Bonnet Surfaces and Painlevé Equations*

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Introduction

Let $\mathcal{F}$ be a surface in Euclidean 3-space without umbilic points. This paper studies the following

Problem : To classify non-trivial one-parameter families $\mathcal{F}_\tau$, $\tau \in (-\epsilon, \epsilon)$ of isometries of $\mathcal{F} = \mathcal{F}_0$ preserving both principal curvatures.

Since the Gaussian curvature is preserved by isometries one can reformulate the problem replacing "both principle curvatures" by "the mean curvature function". Let us specify what do we mean by a non-trivial family. We consider families of surfaces which do not differ by rigid motions. We suppose also that the surfaces and isometries are sufficient smooth. The case of surfaces with constant mean curvature (CMC-surfaces), which all possess non-trivial isometries, is also excluded from our consideration. We suppose that the mean curvature is a non-trivial function on $\mathcal{F}$.

It turns out that the condition of possessing a one-parameter family $\mathcal{F}_\tau$ of isometries, preserving H, implies restrictive conditions on $\mathcal{F}$. Moreover, all the family $\mathcal{F}_\tau$ can be described (see section 2) as a reparametrization of $\mathcal{F}$ itself. The problem is reduced to the problem of classification of surfaces $\mathcal{F}$. Since the problem formulated at the beginning of this introduction was first studied by Bonnet, we call these surfaces *Bonnet surfaces.*

The problem is classical and many mathematicians contributed to its solution. O. Bonnet himself showed in [Bo] that besides the CMC surfaces there is a class of surfaces, depending on finitely many parameters, which allows non-trivial isometries preserving H. These results were developed further by L. Raffy, who proved that the Bonnet surfaces are isothermic (i.e. allow conformal curvature line parametrization) and isometric to surfaces of revolution. J.N. Hazzidakis [H] showed that the mean curvature function H satifies an ordinary differential equation of the third order and was able to integrate it once. Graustein [G] proved that all Bonnet surfaces are Weingarten surfaces, i.e. the

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mean and the Gaussian curvature are related $d\,H \wedge d\,K = 0$. He also found a convenient alternative description for the Bonnet surfaces. Namely, he showed that these surfaces can be characterized as isothermic surfaces, where the function $1/Q$ with

$$Q = \frac{1}{4} < F_{xx} - F_{yy}, N >$$

is harmonic, that means

$$(\partial_{xx} + \partial_{yy}) 1/Q = 0.$$

In modern notations Q is the Hopf differential, written in isothermic coordinates x, y.

Later the problem was treated by E. Cartan in [C], where the most detailed results concerning the Bonnet surfaces are presented. Cartan gave a modern definition of these surfaces and classified them into 3 classes A, B and C. The mean curvature function $H(t)$ satisfies the Hazzidakis equation

$$\left(\frac{H''}{H'}\right)' - H' = |Q|^2 \left(2 - \frac{H^2}{H'}\right), \tag{1}$$

where $|Q|^2$ is a fixed function different for the 3 cases A, B and C (see section 2):

$$\begin{aligned} |Q_A|^2 &= \frac{4}{\sin^2(2\,t)}, \\ |Q_B|^2 &= \frac{4}{\sinh^2(2\,t)}, \\ |Q_C|^2 &= \frac{1}{t^2}. \end{aligned} \tag{2}$$

Equation (1) is the Gauss equation of the Bonnet surfaces. After the result of Hazzidakis, who reduced this equation to equations of the second order for all 3 cases A, B and C there was no progress in investigation of (1). Cartan finished his paper by the phrase: "An investigation of the singularities of the differential equation (1) seems to be difficult." We mention also a more recent paper by S. Chern [Ch], where it was shown in particular that the argument of the Hopf differential written in any conformal coordinates is harmonic.

It turns out that Cartan was right in his estimation of equation (1),(2). In this paper we show that the Hazzidakis equation (1) with $|Q|^2$ given by (2) is isomorphic to the Painlevé equations: namely to the Painlevé VI equation in the cases A and B and to the special case of the Painlevé V equation, which can be reduced to the Painlevé III equation, in the C case. The isomorphism is given by explicit formulas (69), (67) in A and B cases and by (38), (42) in the C case.

Although the formulas establishing the isomorphism can be checked directly, they would hardly be found without using the theory of integrable systems. The starting point of the present paper is an observation made in [B94] that the frame equation for the Bonnet surfaces written via 2×2 matrices have the same structure as the Lax representation of the Painlevé equations given in [J-M], [I-N]. Here we develop this observation and describe the corresponding isomorphism explicitly.

Modern achievements in the global asymptotic analysis of the Painlevé equations make it possible to evaluate in closed from (in terms of elementary functions and their quadratures) asymptotic connection formulae for the corresponding solution manifolds. This is a characteristic analytical property of the special functions. In other words, the current status of the Painlevé transcendents should be considered to be the same as that of the hypergeometric functions and their degenerations. If a problem can be solved in terms of the Painlevé transcendents, the solution should be treated as an explicit one. In more details, this point of view is presented in the review papers [I92], [I94]. Therefore we solve the Bonnet problem mentioned at the beginning of this introduction *explicitly*.

In the appendices two special cases are discussed, when the Bonnet surfaces are described in rational, hyperelliptic and elliptic functions.

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1 Quaternionic description of surfaces in Euclidean 3-space

To study surfaces in $\mathbb{R}^3$ by analytical methods it is convenient to describe them in terms of 2×2 matrices (for more details see [B94]). In section 2 and 3 this description allows us to identify the equations for the moving frame of the Bonnet surfaces with the Lax representation of the Painlevé equations.

Let $F : \mathcal{R} \to \mathbb{R}^3$ be a conformal parametrization of an orientable surface:

$$< F_z, F_z >=< F_{\bar{z}}, F_{\bar{z}} >= 0 \qquad < F_z, F_{\bar{z}} >= \frac{1}{2} e^u .$$

Here $\mathcal{R}$ is a Riemann surface with the induced complex structure,

$$< v, w >= v_1 w_1 + v_2 w_2 + v_3 w_3 \, ,$$

z is a complex coordinate and F_z and $F_{\bar{z}}$ are the partial derivatives

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) .$$

The vectors F_z, $F_{\bar{z}}$ and N define a moving frame on the surface. The fundamental forms are

$$\begin{array}{rcl} < d\,F, d\,F > & = & e^u \, dz \, d\bar{z}, \\ - < d\,F, d\,N > & = & Q \, dz^2 + H \, e^u \, dz \, d\bar{z} + \bar{Q} \, d\bar{z}^2, \end{array} \tag{3}$$

where $Q =< F_{zz}, N >$ is the Hopf differential and H the mean curvature function on F.

The compatibility conditions (the Gauss-Codazzi equations) of the moving frame equations are

$$\begin{array}{rcl} u_{z\bar{z}} + \dfrac{H^2}{2} e^u - 2Q\bar{Q}e^{-u} & = & 0 \, , \\ Q_{\bar{z}} & = & \dfrac{H_z}{2} e^u \, , \\ \bar{Q}_z & = & \dfrac{H_{\bar{z}}}{2} e^u \, . \end{array} \tag{4}$$

Let us denote the algebra of quaterions by $\mathbb{H}$ and the standard basis by $\{\mathbb{1}, \mathbb{i}, \mathbb{j}, \mathbb{k}\}$

$$\mathbb{i}\,\mathbb{j} = -\mathbb{j}\,\mathbb{i} \, , \qquad \mathbb{j}\,\mathbb{k} = -\mathbb{k}\,\mathbb{j} \, , \qquad \mathbb{k}\,\mathbb{i} = -\mathbb{i}\,\mathbb{k} \, . \tag{5}$$

We will use the standard matrix representation of $\mathbb{H}$:

$$\mathbb{i} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad \mathbb{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbb{k} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

We identify 3-dimensional Euclidean space with the space of imaginary quaternions

$$\mathbf{Im}\,\mathbb{H} = \mathrm{su}(2) = \mathrm{span}(\mathbb{i}, \mathbb{j}, \mathbb{k})$$

by

$$X = (x_1, x_2, x_3)^t = x_1 \mathbb{i} + x_2 \mathbb{j} + x_3 \mathbb{k} \in su(2) \tag{7}$$

The scalar product of vectors in terms of matrices is then

$$< X, Y >= -\frac{1}{2}\mathrm{tr}(XY).$$

Let us take $\Phi \in \mathrm{SU}(2)$ which transforms the basis $\mathbb{i}$, $\mathbb{j}$, $\mathbb{k}$ into the frame F_x, F_y, N:

$$F_x = e^{u/2}\mathbf{\Phi}^{-1}\mathbb{i}\mathbf{\Phi}, \quad F_y = e^{u/2}\mathbf{\Phi}^{-1}\mathbb{j}\mathbf{\Phi}, \quad N = \mathbf{\Phi}^{-1}\mathbb{k}\mathbf{\Phi} \tag{8}$$

One can prove [B94] that $\mathbf{\Phi}$ satisfies the following linear system:

$$\mathbf{\Phi}_z\mathbf{\Phi}^{-1} = \begin{pmatrix} \frac{u_z}{4} & -Qe^{-u/2} \\ \frac{H}{2}e^{u/2} & -\frac{u_z}{4} \end{pmatrix} \qquad \mathbf{\Phi}_{\bar{z}}\mathbf{\Phi}^{-1} = \begin{pmatrix} -\frac{u_{\bar{z}}}{4} & -\frac{H}{2}e^{u/2} \\ \bar{Q}e^{-u/2} & \frac{u_{\bar{z}}}{4} \end{pmatrix}. \tag{9}$$

2 Differential equations of Bonnet surfaces

Definition 1 *If a surface $\mathcal{F}$ possess a 1-parameter family of isometries $\mathcal{F}_\tau$*

$$\mathcal{F}_0 = \mathcal{F},\ \tau \in (-\epsilon, \epsilon),\ \epsilon > 0,$$

preserving the mean curvature function, $\mathcal{F}$ is called a Bonnet surface.

Let $F_\tau : \mathcal{R} \to \mathbb{R}^3$ be a conformal parametrization of $\mathcal{F}_\tau$ and

$$z : \mathrm{V} \subset \mathcal{R} \to \mathrm{U} \subset \mathbb{C}$$

be a local parameter. Locally in terms of $z \in \mathrm{U}$ we get a map

$$\begin{array}{rccc} F: & (-\epsilon, \epsilon) \times U \subseteq \mathbb{C} & \longrightarrow & \mathbb{R}^3 \\ & (\tau, z, \bar{z}) & \mapsto & F(\tau, z, \bar{z}) \end{array} \qquad \epsilon > 0.$$

We suppose also that F is umbilic-free and smooth.

Remark 1 *: Since isometries preserve the Gauss curvature, both principal curvatures are preserved.*

Constant mean curvature surfaces as well as surfaces of revolution are examples of Bonnet surfaces. We restrict our discussion to non-constant mean curvature surfaces without outer isometrie[1]).

Let us denote the Hopf differential of $F(\tau, z, \bar{z})$ by $Q_\tau(z, \bar{z})$. The Gauss equation (4) implies that $|Q_\tau(z, \bar{z})|$ is invariant under the deformation.

$$Q_\tau(z, \bar{z}) = e^{i\,\varphi(\tau,z,\bar{z})}\, q_0(z, \bar{z}),$$

where $q_0(z, \bar{z})$ is real-valued. The Codazzi equations in (4) yields that the derivative

$$\frac{\partial}{\partial \tau} Q_\tau(z, \bar{z})$$

is holomorphic. Combining these two observations one gets

$$Q_\tau(z, \bar{z}) = g(\tau, z)\, q_0(z, \bar{z})$$

is a product of the real-valued function q_0 and a holomorphic function $g(\tau, z)$, which proves that the surface is isothermic (see for example [B94]). Indeed, since the quadratic differential $Q_\tau(z, \bar{z})\, dz^2$ is invariant under conformal reparametrisation, this means that there exists conformal curvature line coordinates for any Bonnet surface without umbilics.
The next theorem shows that $1/Q_\tau$ is harmonic.

Theorem 1 (Graustein [G]) *Any umbilic-free Bonnet surface is isothermic. With respect to isothermic coordinates, $1/Q$ is harmonic:*

$$Q(z, \bar{z}) = \frac{1}{h(z) + \bar{h}(\bar{z})},$$

where $h(z)$ is holomorphic.

Proof : Let us consider a conformal curvature line parametrization of $\mathcal{F} = \mathcal{F}_0$. Then $Q_0(z, \bar{z})$ is real. On the other hand our previous observations imply

$$Q_\tau \bar{Q}_\tau = Q_0^2, \qquad Q_\tau = Q_0 + f(\tau, z), \tag{10}$$

with $f(\tau, z)$ holomorphic. Solving (10) one gets

$$Q_0 = -\frac{f\,\bar{f}}{f + \bar{f}}, \qquad Q_\tau = -\frac{f}{\bar{f}} Q_0.$$

Finally, identifying

$$h(z) = -\frac{1}{f(\tau_0, z)} + iT$$

[1]that would form an isometric family with the same curvature properties

we prove the theorem and get the formula

$$Q_T(z,\bar z) = \frac{1-iT\bar h}{1+iTh}\frac{1}{h+\bar h} \quad \forall T \in I\!R \tag{11}$$

for the deformation family in terms of a new deformation parameter $T \in I\!R$. In our argument we do not prefer any special Bonnet surface of a Bonnet family. So all this holds for any Bonnet surface of a Bonnet family but with different functions h and different curvature line coordinates z and $\bar z$. For any such surface the compatibility conditions (4) are as follows:

$$Q(z,\bar z) = \frac{1}{h(z)+\bar h(\bar z)}, \tag{12}$$

$$\left(\frac{H_{z\bar z}}{H_z}\right)_{\bar z} - \frac{H_{\bar z}}{h_z} = 2\frac{|h_z|^2}{(h+\bar h)^2} - \frac{H^2 h_z}{H_{\bar z}(h+\bar h)^2}, \tag{13}$$

$$h_z H_z = h_{\bar z} H_{\bar z}, \tag{14}$$

$$e^{u(z,\bar z)} = -\frac{2h_z}{(h+\bar h)^2 H_{\bar z}}. \tag{15}$$

Theorem 2 *Let $\mathcal F = \mathcal F_0$ be a Bonnet surface with isothermic coordinates $z, \bar z$. Then*

$$w = w(z) = \int \frac{1}{h_z(z)} dz,$$

is also a conformal coordinate, and the mean curvature function H, the metric u and the modulus of the Hopf differential $|Q|$ are functions of

$$t = w + \bar w \tag{16}$$

only.

Proof : By the chain rule we get $H_w = h_z H_z$ and $H_{\bar w} = \bar h_{\bar z} H_{\bar z}$ which implies by (14) the first property. Since $\tilde Q(z,\bar z)dz^2 = Q(w,\bar w)dw^2$ where $\tilde Q$ is the Hopf curvature function with respect to the coordinates $z, \bar z$ we get

$$Q(w,\bar w) = \frac{h_z^2(w^{-1}(w))}{h(w^{-1}(w)) + \bar h(\bar w^{-1}(\bar w))}$$

A simple calculation shows

$$|Q|^2 = -Q_{\bar w} = -\bar Q_w,$$

which implies for the metric function

$$e^u = \frac{2\,Q_{\bar w}}{H_w} = -\frac{2\,|Q|^2}{H'}, \tag{17}$$

where H' is the derivation of H with respect to t defined in (16). If we reformulate the Gauss equation in terms of the mean curvature function H (see (13)) we get the following equation

$$(\frac{H''}{H'})' - H' = |Q|^2(2 - \frac{H^2}{H'}). \tag{18}$$

Here $|Q|$ should be independent of $w - \bar{w}$ because otherwise $(H''/H')' - H'$ and $2 - H^2/H'$ must vanish identically. But this implies $H' > 0$ in contradiction to (17). Finally the metric depends on t only, too.

Remark 2 *An interesting interpretation of the corollary above is that all Bonnet surface are Weingarten surfaces [G]. This should be clear since the dependence of H and u only on one real variable implicies the same property for the Gaussian curvature K and so the (K, H)-diagram must be 1-dimensional,*

$$d\,H \wedge d\,K = 0.$$

Theorem 3 *The holomorphic function $h = h(z)$ satisfies the differential equation*

$$h_{zz}(h + \bar{h}) - h_z^2 = \bar{h}_{\bar{z}\bar{z}}(h + \bar{h}) - \bar{h}_{\bar{z}}^2. \tag{19}$$

Proof : This is a direct result of the property that $|Q|^2$ depends on t only.

Theorem 4 *Up to normalization by linear transformations any solution of (19) is of one of the following five forms:*

$$\begin{aligned} h_1(z) &= z \\ h_2(z) &= -z^2 \\ h_3(z) &= e^z \\ h_4(z) &= 2\cosh(z) \\ h_5(z) &= 2\sinh(z). \end{aligned} \tag{20}$$

Proof : First we reformulate equation (19):

$$h_z^2 - \bar{h}_{\bar{z}}^2 = (h + \bar{h})(h_{zz} - \bar{h}_{\bar{z}\bar{z}}).$$

Since the left hand side is harmonic the same must hold for the right hand side which leads to the condition

$$\frac{\bar{h}_{\bar{z}\bar{z}\bar{z}}}{\bar{h}_{\bar{z}}} = \frac{h_{zzz}}{h_z} = \rho \in I\!R \text{ fixed}. \tag{21}$$

Here $h(z)$ cannot be a constant because this would mean that H is a constant too. So there are two different cases to consider $\rho = 0$ and $\rho \neq 0$. If $\rho = 0$ then all solutions are polynomial. After resetting this in the equation (19) we get some conditions for the coefficient. Reparametrization gives h_1 and h_2. The same procedure for the case $\rho \neq 0$ gives the other 3 solutions. This completes the proof.

By a simple calculation we find for these five cases and $T=0$:

$$\begin{array}{rclrcl} Q_0^1(w,\bar w) &=& \dfrac{1}{w+\bar w} & |Q_0^1(w,\bar w)|^2 &=& \dfrac{1}{t^2}\\ Q_0^2(w,\bar w) &=& 2(\dfrac{\cos(2(w+\bar w))}{\sin(2(w+\bar w))}+i) & |Q_0^2(w,\bar w)|^2 &=& \dfrac{4}{\sin^2(2t)}\\ Q_0^3(w,\bar w) &=& -\dfrac{\bar w}{w}\dfrac{1}{w+\bar w} & |Q_0^3(w,\bar w)|^2 &=& \dfrac{1}{t^2}\\ Q_0^4(w,\bar w) &=& -2\dfrac{\sinh(2\bar w)}{\sinh(2w)}\dfrac{1}{\sinh(2(w+\bar w))} & |Q_0^4(w,\bar w)|^2 &=& \dfrac{4}{\sinh^2(2t)}\\ Q_0^5(w,\bar w) &=& -2\dfrac{\sin(2\bar w)}{\sin(2w)}\dfrac{1}{\sin(2(w+\bar w))} & |Q_0^5(w.\bar w)|^2 &=& \dfrac{4}{\sin^2(2t)}. \end{array} \tag{22}$$

The following theorem classifies umbilic free Bonnet surfaces.

Theorem 5 (Cartan [C]) *There are three different types of Bonnet families classified by whether they contain one, two or four surfaces:*
Type A : $|Q_T^A(w,\bar w)|^2 = \dfrac{4}{\sin^2(2t)}$ *containing four surfaces in each family namely* $F_{T=\pm\frac12}$ *with* Q_0^2 *and* $\bar Q_0^2$*,* $F_{T=0}$ *with* Q_0^5 *and* $F_{T=\infty}$ *with* $Q_0^5(w+\pi/4,\bar w+\pi/4)$ *from (22). The surfaces* $F_{T=\pm\frac12}$ *are helicoids.*
Type B : $|Q_T^B(w,\bar w)|^2 = \dfrac{4}{\sinh^2(2t)}$ *containing only one quasi-periodic surface in each family.*
Type C : $|Q_T^C(w,\bar w)|^2 = \dfrac{1}{t^2}$ *containing two surfaces in each family,namely* $F_{T=0}$ *with* Q_0^1 *and* $F_{T=\infty}$ *with* Q_0^3 *from (22). The surface* $F_{T=0}$ *is a cone or a surface of revolution.*
These surfaces represent all the corresponding one-parameter families of isometries, which are described by the translations

$$w\to w+iT, \qquad \bar w\to\bar w-iT$$

on the surfaces.

Proof : We start with the type **C**. Here we have with (11) and (20)

$$\begin{array}{rclrcl} Q_T^1(w,\bar w) &=& Q_0^3(w-\frac{i}{T},\bar w+\frac{i}{T}) & Q_\infty^1(w,\bar w) &=& Q_0^3(w,\bar w)\\ Q_0^1(w,\bar w) &=& Q_\infty^3(w,\bar w) & Q_\infty^1(w+iT,\bar w-iT) &=& Q_{\frac1T}^1(w,\bar w) \end{array}$$

which shows that the first and the third cases of (22) are the same. We see that there are only two different surfaces (up to translation $w\to w+iT, \bar w\to\bar w-iT$ of the conformal coordinates) belonging to $Q_0^1(w,\bar w)$ and $Q_0^3(w,\bar w)$. Because there is an isometry along one family of curvature lines the surface which we get by integration of the frame equation with $Q_0^1(w,\bar w)$ is a cylinder or a surface of revolution. Any cylinder or surface of revolution is a Bonnet surface. We do not consider them.
Surfaces of type **B**: Here with $h(z)=2\cosh(z)$ and $z=\dfrac{2w+1}{2w-1}$ we get

$$\begin{aligned} Q_T^4(w,\bar w) &= -\frac{2}{\sinh(2(w+\bar w))}\frac{1-2iT\cosh(\bar z)}{1+2iT\cosh(z)}\frac{\sinh(2\bar w)}{\sinh(2w)}\\ &= -\frac{2}{\sinh(2(w+\bar w))}\frac{e^{2\bar w}-\dfrac{1-2iT}{1+2iT}e^{-2\bar w}}{\dfrac{1-2iT}{1+2iT}e^{2w}-e^{-2w}}\\ &= Q_0^4(w+i\hat T,\bar w-i\hat T) \end{aligned} \tag{23}$$

where $\hat{T} = -\frac{i}{2}\log\left(\sqrt{\frac{1-2iT}{1+2iT}}\right) = \frac{1}{2}\arg\left(\sqrt{\frac{1-2iT}{1+2iT}}\right)$. Clearly $\hat{T}(T)$ is unique only up to an addition of $k\pi/2 \quad, k \in \mathbb{Z}$. In particular

$$Q^4_\infty(w,\bar{w}) = Q^4_0(w + (2k+1)i\frac{\pi}{4}, \bar{w} - (2k+1)i\frac{\pi}{4}), \qquad k \in \mathbb{Z},$$

which shows together with the $\frac{\pi}{2}$-periodicity of Q^4_0 that any family of this type contains - up to same reparametrization - only one surface which is totally described by one period of $\hat{T}$ and a fixed motion in $\mathbb{R}^3$.

Surfaces of type **A**. First we recognize

$$\begin{aligned} Q^2_T(w,\bar{w}) &= h_2'^2 \frac{1-iT\bar{h}_2}{1+iTh_2}\frac{1}{h_2+\bar{h}_2} &=& \quad Q^5_0(w+i\hat{T}, w-i\hat{T}) \quad \text{with } \hat{T} = -\tfrac{1}{2}\log(\sqrt{2T}) \\ Q^5_T(w,\bar{w}) &= h_5'^2 \frac{1-iT\bar{h}_5}{1+iTh_5}\frac{1}{h_5+\bar{h}_5} &=& \quad Q^2_{\hat{T}}(w,\bar{w}) \quad \text{with } \hat{T} = \frac{1}{2}\frac{1-2T}{1+2T}. \end{aligned} \tag{24}$$

This shows that both cases coincide. So we can focus only on Q^5_T to describe the hole family. By doing so we get :

$$Q^5_T(w,\bar{w}) = \begin{cases} Q^5_0(w+\hat{T}(T)), \bar{w}-\hat{T}(T)) & \text{if } T \in (\frac{1}{2}, -\frac{1}{2}) \\ Q^5_0(w+\pi/4+\hat{T}(T)), \bar{w}+\pi/4-\hat{T}(T)) & \text{if } T \in \mathbb{R} \setminus \left[\frac{1}{2}, -\frac{1}{2}\right] \\ Q^2_0(w,\bar{w}) & \text{if } T = \frac{1}{2} \\ \bar{Q}^2_0(w,\bar{w}) & \text{if } T = -\frac{1}{2}, \end{cases}$$

with

$$\hat{T}(T) = -\frac{i}{2}\log\left(\sqrt{\frac{1-2T}{1+2T}}\right)$$

Two surfaces with $T = \pm\frac{1}{2}$ are helicoids because the fundamental forms depend only on one real variable: this implies that there exists a motion of $\mathbb{R}^3$ which also forms an inner isometry of the surface. The motion is a combination of translation and rotation. Let $w = x + i\,y$. Then on this helicoids we get for the arc-length parametrized y-curves

$$\|\gamma''(y)\|^2 = \left(\frac{u'(x)}{2}\right)^2 + e^{-u(x)}\left(H(x)e^{u(x)} - 4\cot(4x)\right)^2 = -16e^{-u(x)} + \mu^2 \tag{25}$$

This formula we may later find useful. Since we excluded helicoids in the beginning, we ignore $T = \pm\frac{1}{2}$, too.

Table 1 presents the fundamental functions and the ordinary differential equation to be solved.

3 Bonnet Surfaces of Type C and Painlevé V(III) Equations

Let us return to the description of the moving frame as in (9). In the variables

$$t = w + \bar{w} \qquad \lambda = \frac{w}{w+\bar{w}} \tag{26}$$

Type	Q	H solves	e^u
A_1	$Q_0^5(w,\bar w) = -2\frac{\sin(2\bar w)}{\sin(2w)}\frac{1}{\sin(2(w+\bar w))}$	$\left(\left(\frac{H''}{H'}\right)' - H'\right)\frac{\sin^2(2t)}{4} = 2 - \frac{H^2}{H'}$	$-\frac{8}{\sin^2(2t)H'}$
A_2	$Q_\infty^5(w,\bar w) = 2\frac{\cos(2\bar w)}{\cos(2w)}\frac{1}{\sin(2(w+\bar w))}$		
B	$Q_0^4(w,\bar w) = -2\frac{\sinh(2\bar w)}{\sinh(2w)}\frac{1}{\sinh(2(w+\bar w))}$	$\left(\left(\frac{H''}{H'}\right)' - H'\right)\frac{\sinh^2(2t)}{4} = 2 - \frac{H^2}{H'}$	$-\frac{8}{\sinh^2(2t)H'}$
C	$Q_0^3(w,\bar w) = -\frac{\bar w}{w}\frac{1}{w+\bar w}$	$\left(\left(\frac{H''}{H'}\right)' - H'\right)t^2 = 2 - \frac{H^2}{H'}$	$-\frac{2}{t^2H'}$

Table 1: Table of fundamental functions

this system reads as

$$\begin{aligned}
\Phi_\lambda(\lambda,t)\Phi^{-1}(\lambda,t) &= t\begin{pmatrix} a(t) & \varphi(t) \\ \varphi(t) & -a(t)\end{pmatrix} + e^{-u(t)/2}\begin{pmatrix} 0 & \frac{1}{\lambda} \\ -\frac{1}{\lambda-1} & 0\end{pmatrix} \\
\Phi_t(\lambda,t)\Phi^{-1}(\lambda,t) &= \lambda\begin{pmatrix} a(t) & \varphi(t) \\ \varphi(t) & -a(t)\end{pmatrix} + \begin{pmatrix} -\frac{a(t)}{2} & -\varphi(t) \\ 0 & \frac{a(t)}{2}\end{pmatrix}
\end{aligned} \tag{27}$$

with the function $a(t)$, $\varphi(t)$ and $e^{-u(t)/2}$ given by

$$\begin{aligned}
a(t) &= \frac{u'(t)}{2} = -\frac{1}{t} - \frac{H''(t)}{2H'(t)} \\
\varphi(t) &= \frac{H(t)+tH'(t)}{2}e^{-u(t)/2} \\
e^{u(t)/2} &= t\sqrt{-\frac{H'(t)}{2}}.
\end{aligned} \tag{28}$$

Remark 3 *The equation $A'(s) + [A(s), B(s)]$ with some $B(s)$ implies that the eigenvalues of A are constant.*

Using this remark one can easily see, that the compatibility conditions imply also, that the determinant of the matrix

$$\begin{pmatrix} a(t) & \varphi(t) \\ \varphi(t) & -a(t)\end{pmatrix}$$

is independent of t. Here we come to a result, first obtained by Hazzidakis in [H].

Lemma 1 *Equation (18) in the case C has the first integral μ, given by*

$$a(t)^2 + \varphi(t)^2 = \left(\frac{\mu}{2}\right)^2 \tag{29}$$

with $a(t)$ and $\varphi(t)$ as in (28).

We have a system of matrix dimension 2×2 with the following dependence on λ

$$\begin{aligned}\Phi_\lambda\Phi^{-1} &= tA(t)+\frac{1}{\lambda}A_0(t)+\frac{1}{\lambda-1}A_1(t)\\ \Phi_t\Phi^{-1} &= \lambda A(t)+C(t).\end{aligned} \tag{30}$$

Here specialists in the theory of the Painlevé equations immediately recognize the Lax representation for the Painlevé V equation (see for example [I-N]). In the rest of this section we carry out in detail this identification, showing as a result how the Bonnet surfaces of C-type can be described in terms of the Painlevé transcendents.

Theorem 6 (Jimbo-Miwa [J-M]) *Let us consider the system (30) such that $A(t)$ has two different eigenvalues. Then the compatibility conditions for this system are equivalent to the Painlevé V equation*

$$\begin{aligned}y''(t) &= \left(\frac{1}{2y(t)}+\frac{1}{y(t)-1}\right)y'^2(t)-\frac{y'(t)}{t}+\frac{(y(t)-1)^2}{t^2}\left(\alpha y(t)+\frac{\beta}{y(t)}\right)\\ &\quad+\frac{\gamma y(t)}{t}+\frac{\delta y(t)(y(t)+1)}{y(t)-1}\end{aligned} \tag{31}$$

which implies that the coefficients of the matrices $A(t)$, $A_0(t)$, $A_1(t)$ as well as of the matrix $C(t)$ can be expressed in terms of this function and the constants α, β, γ and δ depend on the eigenvalues of the matrices $A(t)$, $A_0(t)$ and $A_1(t)$ only.

Proof : First let us normalize the matrices in the first equation of (30) to be traceless by

$$\Phi \;\rightarrow\; \Psi = e^{-\tau\, t\,\lambda/2}\,\lambda^{-\tau_0/2}(\lambda-1)^{-\tau_1/2}\Phi \tag{32}$$

with $\tau=\mathrm{tr}(A(t))$, $\tau_0=\mathrm{tr}(A_0(t))$ and $\tau_1=\mathrm{tr}(A_1(t))$. The transformed A-matrix has two not vanishing eigenvalues and we can bring it by another gauge transformation to a diagonal form. So let as assume that the system (30) is of the form

$$\begin{aligned}\Psi_\lambda\Psi^{-1} &= \frac{ti\mu}{2}\mathbb{k}+\frac{1}{\lambda}A_0(t)+\frac{1}{\lambda-1}A_1(t)\\ \Psi_t\Psi^{-1} &= \frac{\lambda i\mu}{2}\mathbb{k}+C(t)\end{aligned} \tag{33}$$

with $\mu\neq 0$. We set $A_\nu(t)=(a^\nu_{ij}(t))$ and $\det(A_\nu(t))=-\theta_\nu^2/4$ for $\nu=0,1$ and $\theta_\infty=-2(a^0_{11}(t)+a^1_{11}(t))$. It is easy to check that these θ's are constants. Now define

$$z(t)=a^0_{11}(t)-\frac{\theta_0}{2}\qquad y(t)=-\frac{a^0_{21}(t)}{a^1_{21}(t)}\left(1+\frac{\theta_0+\theta_1+\theta_\infty}{2z(t)}\right). \tag{34}$$

Finally one can prove that the compatibility condition of (33) are given by

$$\begin{aligned}tz'(t) &= -\frac{1}{y(t)}\left(z(t)+\theta_0\right)\left(z(t)+\frac{\theta_0+\theta_1+\theta_\infty}{2}\right)+y(t)z(t)\left(z(t)+\frac{\theta_0-\theta_1+\theta_\infty}{2}\right)\\ ty'(t) &= \mu t y(t)-2z(t)(y(t)-1)^2-(y(t)-1)\left(\frac{\theta_0-\theta_1+\theta_\infty}{2}y(t)-\frac{3\theta_0+\theta_1+\theta_\infty}{2}\right).\end{aligned} \tag{35}$$

This system can be rewritten as an equation in $y(t)$ only, which gives (31) with

$$\alpha = \frac{1}{2}\left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2}\right)^2, \quad \beta = \frac{1}{2}\left(\frac{\theta_0 - \theta_1 - +\theta_\infty}{2}\right)^2, \quad \gamma = \mu(1 - \theta_0 - \theta_1), \quad \delta = -\frac{1}{2}\mu^2. \tag{36}$$

Now we apply the proof of theorem 6 to our system (27).

Theorem 7 *Let $H(t)$ be a solution of (18) in the C-case. Then*

$$y(t) = \frac{2\,a(t) - \mu}{2\,a(t) + \mu}$$

with $a(t)$ defined in (28) and μ is a root of (29) defines a solution of the Painlevé V equation (31) with

$$\alpha = 0, \qquad \beta = 0, \qquad \gamma = \mu \qquad \text{and} \qquad \delta = -\frac{1}{2}\mu^2. \tag{37}$$

On the other hand let $y(t)$ be an arbitrary solution of (31) with constants as in (37), $\mu \neq 0$. Then

$$H(t) = -\frac{t(y'(t)^2 - \mu^2 y(t)^2)}{2y(t)(y(t)-1)^2} \tag{38}$$

is a solution of (18). If finally $0 \geq y(t)$ and $y(t) \neq C\, e^{\mu t}$, the solution is geometrical and the metric is given by

$$e^{u(t)} = -\frac{4y(t)(y(t)-1)^2}{t^2(y'(t) - \mu y(t))^2}. \tag{39}$$

Proof : First let us normalize (27) to the special form (33) with the gauge transformation

$$\Phi \;\rightarrow\; \Psi = D\,\Phi = \begin{pmatrix} a + \frac{\mu}{2} & \varphi \\ a - \frac{\mu}{2} & \varphi \end{pmatrix} \Phi. \tag{40}$$

Since $\det(D) = \varphi\mu$ this is a regular transformation if $\varphi \neq 0$. If there would be a domain in $\mathbb{C}$ where φ vanishes identically then $H(t) = c/t$ and consequently $a(t)$ vanishes identically. But then necessarily $\mu = 0$ in contradiction to our assumption. By that gauge transformation equation (27) becomes

$$\begin{aligned} \Psi_\lambda \Psi^{-1} &= \frac{i\,t\,\mu}{2}\mathbb{I}\mathrm{k} + \frac{\varphi\, e^{-u/2}}{\mu(\lambda-1)}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \frac{e^{-u/2}}{\varphi\,\mu\,\lambda}\begin{pmatrix} \frac{\mu^2}{4} - a^2 & (a+\frac{\mu}{2})^2 \\ -(a-\frac{\mu}{2})^2 & a^2 - \frac{\mu^2}{4} \end{pmatrix} \\ \Psi_t \Psi^{-1} &= \frac{i\,\lambda\,\mu}{2}\mathbb{I}\mathrm{k} + \begin{pmatrix} -\frac{\mu}{4} & -\frac{1}{2}(a+\frac{\mu}{2}) \\ -\frac{1}{2}(a-\frac{\mu}{2}) & \frac{\mu}{4} \end{pmatrix} + D'(t)D(t)^{-1}. \end{aligned} \tag{41}$$

We see that $\theta_\nu = 0$, for $\nu = 0, 1, \infty$ and by (34) we get

$$y(t) = \frac{2a(t) - \mu}{2a(t) + \mu} \qquad z(t) = -\frac{\varphi\, e^{-u/2}}{\mu}. \tag{42}$$

These function solve

$$
\begin{aligned}
tz'(t) &= z^2(t)\left(y(t)-\frac{1}{y(t)}\right), \\
ty'(t) &= \mu t y(t) - 2z(t)\,(y(t)-1)^2\,, \\
y''(t) &= \left(\frac{3y(t)-1}{2y(t)(y(t)-1)}\right) y'^2(t) - \frac{y'(t)}{t} + \mu\frac{y(t)}{t} - \mu^2\frac{y(t)(y(t)+1)}{2(y(t)-1)}.
\end{aligned}
\tag{43}
$$

This proves the first part of the theorem. We shall remark that we get the following formulas by the theorem 6

$$
\begin{aligned}
a(t) &= -\frac{\mu(y(t)+1)}{2(y(t)-1)}\,, \\
\varphi(t)\,e^{u/2} &= -\frac{2\mu y(t)}{t(y'(t)-\mu y(t))}\,, \\
\varphi(t)\,e^{-u/2} &= \frac{\mu t(y'(t)-\mu y(t))}{2(y(t)-1)^2}\,.
\end{aligned}
\tag{44}
$$

On the other hand $a(t)$, $\varphi(t)\,e^{-u(t)/2}$ and $\varphi(t)\,e^{u(t)/2}$ can be expressed (28) in terms of the functions H, H' and H''. It can be interpreted as a linear system for these functions, which is uniquely solvable since $\varphi \neq 0$. Comparing (28) with (44) we get (38), (39). If $y_\mu(t)$ solves (43) then

$$
\tilde{y}_{-\mu}(t) = \frac{1}{y_\mu(t)}
$$

is a solution of (43) with $-\mu$.

Remark 4 *For another real reduction of the Painlevé V equation the equation in the form (29) appeared already in papers devoted to the calculation of the correlation functions for the Bose gas [M-T].*

Remark 5 *For any solution $H(t)$ of (18) in the C case*

$$
\tilde{H}(\tilde{t}) = \frac{1}{\alpha}H(t), \qquad \text{with} \qquad \tilde{t} = \alpha\, t \tag{45}
$$

is a solution of (18), too. That implies that we can fix μ to some special value in (29). Geometrically this is only a scaling of $\mathbb{R}^3$.

From now on we fix $\mu = 4$. The case $\mu = 0$ is considered in Appendix A.

It turns out that in the case $\alpha = \beta = 0$ the Painlevé V equation can be reduced to the Painlevé III equation. The following three statements can be proved by direct calculations.

Corollary 1 *[Ki] Let $y(t)$ be a solution of (31) with $\alpha = \beta = 0$ and $\mu = 4$. The function $p(t)$ defined by*

$$
y(t) = \left(\frac{p(t)+1}{p(t)-1}\right)^2 \tag{46}
$$

solves the Painlevé III

$$p''(t) = \frac{p'^2(t)}{p(t)} - \frac{p'(t)}{t} - \frac{p^2(t)}{t} + \frac{1}{t} + p^3(t) - \frac{1}{p(t)}. \tag{47}$$

The mean curvature and the metric are

$$\begin{aligned} H(t) &= -\frac{t(p'^2(t) - (p^2(t)-1)^2)}{2p^2(t)} \\ e^{u(t)} &= -\frac{4p^2(t)}{t^2(p'(t)+p^2(t)-1)^2}. \end{aligned} \tag{48}$$

The reduction (46) holds for arbitrary γ and δ but gives another Painlevé III where some constants are involved depending on γ and δ.

Remark 6 *The geometrical solutions $p(t)$ are of modulus 1 and we have to exclude the solutions*

$$p(t) = \tanh(2t + c) \qquad \text{and} \qquad p(t) = \pm 1\,.$$

Corollary 2 *Let $p(t)$ be a solution of (47) of modulus 1. Then its argument $\phi(t)$*

$$p(t) = e^{i\phi(t)}$$

solves

$$t\left(\phi''(t) - 2\sin(2\phi(t))\right) + \phi'(t) + 2\sin(\phi(t))) = 0. \tag{49}$$

The solutions of (49), which are not solutions of $\phi' + 2\sin(\phi) = 0$ are geometrical. The mean curvature and the metric are given by

$$H(t) = 2t\left(\frac{\phi'(t)^2}{4} - \sin^2(\phi(t))\right) \qquad e^{u(t)} = \frac{4}{t^2(\phi'(t) + 2\sin(\phi(t)))^2} \tag{50}$$

Corollary 3 *Let $y(t)$ be a solution of the third equation in (43) and $z(t)$ be defined by the second equation in (43). Then equation (41) is of the form*

$$\begin{aligned} \Psi_\lambda \, \Psi^{-1} &= 2\,i\,t\,\mathbb{k} + \frac{1}{\lambda}\begin{pmatrix} z(t) & -\frac{z(t)}{y(t)} \\ z(t)y(t) & -z(t) \end{pmatrix} + \frac{1}{\lambda - 1}\begin{pmatrix} -z(t) & z(t) \\ -z(t) & z(t) \end{pmatrix} \\ \Psi_t \Psi^{-1} &= 2\,i\,\lambda\,\mathbb{k} + \frac{1}{t}\begin{pmatrix} z(t)(y(t)-1) - t\frac{3y(t)-1}{y(t)-1} & z(t)(1 - \frac{1}{y(t)}) \\ z(t)(y(t)-1) & z(t)(1 - \frac{1}{y(t)}) + t\frac{y(t)-3}{y(t)-1} \end{pmatrix}. \end{aligned} \tag{51}$$

The Ψ-function, which satisfies (51) is related to the geometrical frame Φ by the following transformation:

$$\Phi(t,\lambda) = \frac{1}{4}\begin{pmatrix} 1 & -1 \\ -\sqrt{-y(t)} & -\frac{1}{\sqrt{-y(t)}} \end{pmatrix} \Psi(t,\lambda). \tag{52}$$

4 Bonnet Surfaces of Types A and B and Painlevé VI Equation

In this part we will study Bonnet families of type A and B in more details. First we repeat the ordinary differential equations to be solved by the mean curvature function $H(t)$ in the cases of A and B. These are for $t = w + \bar{w}$ (see Table 1)

$$\left(\frac{H''(t)}{H'(t)}\right)' - H'(t) = \frac{4}{\sin^2(2t)}\left(2 - \frac{H^2(t)}{H'(t)}\right) \tag{53}$$

$$\left(\frac{H''(t)}{H'(t)}\right)' - H'(t) = \frac{4}{\sinh^2(2t)}\left(2 - \frac{H^2(t)}{H'(t)}\right) \tag{54}$$

The solutions of these two equations are simply related. Let $H_B(t)$ a solution of (54). Then

$$H_A(t) \equiv -iH_B(it) \tag{55}$$

solves (53). Now we start again with (9). In all three cases we get:

$$\begin{aligned}
\Phi_\lambda(\lambda,s)\Phi^{-1}(\lambda,s) &= \frac{1}{\lambda}\begin{pmatrix} s\,f'(s) & \varphi_1(s) \\ \varphi_2(s) & -s\,f'(s)\end{pmatrix} \\
&\quad + \frac{\varphi_2(s)-\varphi_1(s)}{\lambda-1}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} + \frac{\varphi_2(s)-\varphi_1(s)}{\lambda-s}\begin{pmatrix} 0 & 0 \\ -1 & 0\end{pmatrix} \\
\Phi_s(\lambda,s)\Phi^{-1}(\lambda,s) &= \begin{pmatrix} -\dfrac{f'(s)}{2} & \dfrac{\varphi_1(s)-s\,\varphi_2(s)}{s(s-1)} \\ \dfrac{\varphi_2(s)-\varphi_1(s)}{s-1} & \dfrac{f'(s)}{2}\end{pmatrix} + \\
&\quad \frac{\varphi_2(s)-\varphi_1(s)}{\lambda-s}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}
\end{aligned} \tag{56}$$

where the coefficients are presented in Table 2.

	$\mathbf{A_1}$	$\mathbf{A_2}$	**B**
s	$e^{4i(w+\bar{w})}$		$e^{4(w+\bar{w})}$
λ	e^{4iw}	$-e^{4iw}$	e^{4w}
$B(s)$	$H_A(-\frac{i}{4}(\log(s)))$		$H_B(\frac{1}{4}(\log(s)))$
$f(s)$	$\frac{1}{2}u_A(-\frac{i}{4}(\log(s)))$		$\frac{1}{2}u_B(\frac{1}{4}(\log(s)))$
$\varphi_1(s)$	$-i\,\frac{B(s)}{8}\,e^{f(s)} - \frac{s}{s-1}\,e^{-f(s)}$		$\frac{B(s)}{8}\,e^{f(s)} - \frac{s}{s-1}\,e^{-f(s)}$
$\varphi_2(s)$	$-i\,\frac{B(s)}{8}\,e^{f(s)} - \frac{1}{s-1}\,e^{-f(s)}$		$\frac{B(s)}{8}\,e^{f(s)} - \frac{1}{s-1}\,e^{-f(s)}$

Table 2: Coefficients in (56)

Let us write down the system (56) in a more general form as

$$\Phi_\lambda \Phi^{-1} = \frac{1}{\lambda - s} A_s + \frac{1}{\lambda - 1} A_1 + \frac{1}{\lambda} A_0 = U, \qquad \Phi_s \Phi^{-1} = -\frac{1}{\lambda - s} A_s + B = V. \tag{57}$$

The compatibility condition for this system for $\lambda \to 0, 1, s$ are

$$\begin{aligned} A_s' + \left[A_s, B + \tfrac{1}{s} A_0 + \tfrac{1}{s-1} A_1\right] &= 0 \\ A_1' + \left[A_1, B + \tfrac{1}{s-1} A_s\right] &= 0 \\ A_0' + \left[A_0, B + \tfrac{1}{s} A_s\right] &= 0 \end{aligned} \tag{58}$$

The last equation implies that the determinant of A_0 is independent of s (see remark 3), and as for the Bonnet surfaces of type C we get the following first integral, first found by Hazzidakis [H].

Lemma 2 *Equation (18) in the case A and B has the first integral μ, given by*

$$s^2 f'(s)^2 + \varphi_1(s)\varphi_2(s) = \left(\frac{\mu}{2}\right)^2 \tag{59}$$

with $f(s)$, $\varphi_1(s)$ and $\varphi_2(s)$ as in Table 2. If we formulate this equation in terms of the function $B(s)$ we get in the B-case

$$s^2 \left(\frac{B''(s)}{2B'(s)} + \frac{1}{s-1}\right)^2 - \frac{sB'(s)}{8} - \frac{B^2(s)}{8B'(s)(s-1)^2} - \frac{B(s)}{8}\frac{s+1}{s-1} = \frac{\mu^2}{4} \tag{60}$$

and in the A-cases

$$s^2 \left(\frac{B''(s)}{2B'(s)} + \frac{1}{s-1}\right)^2 + i\,\frac{sB'(s)}{8} + i\,\frac{B^2(s)}{8B'(s)(s-1)^2} + i\,\frac{B(s)}{8}\frac{s+1}{s-1} = \frac{\mu^2}{4}. \tag{61}$$

In a contrast to the C case the parameter μ seems to be an essential parameter of the surface. In the B case μ can be real as well as pure imaginary and zero. In the A-case equation (61) is only a new form of equation (25). So we know that μ^2 must be strictly positive. We shall also remark that although s is in this case an unitary parameter $s\, f'(s)$ is real.
Now let us return to the more general system (57). Again as in the C-case here one can recognize a Lax representation for a Painlevé equation, but now for the Painlevé VI equation.

Theorem 8 (Jimbo-Miva [J-M]) *: Let us consider the system (57) such that*

$$A_\infty := -A_0 - A_1 - A_t$$

has two different eigenvalues. Then the compatibility conditions of this system are equivalent to the Painlevé VI equation. That implies, that all the coefficients of the matrices in (57) can be expressed *in terms of the function $y(s)$, which satisfies the Painlevé VI equation*

$$
\begin{aligned}
y'' &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-s}\right)y'^2-\left(\frac{1}{s}+\frac{1}{s-1}+\frac{1}{y-s}\right)y'+ \\
&\quad \frac{y(y-1)(y-s)}{s^2(s-1)^2}\left(\alpha+\beta\frac{s}{y^2}+\gamma\frac{s-1}{(y-1)^2}+\delta\frac{s(s-1)}{(y-s)^2}\right)
\end{aligned} \tag{62}
$$

with some constants α, β, γ and δ depending on the traces and determinants of the matrices A_ν, for $\nu = 0, 1, s$.

Prove : Due to remark 3 we know that the determinants of the matrices A_ν are constant. First we normalize that these matrices to be traceless[2]. Then we can reach $\det(A_\nu) = 0$ for $\nu = 0, 1, s$ by the gauge-transformation

$$\Phi \to ((\lambda^{d_0}(\lambda-1)^{d_1}(\lambda-t)^{d_t})\Phi = \tilde{\Phi}$$

with $d_\nu = \sqrt{\det(A_\nu)}$, $\nu = 0, 1, s$. Therefore we can assume that the A_ν-matrices are of the form

$$A_\nu = \begin{pmatrix} z_\nu(s)+\theta_\nu & -u_\nu(s)\,z_\nu(s) \\ \dfrac{z_\nu(s)+\theta_\nu}{u_\nu(s)} & -z_\nu(s) \end{pmatrix} \tag{63}$$

with $\theta_\nu = 2d_\nu$. Now let us consider A_∞ defined as in the theorem. Both gauge-transformations we used to create the special form (63) preserve the inequality of the eigenvalues of this matrix. Let us finally assume that A_∞ is diagonal with the diagonal elements κ_1 and κ_2 as diagonal elements and $\theta_\infty = \kappa_1 - \kappa_2$. The matrix B must be diagonal.

Define

$$
\begin{aligned}
y(s) &= -\frac{s\,u_0(s)\,z_0(s)}{s\,u_s(s)\,z_s(t)+u_1(s)\,z_1(s)}, \\
z(s) &= \frac{z_0(s)+\theta_0}{y(s)}+\frac{z_1(s)+\theta_1}{y(s)-1}+\frac{z_s(s)+\theta_s}{y(s)-s}, \\
\tilde{z}(s) &= z(s)-\frac{\theta_0}{y(s)}-\frac{\theta_1}{y(s)-1}-\frac{\theta_\infty}{y(s)-s}.
\end{aligned} \tag{64}
$$

Now the diagonality of A_∞ implies

$$
\begin{aligned}
z_0 &= \frac{y}{t\theta_\infty}\{y(y-1)(y-t)\tilde{z}^2+(\theta_1(y-t)+t\theta_t(y-1)-2\kappa_2(y-1)(y-t))\tilde{z} \\
&\qquad +\kappa_2^2(y-t-1)-\kappa_2(\theta_1+t\theta_t)\} \\
z_1 &= \frac{1-y}{(t-1)\theta_\infty}\{y(y-1)(y-t)\tilde{z}^2+((\theta_1+\theta_\infty)(y-t)+t\theta_t(y-1) \\
&\qquad -2\kappa_2(y-1)(y-t))\tilde{z}+\kappa_2^2(y-t)-\kappa_2(\theta_1+t\theta_t)-\kappa_1\kappa_2\} \\
z_t &= \frac{y-t}{t(t-1)\theta_\infty}\{y(y-1)(y-t)\tilde{z}^2+(\theta_1(y-t)+t(\theta_t+\theta_\infty)(y-1) \\
&\qquad -2\kappa_2(y-1)(y-t))\tilde{z}+\kappa_2^2(y-1)-\kappa_2(\theta_1+t\theta_t)-t\kappa_1\kappa_2\}.
\end{aligned} \tag{65}
$$

[2] for the proof see the previous section

The compatibility conditions in terms of $z(s)$ and $y(s)$ read as follows:

$$
\begin{aligned}
z' &= \frac{1}{t(t-1)}\Big((-3y^2+2(1+t)y-t)z^2+ \\
&\quad (\theta_0(2y-t-1)+\theta_1(2y-t)+(2y-1)(\theta_t-1))z-\kappa_1(\kappa_2+1)\Big) \\
y' &= \frac{y(y-1)(y-t)}{t(t-1)}\left(2z-\frac{\theta_0}{y}-\frac{\theta_1}{y-1}-\frac{\theta_t-1}{y-t}\right),
\end{aligned} \tag{66}
$$

which can be formulated as (62) with

$$
\alpha=\frac{(\theta_\infty-1)^2}{2} \quad \beta=-\frac{\theta_0^2}{2} \quad \gamma=\frac{\theta_1^2}{2} \quad \delta=\frac{1-\theta_t^2}{2}.
$$

Now we apply this to our case.

Theorem 9 *Let $H(t)=B(s)$ be a solution of (54) or (53), the functions $f(s)$, $\varphi_1(s)$ and $\varphi_2(s)$ be defined as in Table 2 and $\mu\neq 0$ be a root of (61) respectively (60). Then*

$$
y(s)=\frac{s(2s^2f'(s)^2+\varphi_1(s)^2+\varphi_1(s)\varphi_2(s)-\mu s f'(s))}{(2s^3f'(s)^2+\varphi_1(s)^2+s\varphi_1(s)\varphi_2(s)-\mu s^2 f'(s))} \tag{67}
$$

solves the Painlevé VI (62) with the coefficients

$$
\begin{aligned}
\alpha &= \frac{(\mu+1)^2}{2}, & \beta &= -\frac{\mu^2}{2}, \\
\gamma &= 0, & \delta &= \frac{1}{2}.
\end{aligned} \tag{68}
$$

On the other hand let $y(s)$ be any solution of (62) with the above constants. Then

$$
H(w+\bar w)=B(s)=2\frac{\mu^2(s+y^2(s)-y(s)(1+s))^2-(y(s)(1-y(s))+sy'(s)(s-1))^2}{(s-1)y(s)(y(s)-1)(y(s)-s)} \tag{69}
$$

solves the differential equation of the B-case (54). The transformation (55) yields the solution to the differential equation of the A-case (53). In the B-case for $\mu\in\mathbb{R}$ and for $1<y(s)<s$ we get a geometrical solution.The metric is then given by

$$
e^{u(w+\bar w)}=e^{2f(s)}=-\frac{4s(y(s)-1)(y(s)-s)}{(sy'(s-1)-\mu(y(s)-1)(y(s)-s)-y(s)(y(s)-1))^2}. \tag{70}
$$

Prove : By the gauge transformation

$$
\Phi(\lambda,t)\to\lambda^{\mu/2}\begin{pmatrix}\dfrac{2\,\varphi_1(s)}{2\,s\,f'(s)-\mu} & -1 \\ -1 & -\dfrac{2\,\varphi_2(s)}{2\,s\,f'(s)-\mu}\end{pmatrix}\Phi(\lambda,s)=\Psi(\lambda,s) \tag{71}
$$

the system (56) reads as follows

$$
\begin{aligned}
\Psi_\lambda \Psi^{-1} &= \frac{1}{\mu\lambda}\begin{pmatrix} (\varphi_1-\varphi_2)^2+\mu^2 & \mu(\varphi_2-\varphi_1)+\dfrac{2\varphi_1(\varphi_2-\varphi_1)^2}{2\,sf'-\mu} \\ \mu(\varphi_1-\varphi_2)+\dfrac{2\varphi_2(\varphi_1-\varphi_2)^2}{2\,sf'-\mu} & -(\varphi_1-\varphi_2)^2 \end{pmatrix} + \\
&\qquad \frac{\varphi_2-\varphi_1}{\mu(\lambda-1)}\begin{pmatrix} \varphi_1(s) & \dfrac{2\varphi_1^2}{2sf'-\mu} \\ -\frac{1}{2}(2sf'-\mu) & -\varphi_1 \end{pmatrix} + \\
&\qquad \frac{\varphi_2-\varphi_1}{\mu(\lambda-s)}\begin{pmatrix} -\varphi_2 & \frac{1}{2}(2sf'-\mu) \\ -\dfrac{2\varphi_2^2}{2sf'-\mu} & \varphi_2 \end{pmatrix} \\
\Psi_t \Psi^{-1} &= \frac{\varphi_2-\varphi_1}{\mu(\lambda-s)}\begin{pmatrix} \varphi_2 & -\frac{1}{2}(2sf'-\mu) \\ \dfrac{2\varphi_2^2}{2sf'-\mu} & -\varphi_2 \end{pmatrix} + B_0(s).
\end{aligned}
\tag{72}
$$

Here we do not specify the diagonal matrix B_0 because the definition of $y(s)$ is independent of this. With $\kappa_1=-\mu$, $\kappa_2=0$ we get (64), (67) with the coefficients given by (68), which proves the first part for the theorem.

As in the Bonnet C-case we can interpretate the definitions of $f'(s)$, $\varphi_1(s)\,e^{-f(s)}$ and $\varphi_2(s)\,e^{f(s)}$ in Table 2 as a linear system for the functions $B(s)$, $B'(s)$ and $B''(s)$, which can be solved explicitly for these functions. On the other hand we find

$$
\begin{aligned}
z_0(s) &= \frac{(\varphi_1(s)-\varphi_2(s))^2}{\mu} & & & e^{-f} &= +\sqrt{\mu\, z_0} \\
z_1(s) &= \varphi_1(s)\frac{(\varphi_2(s)-\varphi_1(s))}{\mu} & &\text{and so} & \varphi_1\, e^{-f} &= \mu\, z_1 \\
z_s(s) &= \varphi_2(s)\frac{(\varphi_1(s)-\varphi_2(s))}{\mu} & & & \varphi_2\, e^{f} &= -\mu\, z_s\, e^{2f}.
\end{aligned}
\tag{73}
$$

Because of (65) this gives (69) and (70). For the case $\mu=0$ in the B-case Theorem 8 does not hold. But by a simple calculation one can show that for this case (67) as well as (69) and (70) are correct.

Remark 7 *Formulas (69) in the A and B cases deal in general with the complex-valued functions $y(s)$ and $B(s)$. The Bonnet surfaces are characterized by the condition, that for s defined in Table 2, $B(s)$ and $f(s)$ in (69), (70) should be real-valued. It seems to be rather difficult to describe the variety of the geometrical solutions in terms of $y(s)$.*

For integrating the Bonnet surfaces of type A and B we have first to solve our special Painlevé VI under the extra conditions that the functions B and f as defined in (69) and (70) are both real and e^{2f} strictly positive. Then solve the frame equation (72) for Ψ. By the inverse left-multiplication of (71) with the functions given in (73) and (65) we find the geometrical frame, which finally have to be integrated for the surface.

Appendix A Bonnet surfaces of type C with $\mu = 0$

Here we integrate explicitly the equation for the moving frame for the Bonnet surfaces of type C in the case $\mu = 0$ (see section 3). In this case equation (29) with (28) implies

$$H(w,\bar{w}) = \frac{a_0}{w+\bar{w}} \qquad e^{u(w,\bar{w})} \equiv \frac{2}{a_0} \qquad Q(w,\bar{w}) = -\frac{\bar{w}}{w(w+\bar{w})}. \tag{74}$$

Now let us pass to isothermic coordinates as in (12)-(15)

$$z = \log(w) \qquad z = x + iy.$$

In the isothermic coordinates the fundamental functions are :

$$H(z,\bar{z}) = \frac{a_0}{2\,e^{x}\,\cos(y)} \qquad e^{u(z,\bar{z})} = \frac{2}{a_0}\,e^{2x} \qquad Q(z,\bar{z}) = -\frac{e^{x}}{2\cos(y)},$$

and the frame equations are of the following form:

$$\begin{aligned} &F_{xx} = F_x \qquad F_{xy} = F_y \qquad F_{yy} = -F_x + \frac{2e^{x}}{\cos(y)}N \\ &N_x = 0 \qquad N_y = -\frac{a_0}{e^{x}\,\cos(y)}F_y. \end{aligned} \tag{75}$$

One can integrate (75)

$$F(z,\bar{z}) = F_x(z,\bar{z}) = F_z(z,\bar{z}) + F_{\bar{z}}(z,\bar{z}) = w\,F_w(w,\bar{w}) + \bar{w}\,F_{\bar{w}}(w,\bar{w})$$

and so

$$F(w,\bar{w}) = -\mathrm{i}\sqrt{\frac{2}{a_0}}\Phi^{-1}\begin{pmatrix} 0 & \bar{w} \\ w & 0 \end{pmatrix}\Phi = -\mathrm{i}\sqrt{\frac{2}{a_0}}\,t\;\Phi^{-1}(\lambda)\begin{pmatrix} 0 & 1-\lambda \\ \lambda & 0 \end{pmatrix}\Phi(\lambda), \tag{76}$$

where λ and t are defined as in (26). Here Φ depends on λ only and solves (see (27)) the ordinary differential equation

$$\Phi_\lambda(\lambda)\Phi^{-1}(\lambda) = \sqrt{\frac{a_0}{2}}\begin{pmatrix} 0 & \dfrac{1}{\lambda} \\ \dfrac{1}{1-\lambda} & 0 \end{pmatrix} \tag{77}$$

Let us recall that Φ is a quaternion

$$\Phi(\lambda) = \begin{pmatrix} \alpha(\lambda) & -\bar{\beta}(\bar{\lambda}) \\ \beta(\lambda) & \bar{\alpha}(\bar{\lambda}) \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 \equiv const \neq 0$ and especially $\bar{\lambda} = 1 - \lambda$.
In terms of these functions (77) gives:

$$\alpha'(\lambda) = \sqrt{\frac{a_0}{2}}\,\frac{1}{\lambda}\beta(\lambda) \qquad \beta'(\lambda) = \sqrt{\frac{a_0}{2}}\,\frac{1}{1-\lambda}\alpha(\lambda)$$

where $'$ denotes the derivative with respect to λ. This system can be formulated as differential equation of second order for α only:

$$\lambda(1-\lambda)\alpha''(\lambda)+(1-\lambda)\alpha'(\lambda)-\frac{2}{a_0}\alpha(\lambda)=0.$$

This is a hypergeometric equation which can be integrated in terms of special functions. The function ${}_2F_1(a,b,c,\lambda)$ is the solution of the hypergeometric equation

$$\lambda(\lambda-1)\alpha''(\lambda)+((a+b+1)\lambda-c)\alpha'(\lambda)+ab\alpha(\lambda)=0$$

regular at $\lambda=0$, which has for $|\lambda|<1$ the expansion

$${}_2F_1(a,b,c,\lambda)=\sum_{k=0}^{\infty}\frac{\Gamma(c)\Gamma(a+k)\Gamma(b+k)}{k!\Gamma(a)\Gamma(b)\Gamma(c+k)}\,\lambda^k. \tag{78}$$

In our case

$$a=\pm\mathrm{i},\sqrt{\frac{2}{a_0}}\quad b=\mp\mathrm{i}\sqrt{\frac{2}{a_0}}\quad c=1.$$

Let us for short call $\omega=\sqrt{2/a_0}$. The general solution of (77) is given by

$$\alpha(\lambda)=C_1\,{}_2F_1(i\,\omega,-i\,\omega,1;\lambda)+C_2(1-\lambda)\,{}_2F_1(1+i\,\omega,1-i\,\omega,2;1-\lambda) \tag{79}$$

and finally

$$\Phi(\lambda)=\begin{pmatrix}\alpha(\lambda) & -\omega(1-\lambda)\,\alpha'(1-\lambda)\\ \omega\lambda\,\alpha'(\lambda) & \alpha(1-\lambda)\end{pmatrix} \tag{80}$$

which is for geometrical λ in $I\!R\ SU(2)$ if the constants $C_1,C_2\in I\!R$. We can choose $C_2=0$ because any other solution of (77) differs from this special one only by a multiplication on the right by a quaternion independent of w and $\bar{w}$. But this multiplication is a rotation of the surface (76) as a whole.

Theorem 10 *In the isothermic coordinates $(x,\ y)$ the Bonnet cones, which are the surfaces of type C with $\mu=0$, are given by the formulas*

$$F(x,y)=\frac{\omega\,t}{|\alpha(\lambda)|^2+\omega^2\,|\lambda|^2|\alpha'(\lambda)|^2}\begin{pmatrix}\mathrm{Re}\left(\lambda\ \left(\alpha(\lambda)^2-\omega^2\,|\lambda|^2\,\alpha'^2(\lambda)\right)\right)\\ \mathrm{Re}\left(\mathrm{i}\,\lambda\ \left(\alpha(\lambda)^2-\omega^2\,|\lambda|^2\,\alpha'^2(\lambda)\right)\right)\\ -2\,\omega|\lambda|^2\,\mathrm{Re}\left(\alpha'(\lambda)\,\alpha(1-\lambda)\right)\end{pmatrix} \tag{81}$$

where

$$\begin{array}{rclrcl}\alpha(\lambda)&=&C_1\,{}_2F_1(i\omega,-i\omega,1;\lambda) & \alpha'(\lambda)&=&C_1\,{}_2F_1(1+i\,\omega,1-i\,\omega,2;\lambda)\\ \lambda&=&\frac{1}{2}\,(1+\mathrm{i}\,\tan(y)) & t&=&2\,e^x\,\cos(y)\end{array} \tag{82}$$

with some arbitary positive number ω.

For the proof one should use formula (76) and the isomorphism (7) between $I\!R^3$ and $su(2)$. One should also see, that

$$\left(\frac{d}{d\,\lambda}\right)^n\,{}_2F_1(a,b,c;\lambda)=\frac{\Gamma(a+n)\,\Gamma(b+n)\,\Gamma(c)}{\Gamma(a)\,\Gamma(b)\,\Gamma(c+n)}\,{}_2F_1(n+a,a+b,n+c;\lambda).$$

A plot of such a surface which $\omega=1$ is presented at the title page of this preprint.
The other surfaces are cylinder generated by logarithmic spirals as shown in [C-K].

Appendix B Bonnet surface of type B with $\mu = 0$

In this appendix we will give the general solution of the differential equation (54) for $\mu = 0$. To do this we need a simple result on the connection between solutions of different Painlevé VI equations.

Theorem 11 *Let $y_k(s)$ be an arbitrary solution of the Painlevé VI equation with arbitrary constants α_k, β_k, γ_k, δ_k. Then the function*

$$y_{k+1}(s) = \frac{1 - s\ y_k(1/s)}{1 - y_k(1/s)} \tag{83}$$

solves the Painlevé VI with

$$\alpha_{k+1} = \gamma_k, \quad \beta_{k+1} = \delta_k - \frac{1}{2}, \quad \gamma_{k+1} = -\beta_k, \quad \delta_{k+1} = \frac{1}{2} - \alpha_k. \tag{84}$$

In particular

$$y_{k+4}(s) = y_k(s), \quad \alpha_{k+4} = \alpha_k, \quad \beta_{k+4} = \beta_k, \quad \gamma_{k+4} = \gamma_k, \quad \delta_{k+4} = \delta_k. \tag{85}$$

Proof : Here one only has to check the correctness of the formulas for y_{k+1}, which is a simple calculation. To check the second property one has to iterate (83 four times.
A direct consequence of this is that if $y_1(\tau)$, $\tau = 1/s$ solves the Painlevé VI with

$$\alpha = \beta = \gamma = \delta = 0, \tag{86}$$

then $y_0(s)$ solves the Painlevé VI (62), (68) with $\mu = 0$.
The general solution of the Painlevé VI equation with (86) was discovered by Painlevé himself[3]. By introducing the function

$$\omega = \omega(\tau) = \int_\infty^{y_1} \frac{d\,y}{\sqrt{y\,(y-1)(y-\tau)}}$$

the Painlevé VI for y_1 with (86) reduces to the hypergeometric differential equation

$$\tau\,(1-\tau)\,\omega''(\tau) + (1 - 2\,\tau)\,\omega'(\tau) - \frac{\omega(\tau)}{4} = 0 \tag{87}$$

for ω. Since y_1 is the inverse function of ω, we get

$$y_1(\tau) = \wp(\omega;\tau) = \wp(\kappa_1\,\omega_1(\tau) + \kappa_2\,\omega_2(\tau);\tau)$$

where $\omega_1(\tau)$, $\omega_2(\tau)$ are two linear independent solutions of (87).
As a result of this we get the whole 2-parameter- family of solutions for the mean curvature function H and the metric. Therefore abbreviating $\wp(\omega(\tau),\tau)$ to $\phi(\tau)$:

$$\begin{aligned} e^{u(w+\bar{w})} &= -\frac{4\,\phi(\tau)\,(s\,\phi(\tau)-1)^2}{s^2\,(s-1)^2\,(\frac{d}{ds}\phi(\tau))^2} \\ H(w+\bar{w}) &= -2\frac{(s-1)s^2}{\phi(\tau)(\phi(\tau)-1)(s\,\phi(\tau)-1)}\frac{d}{ds}\phi(\tau))^2 \end{aligned} \qquad \text{with } \tau = \frac{1}{s} = e^{-4(w+\bar{w})}. \tag{88}$$

[3] see for example [O]

Because of (88), geometrical solutions ϕ must have the property

$$\phi(\tau) < 0.$$

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