop changes from $-Id$ to $+Id$. We collaps this curve and after a finite number of steps we obtain w.l.o.g. the situation where no holonomy is trivial and holonomy $-Id$ occurs only on seperating loops. It remains to show that on such leaves there is a nonvanishing parallel section iff none of the holonomies are in the center. Assume there is a seperating loop with holonomy $-Id$. Consider the subgraph corresponding to one of the seperated components of the surface. Summing up the values $k(h_{i(\gamma)} + h_{j(\gamma)} + h_{k(\gamma)}) \in Z$ over all nodes of this subgraph we obtain an odd number, because all but the seperating edge occur twice in the sum and the value $kh_i = k$ for this edge is odd. So, there must be a node (or a pant) γ with an odd value. But then none of the holonomies around the boundaries of this pant are $\pm Id$ and the flow of g_γ is defined. According to Lemma 5.2 ∇^k has nontrivial holonomy around ρ_γ.

5.2 The $SO(3)$–case

The characterization in this case is an easy consequence of all we have said so far about its correspondence to the $SU(2)$–case. We summarize this to the following theorem:

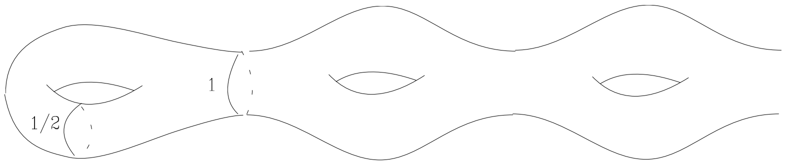
Theorem 5.4 (Theorem 8.1 in [JW1]) *The set of Bohr–Sommerfeld orbits is given by*

$$\left\{ x \in B_g \;\middle|\; \begin{array}{llll} (a) & h_i = l_i/2k & l_i = 0, \ldots, 2k & \textit{for all } i \\ (b) & l_{i(\gamma)} + l_{j(\gamma)} + l_{k(\gamma)} \in 2Z & & \textit{for all pants } \gamma \end{array} \right\} / Z_2^g,$$

for the case of the trivial bundle P_0, and by

$$\left\{ x \in B_{g+1} \;\middle|\; \begin{array}{llll} (a) & h_1 = \frac{1}{2} & h_2 = 1 & \\ (b) & h_i = l_i/2k & l_i = 0, \ldots, 2k & \textit{for all } i > 2 \\ (c) & l_{i(\gamma)} + l_{j(\gamma)} + l_{k(\gamma)} \in 2Z & & \textit{for all pants } \gamma \end{array} \right\} / Z_2^g,$$

for a pant decomposition as indicated in the picture:



This uses the following description of the moduli space: consider the punctured Riemann surface of genus g. $\mathcal{S}_g(1)$ denotes the moduli space of flat $SU(2)$–connections with holonomy $-Id$ around the point. This can be identified with the set of points in $\bar{\mathcal{S}}_{g+1}$ having $h_1 = \frac{1}{2}, h_2 = 1$ factored out by the S^1–action of the s–gradient of h_1 and the $SO(3)$–action defined by the gluing operation along C_2 (see [JW2], Prop.3.12). Now there is again a Z_2^g–action on $\mathcal{S}_g(1)$ and the factor space is $\mathcal{S}_g(P_1)$.

Remark 5.5 *This result raises some questions. The conditions on the Bohr–Sommerfeld leaves correspond exactly to the so–called quantum Clebsch–Gordan conditions (see [JW1]). From this Jeffrey and Weitsman concluded that the Hilbert spaces constructed from the real polarization have the same dimension as the Hilbert spaces obtained in the more familiar Kähler polarization. Is there a natural isomorphism between these Hilbert spaces? In the Kähler case different authors used Hartog's theorem (e.g. [T] to prove finite dimensionality) saying that a holomorphic section of the prequantum line bundle $\mathcal{L}$ is fully determined by its values on the irreducible part $\mathcal{S}_g$. But there are Bohr–Sommerfeld leaves consisiting only of reducible flat connections! What are the holomorphic sections they should correspond to? Finally, we should spell it out again: even restricted to each stratum our foliation is not as good as it has to be in order to define a tidy real polarization (see [Wo] for a definition)! The dimensions of the leaves may jump. Nevertheless, we have to take into account all of these lower dimensional leaves as well. This phenomenon has to be interpreted yet. For more detailed discussions on a construction of a 2+1 dimensional topological field theory (exploiting half densities and BKS pairings) see [JW3].*

6 Chern–Simons Functional and Symplectic Action

In this section we deduce a nice identity (remarked in [DS]) by the methods used above. Let us consider a closed oriented three–manifold M together with a Heegard decomposition along a Riemann surface Σ^g

$$M = N_1 \cup_\Sigma N_2.$$

This determines two Lagrangians L_1, L_2 in $\bar{\mathcal{S}}_g$. Suppose we are given a flat connection A on M. We defined its Chern–Simons action mod$2\pi Z$ by

$$\begin{aligned} CS &: \bar{\mathcal{S}}(M^3) \longrightarrow R/2\pi Z \\ CS(A) &= \frac{1}{4\pi}\int_{M^3} tr(A\wedge dA + \frac{2}{3}A\wedge A\wedge A). \end{aligned}$$

On the other hand restricting A to N_1, N_2 and Σ it corresponds to a point in the intersection $L_1 \cap L_2$. Picking curves joining the trivial connection with A in L_1 and A with the trivial connection in L_2, respectively, we obtain a closed loop γ. As $\pi_1(\bar{\mathcal{S}}_g) = 0$ (see [AB]), we can find a spanning surface δ such that its intersection with the reducible part is of measure zero in the measure induced by the symplectic form on the symplectic stratification. Having chosen another spanning surface δ' we obtain a closed surface σ by gluing $\delta \cup -\delta'$ along γ. It has a volume which is $2\pi k$ for a $k \in Z$ since $\frac{1}{2\pi}\omega$ is the Chern class of $\mathcal{L}|\sigma$. So we may define the symplectic action of A mod$2\pi Z$ as

$$\begin{aligned}\Phi_\omega &: \bar{\mathcal{S}}(M^3) \longrightarrow R/2\pi Z \\ \Phi_\omega(A) &= \int_\delta \omega.\end{aligned}$$

This is easily proved to be independent of all the different choices made (see [S, F] for details). As any line bundle, $\mathcal{L}$ is trivial over δ. In that trivialization ∇ is represented by a one form $i\eta$ with

$$d(i\eta) = R^\nabla.$$

The holonomy is

$$hol_\gamma(\nabla) = \exp(\int_\gamma i\eta).$$

It is now straightforward to see that

$$\exp(i\Phi_\omega(A)) = hol_\gamma(\nabla).$$

But from all we said before (especially the construction of the parallel sections s_{N_i}) we see that this holonomy is just $\exp(iCS(A))$, so we conclude

Proposition 6.1 *Let $M^3 = N_1 \cup_{\Sigma^g} N_2$ be a Heegard splitting of a three manifold along a Riemann surface of genus g. Denote by L_1, L_2 the two Lagrangians in the symplectic stratification $\bar{\mathcal{S}}_g$. A flat connection A may then be considered as a point in the intersection $L_1 \cap L_2$. So we can assign two values $\mathrm{mod} 2\pi Z$ to A: its symplectic action and its Chern–Simons action. Then*

$$\Phi_\omega(A) = CS(A) \bmod 2\pi Z.$$

Remark 6.2 *The interesting side of this theorem is its relation to Floer's homology. Both actions can be defined on bigger spaces: the space of all $SU(2)$–connections and the space of paths from L_1 to L_2, respectively (see [S, F] for a detailed discussion of the latter). This leads, after applying a perturbation theory, to an infinite–dimensional version of Morse theory on these spaces (see [F, F1]). A conjecture of Atiyah and Floer states that these two homologies are canonically isomorphic (see [DS]). The sets of critical points on the other hand are the flat connections and the intersection points, respectively. Now, the theorem states that on these critical points (which can be identified) the values of the functionals are the same. Moreover, one can embed the space of paths connecting L_1 with L_2 where the symplectic action lives into the space of connections on M^3 such that the above identity is satisfied. The bad news: the flows are not the same (see [DS]) and if you perturb in order to obtain Morse functions and regular flow you lose the identity.*

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