ISOMETRIC IMMERSIONS OF SPACE FORMS AND SOLITON THEORY

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1. Introduction

The study of isometric immersions of space forms into space forms is a classical problem of differential geometry. In its simplest form it arises as the study of surfaces in 3-space of constant (non-zero) Gaussian curvature. In this case the integrability condition reduces to the sin resp. sinh-Gordon equations. Due to the complicated structure of these equations one focused over the past decades mainly on non-existence results rather then the construction of explicit examples, the most famous one being a result by Hilbert [14] that there are no complete surfaces of negative curvature in Euclidean 3-space. A slightly different spirit was present in the work of Bianchi and Darboux (see also [3]) where one finds many examples, some of which probably have been forgotten over time and only recently been rediscovered in the framework of what might be called *soliton geometry.* Having its origin in the explicit description of all tori of constant mean curvature [20, 6] the finite gap integration scheme can also be applied to obtain explicit parametrizations of a large class of surfaces of constant Gaussian curvature [16, 15, 5, 4].

In higher dimensions the situation is quite similar: there are a number of interesting non-existence results [21, 9, 17, 18, 19, 13] for isometric immersions $f : M^m(c) \to \widetilde{M}^n(\tilde{c})$ between space forms M of dimension m, curvature c, and space forms $\widetilde{M}^n(\tilde{c})$ of dimension n, curvature $\tilde{c}$, but no general scheme to construct such immersions. The integrability conditions for such an immersion, the Gauss, Codazzi and Ricci equations, are sometimes referred to as a generalized sin-Gordon system. Certain aspects of its inverse scattering theory [1, 24] as well as solutions obtained by Bäcklund transformations (which generate a finite dimensional solution space starting from a known, *trivial*, solution) [23] have recently been studied.

On the other hand, exterior differential calculus and the Cartan-Kähler theory [12] leads to a description of the space of local *real analytic* isometric immersions of space forms (in specific codimensions) in terms of finitely many functions of a single variable. In particular this shows that this space is infinite dimesional.

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In this paper we will develop the finite gap integration theory for the integrability conditions for isometric immersions of space forms. By this we mean that we will construct a hierarchy of commuting *finite dimensional* ODE in Lax form whose solutions will give local real analytic isometric immersions of space forms. We call those isometric immersions of *finite type*. The flows evolve on a certain loop Lie algebra and can, at least in principle, be integrated by theta functions on the Jacobian of an algebraic curve, the *spectral curve* of the Lax flows. In this sense the isometric immersion equations for space forms are a completely integrable system and thus can be regarded as a higher dimesional (w.r.t. the flow variables) soliton equation. The explicit algebraic integration so far has only be carried out in detail in the case of surfaces in 3-space [15, 5] and the higher dimensional case would require substantial more work on the description of Riemann surfaces with two (commuting) involutions and certain reality conditions which will be done elsewhere. The initial condition for the Lax flows is a matrix valued function of one variable so that our solution space has at least functionally the correct dimension given by Cartan-Kähler theory and we expect that an appropiate closure of the finite type solutions will in fact account for all the local real analytic isometric immersions of space forms.

In order to avoid degenerate cases we will make the following assumptions on our isometric immersions $f : M^m(c) \to \widetilde{M}(\tilde{c})$ throughout the paper :

(i) $c \neq 0 \neq \tilde{c}, \quad c \neq \tilde{c}$, and
(ii) the normal bundle of $f : M^m(c) \to \widetilde{M}(\tilde{c})$ is flat.

Note that for $c < \tilde{c}$ and $n = 2m-1$ the second assumption is always satisfied [17] and, in this case, there are no (local) isometric immersions for $n \leq 2m-2$ [21]. It is a classical result due to Hilbert [14, 22] that there are no complete isometric immersions $M^2(c) \to \widetilde{M}^3(\tilde{c})$ if $c < \tilde{c}$, $c < 0$ and it is anybodies guess that this result extends to complete isometric immersions $M^m(c) \to \widetilde{M}^{2m-1}(\tilde{c})$ for $c < \tilde{c}$, $c < 0$. Note that for $c = 0$ one always has the Clifford tori and $c > 0$ cannot occur due to the fact that such immersions induce global Chebycheff coordinates [17, 19]. In contrast, in the case $c > \tilde{c}$ one always has the totally umbilical hypersurfaces. If $n \leq 2m-1$ and the immersion has no umbilic points [18] then $n = 2m-1$ and the normal bundle is again flat. Generally, flatness of the normal bundle is a necessary assumption in higher codimensions to guarantee that those isometric immersions are not too flabby to arise from completely integrable systems [11]. The case $c = \tilde{c}$ is treated exhaustively in [10].

First we discuss how the structural equations for an isometric immersion between space forms can be rewritten as a zero curvature condition involving an auxiliary (*spectral*) parameter. Thus each such isometric immersion is part of a 1-parameter family of isometric immersions and the deformation parameter turns out to be related to the induced curvature. We then express this fact as the flatness condition on a loop algebra valued 1-form with an algebraic constraint. Such a reformulation of the equations at hand is reminiscent in the theory of integrable systems and will be the starting point for their integration. To carry this through we had to slightly modify AKS or R-matrix theory [7], since our equations could not be treated inside the standard setup. This modification was geometrically rooted since the Lax flows so obtained yield framings of the isometric immersions which parallelize the normal bundle and thus are adapted to the geometry of the situation.

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2. Associated family of isometric immersions

Since our discussion will be local, we may (after scaling) assume $\widetilde{M}^n(\tilde{c})$ to be either the Euclidean sphere $S^n \subset \mathbb{R}^{n+1}$ of curvature $\tilde{c} = 1$ or hyperbolic space $H^n \subset \mathbb{R}^{n+1}_1$ of curvature $\tilde{c} = -1$ realized as one sheet of the hyperboloid in Lorentz space. We denote by S either of these two standard spaces.

Let $f : M^m(c) \to S$ be an isometric immersion with adapted framing $F : M^m(c) \to \mathbf{SO}_\epsilon(n+1)$, i.e., $F_0 = f, F_1, \ldots, F_m$ are tangential and $F_{m+1}, \ldots, F_n$ are normal, and denote by

$$A = F^{-1} dF$$

the pull back of the left Maurer-Cartan form of $\mathbf{SO}_\epsilon(n+1)$ by F. (Here and in the sequel $\epsilon = \tilde{c} = \pm 1$ and indicates whether one deals with the Euclidean or Lorentzian version of the corresponding object). Then A is an $\mathbf{so}_\epsilon(n+1)$-valued 1-form and since F is adapted,

$$A = \begin{pmatrix} 0 & -\epsilon\theta^T & 0 \\ \theta & \omega & \beta \\ 0 & -\beta^T & \eta \end{pmatrix}, \tag{2.1}$$

where θ is an $\mathbb{R}^m$-valued 1-form on $M^m(c)$ (the dual frame to $F_1, \ldots, F_m$), ω is an $\mathbf{so}(m)$-valued 1-form (the Levi-Civita connection on $M^m(c)$), η is an $\mathbf{so}(n-m)$-valued 1-form (the connection in the normal bundle) and β is the 2^{nd} fundamental form for f. The integrability conditions for the existence of such a framing F, the Maurer-Cartan equations

$$dA + A \wedge A = 0\,, \tag{2.2}$$

unravel to the fundamental equations for the isometric immersion f, the structure equation and the Gauss, Codazzi and Ricci equations

$$d\theta + \omega \wedge \theta = 0\,, \tag{2.3a}$$

$$d\omega + \omega \wedge \omega - \epsilon\theta \wedge \theta^T - \beta \wedge \beta^T = 0\,, \tag{2.3b}$$

$$d\beta + \omega \wedge \beta + \beta \wedge \eta = 0\,, \tag{2.3c}$$

$$d\eta + \eta \wedge \eta - \beta^T \wedge \beta = 0\,. \tag{2.3d}$$

In addition one has that the induced metric has curvature c,

$$d\omega + \omega \wedge \omega = c\,\theta \wedge \theta^T\,, \tag{2.4}$$

and that the normal bundle is flat,

$$d\eta + \eta \wedge \eta = 0\,. \tag{2.5}$$

The starting point in soliton theory for integrating the equations at hand is their reformulation as a zero curvature condition (Maurer-Cartan equation) involving some auxiliary (spectral) parameter. Let A be an $\mathbf{so}_\epsilon(n+1)$-valued 1-form on $M^m(c)$ of the form (2.1). We define a family of $\mathbf{so}_\epsilon(n+1,\mathbb{C})$-valued 1-forms parametrized by $\lambda \in \mathbb{C}^*$ by

$$\widetilde{A}^\lambda = \begin{pmatrix} 0 & \frac{-\epsilon\sqrt{\epsilon c}}{2}(\lambda+\lambda^{-1})\theta^T & 0 \\ \frac{\sqrt{\epsilon c}}{2}(\lambda+\lambda^{-1})\theta & \omega & \frac{\sqrt{\epsilon c}}{2\sqrt{1-\epsilon c}}(\lambda-\lambda^{-1})\beta \\ 0 & \frac{-\sqrt{\epsilon c}}{2\sqrt{1-\epsilon c}}(\lambda-\lambda^{-1})\beta^T & \eta \end{pmatrix} \tag{2.6}$$

Lemma 2.1. *A solves (2.3), (2.4), and (2.5) if and only if $\widetilde{A}^\lambda$ solves the Maurer-Cartan equation*

$$d\widetilde{A}^\lambda + \widetilde{A}^\lambda \wedge \widetilde{A}^\lambda = 0$$

for all λ (in a set with accumulation point) in $\mathbb{C}^*$.

Proof. Since (2.3a) to (2.3d) are equivalent to the Maurer-Cartan equation (2.2) we have to verify that $\widetilde{A}^\lambda$ solves (2.3a) to (2.3d). Since (2.3a) and (2.3c) are homogeneous in θ and β (and neither ω nor η involve λ) they are trivially satisfied by $\widetilde{A}^\lambda$. Also (2.3d) holds because of (2.5). Now (2.3b) and (2.4) imply

$$(c-\epsilon)\theta \wedge \theta^T = \beta \wedge \beta^T = 0$$

so that, using (2.4), we have

$$c\theta \wedge \theta^T - \frac{c}{2}(\lambda+\lambda^{-1})^2\theta \wedge \theta^T - \frac{\epsilon c}{4(1-\epsilon c)}(\lambda-\lambda^{-1})^2\beta \wedge \beta^T = 0\,,$$

which shows that also (2.3b) holds for $\widetilde{A}^\lambda$. To prove the converse we compute the entries in $d\widetilde{A}^\lambda + \widetilde{A}^\lambda \wedge \widetilde{A}^\lambda = 0$ and compare coefficients of equal powers of λ:

$$\begin{aligned}
&d(\frac{\sqrt{\epsilon c}}{2}(\lambda+\lambda^{-1})\theta) + \omega \wedge (\frac{\sqrt{\epsilon c}}{2}(\lambda+\lambda^{-1})\theta) = 0\\
&d\omega + \omega\wedge\omega - \frac{c}{4}(\lambda+\lambda^{-1})^2\theta\wedge\theta^T - \frac{\epsilon c}{4(1-\epsilon c)}(\lambda-\lambda^{-1})^2\beta\wedge\beta^T = 0\\
&d(\frac{\sqrt{\epsilon c}}{2\sqrt{1-\epsilon c}}(\lambda-\lambda^{-1})\beta) + \omega\wedge(\frac{\sqrt{\epsilon c}}{2\sqrt{1-\epsilon c}}(\lambda-\lambda^{-1})\beta) + (\frac{\sqrt{\epsilon c}}{2\sqrt{1-\epsilon c}}(\lambda-\lambda^{-1})\beta)\wedge\eta = 0\\
&d\eta + \eta\wedge\eta - \frac{c}{4(1-\epsilon c)}(\lambda-\lambda^{-1})^2\beta^T\wedge\beta = 0\,.
\end{aligned}$$

The first and third identity yield (2.3a) and (2.3c) and the last identity gives (2.5) and (2.3d). Expanding the second identity we obtain

$$d\omega+\omega\wedge\omega-\frac{c}{2}\theta\wedge\theta^T+\frac{\epsilon c}{2(1-\epsilon c)}\beta\wedge\beta^T-(\lambda^2+\lambda^{-2})(\frac{c}{4}\theta\wedge\theta^T+\frac{\epsilon c}{4(1-\epsilon c)}\beta\wedge\beta^T)=0$$

and thus

$$\begin{aligned}
(1-\epsilon c)\theta\wedge\theta^T + \epsilon\beta\wedge\beta^T &= 0\\
d\omega+\omega\wedge\omega &= c\theta\wedge\theta^T
\end{aligned}$$

which implies (2.4) and (2.3b). □

In the situation of Lemma 2.1 we can integrate $(F^\lambda)^{-1}dF^\lambda = \widetilde{A}^\lambda$ to a (complex) framing

$$F^\lambda : M^m \to \mathbf{SO}_\epsilon(n+1,\mathbb{C})$$

for each λ. To obtain a real valued framing $\widetilde{A}^\lambda$ has to take values in $\mathbf{so}_\epsilon(n+1)$. This is the case if and only if

(2.7)

	$S=S^n$	$S=H^n$
λ	c^λ	c^λ
real	$(0,1)$	$(-1,0)$
imaginary	$(-\infty,0)$	$(0,\infty)$
unitary	$(1,\infty)$	$(-\infty,-1)$

In either of these cases $F^\lambda : M^m \to \mathbf{SO}_\epsilon(n+1)$ is real and thus we obtain a family of isometric immersions with flat normal bundles (since (2.5) holds for each $\widetilde{A}^\lambda$)

$$f^\lambda = F_0^\lambda : M^m \to S \tag{2.8}$$

from the first column of F^λ. The curvature c^λ of the induced metric is given by (2.4) expressed in the coframe for f^λ:

$$d\omega+\omega\wedge\omega = \epsilon\frac{4}{(\lambda+\lambda^{-1})^2}\frac{\sqrt{c}}{2}(\lambda+\lambda^{-1})\theta\wedge\frac{\sqrt{c}}{2}(\lambda+\lambda^{-1})\theta^T\,,$$

and thus

$$c^\lambda = \epsilon \frac{4}{(\lambda + \lambda^{-1})^2}\,. \tag{2.9}$$

Depending on the domain of λ the induced curvature c^λ ranges over the intervals given in (2.7). Finally, for

$$\lambda_0 = \frac{1}{\sqrt{\epsilon c}}(1 + \sqrt{1 - \epsilon c}) \tag{2.10}$$

we have $\widetilde{A}^{\lambda_0} = A, f^{\lambda_0} = f$ and $c^{\lambda_0} = c$, and thus recover the original immersion. We summarize the discussion so far in the following

Lemma 2.2. *Let $f : M^m \to S$ be an isometric immersion with flat normal bundle, $F : M^m(c) \to \mathbf{SO}_\epsilon(n+1)$ an adapted framing with induced Maurer-Cartan form $A = F^{-1}dF$ and let $\widetilde{A}^\lambda$ be given by (2.6). Then $d\widetilde{A}^\lambda + \widetilde{A}^\lambda \wedge \widetilde{A}^\lambda = 0$ for all λ.*

If $\widetilde{A}^\lambda$ satisfies the reality conditions (2.7) and $\widetilde{F}^\lambda$ integrates $\widetilde{A}^\lambda$ then $\widetilde{F}^\lambda : M^m \to \mathbf{SO}_\epsilon(n+1)$ is an adapted framing for the isometric immersion $f^\lambda = F_0^\lambda : M^m(c^\lambda) \to S$ with flat normal bundle and induced curvature $c^\lambda = \epsilon\frac{4}{(\lambda+\lambda^{-1})^2}$. The original immersion f is recovered at $\lambda = \frac{1}{\sqrt{\epsilon c}}(1 + \sqrt{1 - \epsilon c})$.

Thus isometric immersions of space forms with flat normal bundle come naturally in 1-parameter familes which we call the *associated family*.

From Lemma 2.1 we also obtain the converse to Lemma 2.2:

Lemma 2.3. *Let*

$$\widetilde{A}^\lambda = \begin{pmatrix} 0 & \epsilon(\lambda + \lambda^{-1})\theta^T & 0 \\ (\lambda + \lambda^{-1})\theta & \omega & (\lambda - \lambda^{-1})\beta \\ 0 & -(\lambda - \lambda^{-1})\beta^T & \eta \end{pmatrix} \tag{2.11}$$

be a family of $\mathbf{so}_\epsilon(n+1)$-valued 1-forms on M^m, where the forms θ and β may be imaginary (to fulfill the reality conditions (2.7)), satisfying the Maurer-Cartan equation. If $\theta^T = (\theta^1, \ldots, \theta^m)$ are linearly independent then $F^\lambda : M^m \to \mathbf{SO}_\epsilon(n+1)$ integrating $(F^\lambda)^{-1}dF^\lambda = A^\lambda$ is an adapted framing for the isometric immersion $f^\lambda = F_0^\lambda : M^m(c^\lambda) \to S$ with induced metric $c^\lambda = \epsilon\frac{4}{(\lambda+\lambda^{-1})^2}$ and flat normal bundle.

The gist of this reformulation is that the construction of isometric immersions of space forms with flat normal bundle is equivalent to the construction of a certain family of $\mathbf{so}_\epsilon(n+1)$ valued 1-forms (2.11) satisfying the Maurer-Cartan equation. Notice that such a reformulation is well known in the theory of harmonic maps of Riemann surfaces into Lie groups and symmetric spaces [8].

3. Loop algebra formulation

In the previous section we discussed how the equations for an isometric immersion $f : M^m \to S$ with flat normal bundle can be written as the zero curvature condition for a family ("loop") of Lie algebra valued 1-forms (2.11). The proper setting for this are loop algebras. Let $\mathfrak{g} = \mathbf{so}_\epsilon(n+1)$ with complexification $\mathfrak{g}^{\mathbb{C}} = \mathbf{so}_\epsilon(n+1, \mathbb{C})$ and define the loop algebra

$$\Lambda\mathfrak{g}^{\mathbb{C}} = \{\xi : \mathbb{C}^* \to \mathfrak{g}^{\mathbb{C}}\,; \xi \text{ polynomial in } \lambda \text{ and } \lambda^{-1}\}$$

which is a (complex) Lie algebra under pointwise bracket. If C denotes either of $\mathbb{R}^*, i\mathbb{R}^*$ or S^1 we have the real subalgebras (c.f. also (2.7))

$$\Lambda\mathfrak{g} = \{\xi : C \to \mathfrak{g}\} \subset \Lambda\mathfrak{g}^{\mathbb{C}}$$

corresponding to the conjugations

$$\bar{\xi}(\lambda) = \overline{\xi(\bar{\lambda})}\,, \qquad \bar{\xi}(\lambda) = \overline{\xi(-\bar{\lambda})}\,, \qquad \bar{\xi}(\lambda) = \overline{\xi(1/\bar{\lambda})}\,. \tag{3.1}$$

Note that

$$\widetilde{A}^\lambda = \begin{pmatrix} 0 & \epsilon(\lambda+\lambda^{-1})\theta^T & 0 \\ (\lambda+\lambda^{-1})\theta & \omega & (\lambda-\lambda^{-1})\beta \\ 0 & -(\lambda-\lambda^{-1})\beta^T & \eta \end{pmatrix}$$

for $\lambda \in C$ can be regarded as a $\Lambda\mathfrak{g}$-valued 1-form with the following symmetries:

$$\widetilde{A}^{-\lambda} = Ad\,P\widetilde{A}^\lambda$$
$$\widetilde{A}^{1/\lambda} = Ad\,Q\widetilde{A}^\lambda$$

where

$$P = \left(\begin{array}{c|c|c} \text{-1} & & \\ \hline & 1 & \\ \hline & & \text{-1} \end{array}\right) \text{ and } Q = \left(\begin{array}{c|c|c} 1 & & \\ \hline & 1 & \\ \hline & & \text{-1} \end{array}\right).$$

This motivates to consider the following involutions on $\Lambda\mathfrak{g}$:

$$(\sigma\xi)(\lambda) = Ad\,P\xi(-\lambda)$$
$$(\tau\xi)(\lambda) = Ad\,Q\xi(1/\lambda)\,.$$

We let

$$\Lambda\mathfrak{g}_{\sigma,\tau} = \{\xi \in \Lambda\mathfrak{g}; \sigma\xi = \xi, \tau\xi = \xi\}$$

be the subalgebra fixed under σ and τ.

Lemma 3.1. *Let $\xi = \sum_{k\in\mathbb{Z}} \lambda^k \xi_k \in \Lambda\mathfrak{g}$. Then $\xi \in \Lambda\mathfrak{g}_{\sigma,\tau}$ if and only if $\xi_0 \in \mathfrak{g}^P \cap \mathfrak{g}^Q$, $\xi_{even} \in \mathfrak{g}^P$, $\xi_{odd} \in \mathfrak{g}^{-P}$ and $\xi_{-k} = Ad\,Q\xi_k$. (Here $\mathfrak{g}^{\pm P}, \mathfrak{g}^{\pm Q}$ denote the ± 1-eigenspaces of $Ad\,P$ resp. $Ad\,Q$ on $\mathfrak{g}$).*

Proof. This follows at once by comparing coefficients in $\sigma\xi = \tau\xi = \xi$. $\square$

For $d \in \mathbb{N}$ let

$$\Lambda_d = \{\xi \in \Lambda\mathfrak{g}_{\sigma,\tau};\, \xi = \sum_{|k|\le d} \lambda^k \xi_k\}$$

be the subspace of Laurent polynomial loops of degree at most d.

Corollary 3.1. *$\xi \in \Lambda_1$ if and only if*

$$\xi = \left(\begin{array}{c|c|c} 0 & \epsilon(\lambda+\lambda^{-1})a^T & 0 \\ \hline (\lambda+\lambda^{-1})a & A & (\lambda-\lambda^{-1})C \\ \hline 0 & -(\lambda-\lambda^{-1})C^T & B \end{array}\right)$$

Proof. This follows from Lemma 3.1 together with the fact that

$$\mathfrak{g}^P = \left\{\left(\begin{array}{c|c|c} * & & * \\ \hline & * & \\ \hline * & & * \end{array}\right)\right\}, \quad \mathfrak{g}^{-P} = \left\{\left(\begin{array}{c|c|c} & * & \\ \hline * & & * \\ \hline & * & \end{array}\right)\right\} \tag{3.2}$$

$$\mathfrak{g}^Q = \left\{\left(\begin{array}{c|c|c} * & * & \\ \hline * & * & \\ \hline & & * \end{array}\right)\right\}, \quad \mathfrak{g}^{-Q} = \left\{\left(\begin{array}{c|c|c} & & * \\ \hline & & * \\ \hline * & * & \end{array}\right)\right\} \tag{3.3}$$

□

Corollary 3.1 together with the reformulation of the isometric immersion equations in Lemma 2.2 and Lemma 2.3 now yield

Corollary 3.2. *There is a natural correspondence between isometric immersions $f : M^m \to S$ of space forms with flat normal bundle and Λ_1-valued 1-forms $\tilde{A} : TM \to \Lambda_1$ satisfying the Maurer-Cartan equation (and whose first row $\tilde{A}_{0,-} = (0, \epsilon(\lambda+\lambda^{-1})\theta^T, 0)$ has $\theta^T = (\theta^1, \dots, \theta^m)$ linearly independent).*

The construction of flat loop algebra valued 1-forms satisfying an algebraic constraint (i.e. taking values in Λ_1) is reminiscent in the theory of soliton equations: they appear as solutions to certain completely integrable Lax-type equations [7].

4. Integration of the isometric immersion equations for space forms

We are now going to discuss a recipe for the construction of Λ_1-valued flat 1-forms on $\mathbb{R}^m$ from finite dimensional commuting Lax flows. Following existing nomenclature we will call the isometric immersions so obtained *finite type isometric immersions.* Such immersions will be real analytic by construction (in fact given by theta functions) and thus cannot account for all isometric immersions of space forms. Our solutions will be parametrized by a finite number of functions in one variable which relates to well known results obtained by Cartan-Kähler-theory [12]. We begin by recalling some facts about the integration of certain Lax equations on Lie algebras. Let $\mathcal{G}$ be a Lie algebra (in our case a loop algebra) which has a vector space direct sum decomposition

$$\mathcal{G} = \mathcal{P} \oplus \mathcal{A} \oplus \mathcal{M}$$

with

$$\mathcal{K} = \mathcal{A} \oplus \mathcal{P}, \qquad \mathcal{B} = \mathcal{A} \oplus \mathcal{M} \qquad \text{and } \ \mathcal{A}$$

Lie subalgebras and commutation relations

$$[\mathcal{A}, \mathcal{P}] \subseteq \mathcal{P}, \qquad [\mathcal{A}, \mathcal{M}] \subseteq \mathcal{M}\,.$$

Denote the corresponding projections by $\pi_{\mathcal{P}}$, $\pi_{\mathcal{A}}$ and $\pi_{\mathcal{M}}$. An ad-equivariant vector field on $\mathcal{G}$ is a map

$$V : \mathcal{G} \to \mathcal{G}$$

satisfying

$$d_\xi V[\xi, \eta] = [V(\xi), \eta] \tag{4.1}$$

for $\xi, \eta \in \mathcal{G}$. Natural examples of such maps are the gradients (with respect to an invariant inner product) of ad-invariant functions on $\mathcal{G}$. Notice that (4.1) implies that two ad-equivariant vector fields $V, \widetilde{V} : \mathcal{G} \to \mathcal{G}$ commute pointwisely, i.e., that

$$[V(\xi), \widetilde{V}(\xi)] = 0 \tag{4.2}$$

for $\xi \in \mathcal{G}$. Given an ad-equivariant vector field V on $\mathcal{G}$ we define a vector field X^V on $\mathcal{K}$ by

$$X^V(\xi) = [\xi, \pi_{\mathcal{P}} V(\xi)], \quad \xi \in \mathcal{K}\,.$$

In order to make sense of the derivative of a map on $\mathcal{G}$ without specifying a topology we assume that the images under the map of finite dimensional (vector) subspaces are contained in finite dimensional (vector) subspaces. Note that if V has this property so automatically has X^V.

Lemma 4.1. *Let $V, \widetilde{V} : \mathcal{G} \to \mathcal{G}$ be ad-equivariant vector fields. Then, for $\xi \in \mathcal{K}$,*

$$[X^V, X^{\widetilde{V}}]_{C^\infty}(\xi) = [\xi, \pi_{\mathcal{A}}[\pi_{\mathcal{P}} V(\xi), \pi_{\mathcal{P}} \widetilde{V}(\xi)]]\,.$$

The proof consists of a straightforward calculation using (4.1) and (4.2). Observe that in the case $\mathcal{A} = 0$, i.e., when $\mathcal{G} = \mathcal{K} \oplus \mathcal{B}$ is the (vector space) direct sum of two Lie subalgebras, the vector fields X^V and $X^{\widetilde{V}}$ commute, and one is in the realm of standard AKS resp. R-matrix theory [2, 7].

Lemma 4.2. *Let $V_1, \dots, V_m$ be ad-equivariant vector fields on $\mathcal{G}$ and assume that for all $\xi \in \mathcal{K}$*

$$\pi_{\mathcal{A}}[\pi_{\mathcal{P}} V_i(\xi), \pi_{\mathcal{P}} V_j(\xi)] = 0\,. \tag{4.3}$$

Then the system of ODE

$$d\xi = \sum_{i=1}^{m} X^{V_i}(\xi) dx^i = \sum_{i=1}^{m} [\xi, \pi_{\mathcal{P}} V_i(\xi)] dx^i\,, \quad \xi(0) \in \mathcal{K}$$

has a unique (local) solution $\xi : U \subset \mathbb{R}^m \to \mathcal{K}$, and the $\mathcal{P}$-valued 1-form on U,

$$\widetilde{A} = \sum_{i=1}^{m} \pi_{\mathcal{P}} V_i(\xi) dx^i\,,$$

satisfies the Mauer-Cartan equation

$$d\widetilde{A} + \frac{1}{2}[\widetilde{A} \wedge \widetilde{A}] = 0\,.$$

Proof. By Lemma 4.1 the vector fields X^{V_i} on $\mathcal{K}$ commute. Thus the system

$$d\xi = \sum_{i=1}^{m} X^{V_i}(\xi) dx^i$$

is well-defined and so has a unique solution to any initial condition $\xi(0) \in \mathcal{K}$. The final statement follows from an analagous calculation as in the proof of Lemma 4.1 which gives

$$d\widetilde{A} + \frac{1}{2}[\widetilde{A} \wedge \widetilde{A}] = \sum_{i,j=1}^{m} \pi_{\mathcal{A}}[\pi_{\mathcal{P}} V_i(\xi), \pi_{\mathcal{P}} V_j(\xi)] dx^i \wedge dx^j = 0\,.$$

□

Remark. The previous two Lemmas are simple modifications of standard results in AKS resp. R-matrix theory. Even though one cannot expect the commutativity conditions (4.3) to hold for general ad-equivariant vector fields there are interesting geometric applications where they do in fact hold. One of them are the isometric immersion equations (Corollary 3.3), which do not fit into the standard scheme, but can be treated with this more general setup. A more detailed study of the abstract general situation will perhaps be done elsewhere.

To apply the above considerations to the integration of the isometric immersion equations we first work in the complex setup and put

$$\mathcal{G} = \Lambda\mathfrak{g}^{\mathbb{C}}_{\sigma} = \{\xi \in \Lambda\mathfrak{g}^{\mathbb{C}}; \sigma\xi = \xi\}$$

and

$$\mathcal{K} = \Lambda\mathfrak{g}^{\mathbb{C}}_{\sigma,\tau} = \{\xi \in \Lambda\mathfrak{g}^{\mathbb{C}}; \sigma\xi = \tau\xi = \xi\}\,.$$

A natural vector space complement $\mathcal{M}$ to $\mathcal{K}$ in $\Lambda\mathfrak{g}^{\mathbb{C}}_{\sigma}$ is given as follows: note that

$$\mathcal{G} = \mathcal{G}_- \oplus \mathcal{G}_0 \oplus \mathcal{G}_+ \tag{4.4}$$

with

$$\mathcal{G}_{\pm} = \{\xi \in \Lambda\mathfrak{g}_\sigma^{\mathbb{C}}; \xi = \sum_{k><0} \lambda^k \xi_k\}, \qquad \mathcal{G}_0 = (\mathfrak{g}^{\mathbb{C}})^P .$$

Given $\xi \in \Lambda\mathfrak{g}_\sigma^{\mathbb{C}}$ we write

$$\xi = \xi_- + \xi_0 + \xi_+$$

according to (4.4) and decompose

$$\xi = (\xi_- + \tau\xi_- + \frac{\xi_0 + \tau\xi_0}{2}) + (\xi_+ - \tau\xi_- + \frac{\xi_0 - \tau\xi_0}{2}) \tag{4.5}$$

to obtain the factorization

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{M}$$

where

$$\mathcal{M} = (\mathfrak{g}^P \cap \mathfrak{g}^{-Q})^{\mathbb{C}} \oplus \mathcal{G}_+ .$$

Unfortunately $\mathcal{M}$ is not a Lie subalgebra since

$$0 \neq [\mathfrak{g}^P \cap \mathfrak{g}^{-Q}, \mathfrak{g}^P \cap \mathfrak{g}^{-Q}] \subseteq \mathfrak{g}^P \cap \mathfrak{g}^Q .$$

In fact,

$$\mathfrak{g}^P \cap \mathfrak{g}^Q = \mathbf{so}_\epsilon(1+m) \oplus \mathbf{so}(n-m) =: \mathfrak{a}_1 \oplus \mathfrak{a}_2 = \left\{ \left(\begin{array}{c|c|c} & & \\ \hline & * & \\ \hline & & * \end{array} \right) \right\}$$

and

$$[\mathfrak{g}^P \cap \mathfrak{g}^{-Q}, \mathfrak{g}^P \cap \mathfrak{g}^{-Q}] \subseteq \mathfrak{a}_2 .$$

Thus, setting

$$\mathcal{A} = \mathfrak{a}_2^{\mathbb{C}} = \left\{ \left(\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & & * \end{array} \right) \right\},$$

$$\mathcal{P} = \{\xi \in \mathcal{K}\,;\, \xi_0 \in \mathfrak{a}_1^{\mathbb{C}}\},$$

we arrive at

$$\mathcal{G} = \mathcal{P} \oplus \mathcal{A} \oplus \mathcal{M}$$

with

$$\mathcal{K} = \mathcal{P} \oplus \mathcal{A}, \qquad \mathcal{B} = \mathcal{A} \oplus \mathcal{M} \qquad \text{and} \;\; \mathcal{A}$$

Lie subalgebras satisfying the commutation relations

$$[\mathcal{A}, \mathcal{P}] \subseteq \mathcal{P}, \qquad [\mathcal{A}, \mathcal{M}] \subseteq \mathcal{M} .$$

Finally we introduce the ad-equivariant vector fields relevant to our situation.

Lemma 4.3. *(i) The map $V : \mathcal{G} \to \mathcal{G}$ defined by*

$$V(\xi) = \lambda^{2k}\xi^{2\ell-1}, \qquad k \in \mathbb{Z}, \ell \in \mathbb{N}$$

is an ad-equivariant vector field on $\mathcal{G}$.

(ii) Let $d \in \mathbb{N}$ be odd and define $V_\ell(\xi) = \lambda^{d(2\ell-1)-1}\xi^{2\ell-1}, \ell \in \mathbb{N}$. Then the corresponding vector fields on $\mathcal{K}$

$$X^{V_\ell}(\xi) = [\xi, \pi_{\mathcal{P}} V_\ell(\xi)]\,,$$

are tangential to the finite dimensionsal subspace ${\Lambda_d}^{\mathbb{C}} = \{\xi \in \mathcal{K}\,;\, \xi = \sum_{|k|\le d} \lambda^k \xi_k\,\} \subset \mathcal{K}$ and

$$\pi_{\mathcal{P}} V_\ell(\xi) \in \Lambda_1^{\mathbb{C}}\,.$$

(iii) $[X^{V_i}, X^{V_j}]_{C^\infty} = 0$, i.e., the vector fields X^{V_ℓ} commute.

(iv) (Reality conditions) If $\xi \in \Lambda_d$ is real (w.r.t. any of the three reality conditions (3.1) $\lambda \in \mathbb{R}^*, i\mathbb{R}^*$ or S^1) then also $\pi_{\mathcal{P}} V_\ell(\xi) \in \Lambda_1$ is real (and thus the vector fields X^{V_ℓ} are real, i.e., remain tangent to Λ_d).

Proof. (i) Since $\xi \in \mathcal{G} = \Lambda\mathfrak{g}_\sigma^{\mathbb{C}}$ we have

$$\begin{aligned}\sigma V(\xi)(\lambda) &= (-\lambda)^{2k} AdP\xi(-\lambda)^{2\ell-1} = \lambda^{2k}(AdP\xi(-\lambda))^{2\ell-1}\\ &= \lambda^{2k}\xi(\lambda)^{2\ell-1} = V(\xi)(\lambda)\end{aligned}$$

so that $V : \mathcal{G} \to \mathcal{G}$ is well-defined. To verify (4.1) we simply compute

$$d_\xi V[\xi, \eta] = \lambda^{2k} \sum_{i+j=2\ell-2} \xi^i[\xi, \eta]\xi^j = \lambda^{2k}(\xi^{2\ell-1}\eta - \eta\xi^{2\ell-1}) = [V(\xi), \eta]\,.$$

(ii) Let $\xi = \sum_{|j|\le d} \lambda^j \xi_j \in \Lambda_d^{\mathbb{C}}$ then

$$V_\ell(\xi) = \lambda^{d(2\ell-1)-1}\xi^{2\ell-1} = \lambda^{-1}\xi_{-d}^{2\ell-1} + \sum_{j=0}^{2d(2\ell-1)-1} \lambda^j \tilde{\eta}_j\,,$$

where

$$\tilde{\eta}_0 = \sum_{k=0}^{2l-2} (\xi_{-d})^k \xi_{-d+1} (\xi_{-d})^{2l-2-k}\,. \tag{4.6}$$

Thus, by (4.5),

$$\pi_{\mathcal{P}} V_\ell(\xi) = \lambda^{-1}\xi_{-d}^{2\ell-1} + \eta_0 + \lambda AdQ\xi_{-d}^{2\ell-1} \in \Lambda_1^{\mathbb{C}} \tag{4.7}$$

with $\eta_0 = \pi_{\mathfrak{a}_1^{\mathbb{C}}} \frac{\tilde{\eta}_0 + AdQ\tilde{\eta}_0}{2} \in \mathfrak{a}_1^{\mathbb{C}}$. From this it is clear that

$$X^{V_\ell}(\xi) = [\xi, \pi_{\mathcal{P}} V_\ell(\xi)] \in \Lambda_d^{\mathbb{C}}\,,$$

i.e., X^{V_ℓ} is tangential to $\Lambda_d^{\mathbb{C}}$.

(iii) Due to Lemma 4.1 we have to verify that for $\xi \in \Lambda_d^{\mathbb{C}}$

$$\pi_{\mathcal{A}}[\pi_{\mathcal{P}} V_i(\xi), \pi_{\mathcal{P}} V_j(\xi)] = 0\,.$$

From (4.7) and the fact that $\eta_0 \in \mathfrak{a}_1{}^{\mathbb{C}}$ it suffices to show that

$$\pi_{\mathcal{A}}[\xi_{-d}^{2i-1}, AdQ\xi_{-d}^{2j-1}] = 0\,.$$

But this last is seen from the more general fact that

$$\pi_{\mathcal{A}}[X, Y] = 0$$

for $X, Y \in (\mathfrak{g}^{-P})^{\mathbb{C}}$ implies

$$\pi_{\mathcal{A}}[X, AdQ\, Y] = 0\,,$$

which follows at once from (3.2) and the specific form of Q.

(iv) If λ is real then the statement is obvious. In case λ is purely imaginary $\xi = \sum_{|k|\le d} \lambda^k \xi_k \in \Lambda_d$ is equivalent to $\overline{\xi_k} = (-1)^k \xi_k$. From this and (4.7), (4.6) it is clear that $\pi_{\mathcal{P}} V_\ell(\xi)$ is real. Finally, if λ is unitary the reality condition (3.1) for $\xi \in \Lambda_d$ gives $\xi_{-k} = \overline{\xi_k} = AdQ\xi_k$ so that by (4.7), (4.6) we again conclude that $\pi_{\mathcal{P}} V_\ell(\xi)$ is real. □

Putting together the above discussion with the results in the previous sections we obtain a recipe for the construction of (local) isometric immersions of space forms into S^n and H^n from a hierarchy of finite dimensional ODE.

Theorem 4.1. *Let $d \in \mathbb{N}$ be odd, $m := [\frac{n+1}{2}]$ and recall the ad-equivariant vector fields $V_\ell(\xi) = \lambda^{d(2\ell-1)-1}\xi^{2\ell-1}, \ell = 1, \ldots, m$.*

(i) The system of ODE

$$d\xi = [\xi, \sum_{\ell=1}^{m} \pi_{\mathcal{P}} V_\ell(\xi) dx^\ell], \quad \xi(0) = \mathring{\xi} \in \Lambda_d \tag{4.8}$$

has a unique (local) solution $\xi : U \subset \mathbb{R}^m \to \Lambda_d$ for any initial condition $\mathring{\xi} \in \Lambda_d$.

(ii) If $\xi : U \to \Lambda_d$ is a solution to (4.8) then

$$\widetilde{A} = \sum_{\ell=1}^{m} \pi_{\mathcal{P}} V_\ell(\xi) dx^\ell$$

is a Λ_1-valued 1-form on $U \subset \mathbb{R}^m$ solving the (matrix) Maurer-Cartan equation

$$d\widetilde{A} + \widetilde{A} \wedge \widetilde{A} = 0\,.$$

Thus, integrating $(F^\lambda)^{-1} dF^\lambda = \widetilde{A}^\lambda, F^\lambda(0) = 1$, to a framing $F^\lambda : U \to \mathbf{SO}_\epsilon(n+1)$ gives a family of isometric immersion

$$f^\lambda = F_0^\lambda : U \to S = S^n \text{ or } H^n \tag{4.9}$$

with induced curvature $c^\lambda = \epsilon \frac{4}{(\lambda+\lambda^{-1})^2}$.

Proof. (i) follows from Lemma 4.2 together with Lemma 4.3. (ii) is a consequence of the description of isometric immersions as special loops of flat $\mathbf{so}_\epsilon(1+n)$-valued 1-forms, c.f. Lemma 2.3 and Corollary 3.2, together with Lemmas 4.3 and 4.2. $\square$

Remark. (i) Due to the fact that

$$\tilde{A} = \sum_{\ell=1}^{m} \pi_{\mathcal{P}} V_\ell(\xi) dx^\ell \in \left\{ \left(\begin{array}{c|c|c} & * & \\ \hline * & * & * \\ \hline & * & \end{array} \right) \right\}$$

the framing constructed from the Lax flows (4.8) parallelizes the normal bundle of the isometric immersion (4.9) which makes the flows geometrically adapted.

(ii) Since the ad-equivariant vector fields V_ℓ arise from (shifts of) gradients of ad-invariant functions on the finite dimensional Lie algebra $\mathbf{so}_\epsilon(1+n)$ only rank $= [\frac{n+1}{2}]$ many vector fields are independent, i.e., one cannot construct in this way isometric immersions of space forms of dimension exceeding $m = [\frac{n+1}{2}]$. This fact is also geometrically rooted: it is well-known that there are no isometric immersions $f : M^m(c) \to \tilde{M}^n(\tilde{c})$ in case $c < \tilde{c}$ and $n \leq 2m-2$.

(iii) Using Cartan-Kähler theory one can show that local real analytic isometric immersions $f : M^m(c) \to \tilde{M}^{2m-1}(\tilde{c})$ with $c < \tilde{c}$ depend on finitely many functions in one variable. Our scheme produces such real analytic isometric immersions from an arbitrary initial condition $\mathring{\xi} \in \Lambda_d$, $d \in \mathbb{N}$ odd, which is a certain $\mathbf{so}_\epsilon(1+n)$-valued Laurent polynomial (of arbitrary odd degree) in the variable λ. This indicates that an appropiate closure of the solutions so constructed should account for all real analytic solutions, but at present we have little idea how to adress this issue.

(iv) If the target space form is the sphere S^n and hence the corresponding Lie algebra $\mathfrak{g} = \mathbf{so}_\epsilon(1+n)$ compact, then there exists an ad-invariant positive definite (appropiately weighted L^2) inner product on $\Lambda\mathfrak{g}$ which makes the flows (4.8) evolve on Euclidean spheres in Λ_d and thus complete. But in general we cannot show that the flows (4.8) have global, i.e., defined on all of $\mathbb{R}^m$, solutions, in which case there still remains the question of whether the isometric immersion (4.9) is globally defined or whether its differential drops rank. This is of course related to the conjecture that there are no complete isometric immersions $f : M^m(c) \to \tilde{M}^{2m-1}(\tilde{c})$ for $c < \tilde{c}$, $c < 0$ which is a classical Theorem first proved by Hilbert in the case of surfaces in Euclidean 3-space.

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