# CONNECTION FORMULAS FOR ASYMPTOTICS OF THE FIFTH PAINLEVÉ TRANSCENDENT ON THE REAL AXIS. I

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Abstract. This is the first part of our work in which we collected the connection results for asymptotics as $t \to +0$ and $t \to +\infty$ of the fifth Painlevé transcendent. Keeping in mind already known and possible applications we study in fact a slightly more general object, that is a nonlinear system of ODEs governing isomonodromy deformations for the $2 \times 2$ matrix linear ODE $\frac{d}{d\lambda}\Psi = \left(\frac{t}{2}\sigma_3 + \frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1}\right)\Psi$. In this part we have considered the connection problem for the general isomonodromy deformations corresponding to solutions $y(t)$ of $P_5$ such as $y(t) \to -1$ as $t \to +\infty$.

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## 1. Introduction

In this work we are presenting the asymptotics ($t \to +0$ and $t \to +\infty$) for isomonodromy deformations of the linear ODE

$$\frac{d}{d\lambda}Y = A(\lambda,t)Y = \Big(\frac{t}{2}\sigma_3 + \frac{A_0(t)}{\lambda} + \frac{A_1(t)}{\lambda - 1}\Big)Y. \tag{1.1}$$

Recall the basic facts concerning the isomonodromy deformations of (1.1) [1]. Consider the parametrization of the matrices $A_k = A_k(t)$ $(k = 0,1)$:

$$A_0 = \begin{pmatrix} z + \frac{\Theta_0}{2} & -u(z+\Theta_0) \\ \frac{z}{u} & -z - \frac{\Theta_0}{2} \end{pmatrix}, \; A_1 = \begin{pmatrix} -z - \frac{\Theta_0+\Theta_\infty}{2} & uy\big(z + \frac{\Theta_0-\Theta_1+\Theta_\infty}{2}\big) \\ -\frac{1}{uy}\big(z + \frac{\Theta_0+\Theta_1+\Theta_\infty}{2}\big) & z + \frac{\Theta_0+\Theta_\infty}{2} \end{pmatrix}, \tag{1.2}$$

then the isomonodromy deformations of (1.1) are governed with the following system of non-linear ODEs:

$$t\frac{dy}{dt} = ty - 2z(y-1)^2 - (y-1)\Big(\frac{\Theta_0 - \Theta_1 + \Theta_\infty}{2}y - \frac{3\Theta_0 + \Theta_1 + \Theta_\infty}{2}\Big), \tag{1.3a}$$

$$t\frac{dz}{dt} = yz\big(z + \frac{\Theta_0 - \Theta_1 + \Theta_\infty}{2}\big) - \frac{1}{y}(z+\Theta_0)\big(z + \frac{\Theta_0 + \Theta_1 + \Theta_\infty}{2}\big), \tag{1.3b}$$

$$t\frac{d}{dt}\log u = -2z - \Theta_0 + y\big(z + \frac{\Theta_0 - \Theta_1 + \Theta_\infty}{2}\big) + \frac{1}{y}\big(z + \frac{\Theta_0 + \Theta_1 + \Theta_\infty}{2}\big), \tag{1.3c}$$

where $\Theta_\nu$ $(\nu = 0, 1, \infty)$ are complex constants considered as the parameters. Eliminating the function $z(t)$ from Eqs. (1.3a) and (1.3b) one finds that the function $y(t)$ satisfies the classical fifth Painlevé equation ($P_5$)

$$\frac{d^2y}{d^2t} = \Big(\frac{1}{2y} + \frac{1}{y-1}\Big)\Big(\frac{dy}{dt}\Big)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}\Big(\alpha y + \frac{\beta}{y}\Big) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}, \tag{1.4a}$$

$$\alpha = \frac{1}{2}\Big(\frac{\Theta_0 - \Theta_1 + \Theta_\infty}{2}\Big)^2,\; \beta = -\frac{1}{2}\Big(\frac{\Theta_0 - \Theta_1 - \Theta_\infty}{2}\Big)^2,\; \gamma = 1 - \Theta_0 - \Theta_1,\; \delta = -\frac{1}{2}. \tag{1.4b}$$

Under the scaling of (1.4) $t \to ct$ the coefficients $\gamma$ and $\delta$ are changing as $\gamma \to c\gamma$ and $\delta \to c^2\delta$. For nonvanishing $\delta$ the canonical choice is $\delta = -\frac{1}{2}$.

Together with the functions $y(t)$, $z(t)$ and $u(t)$ we are studying the so-called $\tau$-function [1, 2], which is associated with the Hamiltonian structure for the system (1.3a) and (1.3b). We could define it in the following way. Consider

$$\det A(\lambda,t) = -\frac{1}{2}\operatorname{tr} A^2(\lambda,t) = -\frac{t^2}{4} + \frac{tH_0}{\lambda} + \frac{tH_1}{\lambda-1} - \frac{\Theta_0^2}{4\lambda^2} - \frac{\Theta_1^2}{4(\lambda-1)^2},$$

where

$$H_p = \frac{1}{t}\operatorname*{Res}_{\lambda=p}\det A(\lambda,t) = -\frac{1}{2}\operatorname{tr}\sigma_3 A_p + (-1)^p \operatorname{tr} A_0 A_1, \quad p = 0,1$$

and $H_0 + H_1 = \Theta_\infty/2$. In terms of $H_1$ (or $H_0$) system (1.3a,b) can be written as Hamiltonian system with respect to the canonical variables $y$ and $\frac{z}{y}$. Now, the $\tau$-functions are defined as

$$\frac{d}{dt}\log\tau_k = H_k.$$

Following Jimbo [3] we consider the function

$$\zeta(t) = t\frac{d}{dt}\log\tau_1(t) + \frac{1}{2}(\Theta_0 + \Theta_\infty)t + \frac{1}{4}((\Theta_0 + \Theta_\infty)^2 - \Theta_1^2).$$

The function $\zeta(t)$ in terms of function $y(t)$ and $z(t)$ reads

$$\zeta(t) = \frac{1}{4}((\Theta_0 + \Theta_\infty)^2 - \Theta_1^2) - zt-$$
$$-\big(z - \frac{1}{y}\big(z + \frac{\Theta_0 + \Theta_1 + \Theta_\infty}{2}\big)\big)\big(z + \Theta_0 - y\big(z + \frac{\Theta_0 - \Theta_1 + \Theta_\infty}{2}\big)\big), \tag{1.5}$$
$$\frac{\partial\zeta}{\partial t} = \frac{d\zeta}{dt} = -z. \tag{1.6}$$

The $\tau$-function for $P_5$ is proved to be an important object in physical (correlators and partition functions [4]) and geometrical (mean curvature [5]) applications, in which the connection problem for its asymptotics arises in the natural way.

The original connection problem for asymptotics as $t \to t_0$ with that as $t \to t_1$ of the solution (1.3) we call as $(t_0, t_1)$-connection problem. In our case $\{t_0, t_1\} = \{+0, +\infty\}$. To any solution of (1.3) corresponds one and only one point at the manifold of monodromy data $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ (see Sec. 2 for definitions). Let us call the problem of parametrization (by explicit formulas) of coefficients of asymptotic expansions as $t \to t_0$ of $y(t)$, $z(t)$, and $u(t)$ by the points of $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ as $t_0$*-connection problem.* If the solution of the $t_0$-connection problem is given by invertible formulas we call it the *complete solution.* The master route for solving the $(t_0, t_1)$-connection problem in the framework of Isomonodromy Deformation Method [6] is to find the complete solutions for $t_0$- and $t_1$-connection problems.

The solution of 0-connection problem for system (1.3a,b) with $\Theta_0, \Theta_1 \notin Z$ were obtained by M. Jimbo [3]. The solution of $+\infty$- and $+\imath\infty$-connection problem for system (1.3a,b) with the special choice of the parameters

$$\Theta_0 = \Theta_1 = \Theta \quad \text{and } \Theta_\infty = 2n \in 2\mathbb{Z} \tag{1.7}$$

were studied by B. M. McCoy and Sh. Tang [7, 8, 9]. They applied these results to $(0, +\infty)$- and $(0, +\imath\infty)$-connection problems. The $(0, \imath\infty)$-connection problem for one-parameter solution of $P_5$ in the case (1.7) was studied by B. I. Suleimanov (see [6]).

We mention also the results concerning the complete asymptotic expansions as $t \to 0$ obtained by K. Takano [10] and as $t \to \pm\imath\infty$ by K. Yoshida and K. Takano [11, 12].

Thus, the $(0, \infty)$-connection problem is studied only in the case (1.7). In this case $P_5$ is reducible to the third Painlevé equation $P_3$ and the latter, in its turn, is reducible to the special case of $P_5$ with $\delta = 0$ [13]. For $P_5$ with $\delta = 0$ the $(0, \infty)$-connection problem has been considered in [14].

In our work we deal with $(+0, +\infty)$-connection problem for system (1.3) for general set of parameters $\Theta_0, \Theta_1, \Theta_\infty \in \mathbb{C}$. The difference with the works of M. Jimbo, B. M. McCoy and Sh. Tang [3, 7, 9] is, also, that we parametrized the multiplicative constant of integration in the function $u(t)$ (see (1.3c)) in terms of $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$. This constant of integration is needed in the problems where equation (1.1) is used as the model one in the investigation of asymptotics of more general systems of isomonodromy deformations associated with more complicated linear ODEs (see, for example, [15]). This constant could also appear in physical context (see, for example, [16]), where it represents the phase shift of similarity electromagnetic field in Maxwell–Bloch system with pumping. Note, that having a solution of $(0, \infty)$-connection problem for $y(t)$ and $z(t)$ one can not obtain the solution of the connection problem for $u(t)$ if conditions (1.7) violated.

Actually, this paper is the first part of our work where we present generic solution of $(+0, +\infty)$-connection problem, whereas exceptional and special cases including the connection problem for singular as $t \to +\infty$ solutions we consider in the second part of our work [17].

Organization of this paper is as follows. In Sec. 2 we define the manifold of monodromy data $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$. In Secs. 3 and 4 we present the solutions of $+0$- and $+\infty$-connection problems. In Secs. 5, 6, and Appendix A we give the derivation of the results presented in Secs. 3 and 4. In Appendix B we put in our results conditions (1.7) to compare them with the ones by B. M. McCoy and Sh. Tang [7, 9].

Let us explain more precisely how one can obtain from our results the solution of $(+\infty, +0)$- and $(+0, +\infty)$-connection problems.

Asymptotics of functions $y(t)$, $z(t)$, $u(t)$, and $\zeta(t)$ as $t \to +\infty$ are given in Sec. 3 by formulas (3.1) and (3.2). They include the complex parameters $\beta_0$, $\hat{v}$, and $\hat{u}$. Substitute them into formulas (3.3) to find the coordinates of corresponding point in $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$. Then use the formulas (4.6) of Sec. 4 to find the parameters $\sigma$, $r$, and $\hat{s}$ defining asymptotic expansions of the functions $y(t)$, $z(t)$, $u(t)$, and $\zeta(t)$ as $t \to +0$. If the parameters $\sigma$, $r$ and $\hat{s}$ satisfy the restrictions stated in Theorem 4.3 then the functions $y(t)$, $z(t)$, $u(t)$, and $\zeta(t)$ have asymptotic expansions as $t \to +0$ given by (4.1) and (4.2).

Inversely, to solve the $(+0, +\infty)$-connection problem find the parameters $\sigma$, $r$ and $\hat{s}$ defining the asymptotic expansions as $t \to +0$ for functions $y(t)$, $z(t)$, $u(t)$, and $\zeta(t)$ in the formulas (4.1) and (4.2). Insert these parameters into formulas (4.3) to obtain the coordinates of corresponding point in $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$. Then, formulas (3.5) yield the parameters $\beta_0$, $\hat{v}$, and $\hat{u}$. Check the conditions for them stated in Theorem 3.4. If they are valid then the asymptotics of functions $y(t)$, $z(t)$, $u(t)$, and $\zeta(t)$ are given by (3.1) and (3.2).

Now we run over the proofs of the Theorems presented in Secs. 3 and 4. In Theorems 3.1 and 4.1, the local asymptotics as $t \to +\infty$ and $t \to +0$ for the functions $y(t)$, $z(t)$ and $u(t)$ are stated. Theorem (4.1) was proved by K. Takano [10]. The result for $u(t)$ also could be easily derived from [10] and (1.3c). We believe that the local asymptotics as $t \to +\infty$ for the functions $y(t)$, $z(t)$ and $u(t)$ stated in Theorem 3.1 could also be derived from the general theorems of [11] after a suitable transformation like the way shown in [11, 12]. In our approach, the results stated in Theorems 3.1 and 4.1 are the consequence of justification scheme for the connection formulas in Isomonodromy Deformation Method. We discuss the scheme below.

The local asymptotics as $t \to +0$ for $\zeta(t)$ (Corollary of Sec. 4) is obtained by the direct substitution of the corresponding asymptotics of the functions $y(t)$ and $z(t)$ into (1.5). To obtain the asymptotic result as $t \to +\infty$ stated in Theorem 3.2 one has to obtain two more terms of asymptotic expansions of $y(t)$ and $z(t)$. We found them in Appendix A.

Theorems 3.3 and 4.3 presenting the $+\infty$- and $+0$-connection results are justified by the general scheme proposed in [18]. This scheme was discussed also in the recent work [19].

Briefly, this scheme could be explained as follows. Let $\mathcal{M} \in \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ satisfies the conditions mentioned in Theorem 3.4 (correspondingly, in Theorem 4.3). Calculating the asymptotic parameters by the formulas (3.5) (correspondingly, (4.6)) we construct the asymptotic formulas for $y(t)$, $z(t)$ and $u(t)$ via (3.1) (correspondingly, (4.1)). Then the derivation presented in Sec. 5 (correspondingly, Sec. 6) is the proof that the monodromy data $\mathcal{M}_{\rm as}$ of Eq. (1.1) with the coefficients $y = y_{\rm as}$, $z = z_{\rm as}$ and $u = u_{\rm as}$ differ from the initial ones $\mathcal{M}$ as

$$\|\mathcal{M} - \mathcal{M}_{\rm as}\| \le Ct^{-\varepsilon} \quad \text{as } t \to +\infty \text{ (or } O(t^{\varepsilon}) \text{ as } t \to +0),\ \varepsilon > 0. \tag{1.8}$$

Then as it follows from [18] there exists an exact solution of (1.3) $\{y(t), z(t), u(t)\}$ corresponding to initial $\mathcal{M}$ and $\{y(t), z(t), u(t)\} \to \{y_{\rm as}(t), z_{\rm as}(t), u_{\rm as}(t)\}$. The estimate (1.8) requires in this scheme the property of local uniformness, i.e., the estimate (1.8) holds for all $\tilde{\mathcal{M}}$ in some neighborhood of $\mathcal{M}$ and $C$ and $\varepsilon$ could be chosen just the same for all these $\tilde{\mathcal{M}}$. We miss here the tedious details of proof of the last property. The interested reader could find the details in analogous but simpler situation in [19].

In fact the existence of the independent proof for the local results of Theorem 4.1 provide us with the opportunity to understand the derivation of Secs. 6 as the proof that the monodromy data corresponding to the exact solution differ from that of presented by formulas (4.3) of Theorem 4.2 on the quantities of the order $o(1)$. As the monodromy data of the exact solution are in fact independent of $t$ they should coincide with the left-hand side of (4.3).

The proofs of the last theorems of Secs. 3 and 4 immediately follows from the previous theorems of these sections and general theory of isomonodromy deformation and require no special proof.

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## 2. The manifold of monodromy data

In this Section we define the standard objects [1] of the monodromy theory for equation (1.1).

There are three singular points in the right-hand side of equation (1.1): irregular one at infinity and two regular singularities at 0 and 1. Define the canonical solutions $Y_k = Y_k(\lambda)$ of (1.1) via their large $|\lambda|$ expansions:

$$Y_k(\lambda) \underset{|\lambda|\to\infty}{=} \left(I + \hat{Y}_\infty \frac{1}{\lambda} + O\left(\frac{1}{\lambda^2}\right)\right) \exp\left(\frac{\lambda t}{2}\sigma_3 - \frac{\Theta_\infty}{2} \ln \lambda \sigma_3\right), \quad \hat{Y}_\infty = \begin{pmatrix} -H_1 & * \\ * & H_1 \end{pmatrix}, \tag{2.1}$$

in the sectors

$$-\frac{\pi}{2} + \pi(k-2) < \arg \lambda + \arg t < \frac{3\pi}{2} + \pi(k-2), \quad , k = 1, 2, \ldots . \tag{2.2}$$

Hereinafter, we fix $\ln\lambda$ in a usual way $\ln\lambda = \ln|\lambda| + \imath \arg\lambda$ and consider $\arg t$ as a given number. Actually, throughout this paper $\arg t = 0$.

The canonical solutions are connected by the Stokes matrices

$$Y_{k+1}(\lambda) = Y_k(\lambda) S_k. \tag{2.3}$$

Using (2.1) one proves

$$Y_{k+2}(\lambda e^{2\pi\imath}) = Y_k(\lambda) e^{-\pi\imath\Theta_\infty\sigma_3}. \tag{2.4}$$

Formulas (2.3) and (2.4) yield

$$S_{k+2} = e^{\pi\imath\Theta_\infty\sigma_3} S_k e^{-\pi\imath\Theta_\infty\sigma_3} \tag{2.5}$$

Thus, one can determine all the Stokes matrices having two of them, say $S_1$ and $S_2$. It also follows from (1.1), (2.1), (2.2), and (2.3) that matrices $S_k$ are independent of $\lambda$ and have the following structure

$$S_{2k} = \begin{pmatrix} 1 & s_{2k} \\ 0 & 1 \end{pmatrix}, \quad S_{2k+1} = \begin{pmatrix} 1 & 0 \\ s_{2k+1} & 1 \end{pmatrix}, \quad k \in \mathbb{Z}. \tag{2.6}$$

The complex parameters $s_k$ ($k = 1, 2, \dots$) are called *Stokes multipliers.* The monodromy matrix at infinity point $M_k^\infty$ for function $Y_k$ is defined by equation

$$Y_k(\lambda e^{-2\pi\imath}) = Y_k(\lambda) M_k^\infty. \tag{2.7}$$

Using (2.3) and (2.4) one finds

$$M_k^\infty = S_k S_{k+1} e^{\pi\imath\Theta_\infty\sigma_3}. \tag{2.8}$$

In the following we set

$$M_2^\infty \equiv M^\infty = S_2 e^{\pi\imath\Theta_\infty\sigma_3} S_1.$$

All the others $M_k^\infty$'s can be expressed in terms of $M^\infty$ via the recurrence formula

$$M_{k+1}^\infty = S_k^{-1} M_k^\infty S_k. \tag{2.9}$$

To consider the behavior of the canonical solutions near the regular singularities 0 and 1 let us define a singlevalued branch of $Y_2$. Actually, it can be done by various ways. We find it convenient to restrict $Y_2$ in the domain $\overline{C}\backslash([0,1] \cup [1/2, \infty e^{\frac{-\imath\pi}{2}}])$. Being restricted into $\overline{C}\backslash([0,1] \cup [1/2, \infty e^{-\frac{\imath\pi}{2}}])$, $Y_2(\lambda)$ is a singlevalued analytical function of $\lambda$ with the following expansions in the neighborhoods of the regular singularities ($p = 0, 1$):

$$Y_2(\lambda) \underset{\lambda\to p}{=} \sum_{n=0}^{\infty} \hat{Y}_n^p (\lambda - p)^{n + \Theta_p\sigma_3/2} J E_p, \tag{2.10}$$

$$\hat{Y}_n^p = \hat{Y}_n^p(t, z, y, u, \Theta_0, \Theta_1, \Theta_\infty), \quad E_p = E_p(t, z, y, u, \Theta_0, \Theta_1, \Theta_\infty),$$

$$\det \hat{Y}_0^p = \det E_p = 1, \quad J = I \quad \text{for } \Theta_p \notin \mathbb{Z},$$

$$J = \begin{pmatrix} 1 & j_p^{\Theta_p} \ln(\lambda - p) \\ 0 & 1 \end{pmatrix} \text{ for } \Theta_p = 0, 1, 2 \dots, \; J = \begin{pmatrix} 1 & 0 \\ j_p^{\Theta_p} \ln(\lambda - p) & 1 \end{pmatrix} \text{ for } \Theta_p = 0, -1, -2 \dots$$

$$j_0^0 = z, \quad j_1^0 = z + \frac{\Theta_0 + \Theta_\infty}{2}, \tag{2.11}$$

$$(\hat{Y}_0^p)^{-1} A_p(t) \hat{Y}_0^p = \frac{\Theta_p}{2}\sigma_3 \quad \text{for } \Theta_p \neq 0, \quad (\hat{Y}_0^p)^{-1} A_p(t) \hat{Y}_0^p = \begin{pmatrix} 0 & j_p^0 \\ 0 & 0 \end{pmatrix} \quad \text{for } \Theta_p = 0.$$

Note that series (2.10) is convergent for $|\lambda - p| < 1$.

*Remark 2.1.* The functions $j_p^{\Theta_p}$ can be defined by recurrence relations for every $\Theta_p \in \mathbb{Z}$ by using $j_p^0$ (2.11) and Schlesinger transformations in such a way that the condition $j_p^{\Theta_p} = 0$ defines the special one-parameter solution of $P_5$ for $\Theta_p \in \mathbb{Z}$. In particular, one can find such solutions for $\Theta_0 = 0$ and/or $\Theta_1 = 0$ from (2.10) substituting $z = \Theta_0 = 0$ or $z + \frac{\Theta_0+\Theta_\infty}{2} = \Theta_1 = 0$ into (1.3a).

Using expansions (2.10) we can define the monodromy matrices at regular singularities as follows

$$M^p = E_p^{-1} e^{\pi\imath\Theta_p\sigma_3} \hat{J} E_p, \tag{2.12}$$

where $\hat{J} = I$ for $\Theta_p \notin \mathbb{Z}$, otherwise

$$\hat{J} = \begin{pmatrix} 1 & 2\pi\imath j_p^{\Theta_p} \\ 0 & 1 \end{pmatrix} \text{ for } \Theta_p = 0, 1, 2\ldots, \quad \hat{J} = \begin{pmatrix} 1 & 0 \\ 2\pi\imath j_p^{\Theta_p} & 1 \end{pmatrix} \text{ for } \Theta_p = 0, -1, -2\ldots$$

*Remark 2.2.* Although, the expansion (2.10) does not define $E_p$ uniquely, the matrices $M^p$ are defined properly.

The monodromy matrices are connected by the cyclic relation [3, 15]

$$M^\infty M^1 M^0 = I. \tag{2.13}$$

The matrix elements $m_{ij}^0$, $m_{ij}^1$, $(i, j = 1, 2)$, Stokes multipliers $s_k$, $k = 1, 2$, and parameters $\Theta_0, \Theta_1, \Theta_\infty \in \mathbb{C}$ are called the *monodromy data* of Eq.(1.1). The above definitions yield

$$\det M_0 = 1, \quad \det M_1 = 1, \quad \operatorname{tr} M_0 = 2\cos\pi\Theta_0, \quad \operatorname{tr} M_1 = 2\cos\pi\Theta_1, \quad (M_1 M_0)_{11} = e^{-\pi\imath\Theta_\infty}. \tag{2.14}$$

These equations in

$$\mathbb{C}^8 \ni (m_{11}^0, m_{12}^0, m_{21}^0, m_{22}^0, m_{11}^1, m_{12}^1, m_{21}^1, m_{22}^1) \tag{2.15}$$

for fixed $\Theta_0, \Theta_1, \Theta_\infty \in \mathbb{C}$ define the *manifold of monodromy data* $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$. In terms of $m_{ij}^p$ (2.14) reads as follows

$$\begin{gathered} m_{11}^0 m_{22}^0 - m_{12}^0 m_{21}^0 = 1, \quad m_{11}^1 m_{22}^1 - m_{12}^1 m_{21}^1 = 1, \\ m_{11}^0 + m_{22}^0 = 2\cos\pi\Theta_0, \; m_{11}^1 + m_{22}^1 = 2\cos\pi\Theta_1, \; m_{11}^1 m_{11}^0 + m_{12}^1 m_{21}^0 = e^{-\pi\imath\Theta_\infty}. \end{gathered} \tag{2.16}$$

Note that $\dim_{\mathbb{C}} \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty) = 3$. Given $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ one finds the Stokes multipliers via the relations (2.8) and (2.13). Thus $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ completely defines all the monodromy data.

Given point $(t, z, y, u, \Theta_0, \Theta_1, \Theta_\infty) \in \mathbb{C}^7$ defines equation (1.1) via (1.2). To solve the *direct monodromy problem* is to find the correspondence $(t, z, y, u, \Theta_0, \Theta_1, \Theta_\infty) \to$ (Eq. (1.1)) $\to \mathcal{M} \in \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$.

To find the inverse mapping $\{\mathcal{M}, t\} \to (t, z, y, u, \Theta_0, \Theta_1, \Theta_\infty)$, where $\mathcal{M} \in \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ and $t \in \mathbb{C}$ is to solve the *inverse monodromy problem.* One easily proves the following

**Proposition.** *For a given point $\mathcal{M} \in \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ and $t \in \mathbb{C}$ there exists no more than one analytical function $Y_2$ with the following properties:*

*1. There are only three singular points: 0, 1, and $\infty$ on $\overline{\mathbb{C}}$.*

*2. Asymptotic expansions of $Y_2$ in the neighborhoods of these points are of the forms (2.1), (2.2), and (2.10) with some $\hat{Y}_n^p$, $J$, and $E_p$.*

*3. The monodromy data of $Y_2$ are the coordinates of $\mathcal{M}$.*

Thus, if the monodromy problem is solvable its solution is unique. It is naturally to consider point $\mathcal{M} \in \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ as independent of $t$. In this case the solution of inverse monodromy problem, i.e., the functions $y = y(\mathcal{M}, t)$, $z = z(\mathcal{M}, t)$, and $u = u(\mathcal{M}, t)$, are called the *isomonodromy deformations* and satisfy system (1.3) [1].

Only in exceptional cases the direct and inverse monodromy problems could be solved explicitly. Thus, we are to apply the asymptotical methods. Say, considering the direct monodromy problem we have the asymptotics as $t \to t_0$ with some parameters. To calculate $\mathcal{M} \in \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ in terms of these parameters via the explicit formulas is to solve the $t_0$-connection problem. If the obtained formulas are invertible we say that $t_0$-connection problem is solved completely.

Finally, let us discuss the solvability of the inverse monodromy problem (it is evident that direct monodromy problem is always solvable). A condition for the monodromy data providing the solvability of the inverse monodromy problem for pure imaginary $t$ was obtained in [20]. The results presented in Sec. 4 guarantee the solvability of the inverse monodromy problem for sufficiently small $t \neq 0$ if $\mathcal{M}$ satisfy the conditions of Theorem 4.3. The results formulated in Sec. 3 show the solvability of the inverse monodromy problem for sufficiently large positive $t$ if $\mathcal{M}$ satisfy the conditions of Theorem 3.4. In [17] we describe those $\mathcal{M} \in \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ which correspond to the singular solutions of (1.3). These solutions have infinite sequence of poles $t_n$, $t_n \to +\infty$. For such pairs $\{\mathcal{M}, t_n\}$, the inverse monodromy problem is unsolvable.

## 3. Results as $t \to \infty$

**Theorem 3.1.** *Let $\beta_0 \in \mathbb{C}\backslash 0$, $|\Re\beta_0| < \frac{1}{4}$, $\hat{u}, \hat{v} \in \mathbb{C}\backslash 0$. Then there exists solution of system (1.3) with asymptotic expansion*

$$\begin{gathered} y = -1 + 4\sqrt{\frac{2}{t}} e^{-\frac{\pi \imath}{4}} \sqrt{\beta_0} \sin(\frac{t}{2} + \imath\beta_0 \ln t - \imath \ln \sqrt{\beta_0} + \imath \ln \hat{v}) + O(t^{-1+2|\Re\beta_0|}), \\ z = -\frac{t}{8} - \frac{1}{2}\sqrt{\frac{t}{2}} e^{-\frac{\pi \imath}{4}} \sqrt{\beta_0} \cos(\frac{t}{2} + \imath\beta_0 \ln t - \imath \ln \sqrt{\beta_0} + \imath \ln \hat{v}) + O(t^{2\Re|\beta_0|}), \\ u = \hat{u} e^{\frac{t}{2}} (1 + o(1)). \end{gathered} \tag{3.1}$$

*Hereinafter, branch of $\sqrt{\beta_0}$ is an arbitrary but the same in all the formulas.*

*Remark 3.1.* As it follows from the proof presented in Appendix A formulas (3.1) are valid for all $\beta_0 \in \mathbb{C}\backslash 0$ and $|\Re\beta_0| < \frac{1}{2}$. In the case of $\frac{1}{6} < |\Re\beta_0| < \frac{1}{2}$ it is natural to rewrite (3.1) in the following way
$\frac{1}{6} < \Re\beta_0 < \frac{1}{2}$

$$y = -1 + 2\sqrt{2} e^{\frac{\pi \imath}{4}} \hat{v} t^{\beta_0 - \frac{1}{2}} e^{-\imath\frac{t}{2}} + O(t^{-1+2|\Re\beta_0|}), \; z = -\frac{t}{8} - \frac{\hat{v}}{4\sqrt{2}} e^{-\frac{\pi \imath}{4}} t^{\beta_0 + \frac{1}{2}} e^{-\imath t 2} + O(t^{2\Re|\beta_0|}).$$

$-\frac{1}{2} < \Re\beta_0 < -\frac{1}{6}$

$$y = -1 - 2\sqrt{2}e^{\frac{\pi\imath}{4}}\frac{\beta_0}{\hat{v}}t^{-\beta_0-\frac{1}{2}}e^{\imath\frac{t}{2}} + O(t^{-1+2|\Re\beta_0|}),\ z = -\frac{t}{8} - \frac{\beta_0}{\hat{v}4\sqrt{2}}e^{-\frac{\pi\imath}{4}}t^{-\beta_0+\frac{1}{2}}e^{\imath t2} + O(t^{2\Re|\beta_0|}).$$

The requirement $|\Re\beta_0| < \frac{1}{4}$ originated from our derivation in Sec. 5 of the connection formulas (3.2). The cases $\beta_0 = 0$ and $\frac{1}{4} < |\Re\beta_0| \le \frac{1}{2}$ together with the special reductions $y(t) = \overline{y}(t)$ and $y\overline{y} = 1$ are considered in the second part of this work [17].

**Theorem 3.2.** *The function $\zeta(t)$ (see (1.5)) corresponding to the solution of system (1.3) defined by Theorem 3.1 is*

$$\zeta(t) = \frac{t^2}{16} + (\Theta_\infty + 2\Theta_0 + 2\imath\beta_0)\frac{t}{4} + \sqrt{\frac{t}{2}}e^{-\frac{\pi\imath}{4}}\sqrt{\beta_0}\sin(\frac{t}{2} + \imath\beta_0\ln t - \imath\ln\sqrt{\beta_0} + \imath\ln\hat{v}) + O(1). \tag{3.2}$$

*Remark 3.2.* By substituting (3.1) into the formula (1.5) one finds only the first two terms in (3.2). One could guess for the answer written as the next term of (3.2) by integrating of (1.6) formally. To obtain the result stated in (3.2) in the rigorous manner we found the next two terms in each of the expansions (3.1). In Appendix A the reader can find the formula for $O(1)$ in (3.2) (see (A.7), (A.8), and (A.9)).

**Theorem 3.3.** *The solution of equation (1.3) described in Theorem 3.1 define the isomonodromy deformation of (1.1) with the following monodromy data*

$$\begin{aligned}
m^0_{11} &= -\frac{\sqrt{2\pi}}{\hat{v}\Gamma(\beta_0)}\exp(\beta_0\ln 2 - \frac{\pi\imath\Theta_\infty}{2} + \frac{\pi\imath\beta_0}{2}), \quad m^0_{21} = \frac{1}{\hat{u}}\exp(\pi\imath\beta_0 - \frac{\pi\imath}{2} + 2\Theta_\infty\ln 2 - \pi\imath\Theta_\infty),\\
m^1_{11} &= \frac{\sqrt{2\pi}\hat{v}}{\Gamma(1-\beta_0)}\exp(\frac{\pi\imath}{2} - \beta_0\ln 2 - \frac{\pi\imath\Theta_\infty}{2} + \frac{\pi\imath\beta_0}{2}), \quad m^1_{12} = \hat{u}\exp(\pi\imath\beta_0 + \frac{\pi\imath}{2} - 2\Theta_\infty\ln 2),\\
s_2 &= -\hat{u}\exp(-2\Theta_\infty\ln 2 - \pi\imath\beta_0 + \pi\imath\Theta_\infty)\Big[\frac{\sqrt{2\pi}\hat{v}}{\Gamma(1-\beta_0)}\exp(\frac{\pi\imath\Theta_\infty}{2} - \beta_0\ln 2 + \frac{\pi\imath\beta_0}{2}) +\\
&\quad + \frac{\imath\sqrt{2\pi}}{\hat{v}\Gamma(\beta_0)}\exp(\beta_0\ln 2 - \frac{\pi\imath\Theta_\infty}{2} + \frac{\pi\imath\beta_0}{2}) + 2\imath\cos\pi\Theta_0\Big],\\
s_1 &= \frac{1}{\hat{u}}\exp(2\Theta_\infty\ln 2 - \pi\imath\beta_0)\Big[\frac{\sqrt{2\pi}\hat{v}}{\Gamma(1-\beta_0)}\exp(-\frac{\pi\imath\Theta_\infty}{2} - \beta_0\ln 2 + \frac{\pi\imath\beta_0}{2}) +\\
&\quad + \frac{\imath\sqrt{2\pi}}{\hat{v}\Gamma(\beta_0)}\exp(\beta_0\ln 2 + \frac{\pi\imath\Theta_\infty}{2} + \frac{\pi\imath\beta_0}{2}) + 2\imath\cos\pi\Theta_1\Big].
\end{aligned} \tag{3.3}$$

*Remark 3.3.* The other monodromy data could be computed via the formulas (2.16) of Sec. 2.

**Theorem 3.4.** *To any point $\mathcal{M} \in \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ with the coordinates satisfying the restrictions*

$$\begin{gathered}
m^0_{11}m^1_{11}e^{\pi\imath\Theta_\infty} \ne 1, \quad m^0_{11}m^1_{11}e^{\pi\imath\Theta_\infty} \ne 0,\\
m^0_{21} \ne 0, \quad m^0_{11} \ne 0, \quad m^1_{12} \ne 0, \quad m^1_{11} \ne 0,\\
|\Im\ln(1 - m^0_{11}m^1_{11}e^{\pi\imath\Theta_\infty})| < \frac{\pi}{2}
\end{gathered} \tag{3.4}$$

*there corresponds the unique solution of system (1.3) with asymptotics given by formulas* (3.1) *where the parameters* $\beta_0$, $\hat{u}$, *and* $\hat{v}$ *are as follows*

$$\begin{aligned} \beta_0 &= \frac{1}{2\pi\imath}\ln(1 - m^0_{11}m^1_{11}e^{\pi\imath\Theta_\infty}), \\ \hat{u} &= \frac{1}{m^0_{21}}\exp(-\frac{\pi\imath}{2} - \pi\imath\Theta_\infty + 2\Theta_\infty\ln 2)\exp(\pi\imath\beta_0), \\ \hat{v} &= -\frac{\sqrt{2\pi}}{m^0_{11}\Gamma(\beta_0)}\exp(\beta_0\ln 2 - \frac{\pi\imath}{2}\Theta_\infty + \frac{\pi\imath}{2}\beta_0). \end{aligned} \tag{3.5}$$

## 4. Results as $t \to 0$

**Theorem 4.1.** *Let* $\Theta_0, \Theta_1 \notin \mathbb{Z}$. *Then for any* $\sigma, r, \hat{s} \in \mathbb{C}\backslash 0$, $0 < \Re\sigma < 1$, $\frac{\sigma\pm\Theta_\infty}{2} \notin \mathbb{Z}$, *and* $\frac{\sigma\pm\Theta_0\pm\Theta_1}{2} \notin \mathbb{Z}$ *there exists solution of system (1.3) with asymptotic expansion*

$$\begin{aligned} y &= 1 + \frac{4\sigma^2\hat{s}(\Theta_1+\Theta_0-\sigma)}{(\Theta_\infty+\sigma)(\Theta_0^2-(\Theta_1+\sigma)^2)}t^\sigma + O(t^{2\sigma}) + O(t), \\ z &= \frac{(\Theta_\infty+\sigma)({\Theta_0}^2-(\Theta_1+\sigma)^2)}{8\sigma^2\hat{s}}t^{-\sigma} + \frac{(\Theta_\infty(\Theta_1^2-\Theta_0^2-\sigma^2)-2\sigma^2\Theta_0)}{4\sigma^2} + \\ &\quad + \frac{\hat{s}(\Theta_\infty-\sigma)(\Theta_0^2-(\Theta_1-\sigma)^2)}{8\sigma^2}t^\sigma + O(t^{1-\sigma}), \\ u &= -rt^{\Theta_\infty}[1 - \frac{2\hat{s}\sigma(\Theta_1^2-(\Theta_0-\sigma)^2)}{(\Theta_\infty+\sigma)(\Theta_0^2-(\Theta_1+\sigma)^2)}t^\sigma + O(t^{2\sigma}) + O(t)]. \end{aligned} \tag{4.1}$$

**Corollary.** *The function* $\zeta(t)$ *(see (1.5)) corresponding to the solution of system (1.3) defined by Theorem 4.1 is*

$$\begin{aligned} \zeta(t) &= \frac{1}{4}(\sigma^2 - \Theta_1^2 + \Theta_0^2 + 2\Theta_0\Theta_\infty) + \frac{\hat{s}^{-1}(\Theta_\infty+\sigma)((\Theta_1+\sigma)^2-\Theta_0^2)}{8\sigma^2(1-\sigma)}t^{1-\sigma} + \\ &+ \frac{t}{4\sigma^2}(2\Theta_0\sigma^2 + \Theta_\infty(\Theta_0^2-\Theta_1^2+\sigma^2)) + \frac{\hat{s}(\Theta_\infty-\sigma)((\Theta_1-\sigma)^2-\Theta_0^2)}{8\sigma^2(1+\sigma)}t^{1+\sigma} + O(|t|^{2-2\Re\sigma}). \end{aligned} \tag{4.2}$$

**Theorem 4.2.** *The solution of system (1.3) from Theorem 1 defines the isomonodromy defor-*

*mation of (1.1) with the following monodromy data*

$$
\begin{aligned}
m^p_{11} &= \frac{1}{d}\Big( -\frac{1}{\pi}e^{-\frac{\pi\imath}{2}\sigma}\hat m^p_{11}\sin\frac{\pi}{2}(\sigma-\Theta_\infty) - \frac{1}{\pi}e^{\frac{\pi\imath}{2}\sigma}\hat m^p_{22}\sin\frac{\pi}{2}(\sigma+\Theta_\infty) - \\
&-\frac{1}{\hat s_2}e^{-\frac{\pi\imath}{2}\sigma}\hat m^p_{21}\frac{1}{\Gamma(1-\frac{\sigma-\Theta_\infty}{2})\Gamma(1-\frac{\sigma+\Theta_\infty}{2})} - \hat s_2 e^{\frac{\pi\imath}{2}\sigma}\hat m^p_{12}\frac{1}{\Gamma(\frac{\sigma+\Theta_\infty}{2})\Gamma(\frac{\sigma-\Theta_\infty}{2})}\Big),\\
m^p_{22} &= \frac{1}{d}\Big( -\frac{1}{\pi}e^{\frac{\pi\imath}{2}\sigma}\hat m^p_{11}\sin\frac{\pi}{2}(\sigma+\Theta_\infty) - \frac{1}{\pi}e^{-\frac{\pi\imath}{2}\sigma}\hat m^p_{22}\sin\frac{\pi}{2}(\sigma-\Theta_\infty) + \\
&+\frac{1}{\hat s_2}e^{-\frac{\pi\imath}{2}\sigma}\hat m^p_{21}\frac{1}{\Gamma(1-\frac{\sigma+\Theta_\infty}{2})\Gamma(1-\frac{\sigma-\Theta_\infty}{2})} + \hat s_2 e^{\frac{\pi\imath}{2}\sigma}\hat m^p_{12}\frac{1}{\Gamma(\frac{\sigma-\Theta_\infty}{2})\Gamma(\frac{\sigma+\Theta_\infty}{2})}\Big),\\
m^p_{12} &= \frac{1}{d}re^{\frac{\pi\imath}{2}\Theta_\infty}\Big((\hat m^p_{22}-\hat m^p_{11})\frac{1}{\Gamma(\frac{\sigma-\Theta_\infty}{2})\Gamma(1-\frac{\sigma+\Theta_\infty}{2})} - \\
&-\frac{1}{\hat s_2}\hat m^p_{21}\frac{1}{\Gamma^2(1-\frac{\sigma+\Theta_\infty}{2})} + \hat s_2\hat m^p_{12}\frac{1}{\Gamma^2(\frac{\sigma-\Theta_\infty}{2})}\Big),\\
m^p_{21} &= \frac{1}{d}r^{-1}e^{-\frac{\pi\imath}{2}\Theta_\infty}\Big((\hat m^p_{22}-\hat m^p_{11})\frac{1}{\Gamma(\frac{\sigma+\Theta_\infty}{2})\Gamma(1-\frac{\sigma-\Theta_\infty}{2})} + \\
&+\frac{1}{\hat s_2}\hat m^p_{21}e^{-\pi\imath\sigma}\frac{1}{\Gamma^2(1-\frac{\sigma-\Theta_\infty}{2})} - \hat s_2\hat m^p_{12}e^{\pi\imath\sigma}\frac{1}{\Gamma^2(\frac{\sigma+\Theta_\infty}{2})}\Big),\\
s_2 &= -2\pi\imath re^{\Theta_\infty\pi\imath}\frac{1}{\Gamma(1-\frac{\sigma+\Theta_\infty}{2})\Gamma(\frac{\sigma-\Theta_\infty}{2})},\ s_1 = -2\pi\imath r^{-1}\frac{1}{\Gamma(1-\frac{\sigma-\Theta_\infty}{2})\Gamma(\frac{\sigma+\Theta_\infty}{2})}.
\end{aligned}
\tag{4.3}
$$

*In previous formulas*

$$
\begin{aligned}
d &= -\frac{1}{\pi}e^{\frac{\pi\imath}{2}\Theta_\infty}\sin\pi\sigma,\ \hat m^p_{ij} = \{\hat M^p\}_{ij},\ p=0,1,\ \hat s_2 = -\frac{2\hat s\Gamma(\sigma)}{(\Theta_\infty+\sigma)\Gamma(-\sigma)},\\
\hat M^0 &= \frac{\imath}{\sin\pi\sigma}\begin{pmatrix} -\cos\pi\Theta_1 + e^{\pi\imath\sigma}\cos\pi\Theta_0 & -e^{-\pi\imath\sigma}\hat p\\ -e^{\pi\imath\sigma}\hat q & \cos\pi\Theta_1 - e^{-\pi\imath\sigma}\cos\pi\Theta_0\end{pmatrix},\\
\hat M^1 &= \frac{\imath}{\sin\pi\sigma}\begin{pmatrix} -\cos\pi\Theta_0 + e^{\pi\imath\sigma}\cos\pi\Theta_1 & \hat p\\ \hat q & \cos\pi\Theta_0 - e^{-\pi\imath\sigma}\cos\pi\Theta_1\end{pmatrix},\\
\hat p &= -2\pi^2\frac{\Gamma(1+\sigma)}{\Gamma(1-\sigma)}\frac{1}{\Gamma(1+\frac{\Theta_0+\Theta_1+\sigma}{2})\Gamma(\frac{\sigma-\Theta_0-\Theta_1}{2})\Gamma(1+\frac{\Theta_1-\Theta_0+\sigma}{2})\Gamma(\frac{\Theta_0-\Theta_1+\sigma}{2})},\\
\hat q &= 2\pi^2\frac{\Gamma(1-\sigma)}{\Gamma(1+\sigma)}\frac{1}{\Gamma(1+\frac{\Theta_0+\Theta_1-\sigma}{2})\Gamma(\frac{-\sigma-\Theta_0-\Theta_1}{2})\Gamma(1+\frac{\Theta_1-\Theta_0-\sigma}{2})\Gamma(\frac{\Theta_0-\Theta_1-\sigma}{2})}.
\end{aligned}
\tag{4.4}
$$

**Theorem 4.3.** *To any point $\mathcal{M} \in \mathcal{M}_5(\Theta_0,\Theta_1,\Theta_\infty)$ with the coordinates satisfying the restrictions*

$$s_1s_2 \neq 0, s_1s_2 \neq -\cos\pi\Theta_\infty + \cos\pi(\Theta_0\pm\Theta_1), \tag{4.5a}$$

$$0 < \Re\arccos(\frac{1}{2}e^{-\pi\imath\Theta_\infty}s_1s_2 + \cos\pi\Theta_\infty) < \pi, \tag{4.5b}$$

*there corresponds the unique solution of system (1.3) with asymptotics given by formulas* (4.1), *where the parameters* $\sigma$, $r$, *and* $\hat{s}$ *are as follows*

$$\sigma = \frac{1}{\pi}\arccos\Big(\frac{1}{2}e^{-\pi\imath\Theta_\infty}s_1s_2 + \cos\pi\Theta_\infty\Big), \qquad r = -\frac{2\pi\imath}{s_1}\frac{1}{\Gamma(1+\frac{\Theta_\infty}{2}-\frac{\sigma}{2})\Gamma(\frac{\Theta_\infty}{2}+\frac{\sigma}{2})},$$

$$\hat{s} = \frac{\pi(\Theta_\infty+\sigma)\Gamma(-\sigma)}{2\hat{m}^p_{12}\sin\pi\sigma\Gamma(\sigma)\Gamma(1-\frac{\sigma-\Theta_\infty}{2})\Gamma(1-\frac{\sigma+\Theta_\infty}{2})}\times \tag{4.6}$$

$$\times\Big((m^p_{22}-m^p_{11})e^{-\pi\imath\frac{\sigma+\Theta_\infty}{2}} + m^p_{12}e^{-\pi\imath(\sigma+\Theta_\infty)}\frac{\Gamma(1-\frac{\sigma+\Theta_\infty}{2})}{\Gamma(1-\frac{\sigma-\Theta_\infty}{2})} - m^p_{21}\frac{\Gamma(1-\frac{\sigma-\Theta_\infty}{2})}{\Gamma(1-\frac{\sigma+\Theta_\infty}{2})}\Big)$$

*Remark 4.1.* The results presented in this section were essentially obtained by M. Jimbo [3]. Here we write in more explicit way the asymptotics for the functions $y(t)$, $z(t)$ and $u(t)$, and the complete solution of 0-connection problem, including in it $r$.

*Remark 4.3.* The inequality (4.5b) is also the definition of the branch of $\arccos(\,\cdot\,)$. One proves that the conditions (4.5) guarantee nonvanishing of the denominators in (4.6).

## 5. Large $t$ asymptotics: WKB method, and approximations near the regular singularities

**Preliminaries.** First of all let us exclude from equation (1.1) function $u(t)$ by the following gauge transformation:

$$Y_2 = u^{\frac{1}{2}\sigma_3}Y, \tag{5.1}$$

and carry out the large parameter $t$ from all the entries of matrix $A(\lambda, t)$ (1.1). This yields

$$\partial_\lambda Y = t\mathcal{A}(\lambda,t)Y, \tag{5.2}$$

$$\mathcal{A}(\lambda,t) = \frac{1}{2}\sigma_3 + \hat{A}_0\lambda^{-1} + \hat{A}_1(\lambda-1)^{-1} =$$

$$\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2}\end{pmatrix} + \begin{pmatrix} \frac{z}{t}+\frac{\Theta_0}{2t} & -\frac{z}{t}-\frac{\Theta_0}{t} \\ \frac{z}{t} & -\frac{z}{t}-\frac{\Theta_0}{2t}\end{pmatrix}\frac{1}{\lambda} + \begin{pmatrix} -\frac{z}{t}-\frac{\Theta_0+\Theta_\infty}{2t} & y\big(\frac{z}{t}+\frac{\Theta_0-\Theta_1+\Theta_\infty}{2t}\big) \\ -\frac{1}{y}\big(\frac{z}{t}+\frac{\Theta_0+\Theta_1+\Theta_\infty}{2t}\big) & \frac{z}{t}+\frac{\Theta_0+\Theta_\infty}{2t}\end{pmatrix}\frac{1}{\lambda-1}.$$

In the framework of WKB-method it is natural to consider the situation when $\hat{A}_0, \hat{A}_1 = O(1)$, i.e.,

$$\hat{z} \equiv z/t = O(1), \quad y = O(1), \quad \dot{\hat{z}} = o(1), \quad \dot{y} = o(1) \quad \text{as } t\to+\infty.$$

Looking at system (1.3a,b) we arrive at

$$\hat{z} = -\frac{1}{8} + z_1t^{-\frac{1}{2}}, \quad y = -1 + y_1t^{-\frac{1}{2}}, \tag{5.3a}$$

where functions $z_1(t)$ and $y_1(t)$ satisfy the following conditions

$$z_1 = O(t^\varepsilon), \quad y_1 = O(t^\varepsilon), \quad \varepsilon < 1/2. \tag{5.3b}$$

**WKB-formula.** Let us introduce some notations. We denote

$$l^2 = -\det \mathcal{A}, \quad T = \frac{\imath}{2}\sqrt{\frac{2l}{l+\mathcal{A}_{11}}}(\sigma_3 + \frac{\mathcal{A}}{l}).$$

Matrix $T$ transforms $\mathcal{A}$ to diagonal form

$$T^{-1}\mathcal{A}T = l\sigma_3.$$

The points in $\lambda$-plane, where $l(\lambda) = 0$ are called the *turning points.* At large $t$ outside the points 0, 1, and $\frac{1}{2}$ we have

$$l^2(\lambda) = \frac{1}{4}\lambda^{-1}(\lambda-1)^{-1}[(\lambda-\frac{1}{2})^2 - \frac{2\Theta_\infty}{t}\lambda + \frac{2\imath\beta_0+\Theta_\infty}{t} + o(t^{-1})],$$
$$l(\lambda) = \frac{\lambda-1/2}{2\sqrt{\lambda(\lambda-1)}}[1+\frac{1}{2t}(\lambda-\frac{1}{2})^{-2}(-2\Theta_\infty\lambda + 2\imath\beta_0+\Theta_\infty) + o(t^{-1})],$$

where we denoted

$$2\imath\beta_0 \equiv -(\frac{y_1^2}{16} + 16z_1^2). \tag{5.4a}$$

In what follows we demand

$$\beta_0 = O(1). \tag{5.4b}$$

In our case there are two turning points $\lambda_1$ and $\lambda_2$ with the following behavior as $t\to\infty$

$$\lambda_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{2\imath\beta_0}{t}} + O(t^{-1}).$$

The *Stokes curves* are described by the following equation

$$\Re \int_{\lambda_{1,2}}^{\lambda} l(\lambda)\,d\lambda = 0.$$

*Canonical domain* is an open domain of the complex $\lambda$-plane embodying one and only one of the Stokes curves and bounded with the others. Tops of the *Stokes-graph* are the turning points, singular points, and infinity, while ribs of Stokes-graph are the Stokes curves [21].

In our case the turning points are near $\lambda = \frac{1}{2}$. The Stokes-curves are as follows: $\gamma_0$ from $\lambda_1$ to 0, $\gamma_1$ from $\lambda_2$ to 1, $\gamma_{+1}$ from $\lambda_1$ to $+\imath\infty$, $\gamma_{+2}$ from $\lambda_2$ to $+\imath\infty$, $\gamma_{-1}$ from $\lambda_1$ to $-\imath\infty$, and $\gamma_{-2}$ from $\lambda_2$ to $-\imath\infty$. As $t\to\infty$ the lines $\gamma_{+1}$ and $\gamma_{+2}$ coalesce and become the line $\gamma_+ = [\frac{1}{2}, +\imath\infty)$ while lines $\gamma_{-1}$ and $\gamma_{-2}$ become the line $\gamma_- = [\frac{1}{2}, -\imath\infty)$. To each line $\gamma_q$ corresponds the canonical domain $\Gamma_q$, containing the line $\gamma_q$.

There exists solution $\Psi_q$ of Eq.(5.2) with asymptotics

$$\Psi_q(\lambda) = T\exp(t\sigma_3\int_{\lambda^*}^{\lambda} l(\lambda)\,d\lambda + \int_{\lambda^*}^{\lambda} \operatorname{diag} T^{-1}\partial_\lambda T\,d\lambda), \quad \lambda,\lambda^*\in\Gamma_q \tag{5.5}$$

as $t \to +\infty$. The path of integration $[\lambda^*, \lambda]$ lies in the domain $\Gamma_q$. Asymptotics (5.5) fails in the shrinking (as $t \to \infty$) neighborhoods of turning and singular points.

In our case under assumptions (5.3), the term

$$\int_{\lambda^*}^{\lambda} \operatorname{diag} T^{-1} \partial_\lambda T \, d\lambda \sim o(1) \text{ as } t \to \infty$$

and, thus, may be neglected.

We denote

$$t \int_{\lambda^*}^{\lambda} l(\lambda)\, d\lambda = F(\lambda) - F(\lambda^*) + o(1) \quad \text{as } t \to \infty,$$

$$F(\lambda) = \frac{t}{2}\sqrt{\lambda(\lambda-1)} - \frac{\Theta_\infty}{2}\ln(2\lambda - 1 + 2\sqrt{\lambda(\lambda-1)}) - \beta_0 \ln \frac{-1 + 2\imath\sqrt{\lambda(\lambda-1)}}{2\lambda - 1}.$$

In the subsequent it is supposed that the cuts of $\mathbb{C}$ are made as in the defintion of $M^p$ in Sec. 2. For further calculations agree to fix the branches of the root and logarithms by their asymptotics as $\lambda \to +\imath\infty$: $\sqrt{\lambda(\lambda-1)} \to \lambda$, $\ln(2\lambda - 1 + 2\sqrt{\lambda(\lambda-1)}) \to 2\ln 2 + \ln\lambda$, and $\ln\frac{-1+2\imath\sqrt{\lambda(\lambda-1)}}{2\lambda-1} \to \frac{\pi\imath}{2}$, where $\Im \ln 2 = 0$ and $\ln\lambda$ means the same that in asymptotics (2.1).

Below we present the asymptotics of function $F(\lambda)$ and matrix $T(\lambda)$ near the turning and singular points.

As $\lambda \to +\imath\infty$

$$F(\lambda) = \frac{\lambda t}{2} - \frac{\Theta_\infty}{2}\ln\lambda - \frac{t}{4} - \Theta_\infty \ln 2 - \frac{\pi\imath\beta_0}{2} + O(\frac{t}{\lambda}) + O(\frac{\ln\lambda}{\lambda}),\ T(\lambda) = \imath\sigma_3 + O(\frac{1}{\lambda}). \quad (5.6a)$$

As $\lambda \to +0$

$$F(\lambda) = \frac{\imath}{2}\sqrt{\mu} - \frac{\Theta_\infty}{2}\pi\imath + o(1), \quad \mu = \lambda t^2, \mu \to \infty, \mu \sim t^\varepsilon,\ \varepsilon > 0, l^2 = \hat{l}^2(1 + O(t^{-2+2\varepsilon})),$$

$$T(\lambda) \sim [\hat{l}\sigma_3 + \hat{A}_0\lambda^{-1}](1 + o(1)), \quad \hat{l} = t\mu^{-1}\sqrt{\frac{{\Theta_0}^2}{4} - \frac{\mu}{16}} = \frac{\imath}{4}t\mu^{-\frac{1}{2}}(1 + o(1)). \quad (5.6b)$$

As $\lambda \to 1$, $\lambda < 1$

$$\begin{gathered} F(\lambda) = \frac{\imath}{2}\sqrt{\mu} - \pi\imath\beta_0 + o(1),\ \mu = (1-\lambda)t^2, \mu \sim t^\varepsilon,\ \varepsilon > 0,\ l^2 = \hat{l}^2(1 + O(t^{-2+2\varepsilon})), \\ T(\lambda) \sim [\hat{l}\sigma_3 + \hat{A}_1(\lambda - 1)^{-1}](1 + o(1)), \quad \hat{l} = -\frac{\imath}{4}t\mu^{-\frac{1}{2}}(1 + o(1)). \end{gathered} \quad (5.6c)$$

As $\lambda \to 1/2$

$$\begin{gathered} F(\lambda) = \frac{\imath}{4}t - \frac{\imath}{2}\mu^2 - \frac{\pi\imath}{4}\Theta_\infty - \beta_0\ln(-\Lambda^{-1}) + o(1),\ l = -\imath\Lambda(1 + o(1)), \\ T(\lambda) \sim -\imath\sigma_3 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + o(1),\ \Lambda = \lambda - \frac{1}{2}, \quad \mu = \Lambda\sqrt{t}, \quad \mu \sim t^\varepsilon,\ \varepsilon > 0. \end{gathered} \quad (5.6d)$$

Here, the estimate $o(1)$ means $O(\mu^{-\delta})$ with some $\delta > 0$.

Hereinafter, the symbol $\sim$ means that the expressions are proportional up to nonvanishing scalar factor.

**Asymptotics near the points 0 and 1.** WKB-formula fails as $\lambda \to 0$. We need another approximation valid at the point $\lambda = 0$.

For this purpose let us introduce the function

$$\Phi(x,\Theta) = x\begin{pmatrix} J_{-1+\Theta}(x) & J_{1-\Theta}(x) \\ J_{1+\Theta}(x) & J_{-1-\Theta}(x) \end{pmatrix}, \quad \det\Phi = -\frac{4\Theta\sin\pi\Theta}{\pi}.$$

It has the following properties [7, 24]

$$\Phi(x,\Theta) \underset{x\to 0}{\sim} 2\,\mathrm{diag}\{\frac{2^{-\Theta}}{\Gamma(\Theta)}, \frac{2^{\Theta}}{\Gamma(-\Theta)}\}(1+O(x^2))x^{\Theta\sigma_3}, \tag{7a}$$

$$\Phi^{-1}(x,\Theta) \underset{\substack{x\to\infty \\ \arg|x|<\pi}}{\sim} \begin{pmatrix} \cos(x+\frac{\pi}{4}+\frac{\pi\Theta}{2}) & \cos(x+\frac{\pi}{4}+\frac{\pi\Theta}{2}) \\ \cos(x+\frac{\pi}{4}-\frac{\pi\Theta}{2}) & \cos(x+\frac{\pi}{4}-\frac{\pi\Theta}{2}) \end{pmatrix}(1+O(x^{-2}))-$$

$$-\frac{1}{2x}\begin{pmatrix} ((1+\Theta)^2-\frac{1}{4})\sin(x+\frac{\pi}{4}+\frac{\pi\Theta}{2}) & ((1-\Theta)^2-\frac{1}{4})\sin(x+\frac{\pi}{4}+\frac{\pi\Theta}{2}) \\ ((1+\Theta)^2-\frac{1}{4})\sin(x+\frac{\pi}{4}-\frac{\pi\Theta}{2}) & ((1-\Theta)^2-\frac{1}{4})\sin(x+\frac{\pi}{4}-\frac{\pi\Theta}{2}) \end{pmatrix}. \tag{7b}$$

Let us rewrite Eq.(5.2) as follows

$$\partial_\mu Y = ((\frac{1}{2}\sigma_3 - \hat{A}_1)t^{-1}(1+o(1)) + t\frac{\hat{A}_0}{\mu})Y, \quad \mu = \lambda t^2$$

and consider matrix $T_0 \sim \frac{1}{2}\sigma_3 t^{-1}\Theta_0 + \hat{A}_0$ transforming $\hat{A}_0$ to diagonal form. Now put $Y = T_0\Phi_0$ to obtain

$$\partial_\mu\Phi_0 = [\frac{\varphi}{\Theta_0}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \frac{\Theta_0}{2\mu}\sigma_3]\Phi_0, \quad \varphi \equiv \hat{z}(1-\hat{z}y^{-1}(y-1)^2) = -\frac{1}{16}(1+O(t^{-1}).$$

Fundamental solution of this equation can be written in terms of $\Phi(x,\Theta_0)$

$$\Phi_0 = \Phi(x = \frac{\sqrt{\mu}}{2}, \Theta_0).$$

There exists the unique solution $Y(0,\lambda)$ of Eq.(5.2) with the asymptotic expansion [22]

$$Y(0,\lambda) \to T_0\Phi_0 \text{ as } \lambda \to 0. \tag{5.8}$$

Hereafter, the notation $Y(\lambda_i,\lambda)$ means that this function solves Eq.(5.2) and is appropriately normalized at the point $\lambda_i$.

Near the point $\lambda = 1$ equation (5.2) reads

$$\partial_\mu Y = ((-\frac{1}{2}\sigma_3 - \hat{A}_0)t^{-1}(1+o(1)) + t\frac{\hat{A}_1}{\mu})Y, \quad \mu = (1-\lambda)t^2.$$

We take the matrix $T_1$ transforming $\hat{A}_1$ to diagonal form as $T_1 \sim -\frac{\Theta_1}{2t} - \hat{A}_1\sigma_3$ and define the solution $Y(1,\lambda)$ of Eq.(5.2) by the asymptotic expansion

$$Y(1,\lambda) \to T_1\Phi(\frac{\sqrt{\mu}}{2}, \Theta_1) \text{ as } \lambda \to 1.$$

**Asymptotics near the turning point.** The WKB-formula fails as $\lambda \to 1/2$. So we look for another approximation.

To handle with this problem let us introduce the function (cf. [23])

$$D(x) = \begin{pmatrix} D_{-1-\nu}(\imath x) & D_\nu(x) \\ \dot{D}_{-1-\nu}(\imath x) & \dot{D}_\nu(x) \end{pmatrix},\ \dot{D}_{\tilde{\nu}}(x) = v^{-1}(\frac{\partial D_{\tilde{\nu}}}{\partial x}(x) - \frac{x}{2}D_{\tilde{\nu}}(x)),\ \nu = -vw - 1,$$

where $D_{\tilde{\nu}}(\,\cdot\,)$ is parabolic cylinder function. The function $D(x)$ solves the equation

$$\partial_x D(x) = \big[\frac{x}{2}\sigma_3 + \begin{pmatrix} 0 & v \\ w & 0 \end{pmatrix}\big] D(x). \tag{5.9}$$

and has the following asymptotics as $x \to \infty$, $\arg x = \frac{\pi\imath}{4} + \frac{\pi\imath}{2}k$, $k = -1, 0, 1, 2$,

$$D(x) = (I + O(\frac{1}{x}))\exp((\frac{1}{4}x^2 - (\nu+1)\ln x)\sigma_3)R_k,$$

where

$$R_0 = \begin{pmatrix} e^{-\frac{\pi\imath}{2}(1+\nu)} & 0 \\ -\frac{\imath}{v}\frac{\sqrt{2\pi}}{\Gamma(1+\nu)}e^{-\frac{\pi\imath}{2}(1+\nu)} & -\frac{1}{v} \end{pmatrix}, \quad R_1 = \begin{pmatrix} e^{\frac{3\pi\imath}{2}(1+\nu)} & \frac{\sqrt{2\pi}}{\Gamma(-\nu)}e^{\pi\imath(1+\nu)} \\ -\frac{\imath}{v}\frac{\sqrt{2\pi}}{\Gamma(1+\nu)}e^{-\frac{\pi\imath}{2}(1+\nu)} & -\frac{1}{v} \end{pmatrix},$$

$$R_2 = \begin{pmatrix} e^{\frac{3\pi\imath}{2}(1+\nu)} & \frac{\sqrt{2\pi}}{\Gamma(-\nu)}e^{\pi\imath(1+\nu)} \\ 0 & -\frac{1}{v}e^{-2\pi\imath(1+\nu)} \end{pmatrix}, \quad R_{-1} = \begin{pmatrix} e^{-\frac{\pi\imath}{2}(1+\nu)} & 0 \\ 0 & -\frac{1}{v} \end{pmatrix}.$$

We rewrite Eq.(5.2) as

$$\partial_\Lambda Y = t\{\frac{\sigma_3}{2} + 2(\hat{A}_0 - \hat{A}_1) - 4\Lambda(\hat{A}_0 + \hat{A}_1) + 8\Lambda^2(\hat{A}_0 - \hat{A}_1) + O(\Lambda^3)\}Y,\ \lambda = \frac{1}{2} + \Lambda. \tag{5.10}$$

Define matrix $T_{1/2}$ transforming $(\hat{A}_0 + \hat{A}_1)$ to diagonal form by normalizing its asymptotics as $t \to \infty$

$$T_{1/2} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\imath \\ -\imath & 1 \end{pmatrix}(1 + o(1)).$$

Put

$$v = \frac{1}{\sqrt{2}}e^{-\frac{\pi\imath}{4}}(-4\imath z_1 + \frac{y_1}{4}), \quad w = \frac{1}{\sqrt{2}}e^{-\frac{\pi\imath}{4}}(4\imath z_1 + \frac{y_1}{4}). \tag{5.11}$$

Then, provided the following is valid for some $0 < \varepsilon' < \varepsilon$

$$1 + |y_1| + |z_1| = o(t^{\frac{1}{2}-\varepsilon'}\min\{|v|, |w|\}), \tag{5.12}$$

there exists the unique fundamental solution $Y(\frac{1}{2}, \lambda)$ of Eq.(5.2) with the following asymptotic expansion

$$Y(\frac{1}{2}, \lambda) = (I + O(\frac{1}{x}))T_{1/2}D(x), \quad x = 2^{1/2}e^{\frac{\pi\imath}{4}}t^{\frac{1}{2}}(\lambda - \frac{1}{2})$$

as $\lambda \to \frac{1}{2} + O(t^{\varepsilon-\frac{1}{2}})$, $\frac{1}{6} > \varepsilon > 0$, where $D(x)$ is the solution of Eq.(5.9).

**Matching.** We have defined functions $Y(\lambda)$, $\Psi_q(\lambda)$, $Y(0,\lambda)$, $Y(1,\lambda)$ and $Y(\frac{1}{2},\lambda)$. All of them are solutions of Eq.(5.2). Hence, they differ by matrices independent of $\lambda$. Let us denote

$$C_+ = \Psi_+^{-1}(\lambda)Y(\lambda), \quad L_p = Y^{-1}(p,\lambda)\Psi_p(\lambda), \quad N_q = Y^{-1}(\frac{1}{2},\lambda)\Psi_q(\lambda),\ p=0,1,\ q=0,1,+.$$

We calculate these matrices by taking the following limit passages

$$C_+ = \lim_{\lambda\to+\imath\infty} \Psi_+^{-1}(\lambda)Y(\lambda), \quad \lambda\in\Gamma_+,$$

$$N_q = \lim_{\lambda\to1/2,\lambda-1/2\sim t^{-1/2+\varepsilon}} Y^{-1}(\frac{1}{2},\lambda)\Psi_q(\lambda),\ t\to\infty, 1/6>\varepsilon>0, \lambda\in\Gamma_q,\ q=0,1,+,$$

$$L_p = \lim_{\lambda\to p,(\lambda-p)\sim t^{-2+\varepsilon}} Y^{-1}(p,\lambda)\Psi_p(\lambda),\ t\to\infty, 2>\varepsilon>0,\ \lambda\in\Gamma_p,\ p=0,1.$$

The results are as follows

$$C_+ = -\imath\sigma_3\exp[F(\lambda_+^*)\sigma_3]\exp\left[(\frac{t}{4}+\Theta_\infty\ln2+\frac{\pi\imath}{2}\beta_0)\sigma_3\right]u^{-\frac{1}{2}\sigma_3},$$

$$L_0 = \begin{pmatrix} e^{-\frac{\pi\imath}{2}\Theta_0} & e^{\frac{\pi\imath}{2}\Theta_0} \\ e^{\frac{\pi\imath}{2}\Theta_0} & e^{-\frac{\pi\imath}{2}\Theta_0}\end{pmatrix}\exp[(-\frac{\pi\imath}{4}-\frac{\pi\imath}{2}\Theta_\infty)\sigma_3]\exp[-F(\lambda_0^*)\sigma_3],$$

$$L_1 = \begin{pmatrix} e^{-\frac{\pi\imath}{2}\Theta_1} & e^{\frac{\pi\imath}{2}\Theta_1} \\ e^{\frac{\pi\imath}{2}\Theta_1} & e^{-\frac{\pi\imath}{2}\Theta_1}\end{pmatrix}\exp[(-\frac{\pi\imath}{4}-\pi\imath\beta)\sigma_3]\exp[-F(\lambda_1^*)\sigma_3],$$

$$N_0^{-1}N_+ = \exp[F(\lambda_0^*)\sigma_3]\begin{pmatrix} 1 & \imath\frac{\sqrt{2\pi}}{v\Gamma(\beta)}e^{-\frac{\imath t}{2}+\frac{\pi\imath}{2}\Theta_\infty+\beta_0\ln2+\beta_0\ln t+\frac{\pi\imath}{2}\beta_0} \\ 0 & 1\end{pmatrix}\exp[-F(\lambda_+^*)\sigma_3],$$

where $\lambda_q^*\in\Gamma_q$. In calculation of $C^+$ recall the gauge transformation (5.1) and asymptotics (2.1) of the canonical solution $Y_2$.

Now we are able to calculate matrix $E^0$ (see (2.10) of Sec. 2 for definition). Due to (5.7a) and (5.8) one finds

$$E^0 \underset{t\to\infty}{=} L_0N_0^{-1}N_+C_+(1+o(1)).$$

After straightforward calculations one arrives at

$$E^0 \underset{t\to\infty}{=} \begin{pmatrix} e^{-\frac{\pi\imath}{2}\Theta_0} & e^{\frac{\pi\imath}{2}\Theta_0} \\ e^{\frac{\pi\imath}{2}\Theta_0} & e^{-\frac{\pi\imath}{2}\Theta_0}\end{pmatrix}\times\begin{pmatrix}\alpha_0 & \Delta_0 \\ 0 & \alpha_0\end{pmatrix}u^{-\frac{1}{2}\sigma_3},$$

where

$$\alpha_0 = \exp(-\frac{\pi\imath}{4}-\frac{\pi\imath}{2}\Theta_\infty+\frac{t}{4}+\Theta_\infty\ln2+\frac{\pi\imath}{2}\beta_0),$$

$$\Delta_0 = \imath\frac{\sqrt{2\pi}}{v\Gamma(\beta_0)}\exp\{-\frac{\imath t}{2}+\beta_0\ln2+\beta_0\ln t+\frac{\pi\imath}{2}\beta_0-\Theta_\infty\ln2-\frac{\pi\imath}{4}-\frac{t}{4}\}.$$

For the matrix $M^0$ (see (2.12) of Sec. 2) we obtain

$$M^0 \underset{t\to\infty}{=} \begin{pmatrix} -\Delta_0\alpha_0 & -u(\Delta_0^2+\alpha_0^{-2}+2\Delta_0\alpha_0^{-1}\cos\pi\Theta_0) \\ u^{-1}\alpha_0^2 & \Delta_0\alpha_0+2\cos\pi\Theta_0\end{pmatrix}(1+o(1)). \tag{5.13a}$$

In analogous manner we found the matrix

$$M^1 \underset{t\to\infty}{=} \begin{pmatrix} -\Delta_1\alpha_1^{-1} & u\alpha_1^{-2} \\ -u^{-1}(\Delta_1^2+\alpha_1^2-2\Delta_1\alpha_1\cos\pi\Theta_1) & \Delta_1\alpha_1+2\cos\pi\Theta_1 \end{pmatrix}(1+o(1)),$$

where

$$\begin{aligned} \alpha_1 &= \exp(-\frac{\pi\imath}{2}\beta_0-\frac{\pi\imath}{4}+\Theta_\infty\ln 2+\frac{t}{4}), \\ \Delta_1 &= -\frac{\sqrt{2\pi}\hat{v}}{\Gamma(1-\beta_0)}\exp\{\frac{\pi\imath}{4}-\beta_0\ln 2-\frac{\pi\imath}{2}\Theta_\infty+\Theta_\infty\ln 2+\frac{t}{4}\}. \end{aligned} \tag{5.13b}$$

Requiring now that $M^p$ are constant matrices and taking into account (5.4b) we find that

$$u = \hat{u}e^{\frac{t}{2}}(1+o(1)), \quad v = \hat{v}e^{-\frac{t}{2}t+\beta_0\ln t}(1+o(1)), \tag{5.14}$$

where $\hat{u}$ and $\hat{v}$ are the constants. Now, using (5.13) and (5.14) we arrive at formulas (3.3) of Sec. 3. Formulas (5.3), (5.11), and (5.14) yield the asymptotics of the functions $y(t)$, $z(t)$ and $u(t)$ presented in (3.1) of Sec. 3. Now, taking these asymptotics one goes through all the calculations of this section provided $|\Re\beta_0| < \frac{1}{4}$, which is the consequence of restriction (5.12).

According the scheme explained in Introduction this is enough to justify the results stated in Sec. 3.

## 6. Isomonodromy deformations as $t \to 0$

The isomonodromy deformations as $t \to 0$ were studied by M. Jimbo [3]. However, Jimbo's results do not include the asymptotics of function $u(t)$. The information about these asymptotics in the Jimbo's presentation is hidden in a diagonal matrix $D$ which is not given there in explicit form. To reconstruct this matrix we have to repeat all the Jimbo's calculations but with a more careful attention to diagonal factors of the monodromy matrices. In this section we briefly outline this calculation.

The calculation is organized as follows. We consider the domains of the $\lambda$-complex plane $\Omega_{\mathrm{out}}$ and $\Omega_{\mathrm{in}}$ such that $\overline{C} = \Omega_{\mathrm{out}} \cup \Omega_{\mathrm{in}}$ and $\Omega_{\mathrm{out}} \cap \Omega_{\mathrm{in}} = \Omega_{\mathrm{mat}} \neq \emptyset$. In each the domain the fundamental solution $Y$ of the equation

$$\partial_\lambda Y = (\frac{t}{2}\sigma_3+\frac{A_0}{\lambda}+\frac{A_1}{\lambda-1})Y, \tag{6.1}$$

is approximated with the functions $Y_{\mathrm{out}}$ and $Y_{\mathrm{in}}$ respectively. $Y_{\mathrm{out}}$ and $Y_{\mathrm{in}}$ are the fundamental solutions of equations (6.2) and (6.3) (see below) which are known as solvable in terms of higher transcendental functions. To find these solutions $Y_{\mathrm{out}}$ and $Y_{\mathrm{in}}$ we use the auxiliary functions $Y(x;k,\sigma)$ and $Y(\lambda;\alpha,\beta,\gamma)$ which are defined in [3].

To fulfill the above program we must impose some restrictions on the matrices $A_0$ and $A_1$. First of all, we see that $t = 0$ is the regular singularity of equations (1.3) defining $y(t)$ and $z(t)$. Thus, we will consider the functions $y(t)$ and $z(t)$ with the branching at $t = 0$ and suppose that $A_p(t)$ behave as follows

$$A_0(t),\ A_1(t) \sim O(t^{-\sigma_1}), \quad t \to 0, \quad \sigma_1 > 0.$$

We want to obtain the asymptotics of $A_p(t)$ up to the terms of order $t^{\sigma_2}$ with some $\sigma_2 > 0$.

In $\Omega_{\text{out}}$ (large $\lambda$), equation (6.1) could be reduced to

$$\partial_\lambda Y_{\text{out}} = (\frac{t}{2}\sigma_3 + \frac{A_0 + A_1}{\lambda})Y_{\text{out}}. \tag{6.2}$$

In $\Omega_{\text{in}}$ (small $\lambda t$), equation (6.1) can be reduced to

$$\partial_\lambda Y_{\text{in}} = (\frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1})Y_{\text{in}}. \tag{6.3}$$

More precisely, the limit transition from (6.1) to (6.2) in region $\Omega_{\text{out}} = \{\lambda\colon \lambda \sim t^{-\delta_{\text{out}}}\}$ is true if $\delta_{\text{out}} > \frac{1+\sigma_1}{2}$, $\delta_{\text{out}} > \sigma_2 + \sigma_1$. In particular, it follows that $\delta_{\text{out}} > 0$. The limit transition from (6.1) to (6.3) is correct in region $\Omega_{\text{out}} = \{\lambda\colon \lambda \sim t^{-\delta_{\text{in}}}\}$ if $\delta_{\text{in}} < 1 - \sigma_2$, $\sigma_2 < 1$. Our matching region is $\Omega_{\text{mat}} = \{\lambda\colon \lambda \sim\rightarrow t^{-\delta}$, where $0 < \delta < 1$ and $\delta_{\text{out}} < \delta < \delta_{\text{in}}$. Therefore,

$$\frac{1+\sigma_1}{2} < \delta < 1 - \sigma_2, \quad \sigma_2 + \sigma_1 < \delta. \tag{6.4}$$

We obtain the following restrictions on $\sigma_1$ and $\sigma_2$

$$\sigma_1 < 1 - 2\sigma_2. \tag{6.5}$$

In region $\Omega_{\text{out}}$, to transform Eq.(6.2) to standard form we make the following substitution

$$\begin{gathered} \hat{Y}_{\text{out}}(x) = Rt^{-\frac{\Theta_\infty}{2}\sigma_3}Y_{\text{out}}(x/t)R^{-1}e^{\frac{\lambda t}{2}},\ \lambda t = x, \\ Y_{\text{out}}(\lambda) = t^{\frac{\Theta_\infty}{2}\sigma_3}R^{-1}\hat{Y}_{\text{out}}(\lambda t)Re^{-\frac{\lambda t}{2}}. \end{gathered} \tag{6.6}$$

Then for $\hat{Y}_{\text{out}}(x)$ we obtain

$$\begin{gathered} \partial_x \hat{Y}_{\text{out}}(x) = (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + Rt^{-\frac{\Theta_\infty}{2}\sigma_3}\frac{A_0 + A_1}{x}t^{\frac{\Theta_\infty}{2}\sigma_3}R^{-1})\hat{Y}_{\text{out}}(x), \\ \hat{Y}_{\text{out}}(x) = (1 + O(x^{-1}))x^{-\frac{\Theta_\infty}{2}\sigma_3}e^{\begin{pmatrix} x & \\ & 0 \end{pmatrix}}. \end{gathered} \tag{6.7}$$

The standard equation reads as

$$\begin{gathered} \partial_x \hat{Y}(x;k,\sigma) = (\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \frac{1}{2x}\begin{pmatrix} -k & \sigma - k \\ \sigma + k & k \end{pmatrix})\hat{Y}(x;k,\sigma), \\ Y(x;k,\sigma) = (1 + O(x^{-1}))x^{-\frac{k}{2}\sigma_3}e^{\begin{pmatrix} x & \\ & 0 \end{pmatrix}}, \quad \arg x = \frac{\pi}{2}. \end{gathered} \tag{6.8}$$

Its solution is given by explicit formula (see (3.10) [3]).

$$Y(x;k,\sigma) = \begin{pmatrix} e^{\pi\imath(1-k)/2}W_{(1-k)/2,\sigma/2}(e^{-\pi\imath}x); -\frac{1}{2}(\sigma - k)W_{(-1+k)/2,\sigma/2}(x) \\ -\frac{1}{2}(\sigma + k)e^{\pi\imath(1-k)/2}W_{-(1+k)/2,\sigma/2}(e^{-\pi\imath}x); W_{(1+k)/2,\sigma/2}(x) \end{pmatrix} x^{-\frac{1}{2}}e^{x/2}.$$

Here $W_{\cdot,\cdot}(x)$ is Whittaker's function. We also need the following result (see [3, p.1155])

$$Y(x;k,\sigma)=G_{k\sigma}(1+O(x))x^{\frac{1}{2}\sigma\sigma_3}C_{k\sigma}, \tag{6.9}$$

where

$$G_{k\sigma}=\frac{1}{\sqrt{\sigma}}\begin{pmatrix}\frac{1}{2}(\sigma-k) & -1\\ \frac{1}{2}(\sigma+k) & 1\end{pmatrix},\ C_{k\sigma}=\begin{pmatrix}\frac{-\Gamma(-\sigma)}{\Gamma(1-\frac{\sigma-k}{2})}e^{-\pi\imath(\sigma+k)/2} & \frac{-\Gamma(-\sigma)}{\Gamma(1-\frac{\sigma+k}{2})}\\ \frac{-\Gamma(\sigma)}{\Gamma(\frac{\sigma+k}{2})}e^{\pi\imath(\sigma-k)/2} & \frac{\Gamma(\sigma)}{\Gamma(\frac{\sigma-k}{2})}\end{pmatrix}. \tag{6.10}$$

Now we choose matrix

$$R=\begin{pmatrix}1 & \\ & r\end{pmatrix},\quad \tilde{u}=t^{-\Theta_\infty}u, \tag{6.11}$$

such that $\hat{Y}_{\rm out}(x)$ becomes function $Y(x;k=\Theta_\infty,\sigma)$. From (6.6) and (6.8), one finds that $Y_{\rm out}$ has the canonical asymptotics as $\lambda\to\infty$ (see (2.1) of Sec. 2). Thus, we obtain

$$s_2=-2\pi\imath re^{\Theta_\infty\pi\imath}\frac{1}{\Gamma(1-\frac{\sigma+\Theta_\infty}{2})\Gamma(\frac{\sigma-\Theta_\infty}{2})},\ s_1=-2\pi\imath r^{-1}\frac{1}{\Gamma(1-\frac{\sigma-\Theta_\infty}{2})\Gamma(\frac{\sigma+\Theta_\infty}{2})}.$$

Looking at previous formula we conclude that $r$ and $\sigma$ are constants.

To transform Eq.(6.1) to standard equation in region $\Omega_{\rm in}$ we make the following substitution

$$\begin{gathered}\hat{Y}_{\rm in}(\lambda)=G^{-1}Rt^{-\frac{\Theta_\infty}{2}\sigma_3}Y_{\rm in}(\lambda)R^{-1}C^{-1}\lambda^{\frac{\Theta_0}{2}}(\lambda-1)^{\frac{\Theta_1}{2}},\\ Y_{\rm in}(\lambda)=\lambda^{-\frac{\Theta_0}{2}}(\lambda-1)^{-\frac{\Theta_1}{2}}t^{\frac{\Theta_\infty}{2}\sigma_3}R^{-1}G\hat{Y}_{\rm in}(\lambda)CR.\end{gathered} \tag{6.12}$$

Here matrix $R$ is the same that in (6.6). We insert the matrices $R$ and $C$ for convenience of matching (see below (6.24)). Function $\hat{Y}_{\rm in}(\lambda)$ satisfy the following equation

$$\begin{gathered}\partial_\lambda\hat{Y}_{\rm in}(\lambda)=(\frac{1}{\lambda}G^{-1}R(\tilde{A}_0+\frac{\Theta_0}{2})R^{-1}G+\frac{1}{\lambda-1}G^{-1}R(\tilde{A}_1+\frac{\Theta_1}{2})R^{-1}G)\hat{Y}_{\rm in}(\lambda),\\ \tilde{A}_p(z,y,\Theta_0,\Theta_1,\Theta_\infty,u)=A_p(z,y,u,\Theta_0,\Theta_1,\Theta_\infty,u\to\tilde{u}).\end{gathered} \tag{6.13}$$

The standard equation reads as (see pages 1149-1150 in [3])

$$\begin{gathered}\partial_\lambda Y(\lambda;\alpha,\beta,\gamma)=(\frac{1}{\lambda}\frac{1}{\beta-\alpha}\begin{pmatrix}\beta-\gamma+1\\ \alpha-\gamma+1\end{pmatrix}\times\begin{pmatrix}-\alpha & \beta\end{pmatrix}+\\ +\frac{1}{\lambda-1}\frac{1}{\beta-\alpha}\begin{pmatrix}1\\1\end{pmatrix}\times\begin{pmatrix}\alpha(\alpha-\gamma+1) & -\beta(\beta-\gamma+1)\end{pmatrix})Y(\lambda;\alpha,\beta,\gamma)\end{gathered} \tag{6.14}$$

with the following boundary condition

$$Y(\lambda;\alpha,\beta,\gamma)=(1+O(\lambda^{-1}))\lambda^{-\begin{pmatrix}\alpha & \\ & \beta\end{pmatrix}},\quad \lambda\to\infty. \tag{6.15}$$

The solution of (6.14) and (6.15) is

$$Y(\lambda;\alpha,\beta,\gamma)=$$
$$=\begin{pmatrix} F(\alpha,\alpha-\gamma+1,\alpha-\beta;\frac{1}{\lambda}); & \frac{\beta(\beta-\gamma+1)}{(\beta-\alpha)(\beta-\alpha+1)}\frac{1}{\lambda}F(\beta+1,\beta-\gamma+2,\beta-\alpha+2;\frac{1}{\lambda}) \\ \frac{\alpha(\alpha-\gamma+1)}{(\alpha-\beta)(\alpha-\beta+1)}\frac{1}{\lambda}F(\alpha+1,\alpha-\gamma+2,\alpha-\beta+2;\frac{1}{\lambda}); & F(\beta,\beta-\gamma+1,\beta-\alpha;\frac{1}{\lambda}) \end{pmatrix},$$

where $F$ is hypergeometric function. We need also the local behaviour of this standard function

$$Y(\lambda;\alpha,\beta,\gamma)=G^{(0)}_{\alpha\beta\gamma}(1+O(x))x^{\begin{pmatrix}1-\gamma & \\ & 0\end{pmatrix}}\hat{E}^0=G^{(1)}_{\alpha\beta\gamma}(1+O(x-1))x^{\begin{pmatrix}\gamma-\alpha-\beta-1 & \\ & 0\end{pmatrix}}\hat{E}^1,$$

where

$$\hat{E}^0=\begin{pmatrix} e^{-\pi\imath(\alpha-\gamma+1)}\frac{\Gamma(\gamma-1)\Gamma(\alpha-\beta+1)}{\Gamma(\gamma-\beta)\Gamma(\alpha)} & -e^{-\pi\imath(\beta-\gamma+1)}\frac{\Gamma(\gamma-1)\Gamma(\beta-\alpha+1)}{\Gamma(\gamma-\alpha)\Gamma(\beta)} \\ e^{-\pi\imath\alpha}\frac{\Gamma(1-\gamma)\Gamma(\alpha-\beta+1)}{\Gamma(1-\beta)\Gamma(\alpha-\gamma+1)} & -e^{-\pi\imath\beta}\frac{\Gamma(1-\gamma)\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)\Gamma(\beta-\gamma+1)} \end{pmatrix}$$
$$\hat{E}^1=\begin{pmatrix} -\frac{\Gamma(\alpha+\beta-\gamma+1)\Gamma(\alpha-\beta+1)}{\Gamma(\alpha-\gamma+1)\Gamma(\alpha)} & \frac{\Gamma(\alpha+\beta-\gamma+1)\Gamma(\beta-\alpha+1)}{\Gamma(\beta-\gamma+1)\Gamma(\beta)} \\ -e^{-\pi\imath(\gamma-\alpha-\beta-1)}\frac{\Gamma(\gamma-\alpha-\beta-1)\Gamma(\alpha-\beta+1)}{\Gamma(1-\beta)\Gamma(\gamma-\beta)} & e^{-\pi\imath(\gamma-\alpha-\beta-1)}\frac{\Gamma(\gamma-\alpha-\beta-1)\Gamma(\beta-\alpha+1)}{\Gamma(1-\alpha)\Gamma(\gamma-\alpha)} \end{pmatrix}. \tag{6.16}$$

Now we choose matrix $G$ such that (6.13) becomes (6.14). We denote

$$A^0_{\text{in}}=\frac{1}{\beta-\alpha}\begin{pmatrix}\beta-\gamma+1\\ \alpha-\gamma+1\end{pmatrix}\begin{pmatrix}-\alpha & \beta\end{pmatrix},\ A^1_{\text{in}}=\frac{1}{\beta-\alpha}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}\alpha(\alpha-\gamma+1) & -\beta(\beta-\gamma+1)\end{pmatrix}. \tag{6.17}$$

and demand

$$G^{-1}R(\tilde{A}_0+\frac{\Theta_0}{2})R^{-1}G=A^0_{\text{in}},\ G^{-1}R(\tilde{A}_1+\frac{\Theta_0}{2})R^{-1}G=A^1_{\text{in}}. \tag{6.18}$$

It follows from previous formula that

$$G^{-1}R(\tilde{A}_0+\tilde{A}_1)R^{-1}G=\frac{\beta-\alpha}{2}\sigma_3. \tag{6.19}$$

By definition of $R$ we have

$$R(\tilde{A}_0+\tilde{A}_1)R^{-1}=\frac{1}{2}\begin{pmatrix}-\Theta_\infty & \sigma-\Theta_\infty\\ \sigma+\Theta_\infty & \Theta_\infty\end{pmatrix},\quad \det[\tilde{A}_0+\tilde{A}_1]=-\frac{\sigma^2}{4}. \tag{6.20}$$

From (6.20) and (6.19) we see that $\beta-\alpha=\sigma$ and

$$\frac{1}{2}G^{-1}\begin{pmatrix}-\Theta_\infty & \sigma-\Theta_\infty\\ \sigma+\Theta_\infty & \Theta_\infty\end{pmatrix}G=\frac{\sigma}{2}\sigma_3.$$

Consequently, matrix $G$ is of the form

$$G=\begin{pmatrix}\frac{\sigma-\Theta_\infty}{2\sigma g} & -g\\ \frac{\sigma+\Theta_\infty}{2\sigma g} & g\end{pmatrix}, \tag{6.21}$$

where $g$ is complex function of $t$. It will be defined later from the matching condition. We fix the determinant $\det G = 1$. Using the local behaviour of function $Y(\lambda;\alpha,\beta,\gamma)$ as $\lambda \to 0$ and $\lambda \to 1$ we find

$$\alpha = -\frac{\Theta_0+\Theta_1+\sigma}{2}, \quad \beta = -\frac{\Theta_0+\Theta_1-\sigma}{2}, \quad \gamma = 1-\Theta_0.$$

Finally, the solutions $Y_{\text{out}}$ and $Y_{\text{in}}$ in the region $\lambda \sim t^{-\delta}$, where $\delta$ is from (6.4), should match

$$Y_{\text{out}}(\lambda) = (1+O(t^{\delta'}))Y_{\text{in}}(\lambda), \text{ with some } \delta' > 0. \tag{6.22}$$

We use replacement (6.6) and small $x$ extension (6.9) in the left-hand side and replacement (6.12) and asymptotic (6.15) in the right-hand side. We obtain that

$$G_{\Theta_\infty\sigma}(1+O(\lambda t))t^{\frac{\sigma}{2}\sigma_3}\lambda^{\frac{\sigma}{2}\sigma_3}C_{\Theta_\infty\sigma} = G(1+O(\lambda^{-1}))\lambda^{\frac{\sigma}{2}\sigma_3}C.$$

Hence

$$G = G_{\Theta_\infty\sigma}t^{\frac{\sigma}{2}\sigma_3}s^{\sigma_3}, \quad C = s^{-\sigma_3}C_{\Theta_\infty\sigma}. \tag{6.23}$$

The first equation in (6.23) results in

$$g = \frac{1}{\sqrt{\sigma}}t^{-\frac{\sigma}{2}}s^{-1}. \tag{6.24}$$

Now, by inverting formulas (6.18) we obtain

$$\tilde{A}_0 + \frac{\Theta_0}{2} = R^{-1}GA^0_{\text{in}}G^{-1}R, \; \tilde{A}_1 + \frac{\Theta_1}{2} = R^{-1}GA^1_{\text{in}}G^{-1}R,$$

which gives us results (4.1). In formulas (4.1) we use parameter $\hat{s} = -\frac{\Theta_\infty+\sigma}{2}s^2$ instead of $s$.

Now our aim is to calculate the monodromy data. From (6.12) and (6.16), for matrices $E^p$ we have

$$E^p = \hat{E}^pCR = \hat{E}^p s^{-\sigma_3}C_{\Theta_\infty\sigma}R.$$

Finally, for the monodromy matrices (see (2.12) of Sec. 2) we obtain

$$M^p = R^{-1}C^{-1}_{\Theta_\infty\sigma}s^{\sigma_3}\hat{M}^p s^{-\sigma_3}C_{\Theta_\infty\sigma}R,$$

where the matrices

$$\hat{M}^p = (\hat{E}^p)^{-1}e^{\pi\imath\Theta_p\sigma_3}\hat{E}^p$$

are presented in Sec. 4.

## Appendix A. Higher terms of large $t$ expansions

Here we construct the asymptotic expansions for $y(t)$ and $z(t)$ in the following form

$$y(t) = \sum_{k=0}^{\infty} y_k(t)t^{-\frac{k}{2}}, \quad z(t) = t\sum_{k=0}^{\infty} z_k(t)t^{-\frac{k}{2}} \text{ as } t \to +\infty, \tag{A.1}$$

where the functions $y_k(t)$ and $z_k(t)$ obey the restrictions

$$|y_k(t)| \le O(t^{\delta k/2}), \quad |z_k(t)| \le O(t^{\delta k/2}), \quad 0 < \delta < 1. \tag{A.2}$$

We have all ready found the first terms of the expansion (A.1). They are

$$\begin{gathered} y_0 = -1,\ y_1 = 4\sqrt{2}e^{-\frac{\pi\imath}{4}}\sqrt{\beta_0}\sin\tilde{x},\ z_0 = -\frac{1}{8},\ z_1 = -\frac{1}{2\sqrt{2}}e^{-\frac{\pi\imath}{4}}\sqrt{\beta_0}\cos\tilde{x}, \\ \tilde{x} \equiv \frac{t}{2} + \imath\beta_0\ln t - \imath\ln\sqrt{\beta_0} + \imath\ln\hat{v}. \end{gathered} \tag{A.3}$$

Thus it is naturally to seek $y_k$ and $z_k$ as functions of $\tilde{x}$. To find $y_k$ and $z_k$ for $k \ge 2$ we substitute (A.1) into the system (1.3a,b) and equate the coefficients of the $t^{-\frac{k}{2}}$ in the right- and left-hand sides. The result reads

$$\begin{aligned} \frac{1}{2}\frac{dy_k}{d\tilde{x}} &= -8z_k + g_k(y_1,\ldots,y_{k-1},z_1,\ldots,z_{k-1}), \\ \frac{1}{2}\frac{dz_k}{d\tilde{x}} &= \frac{1}{32}y_k + \frac{1}{16}f_k(y_1,\ldots,y_{k-1},z_1,\ldots,z_{k-1}), \end{aligned} \tag{A.4}$$

where $f_k$ and $g_k$ are polynomials of $y_1,\ldots,y_{k-1},z_1,\ldots,z_{k-1}$. Solving (A.4) one finds

$$y_k = \gamma_k\cos\tilde{x} + \varepsilon_k\sin\tilde{x}, \quad z_k = \frac{1}{16}(-\varepsilon_k\cos\tilde{x} + \gamma_k\sin\tilde{x}) \tag{A.5}$$

and

$$\frac{1}{2}\frac{d\gamma_k}{d\tilde{x}} = g_k\cos\tilde{x} + f_k\sin\tilde{x}, \quad \frac{1}{2}\frac{d\varepsilon_k}{d\tilde{x}} = g_k\sin\tilde{x} - f_k\cos\tilde{x}. \tag{A.6}$$

From (A.6) we can determine $\gamma_k$ and $\varepsilon_k$ only up to constants $\gamma_k \to \gamma_k + \gamma_k^0$ and $\varepsilon_k \to \varepsilon_k + \varepsilon_k^0$. Constants $\varepsilon_k^0$ and $\gamma_k^0$ must be chosen to satisfy conditions (A.2). Requiring $|\Re\beta_0| \equiv \delta$ one finds that (A.2) are equivalent to restriction that $y_k(\tilde{x})$ and $z_k(\tilde{x})$ are periodic functions of $\tilde{x}$.

Functions $f_k$ and $g_k$ are polynomials of $\cos\tilde{x}$ and $\sin\tilde{x}$ of degree $k$. Functions $\varepsilon_k$ and $\gamma_k$ are obtained by integrating these polynomials. Thus, they may contain the terms proportional to $\tilde{x}$. Requiring the vanishing of the coefficients before these terms in $y_{k+2}$ and $z_{k+2}$ we find the linear equations on $\varepsilon_k^0$ and $\gamma_k^0$ with the triangular matrix

$$\begin{pmatrix} 1 & 0 \\ * & 16 \end{pmatrix}.$$

Thus, one proves that asymptotic expansions (A.1) can be constructed.

The first polynomials $f_k$ and $g_k$ are as follows

$$\begin{gathered} g_2 = \frac{1}{4}y_1^2 + 8y_1z_1 - 2(2\Theta_0 + \Theta_\infty),\ f_2 = \frac{1}{4}y_1^2 - 8y_1z_1 - 2(\Theta_1 + \Theta_0 - 1), \\ g_3 = 16\imath\beta_0z_1 + \frac{1}{2}y_1y_2 - 2y_1^2z_1 + 8y_2z_1 + 8y_1z_2 + y_1(3\Theta_0 - \Theta_1 + 2\Theta_\infty + \frac{1}{2}), \\ f_3 = -\imath\beta_0y_1 + 32y_1z_1^2 - 8y_1z_2 - 8y_2z_1 - 4y_1^2z_1 + \\ + \frac{y_1y_2}{2} + \frac{y_1^3}{4} - 2y_1(2\Theta_0 + \Theta_\infty) + 16z_1(\Theta_0 + \Theta_1 - \frac{1}{2}). \end{gathered}$$

The higher polynomials $f_k$ and $g_k$ looks rather cumbersome to write them down here. By performing the procedure described above with the help of symbolic manipulation program REDUCE we found the following

$$\varepsilon_2^0 = \gamma_2^0 = 0$$

and

$$\varepsilon_3^0 = 2^{\frac{3}{2}} e^{\frac{\pi\imath}{4}} \beta_0^{\frac{3}{2}}, \quad \gamma_3^0 = 2^{\frac{3}{2}} e^{\frac{-\pi\imath}{4}} \sqrt{\beta_0}(-3\beta_0^2 + 4\Theta_0^2 + 4\Theta_1^2 - 4\Theta_0 - 4\Theta_1 + 2\Theta_\infty^2 + \frac{3}{2}).$$

Thus, we obtain

$$y_2 = 16\imath\beta_0 \sin^2\tilde{x} + 4(\Theta_0 + \Theta_1) - 4,\ z_2 = -\imath\beta_0 \sin^2\tilde{x} - \frac{1}{4}(2\Theta_0 + \Theta_\infty). \tag{A.7}$$

This is enough to find the term of order $t^{1/2}$ in $\zeta$-function stated in Theorem 3.2. The next terms are

$$\begin{aligned} y_3 &= -16\sqrt{2}\sqrt{\beta_0}e^{-\frac{\pi\imath}{4}} \sin\tilde{x}[\frac{3\imath\beta_0}{2}\sin^2\tilde{x} + \frac{\imath\beta_0}{2} + \Theta_0 + \Theta_1 - 1] + \gamma_3^0 \cos\tilde{x} + \varepsilon_3^0 \sin\tilde{x}, \\ z_3 &= \frac{e^{-\frac{\pi\imath}{4}}\sqrt{\beta_0}}{\sqrt{2}} \sin\tilde{x}[-3\imath\beta_0 \cos\tilde{x}\sin\tilde{x} + \Theta_0 + \Theta_1 - \frac{3}{2}] + \frac{1}{16}(-\varepsilon_3^0 \cos\tilde{x} + \gamma_3^0 \sin\tilde{x}). \end{aligned} \tag{A.8}$$

Now substituting expansions (A.1) into (1.5) of Introduction we find

$$\begin{aligned} \zeta(t) = \frac{t^2}{16} + \frac{t}{4}(2\Theta_0 + \Theta_\infty + \frac{\imath\beta_0}{2}) + e^{-\frac{\pi\imath}{4}}\sqrt{\beta_0}\sqrt{\frac{t}{2}}\sin\tilde{x} - \\ -\imath\beta_0\cos\tilde{x}\sin\tilde{x} - \frac{\beta_0^2}{2} + \frac{3}{4}\Theta_0^2 + \frac{1}{4}\Theta_1^2 + \frac{1}{4}\Theta_\infty^2 + \frac{1}{2}\Theta_0\Theta_\infty - \frac{1}{4} + o(1). \end{aligned} \tag{A.9}$$

Note that it is consistent with the formula (1.6).

## Appendix B. Comparison with the work by B. McCoy and Sh. Tang (MT)

As it was mentioned in Introduction that the case when $\Theta_0 = \Theta_1 \equiv \Theta$ and $\Theta_\infty = 0$ was investigated by MT [7, 9]. Their result reads

$$Y_{\rm V} = -1 + 2t^{-1/2}(ae^{2is} + be^{-2is}) + O(t^{-1}),\ s = t - ab\ln t, \tag{MT.3.3}$$

$$A_0(t) = \begin{pmatrix} -\frac{1}{4}\frac{d\sigma_0}{dt} & \frac{1}{4}\frac{d\sigma_0}{dt} - \frac{\Theta}{2} \\ -\frac{1}{4}\frac{d\sigma_0}{dt} - \frac{\Theta}{2} & \frac{1}{4}\frac{d\sigma_0}{dt} \end{pmatrix}, \tag{MT.2.6a}$$

$$\sigma_0(t) = t^2 - 4abt + O(t^{\frac{1}{2}}), \tag{MT.1.17}$$

where (MT.3.3) means the reference on formula (3.3) from the work [7, 9], etc. Function $Y_{\rm V}$ solves the following equation

$$\frac{d^2y}{d^2t} = (\frac{1}{2y} + \frac{1}{y-1})(\frac{dy}{dt})^2 - \frac{1}{t}\frac{dy}{dt} + \frac{(y-1)^2}{t^2}(\alpha y + \frac{\beta}{y}) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1} \tag{MT.1.9}$$

with

$$\alpha = -\beta = \frac{1}{2}n^2 = 0, \quad \gamma = 4(1-2\Theta), \quad \delta = -8. \tag{MT.1.10}$$

We note that our variable $t$ in previous sections does not coincide with $t$ from [7, 9]. To make the difference, afterwards, we denote $t$ from [7, 9] as $\hat{t}$

$$t_{\rm JM} \equiv t_{\rm our} \equiv t = 4t_{\rm MT} \equiv 4\hat{t}.$$

Comparing our results (3.1) with the above formulas we conclude that for coincidence of the results there should hold the following equations

$$-\frac{1}{2\sqrt{2}}e^{-\frac{\pi\imath}{4}}\sqrt{\beta_0}\cos(\frac{t}{2}+\imath\beta_0\ln t-\imath\ln\sqrt{\beta_0}+\imath\ln\hat{v}) = -\frac{\imath}{4}(ae^{2\imath s}-be^{-2\imath s}),$$
$$4\sqrt{2}e^{-\frac{\pi\imath}{4}}\sqrt{\beta_0}\sin(\frac{t}{2}+\imath\beta_0\ln t-\imath\ln\sqrt{\beta_0}+\imath\ln\hat{v}) = 4(ae^{2\imath s}+be^{-2\imath s}).$$

Thus, we have

$$\beta_0 = 2\imath ab,\ \hat{v} = \sqrt{2}be^{-\pi\imath/4}e^{-4\imath ab\ln 2}. \tag{B.1}$$

Formula (B.1) connect our parameters (left-hand side) with that from [7, 9] (right-hand side).

Let us now turn to the monodromy data. The authors of [7, 9] expressed the monodromy data through parameters $I^p(p=0,1)$. From formulas (MT.2.7), (MT.2.8a), (MT.2.8b) and (MT.2.10a), (MT.2.10b) we find the following

$$I^p = \frac{m^p_{11}(MT)-e^{-\pi\imath\Theta}}{m^p_{11}(MT)-e^{\pi\imath\Theta}},\ p=0,1 \tag{B.2}$$

where $m^p_{11}(MT)$ are the elements of the monodromy matrices of [7, 9]. The result (MT.2.103a) and (MT.2.103b) for $I^p$ reads as

$$I^p = \frac{\overline{C}_p\imath^{-1}e^{-2\pi ab}-e^{\pi\imath\Theta}}{\overline{C}_p\imath^{-1}e^{-2\pi ab}-e^{-\pi\imath\Theta}}, \tag{B.3}$$

where (see (MT.2.101a), (MT.1.101b) and (MT.2.63))

$$\overline{C}_0\imath^{-1}e^{-2\pi ab} = \frac{e^{\pi\imath/4}2b\sqrt{\pi}}{\Gamma(1-2\imath ab)}e^{-\pi ab}e^{-6\imath ab\ln 2}, \quad \overline{C}_1\imath^{-1}e^{-2\pi ab} = -\frac{e^{\pi\imath/4}\sqrt{\pi}}{b\Gamma(2\imath ab)}e^{-\pi ab}e^{6\imath ab\ln 2}. \tag{B.4}$$

Now put in our results (3.3) $\Theta_\infty = 0$ and express them using (B.1) through parameters $a$ and $b$. We find the following:

$$m^0_{11}(MT) = -\frac{e^{\pi\imath/4}\sqrt{\pi}}{b\Gamma(2\imath ab)}e^{-\pi ab}e^{6\imath ab\ln 2}, \quad m^1_{11}(MT) = \frac{e^{\pi\imath/4}2b\sqrt{\pi}}{\Gamma(1-2\imath ab)}e^{-\pi ab}e^{-6\imath ab\ln 2}. \tag{B.5}$$

Since our $m^p_{11}$ and $\overline{C}_0\imath^{-1}e^{-2\pi ab}$ does not contain $\Theta$ then comparing (B.2) and (B.5) with (B.3) and (B.4) one can see that there is a discrepancy between them.

Let us consider the appearance of the contradiction in more details.

The WKB-formulas outside the turning point (MT.2.45a) and (MT.2.45b) are in agreement with ours. The difference comes out from the next section (Section C of [7, 9]), where it becomes clear (see (MT.2.64a) and (MT.2.64b)) that MT fixed their solution at $\arg x = -\pi/2$ ($x$ from [7, 9] is our $\lambda$). The way of calculation of monodromy data in [7, 9] also shows that MT calculated the monodromy data for solution $Y_3$ (in our notations, see Sec. 2) but not $Y_2$ as we do it. This is evident from Figures 2 [7, 9], where the cut is going up (from 1/2 to $+\imath\infty$) and arrows show that the matching is done from below of the cut $(0, 1)$. This speculation means that MT fixed solution at $\arg x = -\pi/2$.

To check this conjecture let us calculate the monodromy data for solution $Y_3$. For brevity, we restrict ourselves with finding $I^0$. We obtain for the monodromy matrices defined in Sec. 2

$$M^p(Y_3) = S_2^{-1} M^p S_2, \quad \text{where } M^p \equiv M^p(Y_2).$$

That is

$$m_{11}^p(Y_3) = m_{11}^p - s_2 m_{21}^p,$$

and (see (3.3))

$$\begin{aligned} m_{11}^0(Y_3) = 2\cos\pi\Theta_0 - \frac{\imath\sqrt{2\pi}}{\Gamma(1-\beta)}\hat{v}e^{\frac{\pi\imath\Theta_\infty}{2} - \beta\ln 2 + \frac{\pi\imath\beta}{2}}\Big|_{\substack{\Theta_0=\Theta \\ \Theta_\infty=0}} = \\ = 2\cos\pi\Theta - \frac{\imath\sqrt{2\pi}}{\Gamma(1-\beta)}\hat{v}e^{-\beta\ln 2 + \frac{\pi\imath\beta}{2}} = 2\cos\pi\Theta - m_{11}^1(\Theta_\infty = 0). \end{aligned} \tag{B.6}$$

By identifying $m_{11}^p(MT)$ with $m_{11}^p(Y_3)$ we read (B.2)

$$I^p = \frac{m_{11}^p(Y_3) - e^{-\pi\imath\Theta}}{m_{11}^p(Y_3) - e^{\pi\imath\Theta}},\ p = 0, 1. \tag{B.7}$$

Now (B.6) and (B.7) yield

$$I^p = \frac{m_{11}^{1-p} - e^{\pi\imath\Theta}}{m_{11}^{1-p} - e^{-\pi\imath\Theta}}. \tag{B.8}$$

Comparing (B.3) and (B.4) with (B.8) and (B.5) we find them in the complete agreement.

The difference in parametrization of monodromy data pointed above is of crucial importance for building the connection formulas. Indeed, the results of [JM] yield the monodromy data for function $Y_2$ but not for $Y_3$. Consequently, the monodromy data as $t \to \infty$ must also be given for function $Y_2$. Thus, the results of MT concerning connection formulas for the case $S_2 \neq I$ seemingly should be revised. In the case of physical interest MT studied in [7, 9] $S_2 = I$ (since $\sigma = 0$ and $\Theta_\infty = 0$, see (4.3)) and our results agree.

## References


1. M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II, Physica **2D** (1981), 407–448.
2. K. Okamoto, On the $\tau$-function of the Painlevé equations, Physica **2D**, 525–535 (1981) .
3. M. Jimbo, Monodromy problem and the boundary condition for some Painlevé equations, Publ. RIMS, Kyoto Univ., **18**, 1137–1161 (1982).
4. B. M. McCoy, Spin systems, statistical mechanics and Painlevé functions, in: Painlevé Transcendents Their Asymptotics and Physical Applications, Ed. Levi D. and P. Winternitz., NATO ASI Series B: Physics, **278**, 377–391 (1992).
5. A. Bobenko and U. Eitner, Bonnet surfaces and Painlevé equations, Sfb 288 Preprint No. 151, TU Berlin (1994).
6. A. R. Its and V. Yu. Novokshenov, The isomonodromic deformation method in the theory of Painlevé equations, Lect. Notes in Math., **1191**, Springer-Verlag, Berlin, Heidelberg, New-York, Tokyo (1986).
7. B. M. McCoy and Sh. Tang, Connection formulae for Painlevé V functions, Physica **19D**, 42–72 (1986).
8. B. M. McCoy and Sh. Tang, Connection formulae for Painlevé V functions II, The $\delta$ function Bose gas problem, Physica **20D**, 187–216 (1986).
9. B. M. McCoy and Sh. Tang, Connection formulae for Painlevé functions, Physica **18D**, 190–196 (1986).
10. K. Takano, Reduction for Painlevé equations at the fixed singular points of the first kind, Funkcial. Ekvac., **29**, 99–119 (1986).
11. K. Takano, Reduction for Painlevé equations at the fixed singular points of the second kind, J.Math. Soc. Jpn., **42**, No. 3, 423–443 (1990).
12. S. Yoshida, 2-parameter family of solutions for Painlevé equations (I)-(V) at an irregular singular point, Funkcial. Ekvac., **28**, 233–248 (1985).
13. V. I. Gromak, On the theory of Painlevé equations, Diff. Eq., **11**, 373–376 (1975).
14. A. V. Kitaev, The method of isomonodromy deformations and the asymptotics of solutions of the "complete" third Painlevé equations, Matem. sb., **134**, No. 3 (1987) (in Russian), English transl.: Math. USSR Sb., **62**, No. 2, 421–444 (1989).
15. A. V. Kitaev, An isomonodromy cluster of two regular singularities, Sfb 288 Preprint No. 149, TU Berlin (1994).
16. A. V. Kitaev, A. V. Rybin, and J. Timonen, Similarity solutions of the deformed Maxwell–Bloch system, J. Phys. A, **26**, 3583–3595 (1993).
17. F. V. Andreev and A. V. Kitaev, Connection formulas for asymptotics of the fifth Painlevé transcendent on the real axis. II, in preparation.
18. A. V. Kitaev, The justification of the asymptotic formulas obtained by the isomonodromy deformation method, Zap. Nauchn. Sem. LOMI, **179** (1989); (English transl.: J. Sov. Math., **57** (1991)).
19. A. R. Its, A. S. Fokas, and A. A. Kapaev, On the asymptotic analysis of the Painlevé equations via the isomonodromy method, Nonlinearity, **7**, No. 5, 1291–1326 (1994).
20. A. S. Fokas, Ugurhan Mugan, and Xin Zhou, On the solvability of Painlevé I, III, and V, INS #192, March 1992.
21. M. V. Fedoruk, Asymptotic methods for linear ordinary differential equations, Nauka, Moscow (in Russian) (1983).
22. F. W. J. Olver, Asymptotics and Special Functions, Academic Press, New York, London (1974).
23. A. R. Its, "Isomonodromic" solutions of equations of zero curvature, Math. USSR Isv., **26** 530–565 (1986).
24. H. Bateman and A. Erdélyi, Higher Transcendental Functions, I, II, McGraw–Hill Book Company Inc. (1953).