# Highest Weight $U_q[sl(n)]$ Modules and Invariant Integrable $n$-State Models with Periodic Boundary Conditions

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**Abstract**

The weights are computed for the Bethe vectors of an RSOS type model with periodic boundary conditions obeying $U_q[sl(n)]$ ($q = \exp(i\pi/r)$) invariance. They are shown to be highest weight vectors. The q-dimensions of the corresponding irreducible representations are obtained.

In the last years considerable progress has been made on the "quantum symmetry" of integrable quantum chain models as the XXZ-Heisenberg model and its generalizations. In [1] we constructed an $sl_q(n)$ invariant RSOS type model with periodic boundary conditions. The model can be regarded as a generalization of the XXZ-model with twisted boundary conditions. Therefore the representational analysis of $U_q[sl(2)]$ as treated e.g. in [2] has to be extended to cases of higher rank as considered first time in [3].

In the present paper we prove for this model the highest weight property of the Bethe states, calculate the weights and the q-dimensions of the representations and classify the irreducible ones. For the case of open boundary conditions see e.g. [4], [5] and [6].

The model of [1] is defined by the transfer matrix $\tau = \tau^{(n)}$ where

$$\tau^{(k)}(x, \underline{x}^{(k)}) = \mathrm{tr}_q(\mathcal{T}^{(k)}(x, \underline{x}^{(k)}) = \sum_\alpha q^{n+1-2\alpha}(\mathcal{T}^{(k)})^\alpha_\alpha(x, \underline{x}^{(k)}), \quad k = 1, \ldots, n. \tag{1}$$

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The "doubled" monodromy matrix is given by

$$\mathcal{T}_0^{(k)}(x,\underline{x}^{(k)}) = \tilde{T}_0^{(k)} \cdot T_0^{(k)}(x,\underline{x}^{(k)}) = (R_{01}\dots R_{0N_k})\cdot(R_{N_k0}(x_{N_k}/x^{(k)})\dots R_{10}(x_1/x^{(k)})). \quad (2)$$

For the purpose of the nested algebraic Bethe ansatz in addition to $\mathcal{T}(x) = \mathcal{T}^{(n)}(x)$ the monodromy matrices for all $k \le n$ are needed. The $sl_q(k)$ R-matrix is given by

$$R(x) = xR - x^{-1}PR^{-1}P, \quad R = \sum_{\alpha\neq\beta} E_{\alpha\alpha}\otimes E_{\beta\beta} + q\sum_{\alpha} E_{\alpha\alpha}\otimes E_{\alpha\alpha} + (q-q^{-1})\sum_{\alpha>\beta} E_{\alpha\beta}\otimes E_{\beta\alpha}, \quad (3)$$

The Yang-Baxter equation reads

$$R_{12}(y/x)\mathcal{T}_1(x)R_{21}\mathcal{T}_2(y) = \mathcal{T}_2(y)R_{12}\mathcal{T}_1(x)R_{21}(y/x). \quad (4)$$

The model is quantum group invariant, i.e. the transfer matrix commutes with the generators of $U_q[sl(n)]$. These are obtained from the monodromy matrices $T(x)$ in the limits $x$ to 0 or $\infty$ (up to normalizations)

$$T = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha E_1 & 1 & \ddots & \vdots \\ & \ddots & \ddots & 0 \\ * & & \alpha E_{n-1} & 1 \end{pmatrix} q^{\mathbf{W}}, \quad T_\infty = q^{-\mathbf{W}} \begin{pmatrix} 1 & -\alpha F_1 & & * \\ 0 & 1 & \ddots & \\ & \ddots & \ddots & -\alpha F_{n-1} \\ 0 & & 0 & 1 \end{pmatrix}, \quad (5)$$

where $\alpha = q - q^{-1}$ and the matrix $\mathbf{W} = \mathrm{diag}\{W_1,\dots,W_n\}$ contains the $U_q[gl(n)]$ Cartan elements. Analogously to eq. (5), we introduce as a limit of $\mathcal{T}^{(n)}(x,\underline{x}^{(n)})$ for $x$ to 0

$$\mathcal{T} = \tilde{T}\cdot T, \quad \text{where} \quad \tilde{T} = T_\infty^{-1}, \quad (6)$$

here and in the following operators without argument denote these limits for $x$ to 0.
We write the doubled monodromy matrices as $k\times k$ block-matrices of operators

$$\mathcal{T}^{(k)}(x) = \begin{pmatrix} \mathcal{A}^{(k)}(x) & \mathcal{B}^{(k)}(x) \\ \mathcal{C}^{(k)}(x) & \mathcal{D}^{(k)}(x) \end{pmatrix}. \quad (7)$$

We also introduce the reference states $\Phi^{(k)}$ with $\mathcal{C}^{(k)}(x)\Phi^{(k)} = 0$. The eigenstates of the transfer matrix $\tau(x)$ are the Bethe ansatz states $\Psi = \Psi^{(n)}$ obtained by the nested procedure

$$\Psi^{(k)} = \mathcal{B}^{(k)}_{\alpha_1}(x_1^{(k-1)})\dots\mathcal{B}^{(k)}_{\alpha_{N_{k-1}}}(x_{N_{k-1}}^{(k-1)})\Phi^{(k)}\Psi^{(k-1)}_{\underline{\alpha}}, \quad (k=2,\dots,n),\ \Psi^{(1)} = 1. \quad (8)$$

The sets of parameters $x_j^{(k)} = \exp\left(i\theta_j^{(k)} - (n-k)\gamma/2\right)$ $(q = e^{i\gamma})$ fulfil the Bethe ansatz equations: for $j = 1,\dots,N_k$ and $k = 1,\dots,n-1$

$$q^{2+w_{n-k}-w_{n-k+1}} \prod_{l=k\pm1}\prod_{i=1}^{N_l} \frac{\sinh\frac{1}{2}\left(\theta_j^{(k)} - \theta_i^{(l)} - i\gamma\right)}{\sinh\frac{1}{2}\left(\theta_j^{(k)} - \theta_i^{(l)} + i\gamma\right)} \prod_{i=1}^{N_k} \frac{\sinh\frac{1}{2}\left(\theta_j^{(k)} - \theta_i^{(k)} + 2i\gamma\right)}{\sinh\frac{1}{2}\left(\theta_j^{(k)} - \theta_i^{(k)} - 2i\gamma\right)} = -1 \quad (9)$$

where, below, the $w_i = N_{n-i+1} - N_{n-i}$ will turn out to be the weights of the state $\Psi$, i.e. the eigenvalues of the $W_i$'s defined by eq. (5).

**Theorem**: *The Bethe ansatz states are highest weight states, i.e.*

$$E_i \Psi = 0 \quad (i = 1, \ldots, n-1);$$

a statement which has been proved first time for rank $> 1$ within the context of the Hubbard model in [7]. The $sl_q(n)$ symmetric case with open boundary conditions has been treated in [5].

<u>*Proof*</u>: First we prove $\mathcal{T}^\alpha_\beta \Psi = 0$ and then $T^\alpha_\beta \Psi = 0$ for $\alpha > \beta$. The Yang-Baxter relation (4) implies

$$\mathcal{T}^\alpha_\beta \mathcal{B}_\gamma(x) = \mathcal{B}_{\gamma'}(x) \mathcal{T}^{\alpha'}_{\beta'} R^{\gamma'\alpha}_{\gamma''\alpha'} R^{\beta'\gamma''}_{\beta\gamma}, \quad \text{(for } \alpha > 1 \text{, else see eq. (12)).} \tag{10}$$

We apply the technique of the nested algebraic Bethe ansatz and commute $\mathcal{T}$ through all the $\mathcal{B}$'s of $\Psi$ in eq. (8)

$$\mathcal{T}^\alpha_\beta \mathcal{B}_{\alpha_1}(x_1) \ldots \mathcal{B}_{\alpha_{N_{n-1}}}(x_{N_{n-1}}) \Phi^{(k)} \Psi^{(n-1)}_{\underline{\alpha}} = \mathcal{B}_{\alpha_1}(x_1) \ldots \mathcal{B}_{\alpha_{N_{n-1}}}(x_{N_{n-1}}) \Phi^{(k)} \left(\mathcal{T}^{(n-1)}\right)^\alpha_\beta \Psi^{(n-1)}_{\underline{\alpha}}. \tag{11}$$

Iterating this procedure $\beta - 1$ times we arrive at $\mathcal{C}^{(k)}_{\alpha'} \Psi^{(k)}$ with $k = n - \beta + 1$ and $\alpha' = \alpha - \beta$. At this stage we use, as usual (see e.g. ref. [8]), the commutation rule

$$\mathcal{C}^{(k)}_{\alpha'} \mathcal{B}^{(k)}_\gamma(x) = q^{-1} R^{\gamma'\alpha'}_{\gamma\alpha''} \mathcal{B}^{(k)}_{\gamma'}(x) \mathcal{C}^{(k)}_{\alpha''} + (1 - q^{-2}) \left( (\mathcal{D}^{(k)})^{\alpha'}_\gamma \mathcal{A}^{(k)}(x) - (\mathcal{D}^{(k)})^{\alpha'}_{\alpha''} (\mathcal{D}^{(k)})^{\alpha''}_\gamma(x) \right), \tag{12}$$

to prove that the Bethe ansatz equations (9) imply

$$\mathcal{C}^{(k)}_{\alpha'} \Psi^{(k)} = 0 \quad (k = 2, \ldots, n) \quad \text{and therefore} \quad \mathcal{T}^\alpha_\beta \Psi = 0 \quad \text{for} \quad \alpha > \beta. \tag{13}$$

Finally we show $T^\alpha_\beta \Psi = 0$ for $\alpha > \beta$. We have with eqs. (5) and (13) for all $\beta < n$

$$\mathcal{T}^n_\beta \Psi = \tilde{T}^n_n T^n_\beta \Psi = 0 \tag{14}$$

and because $\tilde{T}^n_n$ is an invertible operator it follows that $T^n_\beta \Psi = 0$. Now we consider the previous row, where $\beta < n - 1$:

$$\mathcal{T}^{n-1}_\beta \Psi = \left( \tilde{T}^{n-1}_{n-1} T^{n-1}_\beta + \tilde{T}^{n-1}_n T^n_\beta \right) \Psi = 0, \tag{15}$$

and therefrom along with the foregoing result we get $T^{n-1}_\beta \Psi = 0$. By iteration we find $T^\alpha_\beta \Psi = 0$ for $\alpha > \beta$ and hence from eq. (5) $E_i \Psi = 0$ for all $i$, this proves the highest weight property of the Bethe vectors.

Next we compute the weights of the Bethe ansatz states $\Psi$. We consider first

$$\mathcal{T}_\alpha^\alpha \Psi = \tilde{T}_\alpha^\alpha T_\alpha^\alpha \Psi + \sum_{\beta>\alpha} \tilde{T}_\beta^\alpha T_\alpha^\beta \Psi. \tag{16}$$

Again shifting $\mathcal{T}_\alpha^\alpha$ to the right as in eq. (11) we get the operator $(\mathcal{T}^{(n-1)})_\alpha^\alpha$. By iteration we arrive at $\mathcal{A}^{(k)}\Psi^{(k)}$ for $k = n - \alpha + 1$. The Yang-Baxter relation (4) and eqs. (2) and (3) imply

$$\mathcal{A}^{(k)}\mathcal{B}_\beta^{(k)}(x) = q^{-2}\mathcal{B}_\beta^{(k)}(x)\mathcal{A}^{(k)}, \quad \mathcal{A}^{(k)}\Psi^{(k)} = q^{2N_k}\Psi^{(k)} \tag{17}$$

and therefore

$$\mathcal{A}^{(k)}\Psi^{(k)} = q^{2(N_k-N_{k-1})}\Psi^{(k)}. \tag{18}$$

From eqs. (5) and (6) we have

$$\tilde{T}_i^i = T_i^i = q^{W_i} \quad \text{and} \quad \mathcal{T}_i^i = q^{2W_i}. \tag{19}$$

and finally

$$q^{2W_i}\Psi = q^{2w_i}\Psi \quad \text{with} \quad w_i = N_{n-i+1} - N_{n-i}. \tag{20}$$

So any Bethe ansatz solution is characterized by a weight vector

$$w = (w_1, \ldots, w_n) = (N_n - N_{n-1}, \ldots, N_2 - N_1, N_1) \tag{21}$$

with the usual highest weight condition

$$w_1 \geq \cdots \geq w_n \geq 0. \tag{22}$$

Here $N = N^{(n)}$ is the number of lattice sites and $N^{(k)}$ $(k = n-1, \ldots, 1)$ is the number of roots in the $k$-th Bethe ansatz level. The highest weight condition (22) may be shown as usual. The result (21) is consistent with the "ice rule" fulfilled by the $R$-matrix (3). This means that each operator $\mathcal{B}_\alpha^{(k)}(x)$ reduces $w_k$ and lifts $w_\alpha$ by one.

The q-dimension of a representation $\pi$ with representation space $V$ is obtained from the "Markov trace" (see e.g. ref. [3] and [9])

$$\dim_q \pi = \mathrm{tr}_V \left( q^{-\sum_{i>j}(W_j - W_i)} \right). \tag{23}$$

As is well known for the case of $q$ being a root of unity the generators $E_i$ and $F_i$ become nilpotent

$$(E_i)^r = (F_i)^r = 0, \quad q = \exp(i\pi/r), \quad r = n+2, n+3, \ldots. \tag{24}$$

A highest weight module is equivalent to the corresponding one of $sl(n)$, if this relation does not concern it. These representations remain still irreducible and will be called good ones. The other representations are called bad and up to special irreducible cases with vanishing q-dimension they are reducible but not decomposable.

For the irreducible representations $\pi_w$ with highest weight vector $w$ eq. (23) gives

$$\dim_q \pi_w = \prod_{\alpha\in\Phi_+} \frac{[(w+g,\alpha)]_q}{[(g,\alpha)]_q} = \prod_{i>j} \frac{[w_j - w_i + i - j]_q}{[i-j]_q}, \tag{25}$$

where $[x]_q = (q^x - q^{-x})/(q - q^{-1})$ is a q-number, $\Phi_+$ denotes the set of positive roots and $g$ is the Weyl vector $g = \frac{1}{2}\sum_{\alpha\in\Phi_+}\alpha$. The good representations are characterized by positive q-dimensions. From eq. (25) it follows that their weight patterns are restricted in their length

$$w_1 - w_n \leq r - n. \tag{26}$$

The q-dimensions of bad representations vanish.

It is an interesting question, how these good representations are characterized in the language of the Bethe ansatz. In ref. [6] it is shown for $sl_q(2)$ that these are given by all Bethe ansatz solutions with only positive parity strings (in the language of Takahashi [10]) which are additionally restricted in their length. In a forthcoming paper we will show how this classification extends to q-symmetries of higher rank.

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