On the Numerical Approximation of Unstable Minimal Surfaces with Polygonal Boundaries

M. Hinze

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Abstract

This work is concerned with the approximation and the numerical computation of polygonal minimal surfaces in $\mathbb{R}^q$ $(q \geq 2)$. Polygonal minimal surfaces correspond to the critical points of Shiffman's function Θ. Since this function is analytic, polygonal minimal surfaces can be characterized by means of the second derivative of Θ. We present a finite element approximation of quasiminimal surfaces together with an error estimate. In this way we obtain discrete approximations Θ_h of Θ and f_h of $\nabla\Theta$. In particular we prove that the discrete functions converge uniformly on certain compact subsets. This will be the main tool for proving existence and convergence of discrete minimal surfaces in neighbourhoods of non-degenerate minimal surfaces. In the numerical part of this paper we compute numerical approximations of polygonal minimal surfaces by use of Newton's method applied to f_h.

1 Introduction

Let Γ be a Jordan curve in $\mathbb{R}^q$ $(q \geq 2)$ and denote by $B = \{w = (x_1, x_2) \in \mathbb{R}^2;\ |w| < 1\}$ the unit disc in $\mathbb{R}^2$. A mapping $u \in C^0(\overline{B}, \mathbb{R}^q) \cap C^2(B, \mathbb{R}^q)$ is called disc type minimal surface spanning Γ if u satisfies

$$\Delta u = 0 \quad \text{in } B \tag{1}$$

$$|u_{x_1}|^2 - |u_{x_2}|^2 = (u_{x_1}, u_{x_2}) = 0 \quad \text{in } B \tag{2}$$

$$u_{|\partial B} \quad \text{is a (weakly) monotone parametrization of } \Gamma. \tag{3}$$

In the case where Γ is a simple, closed polygon, the minimal surface problem is connected with the critical points of a function of finitely many variables. To be more precise, let $a_1, \ldots, a_{N+3}$ denote the distinct vertices of the polygon Γ $(N \geq 1)$ and consider mappings $v : \overline{B} \to \mathbb{R}^q$ in $C^0(\overline{B}, \mathbb{R}^q) \cap C^1(B, \mathbb{R}^q)$ such that $v(-1) = a_{N+1}$, $v(-i) = a_{N+2}$, $v(1) = a_{N+3}$ and $v(e^{i\kappa_j}) = a_j$ $(1 \leq j \leq N)$, where $\kappa =$

$(\kappa_1, \ldots, \kappa_N)$ is an N-tuple of parameter values κ_j satisfying $0 < \kappa_1 < \ldots < \kappa_N < \pi$. Let $F'(\kappa)$ denote the class of such mappings which map the arc $\overline{\gamma}_j := \{e^{i\phi}; \kappa_j \leq \phi \leq \kappa_{j+1}\}$ in a weakly monotone way onto the segment $[a_j, a_{j+1}]$ on the straight line $a_j + \Gamma_j$ through the points a_j, a_{j+1} $(\kappa_j \equiv \kappa_l \ if \ j \equiv l \ mod \ N+3)$ $(1 \leq j \leq N+2)$. Disc type minimal surfaces spanning Γ now correspond in a one-to-one manner to the critical points of Courant's function

$$d(\kappa) := \inf_{v \in F'(\kappa)} D(v), \tag{4}$$

where Dirichlet's integral is defined by

$$D(v) := \frac{1}{2} \int_B |\nabla v|^2 dx. \tag{5}$$

Problem (4) has a unique solution $v(\kappa)$, called *minimal vector* (see [2, Lemma 6.3]). The function d is known to be of class C^1. Unfortunately, it is not known up to now whether d is of class C^2, therefore Courant's function is not a good candidate to develop a Morse theory for polygonal minimal surfaces.

The Marx-Shiffman function Θ was introduced in the paper of Marx [16], which was based on a joint work of Marx and Shiffman concerning polygonally bounded minimal surfaces. Marx and Shiffman enlarged the class $F'(\kappa)$ to the class $F(\kappa)$ of mappings which map the arc γ_j into the straight lines $a_j + \Gamma_j$ and defined

$$\Theta(\kappa) := \inf_{v \in F(\kappa)} D(v). \tag{6}$$

The unique solution $u(\kappa)$ of problem (6) is called *(normalized) quasiminimal surface*. As a result, one clearly has $d(\kappa) \geq \Theta(\kappa)$ and $d(\kappa) > \Theta(\kappa)$ for certain values of κ (see [15]). In [16], it was stated that Θ is of class C^∞. In [5], Heinz proved that the quasiminimal surfaces $u(\kappa)$ depend analytically on κ. Using this result Heinz was able to prove that Θ is analytic ([6]). Furthermore, according to Heinz the critical points of Θ correspond in a one-to-one manner to solutions of a generalized Plateau problem for Γ. Here, *generalized* means : the solution $u(\kappa)$ can overshoot the vertices and one only knows that $u(\gamma_j) \subset a_j + \Gamma_j$. Note, that d and Θ are closely connected as they coincide at the critical points of d (see [20]).

Sauvigny computed the Morse index of such generalized solutions by studying the second derivative of Θ in [21]. In this context it would be desirable to determine classes of polygons for which d and Θ possess only a finite number of critical points. For arbitrary polygons this question is still open. But for a special class of extreme polygons Sauvigny completely investigated this question for the function d ([20, Eindeutigkeitssatz]). Uniqueness and finiteness results for the set of critical points of the function $\Theta(\kappa)$ have not yet been obtained.

In the last years several numerical approaches to the boundary value problem for parametric minimal surfaces had been made. Here, we mention the works of Jarausch [14], Wohlrab [25], Tsuchija [22],

[23],[24], Dziuk [3], Hutchinson [13], Pinkall/Polthier [18], Dziuk/Hutchinson [4] and refer to the references therein.

In this work we apply a new method to approximate minimal surfaces with polygonal boundaries. The method is based on the numerical treatment of Θ and a suitable approximation of $\nabla\Theta$. Using piecewise linear finite elements, we compute discrete quasiminimal surfaces $u_h(\kappa)$. In this way we obtain an approximation $\Theta_h(\kappa) = D_h(u_h(\kappa))$ of Θ. Based on an explicit expression for $\nabla\Theta$, that only involves the quasiminimal surface $u(\kappa)$, we introduce a discrete approximation f_h of $\nabla\Theta$. Applying Newton's method to the equation $f_h(\kappa) = 0$ we then can compute approximations of critical points of Θ. The corresponding quasiminimal surfaces in certain cases are approximations of generalized minimal surfaces. Since we can compute the eigenvalues of a finite difference approximation of $D^2\Theta_h$, we are able to classify these discrete minimal surfaces.

Our main result can be formulated as follows. Let κ correspond to a non-degenerate critical point of Θ and denote by $u = u(\kappa)$ the associated minimal surface (which might be unstable). Then there are discrete minimal surfaces $u_h = u_h(\kappa_h)$ satisfying

$$||u - u_h||_{1,B_h} \leq ch^{\mu^2}$$

and

$$|D(u) - D_h(u_h)| \leq ch^{2\mu},$$

where $\mu\pi$ denotes the smallest angle at the polygon vertices the minimal surface maps into and κ_h are roots of the function f_h contained in some neighbourhood of κ.

This is the first convergence result for unstable minimal surfaces in non-smooth contours. In [4] Dziuk/Hutchinson applied a linear boundary element method to compute approximations of non-degenerate minimal surfaces that are bounded by smooth curves $\Gamma \subset \mathbb{R}^q$ and proved existence and convergence of discrete minimal surfaces in neighbourhoods of non-degenerate minimal surfaces. The first convergence result for stable minimal surfaces can be found in Tsuchija's papers [23], [24].

For the visualization of grids and discrete surfaces we used the the graphical programing environment GRAPE developed at the Sonderforschungsbereich 256 at the university of Bonn, Germany.

2 Notations

In what follows, N and q denote integers satisfying $N \geq 1$ and $q \geq 2$. Let $B := \{w = x_1 + ix_2; |w| < 1\} \subset \mathbb{C} \equiv \{\mathrm{x} = (\mathrm{x}_1, \mathrm{x}_2); |\mathrm{x}| < 1\} \subset \mathbb{R}^2$ denote the unit disc and let $\Gamma \subset \mathbb{R}^q$ be a closed polygon with $N+3$ distinct vertices $a_1, \ldots, a_{N+3}$ arranged in cyclical order. We set

$$\varphi_k := \frac{a_{k+1} - a_k}{|a_{k+1} - a_k|} \quad (1 \leq k \leq N+3; a_{N+4} := a_1), \tag{7}$$

$$\Gamma_k \;=\; \{x \in \mathbb{R}^q; x = t\varphi_k(\alpha), t \in \mathbb{R}\} \quad (1 \le k \le N+3) \tag{8}$$

and denote by S_k, T_k the reflections at the lines Γ_k and $a_k + \Gamma_k$, respectively. Let $\kappa = (\kappa_1, \dots, \kappa_N) \in \mathbb{R}^N$ denote an $N-tuple$ of parameter values κ_j satisfying

$$0 < \kappa_1 < \dots < \kappa_N < \pi \tag{9}$$

and define

$$T \;:= \{\kappa \in \mathbb{R}^N;\; 0 < \kappa_1 < \dots < \kappa_N < \pi\}. \tag{10}$$

In order to connect problem (6) with the Plateau problem for the polygon Γ we set

$$\kappa_{N+1} := \pi, \quad \kappa_{N+2} := \frac{3\pi}{2}, \quad \kappa_{N+3} := 2\pi. \tag{11}$$

$T' \subset\subset T$ always denotes a compact subset of T. Furthermore, for $\kappa \in T$ we set

$$\gamma_k(\kappa) := \{w = e^{i\phi}; \kappa_k < \phi < \kappa_{k+1}\} \quad (1 \le k \le N+3;\; \kappa_{N+4} \equiv \kappa_1 + 2\pi) \tag{12}$$

and abbreviate

$$\beta_k(\kappa) := e^{i\kappa_k} \quad (1 \le k \le N+3). \tag{13}$$

For $m \in \mathbb{N}_0$ and $1 \le p \le \infty$, $H^{m,p}(\Omega)$ denotes the Sobolev space of all Lebesgue-measurable functions having p-integrable derivatives up to order m equipped with the usual norm $\|\cdot\|_{m,p,\Omega}$. The corresponding seminorms are denoted by $|\cdot|_{m,p,\Omega}$. We abbreviate $H^m(\Omega) \equiv H^{m,2}(\Omega)$, $\|f\|_{m,\Omega} \equiv \|f\|_{m,2,\Omega}$, $|f|_{m,\Omega} \equiv |f|_{m,2,\Omega}$ as well as $\|f\|_\Omega \equiv \|f\|_{0,\Omega}$.

For $k \in \mathbb{N}_0$, $C^k(\Omega)$ denotes the space of all functions having continuous derivatives up to order k. With $C^k(\overline{\Omega})$ we denote the set of all functions in $C^k(\Omega)$, such that, for $|s| \le k$, there exists a continuous extension of $\partial^s f$ onto $\overline{\Omega}$.

$H^{m,p}(\Omega, \mathbb{R}^q)$ is defined as the set of all vector-valued functions $f = (f_1, \dots, f_q)$, such that $f_i \in H^{m,p}(\Omega)$ $(1 \le i \le q)$. The function spaces $C^k(\Omega, \mathbb{R}^q)$ and $C^k(\overline{\Omega}, \mathbb{R}^q)$ are defined analogously.

3 Analytical preliminaries

Introducing the function spaces

$$X(\kappa) \;:=\; \{v \in H^1(B, \mathbb{R}^q);\; v(\gamma_k(\kappa)) \subset a_k + \Gamma_k \;(1 \le k \le N+3)\} \tag{14}$$

and

$$X_0(\kappa, \alpha) := \{v \in H^1(B, \mathbb{R}^q)\;;\; v(\gamma_k(\alpha)) \subset \Gamma_k(\alpha) \quad (1 \le k \le N+3)\}. \tag{15}$$

problem (6) according to [11, Theorem 3.1] alternatively can be written as

$$\Theta(\kappa) = \inf_{v \in X(\kappa)} D(v) \qquad (\kappa \in T). \tag{16}$$

This problem has a unique solution $u = u(\kappa) \in C^2(\overline{B} \setminus \{\beta_1, \ldots, \beta_{N+3}\}, \mathbb{R}^q) \cap C^0(\overline{B}, \mathbb{R}^q)$, satisfies

$$\int_B \nabla u \, \nabla v \; dx \; = \; 0 \qquad \forall \; v \in X_0 \tag{17}$$

and solves the following system of partial differential equations (see [5, Satz 1])

$$\Delta u \;=\; 0 \qquad in \; B \tag{18}$$

$$u(\gamma_k) \;\subset\; a_k + \Gamma_k \qquad 1 \le k \le N+3 \tag{19}$$

$$(\partial_r u, \varphi_k) \;=\; 0 \qquad on \quad \gamma_k, \quad 1 \le k \le N+3. \tag{20}$$

Since u is continuous on $\overline{B}$ one by necessity has

$$u(\beta_k) = a_k \quad (k = 1, \ldots, N+3).$$

At the vertices of Γ the normal to the contour has a jump discontinuity. This causes a lack of regularity of u at β_k. Heinz completely investigated this situation in [5, Satz 2]. In order to formulate the corresponding theorem let us first introduce some notations. Let $\kappa^* \in T$ be arbitrary, but fixed. For $\epsilon > 0$ set

$$U_\epsilon(\kappa^*, \beta_k(\kappa^*)) := \{(\kappa, w) \in \mathbb{C}^{\mathrm{N}} \times \mathbb{C}; |\kappa - \kappa^*| < \epsilon, |\mathrm{w} - \beta_{\mathrm{k}}(\kappa^*)| < \epsilon\} (1 \le \mathrm{k} \le \mathrm{N} + 3)$$

and $\partial_w := \frac{1}{2}(\partial_{x_1} - i\partial_{x_2})$ $(w = x_1 + ix_2)$. Now we extract from [5, Satz 2]

Theorem 3.1 *Let $D = D(\kappa^*) \subset \mathbb{C}$ denote a simply connected region, such that $D \cap B$ is nonempty, connected and the closure $\overline{D}$ does not contain any of the points $\beta_j(\kappa^*)$ $(j = 1, \ldots, N+3)$. Then, there exists a polydisc*

$$U_D(\kappa^*) = \{\kappa \in \mathbb{C}^{\mathrm{N}}; |\kappa_{\mathrm{j}} - \kappa_{\mathrm{j}}^*| < \epsilon_{\mathrm{D}}, \; \mathrm{j} = 1, \ldots, \mathrm{N}\}$$

and a function $f_D(\kappa, w)$, which is analytic in $U_D(\kappa^*) \times D$, such that the solution $u(\kappa)$ of (16) satisfies

$$u_w(\kappa)(w) = f_D(\kappa, w) \quad \forall \; (\kappa, w) \in (U_D(\kappa^*) \cap T) \times (D \cap B). \tag{21}$$

Furthermore, for every $k = 1, \ldots, N+3$ there exists an integer $1 \le p_k \le q$, p_k exponents $-1 < p_1^k < \ldots < p_{p_k}^k \le 0$ and an $\epsilon_0 > 0$, such that for the solutions $u(\kappa)$ of (16) the singular expansions

$$u_w(\kappa)(w) = \sum_{j=1}^{p_k} f_j^k(\kappa, w)(w - \beta_k(\kappa))^{p_j^k} \tag{22}$$

are valid for all $(\kappa, w) \in U_{\epsilon_0}(\kappa^*, \beta_k(\kappa^*)) \cap (T \times B)$. The complex-valued coefficient functions $f_j^k(\kappa, w)$ are analytic in $U_{\epsilon_0}(\kappa^*, \beta_k(\kappa^*))$.

Integration of (22) yields for some suitable $\tilde{\epsilon}_0 > 0$ and forall $(\kappa, w) \in U_{\tilde{\epsilon}_0}(\kappa^*, \beta_k(\kappa^*)) \cap (T \times B)$

$$u(\kappa)(w) = a_k + Re(\sum_{j=1}^{p_k} g_j^k(\kappa, w)\ (w - \beta_k(\kappa))^{p_j^k+1}) \tag{23}$$

with suitable coefficient functions $g_j^k(\kappa, w)$. The exponents p_j^k are directly related to the eigenvalues of $S_{k-1}S_k^{-1}$ and therefore to the angles of the polygon Γ at a_k $(1 \le k \le N+3)$ (see [5, (3.26),(3.28),(3.49),(3.63)]). In the case $q = 3$ one has from [7, (4.2)]

$$p_3^k = 0, \quad p_2^k = -\alpha_k, \quad p_1^k = \alpha_k - 1 \qquad (1 \le k \le N+3), \tag{24}$$

where

$$\cos \alpha_k \pi := \varphi_{k-1}^t \varphi_k \quad (k = 2, \ldots, N+3). \tag{25}$$

Using the exponents of Theorem 3.1 set

$$p_{min} := \min_{1 \le k \le N+3} \min_{1 \le j \le p_k} p_j^k \in (-1, 0]. \tag{26}$$

Theorem 3.1 then implies for a suitable $\tilde{\epsilon} > 0$ (compare also [8, (2.1)])

$$|u_w(\kappa)(w)| \le\ c \sum_{k=1}^{N+3} |w - \beta_k(\kappa)|^{p_{min}} \quad (|w| < 1; |\kappa - \kappa^*| < \tilde{\epsilon}) \tag{27}$$

and

$$|u_{ww}(\kappa)(w)| \le\ c \sum_{k=1}^{N+3} |w - \beta_k(\kappa)|^{p_{min}-1} \quad (|w| < 1; |\kappa - \kappa^*| < \tilde{\epsilon}), \tag{28}$$

with some $c > 0$.

Next, define for $\epsilon > 0$

$$\Omega_\epsilon(\kappa) := \overline{B} \setminus \cup_{k=1}^{N+3} B_\epsilon(\beta_k(\kappa)), \quad \kappa \in T \tag{29}$$

and assign to every $\kappa \in T$ a simply connected domain $D(\kappa)$ in such a way, that Theorem 3.1 is valid for $\kappa = \kappa^*$. Let $T' \subset\subset T$ and apply Theorem 3.1 for every $\kappa \in T'$ together with suitable domains $D(\kappa)$. Then the corresponding polydiscs in Theorem 3.1 induce an open covering of T'. Thus, the Heine-Borel theorem together with (27) and (28) proves

Corollary 3.1 *For every $T' \subset\subset T$ there exist positive constants $c_1(T')$, $c_2(T', \epsilon)$ and $c_3(T')$ such that*

$$||u(\kappa)||_{1,p,B} \ \le c_1(T') \qquad \forall\ \kappa \in T'\ \forall\ 2 \le p < \frac{-2}{p_{min}}, \tag{30}$$

$$||u(\kappa)||_{2,\Omega_\epsilon(\kappa)} \le c_2(T', \epsilon) \qquad \forall\ \kappa \in T'\ \forall\ \epsilon > 0,\ \Omega_\epsilon(\kappa) \ne \emptyset \tag{31}$$

and

$$|u_{ww}(\kappa)(w)| \le c_3(T') \sum_{k=1}^{N+3} |w - \beta_k(\kappa)|^{p_{min}-1} \qquad \forall\ \kappa \in T' \quad (|w| < 1). \tag{32}$$

In particular for every $T' \subset\subset T$, there exists some $\epsilon(T') > 0$ so that the assertion in the second part of Theorem 3.1 is valid for every $\kappa = \kappa^* \in T'$ and $\epsilon_0 = \epsilon(T')$. For $\epsilon(T')$ inequality (31) reads

$$\|u(\kappa)\|_{2,\Omega_{\epsilon(T')}(\kappa)} \leq c_2(T') \qquad \forall\ \kappa \in T'.$$

This Corollary will be in important tool in proving a uniform error estimate for discrete quasiminimal surfaces (see Sect. 3).

The next Theorem was proved by Heinz in [6, Satz] and characterizes minimal surfaces in terms of Θ.

Theorem 3.2 *For $\kappa \in T$ the surface $u(\kappa)$ is a minimal surface iff $\nabla\Theta(\kappa) = 0$. Θ is analytic.*

Consequently, Plateau's problem for Γ is analytically embedded into a N-parameter family of quasiminimal surfaces.

Let

$$a(x) \ := \ \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2}, \quad b(x) \ := \ \frac{x_1 x_2}{x_1^2 + x_2^2}, \quad x \in \overline{B} \setminus \{0\},$$

denote by $S(\kappa_{j-1}, \kappa_j)$ the sector of B defined by the points $0, \beta_{j-1}, \beta_j$ $(j = 2, \ldots, N+3)$ and set

$$\begin{aligned} &V(S(\kappa_{j-1}, \kappa_j), S(\kappa_{j-1}, \kappa_j), u(\kappa)) := \\ -\frac{1}{2}\Big\{ &\frac{1}{\kappa_{j+1} - \kappa_j} \int\limits_{S(\kappa_j, \kappa_{j+1})} (u_{x_1}^2(\kappa) - u_{x_2}^2(\kappa))\ a + 4 u_{x_1}(\kappa) u_{x_2}(\kappa)\ b\ dx \\ -&\frac{1}{\kappa_j - \kappa_{j-1}} \int\limits_{S(\kappa_{j-1}, \kappa_j)} (u_{x_1}^2(\kappa) - u_{x_2}^2(\kappa))\ a + 4 u_{x_1}(\kappa) u_{x_2}(\kappa)\ b\ dx \Big\}. \end{aligned} \tag{33}$$

In [12, Theorem 3.3.3] we proved that

$$\Theta_{\kappa_j}(\kappa) = V(S(\kappa_{j-1}, \kappa_j), S(\kappa_{j-1}, \kappa_j), u(\kappa))\ (j = 1, \ldots, N+3) \tag{34}$$

holds. Due to results of Heinz ([9, Satz 2]) and Sauvigny ([21, Theorem II]) the following Definition makes sense.

Definition 3.1 *A minimal surface $u(\kappa)$ is called non-degenerate iff* $\mathrm{rank} D^2\Theta(\kappa) = N$.

Lemma 3.1 *Non-degenerate minimal surfaces are isolated.*

Proof. Let $u(\kappa)$ be a non-degenerate minimal surface. By use of the Inverse-function theorem one obtains an open neighbourhood $V(\kappa)$ so that $\nabla\Theta(\tilde\kappa) \neq 0,\ \tilde\kappa \in V(\kappa) \setminus \{\kappa\}$. This proves the Lemma.

We close this section with stating a Poincaré inequality for functions in

$$V := \{v \in H^1(B, \mathbb{R}^q);\ v(\gamma_1) \subset \Gamma_1, v(\gamma_{N+3}) \subset \Gamma_{N+3}\}$$

(compare also [5, (2.12)]).

Lemma 3.2

$$\exists\ c > 0\ \forall\ v \in V\ :\ \|v\|_B\ \leq\ c\ |v|_{1,B}. \tag{35}$$

This lemma can be proved as *Hilfsatz 2.6* in [10].

4 The discrete approach

Let $\kappa \in T$ be fixed. To obtain a conforming finite element method approximating problem (16) we first construct a finite dimensional affine space $X_h = X_h(\kappa) \subseteq X = X(\kappa)$. For this purpose denote by $\tau = \tau(\kappa)$ triangulations of B that are obtained by one of the following refinement strategies applied to macro-triangulations of the unit disc of the form as indicated in Fig. 1. Let $k \in \mathbb{N}$ denote the refinement level.

S1 :

1. (a) $\tau_0 := \cup_{j=1}^{N+3} \Delta(0, e^{i\kappa_{j-1}}, e^{i\kappa_j})$ ($\kappa_0 \equiv \kappa_{N+3} - 2\pi$) (macro-triangulation, Fig. 1)
   (b) Mark the edge $[e^{i\kappa_{j-1}}, e^{i\kappa_j}]$, called refinement edge, of the triangle $\Delta(0, e^{i\kappa_{j-1}}, e^{i\kappa_j})$ ($j = 1, \ldots, N+3$; $\kappa_0 \equiv \kappa_{N+3} - 2\pi$)
   (c) $l = 0$
2. Bisect each $\Delta \in \tau_l$ along the refinement edge and mark the new triangles as indicated in Fig. 3. If a new node V belongs to the discrete boundary, proceed as indicated in Fig. 4, i.e. set $V = \frac{V}{|V|}$.
3. $l = l + 1$, denote the new set of triangles with τ_l
4. If $l = 2k$ then stop
5. Goto 2.

S2 :

1. (a) $\tau_0 := \cup_{j=1}^{N+3} \Delta(0, e^{i\kappa_{j-1}}, e^{i\kappa_j})$ (macro-triangulation, Fig. 1)
   (b) $l = 0$
2. Divide each $\Delta \in \tau_l$ into 4 congruent triangles as indicated in Fig 2. If a new node V belongs to the discrete boundary, proceed as indicated in Fig. 4, i.e. set $V = \frac{V}{|V|}$.
3. $l = l + 1$, denote the new set of triangles with τ_l
4. If $l = k$ then stop
5. Goto 2.

Clearly, both refinement strategies yield regular and conforming triangulations of B.

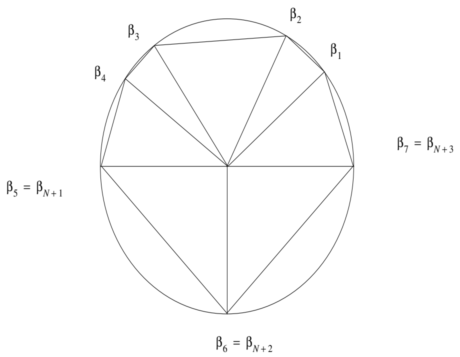
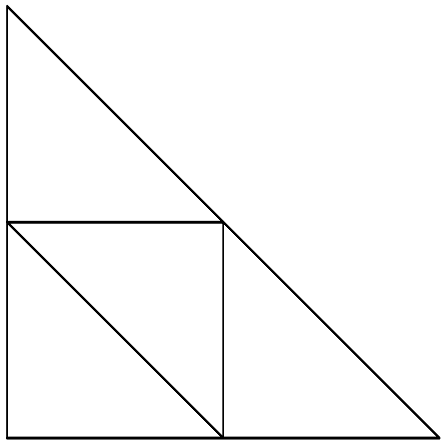


Figure 1: Macro triangulation



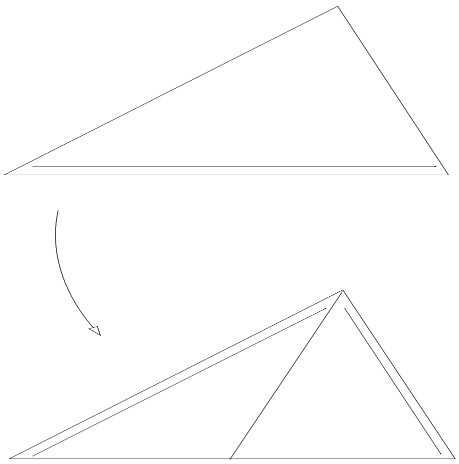


Figure 3: Bisection, refinement edges

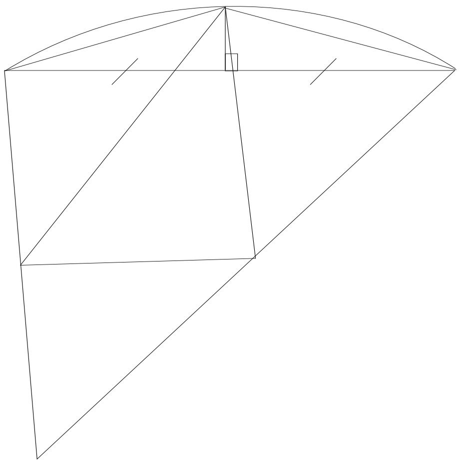


Figure 4: Boundary refinement

Set

$$B_h \quad := \quad \cup\{\Delta \in \tau\}, \tag{36}$$

$$Y_h \quad := \quad \{v \in C^0(\overline{B}_h, \mathbb{R}^q)\ ;\ v_{|\Delta} \text{ linear } \forall\ \Delta \in \tau\}, \tag{37}$$

and consider Y_h as a subspace of $H^1(B_h, \mathbb{R}^q)$. The discrete analogues of X and X_0 are

$$X_h \ := \{v \in Y_h\ ; v(\overline{\gamma}_k \cap \partial B_h) \subset a_k + \Gamma_k\ \ (1 \leq k \leq N+3)\} \tag{38}$$

and

$$X_{h0} \ := \ \{v \in Y_h\ ; v(\overline{\gamma}_k \cap \partial B_h) \subset \Gamma_k\ \ (1 \leq k \leq N+3)\}, \tag{39}$$

respectively. For functions in Y_h the discrete Dirichlet integral is defined by

$$D_h(v) := \frac{1}{2}\int_{B_h} |\nabla v|^2 dx. \tag{40}$$

In order to obtain the inclusion $X_h \subseteq X$ one has to define a suitable prolongation operator for functions in Y_h. Let η denote the outer unit normal vector to ∂B_h. For $x \in \mathbb{R}^2$, let $a(x) := dist(x, \partial B_h)$ measure the distance between x and ∂B_h. The prolongation operator is defined as follows

$$\begin{aligned} Pro: \quad Y_h \quad & \rightarrow H^1(B, \mathbb{R}^q) \\ Pro(v)(x) \quad & := \begin{cases} v(x) \quad , & x \in \overline{B}_h \\ v(x - a(x)\eta) \quad , & x \in \overline{B} \setminus \overline{B}_h . \end{cases} \end{aligned} \tag{41}$$

The definition of Pro implies that $Pro(v) \in X$, if $v \in X_h$. Moreover, Pro is a continuous operator with respect to the H^1-topology (see [10, Bemerkung 3.8]). For what follows we introduce the convention of not distinguishing between v and $Pro(v)$.

The discrete analogue to the variational problem (16) is

$$\Theta_h(\kappa) \;=\; \inf_{v \in X_h(\kappa)} D_h(v). \tag{42}$$

This problem has a unique solution $u_h = u_h(\kappa) \in X_h$, called *discrete quasiminimal surface* that satisfies (see [11, Theorem 3.1])

$$\int_{B_h} \nabla u_h \nabla v \; dx \;=\; 0 \qquad \forall\; v \in X_{h0}. \tag{43}$$

Next we prove a H^1-error estimate for discrete quasiminimal surfaces. For this purpose let $m \in \mathbb{N}$, $2 < p \leq \infty$, $m - \frac{2}{p} > 0$, let $h > 0$ denote the gridsize of τ and set

$$\begin{aligned} I_h: \quad H^{m,p}(B, \mathbb{R}^q) \quad & \rightarrow Y_h \\ I_h v(x) \quad & := v(x) \quad \text{at the nodes } x \text{ of } \tau. \end{aligned} \tag{44}$$

The definition of I_h clearly implies

$$I_h v \;\in\; X_h \; (X_{h0}) \quad if \quad v \;\in\; X \; (X_0) \cap C^0(\overline{B}, \mathbb{R}^q). \tag{45}$$

Now let $u = u(\kappa)$ denote the solution of (16). The I_h-interpolant of u satisfies (see [11, Theorem 5.3])

$$\|u - I_h u\|_{1,B_h} \;\leq\; c\; h^{\frac{p-2}{p}} \qquad \forall\; 2 \leq p < -\frac{2}{p_{min}} \tag{46}$$

with some positive constant $c = c(|u|_{1,p,B}, \tau, N)$ and p_{min} from (26). Note that $p < -\frac{2}{p_{min}}$ since $u \notin H^{1,-\frac{2}{p_{min}}}(B)$. Estimate (46) is the main tool in proving

Theorem 4.1 *Let $h > 0$ denote the gridsize of τ of and let u, u_h be the unique solutions of (16) and (42), respectively. Then one has*

$$\|u - u_h\|_{1,B_h} \;\leq\; c\; h^{\frac{p-2}{p}} \qquad \forall\; 2 \leq p < -\frac{2}{p_{min}} \tag{47}$$

where $c = c(|u|_{1,p,B}, \tau, N) > 0$.

Proof. With the help of Lemma 3.2, the continuity of the operator Pro and Eq. (46) one obtains (compare [10, p.38])

$$\begin{aligned}\|u-u_h\|_{B_h} &\leq \|u-I_h u\|_{B_h} + \|I_h u - u_h\|_{B_h} \\ &\leq c\left(h^{\frac{p-2}{p}} + |u-u_h|_{1,B_h}\right).\end{aligned}$$

Obviously it remains to estimate $|u-u_h|_{1,B_h}$. The variational equalities (17) and (43) imply for $v \in X_h$

$$|u-u_h|^2_{1,B_h} = \int_{B_h} \nabla(u-u_h)\nabla(u-v)\ dx - \int_{B\setminus B_h} \nabla u\nabla(v-u_h)\ dx.$$

Using Young's inequality, a straightforward estimate yields

$$|u-u_h|_{1,B_h} \leq c \inf_{v\in X_h} (|u-v|_{1,B_h} + |u|_{1,B\setminus B_h}). \tag{48}$$

Since $\text{meas}(B\setminus B_h) \leq ch^2$, it follows from Hölder's inequality that

$$\begin{aligned}|u|_{1,B\setminus B_h} &\leq \text{meas}(B\setminus B_h)^{\frac{p-2}{2p}}\ |u|_{1,p,B\setminus B_h} \\ &\leq c\ h^{\frac{p-2}{p}}\ |u|_{1,p,B} \qquad \forall\ 2\leq p < -\frac{2}{p_{min}}.\end{aligned}$$

Setting $v := I_h u$ in (48), the assertion follows from (46).

For the convergence of Dirichlet's integral one can say more.

Corollary 4.1 *With the notation of Theorem 4.1 the difference $D(u) - D(u_h)$ satisfies the estimate*

$$|D(u) - D_h(u_h)| \leq c\ h^{2\frac{p-2}{p}} \quad \forall\ 2 \leq p < \frac{-2}{p_{min}} \tag{49}$$

with some $c = c(|u|_{1,p,B}, \tau, N) > 0$.

Proof. We write

$$\begin{aligned}D(u) - D_h(u_h) = &\overbrace{-\frac{1}{2}\int_{B_h} |\nabla(u-u_h)|^2 dx}^{[1]} + \\ &+ \underbrace{\int_{B_h} \nabla u\nabla(u-u_h)dx}_{[2]} + \underbrace{\frac{1}{2}\int_{B\setminus B_h} |\nabla u|^2 dx}_{[3]}\end{aligned}$$

and set $\mu := \frac{p-2}{p}$. Equation (46) together with $meas(B\setminus B_h) = O(h^2)$ yields

$$|[1],[3]| \leq c\ h^{2\mu} \quad \forall\ 2 \leq p < \frac{-2}{p_{min}}.$$

In order to estimate (2) we recall that the operator Pro defined in (41) is continuous, satisfies $Pro(u_h) \in X$ and $\nabla u_{h_{|T}} \equiv const$ on every triangle T of the triangulation τ_h. The variational equality (17) implies

$$\int_{B_h} \nabla u\nabla(u-u_h)dx = -\int_{B\setminus B_h} |\nabla u|^2 dx + \int_{B\setminus B_h} \nabla u\nabla u_h dx.$$

Thus

$$|[2]| \le c\ h^{2\mu} + \underbrace{\int_{B\setminus B_h} |\nabla u \nabla u_h| dx}_{[4]}$$

and, using the Cauchy-Schwarz inequality

$$\begin{aligned}[4] &\le (\int_{B\setminus B_h} |\nabla u|^2 dx)^{\frac{1}{2}} (\int_{B\setminus B_h} |\nabla u_h|^2 dx)^{\frac{1}{2}} \\ &\le ch^{\mu}\ \underbrace{(\int_{B\setminus B_h} |\nabla u_h|^2 dx)^{\frac{1}{2}}}_{[5]}.\end{aligned}$$

In order to estimate [5] define Ω^* to be the union of triangles of the triangulation τ_h having a boundary edge and denote for a triangle $T \subset \Omega^*$ by Ω_T the segment defined by T and ∂B. Then one has

$$\begin{aligned}[5]^2 &= \sum_{T\subset\Omega^*} \int_{\Omega_T} |\nabla u_h|^2 dx \le \sum_{T\subset\Omega^*} \underbrace{meas(\Omega_T)}_{=O(h^3)} |\nabla(u_{h_{|T}})|^2 \\ &\le c\ h \sum_{T\subset\Omega^*} \int_T |\nabla u_h|^2 dx = ch\ \underbrace{\int_{\Omega^*} |\nabla u_h|^2 dx}_{[6]}.\end{aligned}$$

Since $meas(\Omega^*) \le c(N)\ h$ Hölder's inequality yields

$$\begin{aligned}[6] &= \int_{\Omega^*} |\nabla u|^2 dx + \int_{\Omega^*} |\nabla(u-u_h)|^2 dx - 2\int_{\Omega^*} \nabla u \nabla(u-u_h) dx \\ &\le c\ h^{\mu} + c\ h^{2\mu} + 2(\int_{\Omega^*} |\nabla u|^2 dx)^{\frac{1}{2}} (\int_{\Omega^*} |\nabla(u-u_h)|^2 dx)^{\frac{1}{2}} \\ &\le c\ h^{\mu} \quad \forall\ 2 \le p < \frac{-2}{p_{min}}.\end{aligned}$$

One concludes

$$[5] \le c\ h^{\frac{1+\mu}{2}} \quad \forall\ 2 \le p < \frac{-2}{p_{min}},$$

$$[4] \le c\ h^{\frac{1+3\mu}{2}} \quad \forall\ 2 \le p < \frac{-2}{p_{min}}$$

and finally, since $h^{\frac{1+3\mu}{2}} \le h^{2\mu}\ \forall\ 0 < h < 1$

$$|[2]| \le c\ h^{2\frac{p-2}{p}} \quad \forall\ 2 \le p < \frac{-2}{p_{min}}.$$

This completes the proof.

Remark 4.1 *In the case $q \le 3$ one has $p_{min} = \min_{1\le k\le N+3}\{-\alpha_k, \alpha_k - 1\}$, where $\alpha_k\pi (1 \le k \le N+3)$ has been defined in (25). In this case there is a direct relation between the smallest angle of the polygon Γ and the order of convergence.*

By our choice of triangulations $\tau(\kappa)$ of the unit disc (note that the points $\beta_j(\kappa) = e^{i\kappa_j}$ are nodes of the macro-triangulation) and Corollary 3.1 the constants in (46), (47) and (49) can be controlled uniformly for $\kappa \in T' \subset\subset T$ ([11, Sect. 4.5]). This proves the following theorem that will be the main tool in establishing existence and convergence of discrete minimal surfaces in the next section.

Theorem 4.2 *Let $T' \subset\subset T$ and let $h > 0$ denote the gridsize of $\tau_h(\kappa)$. Let $u_h(\kappa), u(\kappa)$ denote the solutions of (42) and (16), respectively. Then one has*

$$\sup_{\kappa\in T'} \|u(\kappa) - u_h(\kappa)\|_{1,B_h} \ \leq\ c_1\ h^{\frac{p-2}{p}} \quad \forall\ 2 \leq p < -\frac{2}{p_{min}} \tag{50}$$

and

$$\|\Theta - \Theta_h\|_{\infty,T'} \ \leq\ c_2 h^{2\frac{p-2}{p}} \quad \forall\ 2 \leq p < \frac{-2}{p_{min}} \tag{51}$$

with constants $c_i = c_i(T') > 0$ $(i = 1, 2)$.

5 Existence and convergence of discrete minimal surfaces

Let us start with defining an approximation of $\nabla\Theta$ in (34). For this purpose let $S_h(\kappa_k, \kappa_{k+1}) := S(\kappa_k, \kappa_{k+1}) \cap B_h$ and define a function $f_h : T \to \mathbb{R}^N$, $f_h = (f_h^1, \ldots, f_h^N)^t$ in terms of their components by (compare (34))

$$f_h^k(\kappa) := V(S_h(\kappa_{j-1}, \kappa_j), S_h(\kappa_{j-1}, \kappa_j), u_h(\kappa)). \tag{52}$$

In this expression $u_h(\kappa)$ denotes the discrete quasiminimal surfaces corresponding to a $\kappa \in T$.

For f_h one has

Theorem 5.1 *Let f_h be defined by (52). Then one has for every $T' \subset\subset T$*

$$\|\nabla\Theta - f_h\|_{\infty,T'} \ \leq\ c h^{\frac{p-2}{p}} \quad \forall\ 2 \leq p < \frac{-2}{p_{min}} \tag{53}$$

with a positive constant $c = c(T')$ and $p_{min} > -1$ from (26).

Proof. Taking the difference of (34) and (52), it remains to estimate integrals of the form

$$\int_{S_h} (u_{hx_i}^2(\kappa) - u_{x_i}^2(\kappa))\ a\ dx \quad , \int_{S\setminus S_h} u_{x_i}^2(\kappa)\ a\ dx \quad (i = 1, 2),$$

and

$$\int_{S_h} (u_{hx_1}(\kappa)u_{hx_2}(\kappa) - u_{x_1}(\kappa)u_{x_2}(\kappa))\ b\ dx, \quad \int_{S\setminus S_h} u_{x_1}(\kappa)u_{x_2}(\kappa))\ b\ dx$$

respectively. But

$$u_{hx_i}^2(\kappa) - u_{x_i}^2(\kappa) = (u_{hx_i}(\kappa) - u_{x_i}(\kappa))^2 + 2u_{x_i}(\kappa)(u_{hx_i}(\kappa) - u_{x_i}(\kappa)) \quad (i = 1, 2),$$

$$u_{hx_1}(\kappa)u_{h_{x_2}}(\kappa) - u_{x_1}(\kappa)u_{x_2}(\kappa) = (u_{hx_1}(\kappa) - u_{x_1}(\kappa))(u_{h_{x_2}}(\kappa) - u_{x_2}(\kappa)) + \\ +(u_{hx_2}(\kappa) - u_{x_2}(\kappa))u_{x_1}(\kappa) + (u_{hx_1}(\kappa) - u_{x_1}(\kappa))u_{x_2}(\kappa)$$

and $meas(S \setminus S_h) = O(h^2)$. Thus, estimate (50) gives the desired result.

According to Theorem 5.1 it makes sense to define discrete minimal surfaces spanning a polygon $\Gamma \subset \mathbb{R}^q$ in the following way.

Definition 5.1 *Let $\kappa_h^* \in T$ satisfy $f_h(\kappa_h^*) = 0$. The corresponding discrete quasiminimal surface $u_h(k_h^*)$ is called discrete minimal surface.*

In what follows existence of discrete minimal surfaces in neighbourhoods of minimal surfaces corresponding to regular critical points of the function Θ will be proved. For this purpose let $\kappa^* \in T$ with $\mathrm{rank} D^2\Theta(\kappa^*) = N$ and set

$$s_h(\kappa) \ := \ \kappa - D^2\Theta(\kappa^*)^{-1} f_h(\kappa), \qquad \kappa \in T.$$

Then clearly every fixed point of s_h in T is a root of the vector function f_h.

By Lemma 3.1 an open ball $V_{\varrho_0}(\kappa^*) \subset T$ exists, such that

$$\nabla\Theta(\kappa) \neq 0 \quad \forall\ \kappa \in V_{\varrho_0}(\kappa^*) \setminus \{\kappa^*\}.$$

In order to apply the Brouwer fixed point theorem to s_h we prove

Lemma 5.1 *There exists a constant $\tilde{c} > 0$ and a $h_0 > 0$ such that for all $0 < h \leq h_0$*

$$s_h(\overline{V_{\varrho_h}(\kappa^*)}) \subset V_{\varrho_h}(\kappa^*) \subset V_{\varrho_0}(\kappa^*)$$

holds, where $\varrho_h := \tilde{c}h^{\frac{p-2}{p}}$, $2 < p < \frac{-2}{p_{min}}$, p_{min} from (26)

Proof. The definition of s_h implies

$$s_h(\kappa) - \kappa^* = \overbrace{D^2\Theta(\kappa^*)^{-1}\{D^2\Theta(\kappa^*)(\kappa - \kappa^*) - \nabla\Theta(\kappa)\}}^{[1]} + \\ + \underbrace{D^2\Theta(\kappa^*)^{-1}\{\nabla\Theta(\kappa) - f_h(\kappa)\}}_{[2]}.$$

Estimate (53) applied to the difference [2] yields the existence of a positive constant $c = c(\overline{V_{\varrho_0}(\kappa^*)})$ such that

$$\|[2]\|_{\infty, V_{\varrho_0}(\kappa^*)} \leq ch^{\frac{p-2}{p}} \quad \forall\ 2 \leq p < \frac{-2}{p_{min}}. \tag{54}$$

Now define $\tilde{c} := 2c$ and note that since Θ is analytic in T and $\nabla\Theta(\kappa^*) = 0$

$$D^2\Theta(\kappa^*)(\kappa - \kappa^*) - \nabla\Theta(\kappa) = o(|\kappa - \kappa^*|) \ (\kappa \to \kappa^*)$$

holds. Therefore one can choose $0 < h_0 < \varrho_0$ in such a way that for $0 < h \leq h_0$, $2 \leq p < \frac{-2}{p_{min}}$

$$|\kappa - \kappa^*| \leq \tilde{c} h^{\frac{p-2}{p}} \Rightarrow |[1]| < \frac{\tilde{c}}{2} h^{\frac{p-2}{p}}. \tag{55}$$

Now set $\varrho_h := \tilde{c} h^{\frac{p-2}{p}}$ and let $\kappa \in V_{\varrho_h}(\kappa^*)$. Then Eqs. (54) and (55) imply

$$|s_h(\kappa) - \kappa^*| < (\frac{\tilde{c}}{2} + c) h^{\frac{p-2}{p}} = \tilde{c} h^{\frac{p-2}{p}} \quad \forall\, 2 \leq p < \frac{-2}{p_{min}}, \tag{56}$$

i.e. $s_h(\overline{V_{\varrho_h}(\kappa^*)}) \subset V_{\varrho_h}(\kappa^*) \subset V_{\varrho_0}(\kappa^*)$. This proves the lemma.

Since the function $f_h(\kappa)$ is continuous and $\overline{V_{\varrho_h}(\kappa^*)}$ is compact and convex, Brouwer's fixed point theorem applied to s_h yields that there exists at least one fixed point κ_h^* of s_h in $V_{\varrho_h}(\kappa^*)$, i.e. $f_h(\kappa_h^*) = 0$. For $\kappa = \kappa_h^*$ Eq. (56) now gives

$$|s_h(\kappa_h^*) - \kappa^*| = |\kappa_h^* - \kappa^*| \leq \tilde{c}\ h^{\frac{p-2}{p}} \quad \forall\, 2 \leq p < \frac{-2}{p_{min}}. \tag{57}$$

This proves

Theorem 5.2 *Let $\kappa^* \in T$ be a critical point of Θ such that rank $D^2\Theta(\kappa^*) = N$. Then there exists an open ball $V(\kappa^*) \subset T$ and a $h_0 > 0$ so that for all $0 < h \leq h_0$ the equation*

$$f_h(\kappa) \;=\; 0$$

has at least one solution $\kappa_h^* \in V(\kappa^*)$ which satisfies (57).

Remark 5.1 *Investigating the question of uniqueness of κ_h^* in Theorem 5.2 would require a deeper analysis of the functions f_h and Df_h (compare [19, Theorem 2]).*

Theorem 5.2 now implies the convergence of every sequence $\{\kappa_h^*\}_{0<h\leq h_0}$ to κ^*. The convergence of the discrete minimal surfaces $u_h(\kappa_h^*)$ to $u(\kappa^*)$ $(h \to 0)$ in the H^1-norm and the convergence of the corresponding Dirichlet integrals will be proved in the following theorem.

Theorem 5.3 *Let $\kappa^* \in T$ be a regular critical point of Θ and let $u(\kappa^*)$ denote the corresponding minimal surface. Furthermore, let $\{\kappa_h^*\}_{0<h\leq h_0}$ denote the sequence from Theorem 5.2 and let $u_h(\kappa_h^*)$ denote the corresponding discrete minimal surfaces. Then*

$$\|u(\kappa^*) - u_h(\kappa_h^*)\|_{1,B_h} \to 0 \quad (h \to 0) \tag{58}$$

and

$$|D(u(\kappa^*)) - D_h(u_h(\kappa_h^*))| \leq c\ h^{2\frac{p-2}{p}} \quad \forall\, 2 \leq p < \frac{-2}{p_{min}} \tag{59}$$

holds. The constant c is positive and depends on $V(\kappa^*)$ from Theorem 5.2.

Proof. We write

$$\|u(\kappa^*) - u_h(\kappa_h^*)\|_{1,B_h} \le \underbrace{\|u(\kappa^*) - u(\kappa_h^*)\|_{1,B}}_{[1]} + \underbrace{\|u(\kappa_h^*) - u_h(\kappa_h^*)\|_{1,B_h}}_{[2]}.$$

Since $\{\kappa_h^*\}_{0<h\le h_0} \subset V(\kappa^*) \subset\subset T$ the second term can be estimated using (50),

$$[2] \le ch^{\frac{p-2}{p}} \quad \forall\ 2 \le p < \frac{-2}{p_{min}}. \tag{60}$$

$\{u(\kappa_h^*)\}_{0<h\le h_0} \subset H^1(B, \mathbb{R}^q)$ is uniformly bounded by (30). Therefore, at least for subsequences, which will also be denoted by $\{u(\kappa_h^*)\}_{0<h\le h_0}$ one has

$$\begin{aligned} u(\kappa_h^*) &\rightharpoonup v \quad \text{in } H^1(B, \mathbb{R}^q), \\ u(\kappa_h^*) &\to v \quad \text{in } L^2(B, \mathbb{R}^q) \end{aligned}$$

and

$$u(\kappa_h^*) \to v \quad \text{in } L^2(\partial B, \mathbb{R}^q) \quad (\Rightarrow v \in X(\kappa^*)).$$

Since

$$D(u(\kappa^*)) = \Theta(\kappa^*) = \lim_{h\to 0} \Theta(\kappa_h^*) = \lim_{h\to 0} D(u(\kappa_h^*)) \ge \liminf_{h\to 0} D(u(\kappa_h^*)) \ge D(v)$$

and $u(\kappa^*)$ is uniquely determined by κ^*, one concludes that $v = u(\kappa^*)$. A well known theorem ([1, Übung 5.5]) now implies

$$u(\kappa_h^*) \to u(\kappa^*) \quad \text{in } H^1(B, \mathbb{R}^q),\ i.e.\ [1] \to 0. \tag{61}$$

Moreover, the whole sequence $\{u(\kappa_h^*)\}_{0<h\le h_0}$ converges to $u(\kappa^*)$, for otherwise one could select a subsequence converging to another limit. Combining (60) and (61) finally yields

$$\|u(\kappa^*) - u_h(\kappa_h^*)\|_{1,B_h} \to 0 \quad (h \to 0).$$

With respect to the convergence of the Dirichlet integrals $D(u_h(\kappa_h^*))$ one can show more. We write

$$\begin{aligned} |D(u(\kappa^*)) - D_h(u_h(\kappa_h^*))| &\le \overbrace{|D(u(\kappa^*)) - D(u(\kappa_h^*))|}^{[1]} + \\ &+ \underbrace{|D_h(u(\kappa_h^*)) - D_h(u_h(\kappa_h^*))|}_{[2]} + \underbrace{\int_{B\setminus B_h} |\nabla u(\kappa_h^*)|^2 dx}_{[3]}. \end{aligned}$$

Since Θ is analytic in T and $\nabla\Theta(\kappa^*) = 0$ Taylor's theorem implies

$$\begin{aligned} [1] &\le \frac{1}{2}|D^2\Theta(\kappa^*)||\kappa_h^* - \kappa^*|^2 + o(|\kappa_h^* - \kappa^*|^2)\ (h \to 0) \\ &\le c\ h^{2\frac{p-2}{p}} \quad \forall\ 2 \le p < \frac{-2}{p_{min}} \end{aligned}$$

with some positive constant $c = c(\overline{V(\kappa^*)})$.

The second term can be estimated using (51),

$$[2] \le c\ h^{2\frac{p-2}{p}} \quad \forall\ 2 \le p < \frac{-2}{p_{min}}.$$

Applying Hölder's inequality, (30) finally yields

$$[3] \le c\ \underbrace{meas(B \setminus B_h)}_{\le c\ h^2}^{\frac{p-2}{p}}\ \ \underbrace{|u(\kappa_h^*)|_{1,p,B}}_{\le c}$$

with some positive constant $c = c(\overline{V(\kappa^*)})$. Hence,

$$|D(u(\kappa^*)) - D_h(u_h(\kappa_h^*))| \le c\ h^{2\frac{p-2}{p}} \quad \forall\ 2 \le p < \frac{-2}{p_{min}}.$$

This completes the proof.

The following corollary is a refinement of (58).

Corollary 5.1 *Under the assumptions of Theorem 5.3*

$$\|u(\kappa^*) - u_h(\kappa_h^*)\|_{1,B_h} \le ch^{(\frac{p}{p-2})^2} \quad \forall\ 2 \le p < \frac{-2}{p_{min}} \tag{62}$$

holds. The constant $c = c(V(\kappa^*))$ is positive.

Proof. Since $\beta_k(\kappa_{N+i}^*) = \beta_k(\kappa_{h_{N+i}}^*)$ $(i = 1, 2, 3)$ Lemma 3.2 gives

$$\|u(\kappa^*) - u_h(\kappa_h^*)\|_{1,B_h} \le c\,\underbrace{|u(\kappa^*) - u(\kappa_h^*)|_{1,B}}_{[1]} + \underbrace{\|u(\kappa_h^*) - u_h(\kappa_h^*)\|_{1,B_h}}_{[2]}.$$

For [2] estimate (60) holds. In order to estimate [1] let us write

$$\frac{1}{2}|[1]|^2 \le \underbrace{|D(u(\kappa_h^*)) - D(u(\kappa^*))|}_{[3]} + \underbrace{|\int_B \nabla u(\kappa^*)\nabla(u(\kappa^*) - u(\kappa_h^*))dx\,|}_{[4]}.$$

As in the proof of the previous theorem one concludes

$$|[3]| \le ch^{2\frac{p}{p-2}} \quad \forall\ 2 \le p < \frac{-2}{p_{min}}.$$

Set

$$I_1^k := \gamma_k(\kappa^*) \cap \gamma_k(\kappa_h^*),\ \ I_2^k := (\overline{\gamma_k(\kappa^*)} \setminus \gamma_k(\kappa_h^*)) \cup (\overline{\gamma_k(\kappa_h^*)} \setminus \gamma_k(\kappa^*))$$

and write

$$\partial B = \cup_{k=1}^{N+3} \left\{ I_1^k \cup I_2^k \right\}.$$

Integration by parts applied to [4] together with (18) yields

$$[4] = \int_{\partial B} u_r(\kappa^*)(u(\kappa^*) - u(\kappa_h^*))ds \quad = $$
$$\sum_{k=1}^{N+3} \Big(\underbrace{\int_{I_1^k} u_r(\kappa^*)(u(\kappa^*) - u(\kappa_h^*))ds}_{[5]_k} \quad + \quad \underbrace{\int_{I_2^k} u_r(\kappa^*)(u(\kappa^*) - u(\kappa_h^*))ds}_{[6]_k} \Big).$$

Since

$$u(\kappa^*) - u(\kappa_h^*) \in a_k + \Gamma_k, \; w \in \gamma_k(\kappa^*) \cap \gamma_k(\kappa_h^*),$$

Eq. (20) gives $[5]_k = 0$ ($k = 1, \ldots, N+3$). To estimate $[6]_k$ note that by (57)

$$meas(I_2^k) \leq ch^{\frac{p-2}{p}} \quad \forall\; 2 \leq p < \frac{-2}{p_{min}}$$

holds. Using expansion (23) one arrives at

$$|[4]| \leq c(N)h^{(\frac{p-2}{p})^2} \underbrace{\|u(\kappa^*) - u(\kappa_h^*)\|_{\infty,B}}_{[7]}.$$

Combination of (23), Taylor's theorem and (57) finally gives

$$[7] \leq ch^{(\frac{p-2}{p})^2} \quad \forall\; 2 \leq p < \frac{-2}{p_{min}},$$

and thus the desired result.

6 Examples of computed discrete minimal surfaces

In the present section numerical investigations on generalized polygonal minimal surfaces are presented. The starting point is the vector function $\nabla\Theta$. Due to Theorem 3.2 polygonal minimal surfaces $u(\kappa^*)$ (with Shiffman boundary conditions) are characterized by the equation $\nabla\Theta(\kappa^*) = 0$. For this reason the computation of a polygonal minimal surface $u(\kappa^*)$ is equivalent to the computation of the corresponding critical value $\kappa^* \in T$ of Θ. Moreover, all minimal surfaces spanning a polygon $\Gamma \subset \mathbb{R}^q$ can be obtained in this way.

According to Theorem 5.1 the function f_h defined in (52) is well suited to substituting $\nabla\Theta$.

All computations in the present section were carried out by applying a damped Newton method to the equation

$$f_h(\kappa) = 0, \quad (\kappa \in T). \tag{63}$$

For an extensive discussion of the numerical implementation of (63) see [12, Chapter 5].

Table 1: **EOC for the Riemann - Schwarz Surface**

RL	NV	NT	NR	$A_h(u_h) - A(u)$	EOC	$D_h(u_h) - D(u)$	EOC
1	13	16	8	2.3280e-2	-	2.0682e-1	-
2	41	64	16	7.9732e-3	1.79	1.5917e-1	0.44
3	145	256	32	2.9080e-3	1.58	1.0698e-1	0.62
4	545	1024	64	1.0978e-3	1.46	6.8946e-2	0.66
5	2113	4096	128	4.2100e-4	1.41	4.3741e-2	0.67
6	8321	16 384	256	1.6290e-4	1.38	2.7585e-2	0.67
7	33 025	65 536	512	6.3600e-5	1.36	1.7360e-2	0.67

Example 6.1 *The Riemann - Schwarz Surface, experimental order of convergence (EOC)*

The uniquely determined Plateau solution u spanning the polygon $\Gamma \subset \mathbb{R}^3$ defined by the vertices

$$a_1(\frac{1}{2}, 0, \frac{1}{2\sqrt{2}}), a_2(0, -\frac{1}{2}, -\frac{1}{2\sqrt{2}}), a_3(-\frac{1}{2}, 0, \frac{1}{2\sqrt{2}}), a_4(0, \frac{1}{2}, -\frac{1}{2\sqrt{2}}).$$

is called *Riemann - Schwarz Surface.* This surface was already been investigated numerically by Jarausch in [14, Abb. 12, Abb. 27] and is therefore well suited for a comparismn of numerical results.

The area of the surface is $A(u) = 0.63963 \pm 3 \cdot 10^{-5}$ and coincides with the value of Dirichlet's integral $D(u)$. At the corners of the polygon u maps into the angle $\sphericalangle\,(\varphi_{\mathrm{k}}, -\varphi_{\mathrm{k}-1})$, where φ_k $(k = 1, \ldots, N+3)$ was defined in (7). For the Dirichlet integrals of discrete minimal surfaces $u_h(\kappa_h^*)$ corresponding to parameter tuples $\kappa_h^* \in T$ satisfying $f_h(\kappa_h^*) = 0$ asymptotic convergence of order $O(h^{2\mu-\epsilon})$ $(\epsilon > 0)$ was proved, where

$$\mu = \frac{1}{\pi}\sphericalangle\,(\varphi_{\mathrm{k}}, -\varphi_{\mathrm{k}-1}) = \frac{1}{3}.$$

For the error

$$E(h) = |D(u) - D_h(u_h(\kappa_h^*))|$$

the EOC between two gridsizes $h_1 \neq h_2$ is given by

$$EOC := \frac{\ln \frac{E(h_1)}{E(h_2)}}{\ln \frac{h_1}{h_2}} \approx 2\mu.$$

As Table 1 shows, the numerical results confirm the asymptotic convergence of Dirichlet's integral. The areas $A_h(u_h)$ of the discrete surfaces seem to converge faster than the values of the corresponding Dirichlet integrals.

Jarausch introduced polar coordinates on B and approximated this surface using piecewise bilinear continuous finite elements. Jarausch's best approximation of the *Riemann - Schwarz Surface* has the area $A = 0.640007$, the corresponding value of Dirichlet's integral is $D = 0.675515$. This values nearly

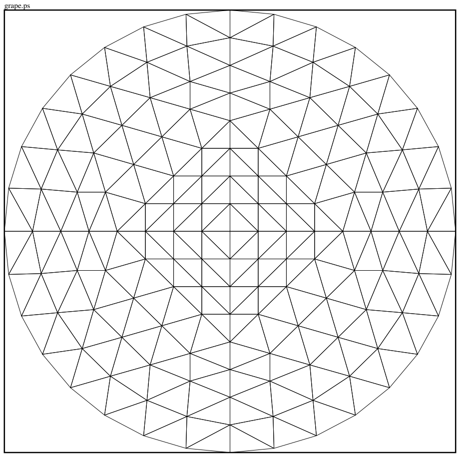


Figure 5: Triangulation

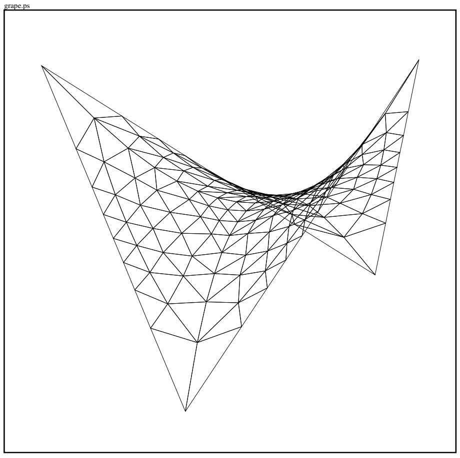


Figure 6: *Riemann - Schwarz Surface*

coincides with our values for $RL = 5$.

In Figs. 6, 5 an approximation $u_h(\kappa_h^*)$ of the *Riemann - Schwarz Surface* together with corresponding triangulation are shown.

Example 6.2 *Minimal surface spanning a polygon of Lewerenz*

Consider the family of contours $\{P_\epsilon^2\}_{\epsilon>0}$ defined by the vertices

$$a_1(0,0,\epsilon), a_2(1,1,-1), a_3(0,2,0), a_4(-1,1,1), a_5(0,0,-\epsilon) \ (\epsilon > 0),$$

that was constructed by Lewerenz to prove that Courant's function d and Shiffman's function Θ do not coincide for certain values of κ (see [15, Satz]). Note that quasiminimal surfaces $u(\kappa)$ are not required to map the closed arc $\overline{\gamma}_k$ onto the segment $[a_k, a_{k+1}]$ on the line $a_k + \Gamma_k$ $(1 \le k \le N+3)$ in a monotone way, but instead may overshoot the vertices a_k, a_{k+1}.

In Figs. 7, 8 the polygon P_{2-6} is presented that bounds a discrete quasiminimal surface $u(\kappa_h)$ that overshoots the vertices of the polygon and is not order preserving on the boundary nodes of the corresponding triangulation of the unit disc. The parameter tuple κ_h was used as the initial guess for the Newton process. In Figs. 9, 10 the discrete minimal surface $u_h(\kappa_h^*)$ spanning P_{2-6} together with the triangulation $\tau_h(\kappa_h^*)$ are presented. The area of the discrete surface is $A = 2.8136$, Dirichlet's integral has the value $D = 2.9426$. The triangulation contains $NT = 1280$ triangles, $NV = 681$ nodes and $NR = 80$ boundary nodes.

As Fig. 11 shows, the discrete minimal surface does not overshoot the vertices of the polygon and is order preserving on $\{boundary\ nodes\}$.

No overshooting of the vertices a_j $(j = 1, \ldots, N+3)$ of a polygon Γ has been observed in any numerical computations of generalized discrete minimal surfaces. Moreover, all computed discrete surfaces u_h satisfy Plateau's boundary condition for Γ in the sense that

$$u_h\{\text{boundary nodes}\} \subset \Gamma$$

and

$$u_h \text{ is order preserving on } \{\text{boundary nodes}\}.$$

This fact justifies the use of our numerical approximation of the Marx-Shiffman function Θ for the computation of discrete minimal surfaces that are bounded by a polygon Γ and might lead to the conjecture that d and Θ coincide at their critical point set.

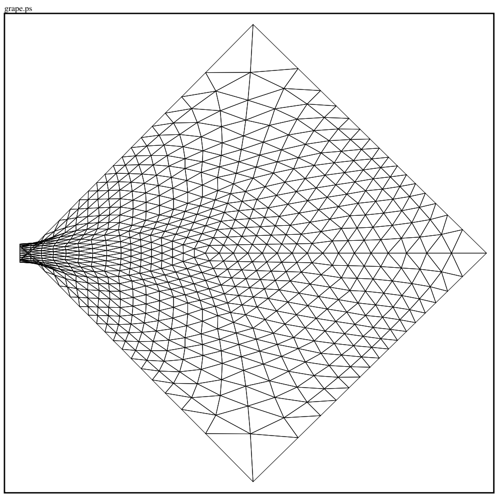


Figure 7: Initial Surface

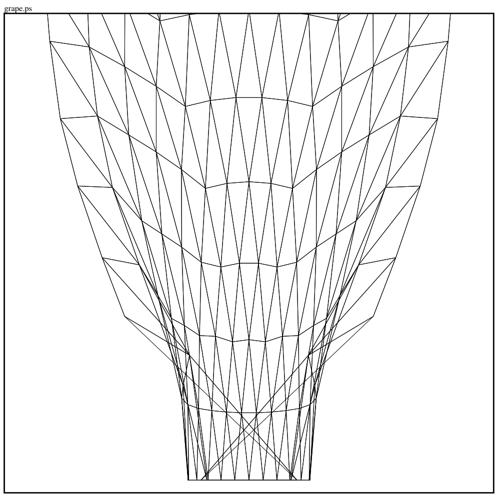


Figure 8: Zoom on $a_1 + \Gamma_1$

Example 6.3 *Polygonal approximations of minimal surfaces bounded by Enneper contours*

The one-parameter family of Enneper contours is given by

$$E_r(\varphi) \;=\; \begin{cases} r cos\varphi \; - \frac{1}{3} r^3 cos 3\varphi & \\ -r sin\varphi - \frac{1}{3} r^3 sin 3\varphi & \varphi \in [0, 2\pi] \\ r^2 cos 2\varphi. & \end{cases} \tag{64}$$

Several authors investigated these contour-family and the corresponding minimal surfaces problem. The best general reference here is the famous book of Nitsche [17]. For a comprehensive numerical treatment see [12, Section 5.4.2].

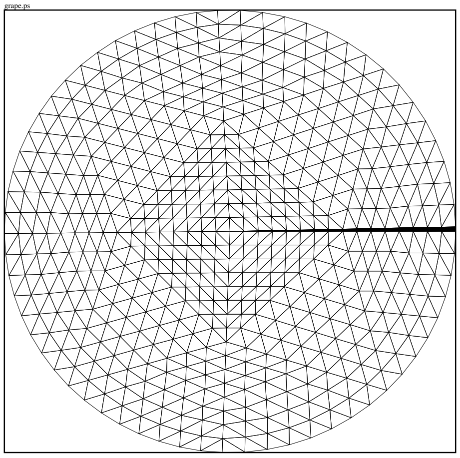


Figure 9: Final Triangulation

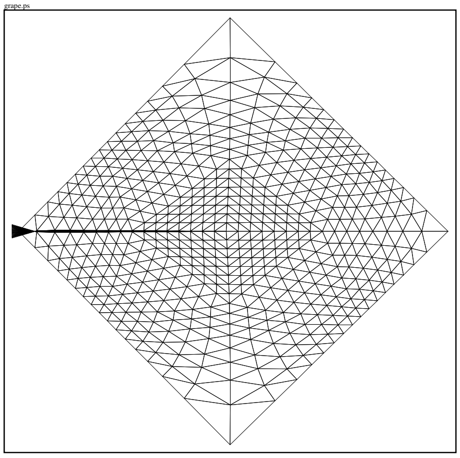


Figure 10: Discrete minimal surface

Polygonal approximations of Enneper contours can be obtained as follows. For $n \in \mathbb{N}$, $n \geq 4$ subdivide the interval $[0, 2\pi]$ into n parts

$$[0, \varphi_1], [\varphi_1, \varphi_2], \ldots, [\varphi_{n-1}, 2\pi],$$

where

$$\varphi_i \;:=\; 2\pi \frac{i}{n} \qquad 0 \leq i \leq n-1.$$

The polygon Γ_n^r then is defined by the $n =: N+3$ vertices $a_i := E_r(\varphi_i)$ $(0 \leq i \leq n-1)$. In Figs. 12, ..., 13 numerical results in the case $n = 18$, $r = 1.2$ are presented. Since in numerical practice the three-point-condition (11) is very restrictive in the sense that triangulations of the unit disc obtained by the refinement strategies **S1** or **S2**, respectively are not uniform whenever the number $N+3$ of vertices of the polygon is large, we used more appropriate three-point-conditions for the following computations. For the approximation triangulations containing 3816 triangles and 2197 nodes, where 576 nodes are boundary nodes were used. The energies and areas, respectively of the surfaces are (in decending order) 13.0797, 13.0797, 13.1128, 12.9218, 13.1128, 12.9218, respectively. The matrix $D^2\Theta_h(\kappa_h^1, 1.2)$ has one negative eigenvalue, the matrices $D^2\Theta_h(\kappa_h^i), 1.2)$ $(i = 2, 3)$ are positive definite. Therefore, these discrete minimal surfaces might be approximations of an unstable critical point and two local minima of the area functional.

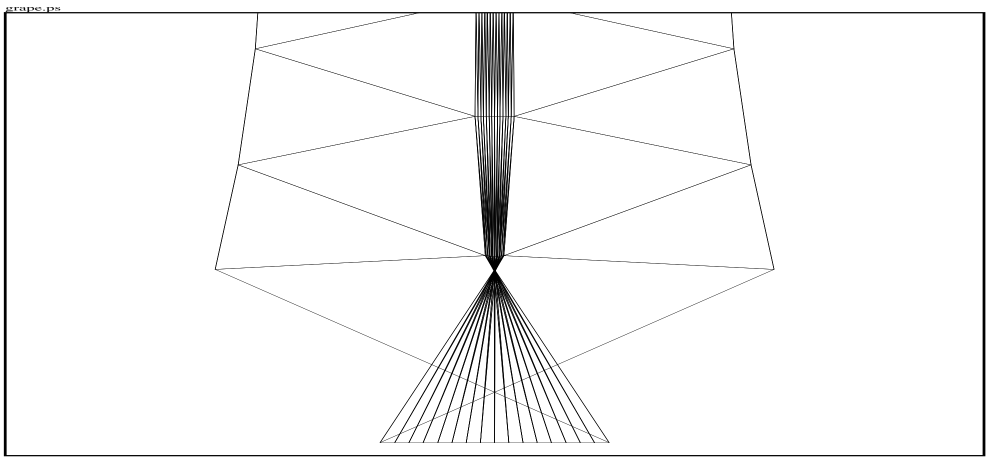


Figure 11: Discrete minimal surface, Zoom on $a_1 + \Gamma_1$

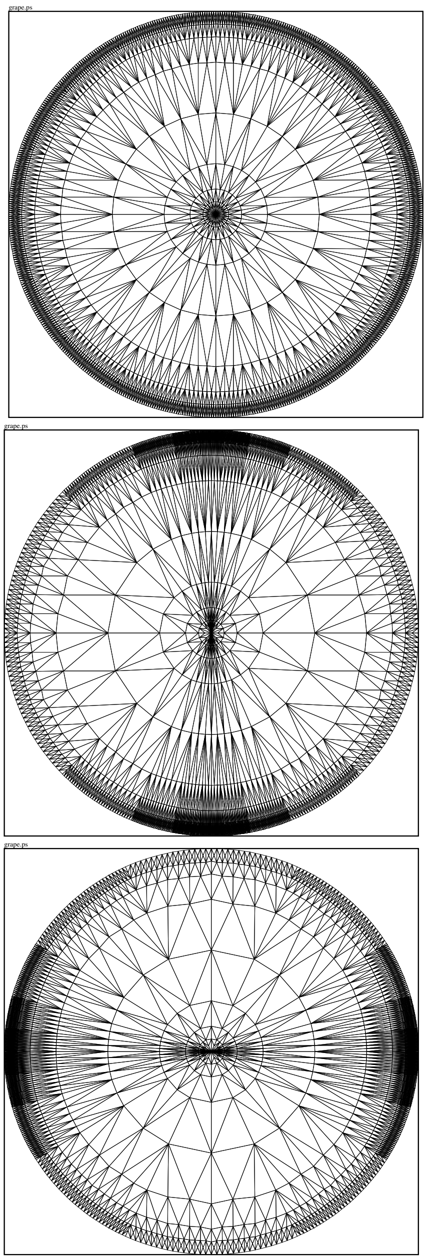


Figure 12: Triangulations, 3816 triangles

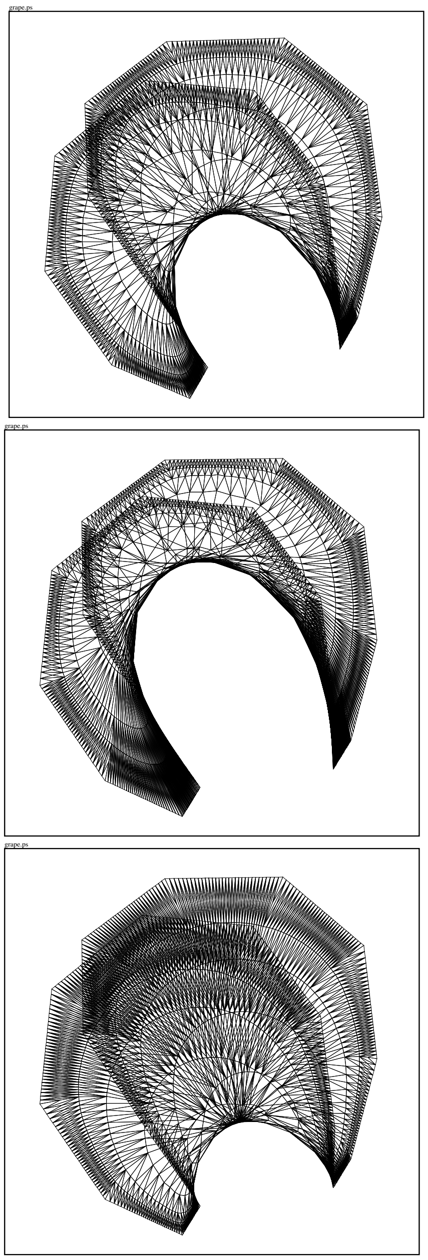


Figure 13: Corresponding minimal surfaces

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