Spin generalization of the Ruijsenaars-Schneider model, non-abelian 2D Toda chain and representations of Sklyanin algebra

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Abstract

Action-angle type variables for spin generalizations of the elliptic Ruijsenaars-Schneider system are constructed. The equations of motion of these systems are solved in terms of Riemann theta-functions. It is proved that these systems are isomorphic to special elliptic solutions of the non-abelian 2D Toda chain. A connection between the finite gap solutions of solitonic equations and representations of the Sklyanin algebra is revealed and discrete analogs of the Lame operators are introduced. A simple way to construct representations of the Sklyanin algebra by difference operators is suggested.

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1 Introduction

In a sense this paper is a preliminary result of our recent attempt to analyse representations of the Sklyanin algebra. This is the algebra with four generators $S_0, S_\alpha,\ \alpha = 1,2,3$, subject to homogeneous quadratic relations

$$[S_0, S_\alpha]_- = iJ_{\beta\gamma}[S_\beta, S_\gamma]_+, \tag{1.1}$$

$$[S_\alpha, S_\beta]_- = i[S_0, S_\gamma]_+ \tag{1.2}$$

($[A,B]_\pm = AB \pm BA$, a triple of latin indices α, β, γ in (1.1, 1.2) stands for any *cyclic* permutation of $(1,2,3)$). Structure constants of the algebra $J_{\alpha\beta}$ have the form

$$J_{\alpha\beta} = \frac{J_\beta - J_\alpha}{J_\gamma}, \tag{1.3}$$

where J_α are arbitrary constants. Therefore, the relations (1.1-1.3) define a two-parametric family of quadratic algebras. These relations (1.1-1.3) were introduced in the paper [1] as the minimal set of conditions under which operators

$$L(u) = \sum_{a=0}^{3} W_a(u) S_a \otimes \sigma_a, \tag{1.4}$$

satisfy the equation

$$R^{23}(u-v)L^{13}(u)L^{12}(v) = L^{12}(v)L^{13}(u)R^{23}(u-v). \tag{1.5}$$

Here σ_α are Pauli matrices, σ_0 is the unit matrix; $W_a(u) = W_a(u|\eta,\tau),\ a = 0,\ldots,3$ are functions of the variable u with parameters η and τ:

$$W_a(u) = \frac{\theta_{a+1}(u)}{\theta_{a+1}(\eta/2)} \tag{1.6}$$

($\theta_a(x) = \theta_a(x|\tau)$ are standard Jacobi theta-functions with characteristics and the modular parameter τ; for the definitions see Appendix to Section 6);

$$R(u) = \sum_{a=0}^{3} W_a(u + \frac{\eta}{2})\sigma_a \otimes \sigma_a \tag{1.7}$$

is the elliptic solution of the quantum Yang-Baxter equation

$$R^{23}(u-v)R^{13}(u)R^{12}(v) = R^{12}(v)R^{13}(u)R^{23}(u-v) \tag{1.8}$$

that corresponds to the 8-vertex model. (The limit $\tau \to 0$ yields the R-matrix of the 6-vertex model.)

In (1.5), (1.8) we use the following standard notation. For any module M over the Sklyanin algebra eq. (1.4) defines an operator in the tensor product $M \otimes \mathbf{C}^2$. The operator $L^{13}(u)$ ($L^{12}(u)$) in the tensor product $M \otimes \mathbf{C}^2 \otimes \mathbf{C}^2$ acts as $L(u)$ on the first and the third spaces and as the identity operator on the second one (acts as $L(u)$ on the first and the second spaces and as the identity operator on the third one). Similarly, R^{23} acts identically on M and coincides with the operator (1.7) on the last two spaces.

Classification of discrete quantum systems solvable by the quantum inverse scattering method (see the reviews [2], [3], [4]) is equivalent to solving eq. (1.5), where $R(u)$ is a fixed solution to the Yang-Baxter equation (1.8).

More general elliptic solutions to (1.8) were found in [5]. Corresponding generalizations of the Sklyanin algebra were introduced in [6], [7]. At present time only the simplest finite-dimensional representations of these generalized Sklyanin algebras are known. It would be very interesting to construct representations of these algebras in terms of difference operators similar to those found in [8] for the original Sklyanin algebra.

As it was shown in the paper [8], the operators $S_a,\ a = 0, \dots, 3$ admit representations in the form of second order difference operators acting in the space of meromorphic functions $f(x)$ of one complex variable x. One of the series of such representations has the form

$$(S_a f)(x) = (i)^{\delta_{2,a}} \theta_{a+1}(\eta/2) \frac{\theta_{a+1}(x - l\eta) f(x + \eta) - \theta_{a+1}(-x - l\eta) f(x - \eta)}{\theta_1(x)}. \tag{1.9}$$

By a straightforward but tedious computation one can check that for any τ, η, l the operators (1.9) satisfy commutation relations (1.1-1.3), the values of the structure constants being

$$J_\alpha = \frac{\theta_{\alpha+1}(\eta)\theta_{\alpha+1}(0)}{\theta^2_{\alpha+1}(\eta/2)}. \tag{1.10}$$

Therefore, the values of τ and η parametrize the structure constants, while l is the parameter of the representation. Note that the original Sklyanin's parameter denoted by η in the paper [8] is equal to *half* of our η entering (1.6), (1.7), (1.9) and (1.10).

Putting $f_n = f(n\eta + x_0)$, to the operators (1.9) we assign difference Schrödinger operators

$$S_a f_n = A^a_n f_{n+1} + B^a_n f_{n-1} \tag{1.11}$$

with quasiperiodic coefficients. The spectrum of a generic operator of this form in the space $l^2(\mathbf{Z})$ (square integrable sequences f_n) has the structure of the Cantor set type. If η is a rational number, $\eta = p/q$, then operators (1.11) have q-periodic coefficients. In general q-periodic difference Schrödinger operators have q unstable bands in the spectrum.

In Sect. 5 of this paper we show that the spectral properties of the operator S_0 given by eq. (1.9) are in this sense extremely unusual:

Theorem 1.1 *The operator S_0 given by eq. (1.9) for positive integer values of "spin" l and arbitrary η has $2l$ unstable bands in the spectrum. Its Bloch functions are parametrized by points of a hyperelliptic curve of genus $2l$ defined by the equation*

$$y^2 = R(\varepsilon) = \prod_{i=1}^{2l+1} (\varepsilon^2 - \varepsilon_i^2). \tag{1.12}$$

Bloch eighenfunctions $\psi(x, \pm\varepsilon_i)$ of the operator S_0 at the edges of bands span an invariant functional subspace for all operators S_a. The corresponding $4l + 2$-dimensional representation of the Sklyanin algebra is a direct sum of two equivalent $2l+1$-dimensional representations of the Sklyanin algebra.

Remark. In section 5 we show that there is a unique choice of signs for ε_i such that the space of *irreducible* representation of the Sklyanin algebra is spanned by Bloch eigenfunctions $\psi(x, \varepsilon_i)$. Unfortunately, at this stage we do not know at this stage any explicit constructive description of the corresponding splitting of the edges in two parts. We conjecture that for real structure constants (when all ε_i are real) one has to choose all positive edges of bands: $\varepsilon_i > 0$.

This theorem indicates a connection between representations of the Sklyanin algebra and the theory of finite gap integration of solitonic equations. (The theory of finite difference Schrödinger operators [9], [10], [11], [12] was developed in the context of solving the Toda chain and difference KdV equations.) Furthermore, this theorem suggests that S_0 is the proper difference analog of the classical Lame operator

$$L = -\frac{d^2}{dx^2} + l(l+1)\wp(x), \tag{1.13}$$

which can be obtained from S_0 in the limit $\eta \to 0$. Finite gap properties of higher Lame operators (for arbitrary integer values of l) were established in [13].

In Sect. 6 of this paper we suggest a relatively simple way to derive the realization (1.9) of the Sklyanin algebra by difference operators. This approach partially explains the origin of these operators. The basic tool is a key property of the elementary R-matrix (1.7), which was used by Baxter in his solution of the eight-vertex model and called by him "pair-propagation through a vertex" [14]. A suitable generalization of this property for the arbitrary spin L-operator (1.4) leads to formulas (1.9). This derivation needs much less amount of computations than the direct substitution of the operators (1.9) into the commutation relations (1.1, 1.2). This method gives automatically the three representation series obtained by Sklyanin and an extra one which, presumably, was unknown.

In the paper [15] a remarkable connection between the motion of poles of the elliptic solutions of KdV equation (which are isospectral deformations of the higher Lame potentials) and the Calogero-Moser dynamical system was revealed. As it has been shown in [16], [17], this relation becomes an isomorphism in the case of the elliptic solutions of the Kadomtsev-Petviashvili (KP) equation. The methods of finite gap integration of the KP equation were applyed to the Calogero-Moser system in the paper [18], where the complete solution in terms of Riemann theta-functions was obtained. These results have been extended to spin generalizations of the Calogero-Moser system in the paper [19].

The main goal of this work is to extend this theory to elliptic solutions of the two-dimensional (2D) Toda chain and its non-abelian analogs. The equations of the 2D Toda chain have the form

$$\partial_+\partial_-\varphi_n = e^{\varphi_n - \varphi_{n-1}} - e^{\varphi_{n+1} - \varphi_n}, \quad \partial_\pm = \frac{\partial}{\partial t_\pm}. \tag{1.14}$$

Let us consider *elliptic solutions with respect to the discrete variable* n, specifically,

$$\varphi_n(t_+, t_-) = \varphi(n\eta + x_0, t_+, t_-) \tag{1.15}$$

such that

$$c(x, t_+, t_-) = \exp(\varphi(x, t_+, t_-) - \varphi(x - \eta, t_+, t_-)) \tag{1.16}$$

is an elliptic function of the variable x. In what follows we show that the function $(\exp\varphi)$ has the form

$$\exp\varphi(x,t_+,t_-)=\prod_{i=1}^{n}\frac{\sigma(x-x_i+\eta)}{\sigma(x-x_i)},\qquad x_i=x_i(t_+,t_-), \tag{1.17}$$

($\sigma(x|\omega_1,\omega_2)$ is the standard Weierstrass σ-function) and the dynamics of its poles x_i with respect to the time flows t_+,t_- coincides with the equations of motion of the Ruijsenaars-Schneider dynamical system: [20]:

$$\ddot{x}_i=\sum_{s\neq i}\dot{x}_i\dot{x}_s(V(x_i-x_s)-V(x_s-x_i)), \tag{1.18}$$

where

$$V(x)=\zeta(x)-\zeta(x+\eta),\quad \zeta(x)=\frac{\sigma(x)'}{\sigma(x)}. \tag{1.19}$$

This system is the relativistic analog of the Calogero-Moser model. Hamiltonians generating the commuting $t_\pm$-flows have the form

$$H_\pm=\sum_{j=1}^{n}e^{\pm p_j}\prod_{s\neq j}^{n}\left(\frac{\sigma(x_j-x_s+\eta)\sigma(x_j-x_s-\eta)}{\sigma^2(x_j-x_s)}\right)^{1/2} \tag{1.20}$$

with canonical Poisson brackets: $\{p_i,x_k\}=\delta_{ik}$.

Our proof of this statement allows us to construct the action-angle variables for the system (1.18) and to solve it explicitly in terms of theta-functions. Being applyed to the non-abelian analog of the 2D Toda chain, this approach leads to spin generalization of the Ruijsenaars-Schneider model.

This generalized model is a system of N particles on the line with coordinates x_i and internal degrees of freedom given by l-dimensional vectors $a_i=(a_{i,\alpha})$ and l-dimensional covectors $b_i^+=(b_i^\alpha)$, $\alpha=1,\ldots,l$. The equations of motion have the form

$$\ddot{x}_i=\sum_{j\neq i}(b_i^+a_j)(b_j^+a_i)(V(x_i-x_j)-V(x_j-x_i)), \tag{1.21}$$

$$\dot{a}_i=\sum_{j\neq i}a_j(b_j^+a_i)V(x_i-x_j), \tag{1.22}$$

$$\dot{b}_i^+=-\sum_{j\neq i}b_j^+(b_i^+a_j)V(x_j-x_i). \tag{1.23}$$

The potential $V(x)$ is given by the function (1.19) or its trigonometric or rational degenerations $(V(x)=(\coth x)^{-1}-(\coth(x+\eta))^{-1}, V(x)=x^{-1}-(x-\eta)^{-1}$, respectively). Hamiltonian formalism for this system needs special consideration and will not be discussed in this paper.

Let us count the number of non-trivial degrees of freedom. The original system has $2N+2Nl$ dynamical variables $x_i,\ \dot{x}_i,\ a_{i,\alpha},\ b_i^\alpha$. The equations of motion (1.21-1.23) are symmetric under rescaling

$$a_i\rightarrow\lambda_i a_i,\quad b_i\rightarrow\frac{1}{\lambda_i}b_i \tag{1.24}$$

The corresponding integrals of motion have the form $I_i = \dot{x}_i - (b_i^+ a_i)$. Let us put them equal to zero:

$$\dot{x}_i = (b_i^+ a_i), \tag{1.25}$$

The reduced system is defined by N extra constraints $\sum_\alpha b_i^\alpha = 1$ (they destroy the symmetry (1.24)). Therefore, the phase space of the reduced system has dimension $2Nl$. Moreover, the system has a further symmetry:

$$a_i \to W^{-1} a_i, \quad b_i^+ \to b_i^+ W, \tag{1.26}$$

where W is any matrix in $GL(r, \mathbf{R})$ (independent of i) preserving the above condition on b_i's. This means that W must leave the vector $v = (1, \cdots, 1)$ invariant. Taking this symmetry into account, it is easy to see that dimension of the completely reduced phase space $\mathcal{M}$ is

$$\dim \mathcal{M} = 2\left[Nl - \frac{l(l-1)}{2}\right]. \tag{1.27}$$

In the next three sections of this paper we derive explicit formulas for general solutions of the system (1.21-1.23) in terms of theta-functions. We would like to stress that these formulas are identical to those obtained for spin generalizations of the Calogero-Moser model in the paper [19]. At the same time the class of auxiliary spectral curves (in terms of which the theta-functions are constructed) is different. These curves can be described purely in terms of algebraic geometry.

To each smooth algebraic curve Γ of genus N it corresponds a N-dimensional complex torus $J(\Gamma)$ (Jacobian of the curve). A pair of points $P^{\pm} \in \Gamma$ defines a vector U in the Jacobian. Let us consider a class of curves having the following property: there exists a pair of points on the curve such that the complex linear subspace generated by the corresponding vector U *is compact*, i.e., it is an elliptic curve $\mathcal{E}_0$. This means that there exist two complex numbers $2\omega_\alpha$, Im $\omega_2/\omega_1 > 0$, such that $2\omega_\alpha U$ belongs to the lattice of periods of holomorphic differentials on Γ. From pure algebraic-geometrical point of view the problem of the description of such curves is transcendental. It turns out that this problem has an explicit solution and algebraic equations that define such curves can be written as a characteristic equations for the Lax operator corresponding to the Ruijsenaars-Shneider system. Moreovere, it turns out that in general position $\mathcal{E}_0$ intersects theta-divisor at N points x_i and if we move $\mathcal{E}_0$ in the direction that is defined by the vector V^+ (V^-) tangent to $\Gamma \in J(\Gamma)$ at the point P^+ (P^-), then the intersections of $\mathcal{E}_0$ with the theta-divisor move according to the Ruijsenaars-Shneider dynamics. An analogous description of spin generalisations of this system is very similar. The corresponding curves have two sets of points $P_i^{\pm}$, $i = 1, \ldots, l$ such that in the linear subspace spanned by the vectors corresponding to each pair there exist a vector U with the same property as above.

The following remark is in order. The geometric interpretation of integrable many body systems of Calogero-Moser-Sutherland type consists in the representation of the models as reductions of geodesic flows on symmetric spaces [21]. Equivalently, these models can be obtained from free dynamics in a larger phase space possessing a rich symmetry by means of the hamiltonian reduction [22]. A generalization to infinite-dimensional phase spaces (cotangent bundles to current algebras and groups) was suggested in [23], [24]. The infinite-dimensional gauge symmetry allows one to make a reduction to finite degrees of freedom. Among systems having appeared this way, there are Ruijsenaars-Schneider-type models and the elliptic Calogero-Moser model.

A further generalization of this approach should consist in considering dynamical systems on cotangent bundles to moduli spaces of stable holomorphic vector bundles on Riemann surfaces. Such systems were introduced by Hitchin in the paper [25], where their integrability was proved. An attempt to identify the known many body integrable systems in terms of the abstract formalism developed by Hitchin was recently made in [26]. To do this, it is necessary to consider vector bundles on algebraic curves with singular points. It turns out that the class of integrable systems corersponding to the Riemann sphere with marked points includes spin generalizations of the Calogero-Moser model as well as integrable Gaudin magnets [27] (see also [28]).

However, Hitchin's approach, explaining the algebraic-geometrical origin of integrable systems, does not allow one to obtain explicit formulas for solutions of equations of motion. Furthermore, in general case any explicit form of the equations of motion is unknown. We hope that the method suggested for the first time in [18]) (for the elliptic Calogero-Moser system) and further developed in the present paper, could give an alternative approach to Hitchin's systems; may be less invariant but yielding more explicit formulas. We suspect that this method has not yet been used in its full strength. Conjecturelly, to each Hitchin's system one can assign a linear problem having solutions of a special form. In terms of these solutions one might construct explicit formulas solving the initial system.

Concluding the introduction, we remark that this paper can be devided into three parts which are relatively independent. The structure of the first one (Sects. 2-4) is very similar to that of the paper [19]. Furthermore, in order to make our paper self-contained and to stress the universal character of the method suggested in [18], we sometimes use the literal citation of the paper [19]. At the same time we skip some technical detailes common for both cases, trying to stress the specifics of difference equations. In the second part (Sect. 5) discrete analogs of Lame operators are introduced and studied. Finally, in the third part (Sect. 6) we give a simple derivation of difference operators representing the Sklyanin algebra and explain the origin of these operators. Actually, we expect a deeper connection between the three main topics of this paper; a discussion on this point is given in Sect. 7.

2 Generating linear problem

The equations of the non-abelian 2D Toda chain have the form

$$\partial_+((\partial_- g_n)g_n^{-1}) = g_n g_{n-1}^{-1} - g_{n+1}g_n^{-1}. \tag{2.1}$$

These equations are equivalent to the compatibility condition for the overdetermined system of the linear problems

$$\partial_+\psi_n(t_+,t_-) = \psi_{n+1}(t_+,t_-) + v_n(t_+,t_-)\psi_n(t_+,t_-), \tag{2.2}$$

$$\partial_-\psi_n(t_+,t_-) = c_n(t_+,t_-)\psi_{n-1}(t_+,t_-), \tag{2.3}$$

where

$$c_n = g_n g_{n-1}^{-1}, \qquad v_n = (\partial_+ g_n) g_n^{-1} \tag{2.4}$$

(g_n is l by l matrix). Similarly to the case of the Calogero-Moser model and its spin generaliztions [18], [19], the isomorphism between the system (1.21-1.23) and the pole dynamics of elliptic solutions to the non-abelian 2D Toda chain is based on the fact that the auxiliary linear problem with elliptic coefficients has infinite number of *double-Bloch* solutions.

We call a meromorphic vector-function $f(x)$ that satisfies the following monodromy properties:

$$f(x + 2\omega_\alpha) = B_\alpha f(x), \quad \alpha = 1, 2, \tag{2.5}$$

a *double-Bloch function.* Here ω_α are two periods of an elliptic curve. The complex numbers B_α are called *Bloch multipliers.* (In other words, f is a meromorphic section of a vector bundle over the elliptic curve.) Any double-Bloch function can be represented as a linear combination of elementary ones.

Let us define a function $\Phi(x, z)$ by the formula

$$\Phi(x, z) = \frac{\sigma(z + x + \eta)}{\sigma(z + \eta)\sigma(x)} \left[\frac{\sigma(z - \eta)}{\sigma(z + \eta)}\right]^{x/2\eta}. \tag{2.6}$$

Using addition theorems for the Weierstrass σ-function, it is easy to check that this function satisfies the difference analog of the Lame equation:

$$\Phi(x + \eta, z) + c(x)\Phi(x - \eta, z) = E(z)\Phi(x, z), \tag{2.7}$$

where

$$c(x) = \frac{\sigma(x - \eta)\sigma(x + 2\eta)}{\sigma(x + \eta)\sigma(x)}. \tag{2.8}$$

Here z plays the role of spectral parameter and parametrizes eigenvalues $E(z)$ of the difference Lame operator:

$$E(z) = \frac{\sigma(2\eta)}{\sigma(\eta)} \frac{\sigma(z)}{(\sigma(z - \eta)\sigma(z + \eta))^{1/2}}. \tag{2.9}$$

The Riemann surface $\hat{\Gamma}_0$ of the function $E(z)$ is a two-fold covering of the initial elliptic curve Γ_0 with periods $2\omega_\alpha$, $\alpha = 1, 2$. Its genus is equal to 2.

Considered as a function of z, $\Phi(x, z)$ is double-periodic:

$$\Phi(x, z + 2\omega_\alpha) = \Phi(x, z). \tag{2.10}$$

For values of x such that $x/2\eta$ is an integer, Φ is well defined meromorphic function on Γ_0. If $x/2\eta$ is a half-integer number, then Φ becomes single-valued on $\hat{\Gamma}_0$. For general values of x one can define a single-valued branch of $\Phi(x, z)$ by cutting the elliptic curve Γ_0 between the points $z = \pm\eta$.

As a function of x the function $\Phi(x, z)$ is double-Bloch function, i.e.

$$\Phi(x + 2\omega_\alpha, z) = T_\alpha(z)\Phi(x, z), \tag{2.11}$$

where Bloch multipliers are equal to

$$T_\alpha(z) = \exp(2\zeta(\omega_\alpha)(z+\eta))\left(\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right)^{\omega_\alpha/\eta}. \tag{2.12}$$

In the fundamental domain of the lattice defined by $2\omega_\alpha$ the function $\Phi(x,z)$ has a unique pole at the point $x=0$:

$$\Phi(x,z) = \frac{1}{x} + A + O(x), \quad A = \zeta(z+\eta) + \frac{1}{2\eta}\ln\frac{\sigma(z-\eta)}{\sigma(z+\eta)}. \tag{2.13}$$

That implies that any double-Bloch function $f(x)$ with simple poles at points x_i in the fundamental domain and with Bloch multipliers B_α such that at least one of them is not equal to 1 may be represented in the form:

$$f(x) = \sum_{i=1}^{N} s_i \Phi(x-x_i, z) k^{x/\eta}, \tag{2.14}$$

where z and a complex number k are related by

$$B_\alpha = T_\alpha(z) k^{2\omega_\alpha/\eta}. \tag{2.15}$$

(Any pair of Bloch multipliers may be represented in the form (2.15) with an appropriate choice of parameters z and k.)

Indeed, let $x_i,\ i=1,\ldots,m$, be poles of $f(x)$ in the fundamental domain of the lattice with periods $2\omega_1, 2\omega_2$. Then there exist vectors s_i such that the function

$$F(x) = f(x) - \sum_{i=1}^{m} s_i \Phi(x-x_i, z) k^{x/\eta}$$

is *holomorphic* in x in the fundamental domain. It is a double-Bloch function with the same Bloch multipliers as the function f. Any non-trivial double-Bloch function with at least one of the Bloch multipliers that is not equal to 1 has at least one pole in the fundamental domain. Hence $F=0$.

The gauge transformation

$$f(x) \longmapsto \tilde{f}(x) = f(x)e^{ax}, \tag{2.16}$$

where a is an arbitrary constan does not change poles of any functions and transform double Bloch-function into double-Bloch function. If B_α are Bloch multipliers for f than the Bloch multipliers for $\tilde{f}$ are equal to

$$\tilde{B}_1 = B_1 e^{2a\omega_1}, \quad \tilde{B}_2 = B_2 e^{2a\omega_2}. \tag{2.17}$$

We call two pairs of Bloch multipliers that are connected with each other through the relation (2.17) for some a by equivalent. Note that for all equivalent pairs of Bloch multipliers the product

$$B_1^{\omega_2} B^{-\omega_1} = B. \tag{2.18}$$

is a constant depending on the equivalnce class, only.

Theorem 2.1 *The equations*

$$\partial_t \Psi(x,t) = \Psi(x+\eta,t) + \sum_{i=1}^{N} a_i(t) b_i^+(t) V(x - x_i(t)) \Psi(x,t) \tag{2.19}$$

$$-\partial_t \Psi^+(x,t) = \Psi^+(x-\eta,t) + \Psi^+(x,t) \sum_{i=1}^{N} a_i(t) b_i^+(t) V(x - x_i(t)) \tag{2.20}$$

have N pairs of linearly independent double-Bloch solutions $\Psi_{(s)}(x,t)$, $\Psi^+_{(s)}(x,t)$ with simple poles at points $x_i(t)$ and $(x_i(t)-\eta)$, respectively,

$$\Psi_{(s)}(x+2\omega_\alpha,t) = B_{\alpha,s}\Psi_{(s)}(x,t), \quad \Psi^+_{(s)}(x-2\omega_\alpha,t) = B_{\alpha,s}\Psi_{(s)}(x,t), \tag{2.21}$$

with equivalent Bloch multipliers (i.e. such that the value

$$B_{1,s}^{\omega_2} B_{2,s}^{-\omega_1} = B, \tag{2.22}$$

does not depend on s) if and only if $x_i(t)$ satisfy the equations (1.21) and the vectors a_i, b_i^+ satisfy the constraints (1.25) and the system of equations

$$\dot a_i = \sum_{j\neq i} a_j (b_j^+ a_i) V(x_i - x_j) - \lambda_i a_i, \tag{2.23}$$

$$\dot b_i^+ = -\sum_{j\neq i} b_j^+ (b_i^+ a_j) V(x_j - x_i) + \lambda_i b_i^+, \tag{2.24}$$

where $\lambda_i = \lambda_i(t)$ are scalar functions.

Remark. The system (1.21), (2.23), (2.24) is "gauge equivalent" to the system (1.21)-(1.23). This means that if (x_i, a_i, b_i^+) sartisfy the equations (1.21), (2.23), (2.24), then x_i and the vector-functions

$$\hat a_i = a_i q_i, \;\; \hat b_i^+ = b_i q_i^{-1}, \quad q_i = \exp(\int^t \lambda_i(t')dt') \tag{2.25}$$

are solutions of the system (1.21-1.23).

Theorem 2.2 *If the equations (2.19), (2.20) have N linearly independent double-Bloch solutions with Bloch multipliers satisfying (2.22) then they have infinite number of them. All these solutions can be represented in the form:*

$$\Psi = \sum_{i=1}^{N} s_i(t,k,z) \Phi(x - x_i(t), z) k^{x/\eta}, \tag{2.26}$$

$$\Psi^+ = \sum_{i=1}^{N} s_i^+(t,k,z) \Phi(-x + x_i(t) - \eta, z) k^{-x/\eta}, \tag{2.27}$$

where s_i is l-dimensional vector $s_i = (s_{i,\alpha})$, s_i^+ is l-dimensional covector $s_i^+ = (s_i^\alpha)$. The set of corresponding pairs (z,k) are parametrized by points of algebraic curve defined by the equation of the form

$$R(k,z) = k^N + \sum_{i=1}^{N} r_i(z) k^{N-i} = 0$$

Proof of Theorem 2.1. As it was mentioned above, $\Psi_{(s)}(x,t)$ (as any double-Bloch function) may be written in the form (2.26) with some values of the parameters z_s, k_s. The conditions (2.22) imply that these parameters can be choosen as follows:

$$z_s = z, \quad s = 1, \ldots, N.$$

Let us substitute the function $\Psi(x,t,z,k)$ of the form (2.26) with this particular value of z into eq. (2.19). Since any function with such monodromy properties has at least one pole, it follows that the equation (2.19) is satisfied if and only if the right and left hand sides of this equality have the same singular parts at the points $x = x_i$ and $x = x_i - \eta$.

Comparing the coefficients in front of $(x - x_i)^{-2}$ in (2.19) gives the equalities

$$\dot{x}_i s_i = a_i(b_i^+ s_i), \tag{2.28}$$

hence the vector s_i is proportional to a_i:

$$s_{i,\alpha}(t,k,z) = c_i(t,k,z) a_{i,\alpha}(t), \tag{2.29}$$

and a_i, b_i satisfy the constraints (1.25). Cancellation of the coefficients in front of $(x - x_i + \eta)^{-1}$ gives the conditions

$$-k s_i + \sum_{j \neq i} a_i b_i^+ s_j \Phi(x_i - x_j - \eta, z) = 0. \tag{2.30}$$

Taking into account (1.25) and (2.29), we rewrite them as a matrix equation for the vector $C = (c_i)$:

$$(L(t,z) - kI)C = 0\,, \tag{2.31}$$

where I is the unit matrix and the Lax matrix $L(t,z)$ is defined as follows:

$$L_{ij}(t,z) = (b_i^+ a_j)\Phi(x_i - x_j - \eta, z). \tag{2.32}$$

Finally, cancellation of the poles $(x - x_i)^{-1}$ gives the conditions

$$\dot{s}_i - \left(\sum_{j \neq i} a_j b_j^+ V(x_i - x_j) + (A - \zeta(\eta)) a_i b_i^+ \right) s_i - a_i \sum_{j \neq i} (b_i^+ s_j)\Phi(x_i - x_j, z) = 0. \tag{2.33}$$

Using (2.29), we get the equations of motion (2.23), where

$$\lambda_i(t) = \frac{\dot{c}_i}{c_i} + (\zeta(\eta) - A)\dot{x}_i - \sum_{j \neq i} (b_i^+ a_j)\Phi(x_i - x_j, z)\frac{c_j}{c_i}. \tag{2.34}$$

Rewriting (2.34) in the matrix form,

$$(\partial_t + M(t,z))C = 0\,, \tag{2.35}$$

we find the second operator of the Lax pair:

$$M_{ij}(t,z) = (-\lambda_i + (\zeta(\eta) - A)\dot{x}_i)\delta_{ij} - (1 - \delta_{ij}) b_i^+ a_j \Phi(x_i - x_j, z). \tag{2.36}$$

Substituting in the same way the vector Ψ^+ (2.27) into the equation (2.20), we get:

$$s_i^\alpha(t,k,z) = c_i^+(t,k,z) b_i^\alpha(t), \tag{2.37}$$

$$C^+(L(t,z)-kI)=0\,, \tag{2.38}$$

$$-\partial_t C^+ + C^+M^{(+)}(t,z)=0\,, \tag{2.39}$$

where $C^+=(c_i^+)$, L is given by (2.32), and $M^{(+)}$ has the form (2.36) with changing $\lambda_i(t)$ to

$$\lambda_i^+(t)=-\frac{\dot c_i^+}{c_i^+}+(\zeta(\eta)-A)\dot x_i-\sum_{j\neq i}(b_j^+a_i)\Phi(x_j-x_i,z)\frac{c_j^+}{c_i^+}. \tag{2.40}$$

Moreover, it follows that the covector b_i^+ satisfies the equations

$$\dot b_i^+ = -\sum_{j\neq i} b_j^+(b_i^+a_j)V(x_j-x_i)+\lambda_i^+b_i^+\,. \tag{2.41}$$

The assumption of the theorem imply that the equations (2.31), (2.35) and (2.38), (2.39) have N linear independent solutions (corresponding to different values of k). The compatibility conditions of these equations have the form of the Lax equations

$$\dot L+[M,L]=0,\qquad \dot L+[M^{(+)},L]=0\,. \tag{2.42}$$

In particular, they imply that $\lambda_i=\lambda_i^+$.

The function $\Phi(x,z)$ satisfies the following functional relations:

$$\Phi(x-\eta,z)\Phi(y,z)-\Phi(x,z)\Phi(y-\eta,z)=\Phi(x+y-\eta,z)(V(-x)-V(-y)), \tag{2.43}$$

$$\Phi'(x-\eta,z)=-\Phi(x-\eta,z)(V(-x)+\zeta(\eta)-A)-\Phi(-\eta,z)\Phi(x,z) \tag{2.44}$$

(the constant A is defined in (2.13)). The first relation is equivalent to the three-term functional equation for the Weierstrass σ-function. The second relation follows from the first one in the limit $y\to 0$.

The identities (2.43) and (2.44) allow one to prove by direct computations the following lemma completing the proof:

Lemma 2.1 *For the matrices L and M defined by (2.32) and (2.36) respectively, where a_i and b_i^+ satisfy the equations (2.23) and the constraints (1.25), the Lax equations (2.42) hold if and only if $x_i(t)$ satisfy the equations (1.21).*

■

Proof of Theorem 2.2. As it was proved above, eqs. (2.19), (2.20) have N linearly independent solutions if eqs. (2.23) and (2.24), constraints (3.9) and Lax equations (2.42) are fulfilled for some value of the spectral parameter z. But according to the statement of Lemma 2.1 the Lax equations are then fulfilled for all values of the spectral parameter z. Therefore, for each value of z we can define the double-Bloch solutions of eq. (2.19) with the help of formulas (2.26 and (2.29), where c_i are components of the common solution of eqs. (2.31) and (2.35).

As it follows from (2.31), all addmissible pairs of the spectral parameters z and k satisfy the characteristic equation

$$R(k,z)\equiv \det\left(kI-L(t,z)\right)=0.$$

At the begining of the next section we prove that this equation defines an algebraic curve $\hat{\Gamma}$ of finite genus. That will complete the proof of the theorem. ■

Remark 1. In the abelian case ($l = 1$) the operators of the Lax pair by a "gauge" transformation with the matrix $U_{ij} = a_i\delta_{ij}$ can be represented in the form

$$L_{ij}^{(l=1)} = \dot{x}_i\Phi(x_i - x_j - \eta, z), \tag{2.45}$$

$$M_{ij}^{(l=1)} = \left((\zeta(\eta) - A)\dot{x}_i - \sum_{s\neq i} V(x_i - x_s)\dot{x}_s\right)\delta_{ij} - (1-\delta_{ij})\dot{x}_i\Phi(x_i - x_j, z). \tag{2.46}$$

These formulas are equivalent to the Lax pair obtained in [29], [30].

Remark 2. In the abelian case it is ehough to require that only one of eqs. (2.19), (2.20) has N linearly independent double-Bloch solutions with Bloch multipliers satisfying conditions (2.22).

3 The direct problem

As it follows from the Lax equation (2.42), the coefficients of the characteristic equation

$$R(k, z) \equiv \det\left(kI - L(t, z)\right) = 0 \tag{3.1}$$

do not depend on time. Note that they are invariant with respect to the symmetries (1.24), (1.26).

Theorem 3.1 *The coefficients $r_i(z)$ of the characteristic equation (3.1)*

$$R(k, z) = k^N + \sum_{i=1}^{N} r_i(z)k^{N-i} \tag{3.2}$$

do not depend on t and have the form

$$r_i(z) = \phi_i(z)(I_{i,0} + (1-\delta_{l,1})I_{i,1}\tilde{s}_i(z) + \sum_{s=2}^{m_i} I_{i,s}\partial_z^{s-2}\wp(z+\eta)), \tag{3.3}$$

where

$$m_i = i - 1, \quad i = 1, \ldots, l; \quad m_i = l - 1, \quad i = l+1, \ldots, N; \tag{3.4}$$

$$\phi_i(z) = \frac{\sigma(z+\eta)^{(i-2)/2}\sigma(z-(i-1)\eta)}{\sigma(z-\eta)^{i/2}}. \tag{3.5}$$

$$\tilde{s}_i(z) = \frac{\sigma(z-\eta)\sigma(z-(i-3)\eta)}{\sigma(z+\eta)\sigma(z-(i-1)\eta)}, \tag{3.6}$$

In a neighbourhood of $z = -\eta$ the function $R(k, z)$ can be represented in the form

$$R(k, z) = \prod_{i=1}^{l}(k + (z+\eta)^{-1/2}h_i(z+\eta)) \prod_{i=l+1}^{N}(k + (z+\eta)^{1/2}h_i(z+\eta)), \tag{3.7}$$

where $h_i(z)$ are regular functions of z in a neighbourhood of $z = 0$.

Proof. Due to (2.10) the matrix elements L_{ij} are double-periodic functions of z. Therefore, we can consider them as single-valued functions on the original elliptic curve Γ_0 with a branch cut between the points $z = \pm\eta$. First of all let us show that the coefficients $r_i(z)$ of the characteristic polynomial (3.1) are meromorphic functions on the Riemann surface $\hat{\Gamma}_0$ of the function $E(z)$ defined by (2.9). (This means that $r_i(z)$ are double-valued functions of z with square root branching at the points $z = \pm\eta$.)

This fact can be immediately seen from the "gauge equivalent" form of $L(t,z)$:

$$L(t,z) = G(t,z)\tilde{L}(t,z)G^{-1}(t,z),\ G_{ij} = \delta_{ij}\left[\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right]^{x_i(t)/2\eta}, \tag{3.8}$$

where the matrix elements of $\tilde{L}(t,z)$ have square root branching at the points $z = \pm\eta$. Moreover, writing them explicitly,

$$\tilde{L}_{ij}(t,z) = \frac{(b_i^+ a_j)}{[\sigma(z-\eta)\sigma(z+\eta)]^{1/2}}\frac{\sigma(z+x_i-x_j)}{\sigma(x_i-x_j-\eta)}, \tag{3.9}$$

we conclude that $r_{2i}(z)$ are single-valued meromorphic functions of z (i.e., elliptic functions). Further, $r_{2i+1}(z)$ are meromorphic functions on $\hat{\Gamma}_0$, which are odd with respect to the involution $\hat{\tau}_0 : \hat{\Gamma}_0 \to \hat{\Gamma}_0$, interchanging sheets of the covering $\hat{\Gamma}_0 \to \Gamma_0$ (this involution corresponds to changing sign of the square root: $E(z) \to -E(z)$). Thus the curve $\hat{\Gamma}$ is invariant with respect to the involution

$$\hat{\tau} : \hat{\Gamma} \longmapsto \hat{\Gamma},\quad \hat{\tau}(k,E) \longmapsto (-k,-E), \tag{3.10}$$

which covers the involution $\hat{\tau}_0$.

Let us consider the factor-curve:

$$\Gamma := \{\hat{\Gamma}/\hat{\tau}\}. \tag{3.11}$$

This curve is a N-fold ramified covering of the initial elliptic curve,

$$\Gamma \longmapsto \Gamma_0\,. \tag{3.12}$$

It can be defined by the equation

$$\hat{R}(K,z) = K^N + \sum_{i=1}^{N} R_i(z)K^{N-i} = 0, \tag{3.13}$$

where

$$K = k\left[\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right]^{1/2},\quad R_i(z) = r_i(z)\left[\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right]^{i/2}. \tag{3.14}$$

Let us give some more comments on this statement. The coefficients $R_j(z)$ of the equation (3.13) are meromorphic functions of the complex variable z obeying the following monodromy properties:

$$R_j(z+2\omega_\alpha) = R_j(z)e^{-2j\zeta(\omega_\alpha)\eta}. \tag{3.15}$$

Eq. (3.13) defines a Riemann surface $\tilde{\Gamma}$, which is a N-fold covering of the complex plane. Due to (3.15) this surface is invariant with respect to the transformations

$$z \longmapsto z + 2\omega_\alpha,\quad K \longmapsto Ke^{-2\zeta(\omega_\alpha)\eta}. \tag{3.16}$$

The corresponding factor-surface is an algebraic curve Γ, which is the covering of the elliptic curve with periods $2\omega_\alpha$.

Now we switch to (3.7). This equality follows from the fact that the leading term of $\tilde{L}(t,z)$ in a neighbourhood of the point $z=-\eta$,

$$\tilde{L}_{ij}(t,z)=\frac{(b_i^+a_j)}{[\sigma(-2\eta)(z+\eta)]^{1/2}}+O((z+\eta)^{1/2}), \tag{3.17}$$

has rank l. The corresponding $N-l$-dimensional subspace of eigenvectors $C=(c_1,\ldots,c_N)$ with zero eigenvalue is defined by the equations

$$\sum_{j=1}^{N} c_j a_{j,\alpha}=0, \quad \alpha=1,\ldots,l. \tag{3.18}$$

Let us turn to determination of the coefficients in the characteristic equation (3.1). Since the matrix elements of $\tilde{L}(t,z)$ have simple poles at the point $z=\eta$, then in case of general position the function $r_i(z)$ has pole of order i at this point. It follows from (3.7) that at the point $z=-\eta$ the function r_i, $i=1,\ldots,l$, has pole of order i. If $i=l+1,\ldots,2l$ it has pole of order $2l-i$. At last, if $i=2l+1,\ldots,N$ it has zero of order $i-2l$, i.e.,

$$r_i(z)=(z+\eta)^{-i/2}\rho_i(z+\eta), \quad i=1,\ldots,l, \tag{3.19}$$

$$r_i(z)=(z+\eta)^{i/2-l}\rho_i(z+\eta), \quad i=l+1,\ldots,N, \tag{3.20}$$

where $\rho_i(z)$ are regular functions. All these properties of the functions $r_i(z)$ allow one to represent them in the form (3.3, 3.4). (Note that the function $\phi_i(z)$ defined by eq. (3.5) has pole of order i at the point $z=\eta$ and zero of order $i-2$ at the point $z=-\eta$.) ■

Important remark. It is necessary to emphasize that (3.7) implies that the characteristic equation (3.1) defines a singular algebraic curve. Indeed, (3.7) implies that $(N-l)$ sheets of the corresponding ramified covering intersect at the point ($z=-\eta$, $k=0$). We keep the same notation Γ for the algebraic curve with the resolved singularity at this point.

The coefficients $I_{i,s}$ in (3.3) are integrals of motion. The total number of them is equal to $Nl-l(l-1)/2$ which is exactly half the dimension of the reduced phase space. It follows from the results of the next section that they are independent.

Lemma 3.1 *In general position genus g of the spectral curve Γ (defined by eq. (3.13)) is equal to $Nl-l(l+1)/2+1$.*

Proof. First let us determine genus $\hat{g}$ of the curve $\hat{\Gamma}$ defined by eq. (3.1). By the Riemann-Hurwitz formula $\hat{g}$ we have $2\hat{g}-2=2N+\nu$, where ν is the number of branch points of $\hat{\Gamma}$ over $\hat{\Gamma}_0$, i.e., the number of values of z for which $R(k,z)=0$ has a double root. This is equal to the number of zeros of $\partial_k R(k,z)$ on the surface $R(k,z)=0$ outside the points located above the point $z=-\eta$ (due to the singularity of the original curve mentioned above). The function $\partial_k R(k,z)$ has poles of order $N-1$ above the point $z=\eta$ It has also poles of the same order in l points located above the point $z=-\eta$

that correspond to first l factors in (3.7). In the other $N-l$ points above $z=-\eta$ it has zeros of order $(N-l)(N-2l-1)$. Therefore, $\nu = 4lN - 2l(l+1)$. The curve $\hat{\Gamma}$ is the two-fold branched covering of the spectral curve Γ, the number of branch points being equal to $2N$ (they are located above $z=\pm\eta$). The Riemann-Hurwitz formula gives then the relation $2\hat{g}-2 = 2(2g-2)+2N$, which proves the lemma. ∎

The characteristic equation (3.1) allows us to define two sets of distinguished points on the spectral curve (3.13). It follows from the factorization of $R(k,z)$ (3.7) that the function k has poles at l points lying on the different sheets over the point $z=-\eta$ (they correspond to the first l factors in (3.7)). Let us denote them by P_i^+, $i=1,\dots,l$,. Since a meromorphic function has as many zeros as poles, we conclude from (3.14) and (3.7) that the function k on the unreduced spectral curve $\hat{\Gamma}$ has $2l$ zeros which do not lie above the point $z=-\eta$. These zeros correspond to l points $P_i^-, i=1,\dots,l$, on the spectral curve Γ:

$$k(P_i^-)=0. \tag{3.21}$$

In general position there is such a point P_i^- above each zero z_i^- of the function $r_N(z)$ different from its apparent zero $z=-\eta$:

$$r_N(z) = \tilde{I}_{N,0}\frac{\sigma^{(N-l)/2}(z+\eta)}{\sigma^{N/2}(z-\eta)}\prod_{i=1}^{l}\sigma(z-z_i^-). \tag{3.22}$$

In the abelian case ($l=1$) the second marked point P_1^- lies above the point $z=(N-1)\eta$.

Theorem 3.2 *The components $\Psi_\alpha(x,t,P)$ of the solution $\Psi(x,t,P)$ to the equation (2.19) are defined on the N-fold covering Γ of the initial elliptic curve cutted between the points P_i^+ and P_i^-, $i=1,\dots,l$. Outside these branch cuts they are meromorphic. For general initial conditions the curve Γ is smooth. Its genus equals $g = Nl - \frac{l(l+1)}{2}+1$ and Ψ_α have $(g-1)$ poles $\gamma_1,\dots,\gamma_{g-1}$ which do not depend on the variables x,t. In a neighbourhood of P_i^+, $i=1,\dots,l$ the function Ψ_α has the form*

$$\Psi_\alpha(x,t,P) = (\chi_0^{\alpha i} + \sum_{s=1}^{\infty}\chi_s^{\alpha i}(x,t)(z+\eta)^s)(\kappa_i(z+\eta)^{-1})^{x/\eta}e^{\kappa_i(z+\eta)^{-1}t}\ \Psi_1(0,0,P), \tag{3.23}$$

where $\chi_0^{\alpha i}$ are constants independent of x,t, and κ_i are non-zero eigenvalues of the matrix $(b_i^+a_j)$. In a neighbourhood of P_i^- the function Ψ_α has the form

$$\Psi_\alpha(x,t,P) = (z-z_i^-)^{x/\eta}\ (\sum_{s=0}^{\infty}\tilde{\chi}_s^{\alpha i}(x,t)(z-z_i^-)^s)\ \Psi_1(0,0,P) \tag{3.24}$$

(z_i^- are projections of the points P_i^- to the initial elliptic curve; they are defined by eq. (3.22)). The boundary values $\Psi_\alpha^{(\pm)}$ of the function Ψ_α at the opposite sides of the cuts are connected by the relation

$$\Psi_\alpha^{(+)} = \Psi_\alpha^{(-)}e^{2\pi i x/\eta}. \tag{3.25}$$

Proof. We start with analitic properties of the eigenvectors of the Lax matrix.

By $\hat{\Gamma}^*$ denote the curve $\hat{\Gamma}$ with cuts between the pre-images P_i^+ of the point $z=-\eta$ and the pre-images Q_i^- of the point $z=\eta$, $i=1,\dots,N$. For a generic point $\hat{P}$ of the curve

$\hat{\Gamma}$, i.e., for the pair $(k,z)=\hat{P}$, which satisfies the equation (3.1), there exists a unique eigenvector $C(0,\hat{P})$ of the matrix $L(0,z)$ normalized by the condition $c_1(0,P)=1$. All other components $c_i(0,\hat{P})$ are given by $\Delta_i(0,\hat{P})/\Delta_1(0,\hat{P})$, where $\Delta_i(0,P)$ are suitable minors of the matrix $kI-L(0,z)$. Thus they are meromorphic on $\hat{\Gamma}^*$. The poles of $c_i(0,\hat{P})$ are the zeros on $\hat{\Gamma}^*$ of the first principal minor

$$\Delta_1(0,\hat{P})=\det(k\delta_{ij}-L_{ij}(0,z))=0,\quad i,j>1. \tag{3.26}$$

Therefore, these poles only depend on the initial data.

Lemma 3.2 *The coordinates $c_j(0,\hat{P})$ of the eigenvector $C(0,\hat{P})$ are meromorphic functions on $\hat{\Gamma}^*$. The boundary values $c_j^{\pm}$ of the functions $c_j(0,\hat{P})$ at opposite sides of the cuts satisfy the relation*

$$c_j^+=c_j^-e^{\pi i(x_j(0)-x_1(0))/\eta}. \tag{3.27}$$

In a neighbourhood of the point P_i^+ the functions $c_j(0,\hat{P})$ have the form

$$c_j(0,\hat{P})=(c_j^{(i,+)}(0)+O(z+\eta))(z+\eta)^{(x_1(0)-x_j(0))/2\eta}, \tag{3.28}$$

where $c_j^{(i,+)}(t)$ are eigenvalues of the residue of the matrix $\tilde{L}(0,z)$ at $z=-\eta$, i.e.,

$$\sum_{j=1}^{N}(b_k^+a_j)c_j^{(i,+)}(t)=-\kappa_i c_k^{(i,+)}(t). \tag{3.29}$$

In a neighbourhood of the point Q_i^- the functions $c_j(0,\hat{P})$ have the form

$$c_j(0,\hat{P})=(c_j^{(i,-)}(0)+O(z-\eta))(z-\eta)^{(x_j(0)-x_1(0))/2\eta}, \tag{3.30}$$

where $c_j^{(i,-)}(t)$ are eigenvectors of the residue of the matrix $\tilde{L}(0,z)$ at $z=\eta$.

The proof follows from the representation of $C(0,\hat{P})$ in the form (see (3.8)):

$$C(0,\hat{P})=G(0,z)\tilde{C}(0,P), \tag{3.31}$$

where $G(t,z)$ is defined in (3.8), and $\tilde{C}(0,P)$ is the eigenvector of the matrix $\tilde{L}(0,z)$. Matrix elements of $\tilde{L}(0,z)$ are analitic on the cuts between $z=\pm\eta$. Hence $\tilde{C}(0,P)$ has no discontinuity on the cuts. This proves (3.27). The equations (3.28) and (3.30) are direct consequences of the fact that $\tilde{L}(0,z)$ has simple poles at the points $z=\pm\eta$.

Remark. The vector $\tilde{C}(0,P)$ is invariant under the involution (3.10) (that's why its argument is a point P of the spectral curve Γ rather than a point $\hat{P}\in\hat{\Gamma}$). However, this notation is somewhat misleading since both factors in (3.31) are multi-valued on $\hat{\Gamma}$, and only their product is well-defined.

Lemma 3.3 *The poles of $C(0,\hat{P})$ are invariant under the involution $\hat{\tau}$. The number of them is equal to $2Nl-l(l+1)$.*

To prove the lemma we use the following standard argument. Consider the function of the complex variable z defined by:

$$F(z) = \left(\mathrm{Det}\,|c_i(0, M_j)|\right)^2,$$

where $M_j,\ j = 1, \ldots, N$, are the points above z. It is well defined as a function of z since it does not depend on the ordering of the M_j's. The analitic properties of c_j allow us to represent F in the form

$$F(z) = \tilde{F}(z)\left[\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right]^{\sum(x_i(0)-x_1(0))}, \tag{3.32}$$

where $\tilde{F}$ is a meromorphic function. This means that F has as many zeros as poles. The number of its poles is twice the number of zeros of the vector $C(0, \hat{P})$ whereas the number of zeros of F is equal to the number of branch points ν of the covering $\hat{\Gamma}$ over $\hat{\Gamma}_0$ (3.1). In the proof of Lemma 3.1 we showed that $\nu = 4Nl - 2l(l+1)$. The invariance of poles of $C(0, \hat{P})$ under the involution $\hat{\tau}$ follows from the $\hat{\tau}$-invariance of eq. (3.26) which determines positions of the poles. This completes the proof.

Let $\gamma_1, \ldots, \gamma_{g-1}$ be the points of the spectral curve Γ whose pre-images are poles of $C(0, \hat{P})$. Note that if Γ is smooth, then $g = Nl - l(l+1)/2 + 1$ coincides with its genus.

Let $C(t, \hat{P})$ be the vector obtained from $C(0, \hat{P})$ by the time evolution according to eq. (2.35).

Lemma 3.4 *The coordinates $c_j(t, \hat{P})$ of the vector $C(t, \hat{P})$ are meromorphic on $\hat{\Gamma}^*$. Their poles are located above the points $\gamma_1, \ldots, \gamma_{g-1}$ and do not depend on t. The boundary values $c_j^{\pm}$ of $c_j(t, \hat{P})$ at opposite sides of the cuts satisfy the relation*

$$c_j^+ = c_j^- e^{\pi i(x_j(t)-x_1(0))/\eta}. \tag{3.33}$$

In a neighbourhood of P_i^+ the functions $c_j(t, \hat{P})$ have the form

$$c_j(0, \hat{P}) = (c_j^{(i,+)}(t) + O(z+\eta))(z+\eta)^{(x_1(0)-x_j(t))/2\eta}\exp(\kappa_i(z+\eta)^{-1}t), \tag{3.34}$$

where κ_i and $c_j^{(i,+)}(t)$ are defined in (3.29). In a neighbourhood of Q_i^- the functions $c_j(t, \hat{P})$ have the form

$$c_j(t, \hat{P}) = (c_j^{(i,-)}(t) + O(z-\eta))(z-\eta)^{(x_j(t)-x_1(0))/2\eta}. \tag{3.35}$$

Proof. The fundamental matrix of solutions $S(t, z)$ to the equation

$$(\partial_t + M(t, z))S(t, z) = 0, \quad S(0, z) = 1, \tag{3.36}$$

is a holomorphic function of the variable z outside the cut connecting the points $z = \pm\eta$. We have $L(t, z) = S(t, z)L(0, z)S^{-1}(t, z)$. Therefore, the vector $C(t, z)$ equals $C(t, z) = S(t, z)C(0, z)$ hence it has the same poles as $C(0, P)$.

Let us consider the vector $\tilde{C}(t, \hat{P})$ such that

$$C(t, \hat{P}) = G(t, z)\tilde{C}(t, \hat{P}), \tag{3.37}$$

where $G(t,z)$ is the same diagonal matrix as in (3.8). This vector is an eigenvector of the matrix $\tilde{L}(t,z)$ and satisfies the equation

$$(\partial_t + \tilde{M}(t,z))\tilde{C}(t,P) = 0, \quad \tilde{M} = G^{-1}\partial_t G + G^{-1}MG. \tag{3.38}$$

The matrix elements of $\tilde{M}$ are analitic at the cuts between $z = \pm\eta$. Thus $\tilde{C}(t,\hat{P})$ is analitic at the cuts on $\hat{\Gamma}$. Therefore, the multi-valuedness of $C(t,\hat{P})$ is fully caused by the multi-valuedness of $G(t,z)$. This proves eq. (3.33). Eq. (3.35) follows rom the analiticity of $\tilde{M}$ at the point $z = \eta$. In a neighbourhood of $z = -\eta$ we have:

$$\tilde{M}_{ij}(t,z) = \frac{(b_i^+ a_j)}{(z+\eta)} + O((z+\eta)^0). \tag{3.39}$$

Therefore, in a neighbourhood of P_i^+ it holds

$$\partial_t \tilde{C}(t,\hat{P}) = (\mu_i(t,z) + O(z^0))\tilde{C}(t,\hat{P}),$$

where

$$\mu_i(t,z) = \kappa_i(z+\eta)^{-1} + O(1), \tag{3.40}$$

are eigenvalues of the matrix $\tilde{M}$. This proves eq. (3.34). ∎

Let us now continue the proof. By the initial definition,

$$\Psi(x,t,\hat{P}) = \sum_{j=1}^{N} s_j(t,\hat{P})\Phi(x - x_j(t), z)k^{x/\eta}, \quad s_j(t,\hat{P}) = c_j(t,\hat{P})a_j(t),$$

The solutions Ψ of the linear problem (2.19) are defined on the curve $\hat{\Gamma}$. In order to show that Ψ is well defined on the spectral curve Γ we use the equality

$$c_j(t,\hat{P})\Phi(x - x_j(t), z)k^{x/\eta} = \tilde{c}_j(t,P)\frac{\sigma(z + x - x_j + \eta)}{\sigma(z+\eta)\sigma(x - x_j)}\left[k\left(\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right)^{1/2}\right]^{x/\eta}. \tag{3.41}$$

Recall that components of the vector $\tilde{C}$ are even with respect to the involution $\hat{\tau}$ (3.10). The factor

$$K(P) = k\left[\frac{\sigma(z-\eta)}{\sigma(z+\eta)}\right]^{1/2}, \tag{3.42}$$

is $\hat{\tau}$-invariant too. Thus $\Psi(x,t,P)$ is well defined on the spectral curve Γ. At the same time we see that poles of $\Psi(x,t,P)$ coincide with poles of $\tilde{C}(0,P)$, i.e., they are located at the points $\gamma_1, \ldots, \gamma_{g-1}$.

Note that $K(P)$ is a multi-valued meromorphic function on Γ with zeros and poles (which are nevertheless well defined) at the points P_i^- and P_i^+, $i = 1, \ldots, l$, respectively. Therefore, cutting Γ between $P_i^\pm$, $i = 1, \ldots, l$, we can choose the branch of the third factor in (3.42) in such a way that Ψ becomes single-valued outside these cuts, and its boundary values at the sides of the cuts satisfy the relation (3.25). Consider now the behaviour of Ψ in neighbourhoods of P_i^+. In a neighbourhood of $z = -\eta$ we have:

$$\frac{\sigma(z+x+\eta)}{\sigma(z+\eta)\sigma(x)} = \frac{1}{z+\eta} + O(1). \tag{3.43}$$

Hence in a neighbourhood of P_i^+ it holds

$$\Psi_\alpha = \sum_{j=1}^{N} \left(\frac{a_{j,\alpha} c_j^{(i,+)}(t)}{z+\eta} + O(1) \right) \left[k_i(z) \left(\frac{\sigma(z-\eta)}{\sigma(z+\eta)} \right)^{1/2} \right]^{x/\eta}, \tag{3.44}$$

where $k_i(z)$ is the branch of $k(P)$ defined by i-th factor of the equality (3.7). For $i > l$ the product of the second and the third factors in (3.44) is regular in a neighbourhood of P_i^+. Since the eigenvalues κ_i in (3.29) equal zero for $i > l$, then the first factor is regular in a neighbourhood of $P_i^+, i > l$, too. Therefore, the functions Ψ_α are regular at these points. Similar arguments for $i = 1, \ldots, l$, prove the equality (3.23). Note also that $\Psi_1(0,0,P)$ in (3.23) has simple poles at the points $P_i^+,\ i = 1, \ldots, l$.

Consider now the term $\chi_0^{\alpha i}$ in (3.23). By construction, it does not depend on x. Substituting the series (3.23) into (2.19), we see that it does not depend on t as well. ■

The following theorem can be proved by the same arguments.

Theorem 3.3 *The components $\Psi^{+,\alpha}(x,t,P)$ of the solution $\Psi^+(x,t,P)$ to the equation (2.20) are defined on N-fold covering Γ of the initial elliptic curve with branch cuts between the points P_i^+ and P_i^-, $i = 1, \ldots, l$. Outside these cuts they are meromorphic. In general position, the curve Γ is a smooth algebraic curve. Its genus equals $g = Nl - \frac{l(l+1)}{2} + 1$ and $\Psi^{+,\alpha}$ has $(g-1)$ poles $\gamma_1^+, \ldots, \gamma_{g-1}^+$, which do not depend on x, t. In a neighbourhood of P_i^+, $i = 1, \ldots, l$ the function $\Psi^{+,\alpha}$ has the form*

$$\Psi^{+,\alpha}(x,t,P) = (\chi_0^{+,\alpha i} + \sum_{s=1}^{\infty} \chi_s^{+,\alpha i}(x,t)(z+\eta)^s)(\kappa_i(z+\eta)^{-1})^{-x/\eta} e^{-\kappa_i(z+\eta)^{-1}t}\ \Psi^{+,1}(0,0,P), \tag{3.45}$$

where $\chi_0^{\alpha i}$ are constants independent of x, t. In a neighbourhood of P_i^- the function $\Psi^{+,\alpha}$ has the form

$$\Psi^{+,\alpha}(x,t,P) = (z - z_i^-)^{-x/\eta}\ (\sum_{s=0}^{\infty} \tilde{\chi}_s^{\alpha i}(x,t)(z-z_i^-)^s)\ \Psi^{+,1}(0,0,P). \tag{3.46}$$

The boundary values $\Psi^{+,\alpha;(\pm)}$ of the function $\Psi^{+,\alpha}$ at opposite sides of the cuts are connected by the relation

$$\Psi^{+,\alpha;(+)} = \Psi^{+,\alpha;(-)} e^{-2\pi i x/\eta}. \tag{3.47}$$

Remark. Theorem 3.2 shows, in particular, that the solution Ψ to the equation (2.19) is (up to normalization) the Baker-Akhiezer function. In the next section we show that this function is uniquely defined by the curve Γ, the poles γ_s, the matrix χ_0 and the value $x_1(0)$. All these quantities are defined by initial Cauchy data and do not depend on t. However, it is necessary to emphasize that part of them depend on the choice of the normalization point t_0, that we have chosen as ($t_0 = 0$). Any initial data $\{x_i, \dot{x}_i, a_i, b_i^+ | (b_i^+, a_i) = \dot{x}_i\}$ define the matrix L by means of eq. (2.32). The characteristic equation (3.1) defines the curve Γ. Eq. (3.26) defines a set of $g-1$ points γ_s on Γ. Therefore, there exists a map

$$\{x_i, \dot{x}_i, a_i, b_i^+ | (b_i^+, a_i) = \dot{x}_i\} \longmapsto \{\Gamma,\quad D \in J(\Gamma)\}, \tag{3.48}$$

$$D = \sum_{s=1}^{g-1} A(\gamma_s) + x_1 U^{(0)}, \tag{3.49}$$

where $A : \Gamma \to J(\Gamma)$ is the Abel map and $U^{(1)}$ is a vector depending on Γ only (see (4.13)). The coefficients of the equation (3.1) are integrals of the system (1.21)-(1.23). As we shall see in the next section the second part of the data (3.48) define angle-type variables, i.e., the vector $D(t)$ evolves linearly, $D(t) = D(t_0) + (t - t_0)U^{(+)}$, if a point of the phase space evolves according to eqs. (1.21)-(1.23). These equations have the obvious symmetries:

$$a_i, b_i^+ \to q_i a_i, \ q_i^{-1} b_i^+, \quad a_i, b_i^+ \to W^{-1} a_i, \ b_i^+ W, \tag{3.50}$$

where q_i are constants and W is an arbitrary constant matrix. In the next section we prove that the data Γ, D uniquely define a a point of the phase space up the symmetry transformations (3.50).

4 Finite-gap solutions of the non-abelian Toda chain

Finite-gap solutions of the non-abelian Toda chain were found in the paper [31] by one of the authors. To construct the inverse spectral transformation for the spin generalizations of the Ruijsenaars-Schneider model, we recall the main points of this theory. Since we are working with a continuous variable x rather than the discrete variable n, some minor modifications of the construction are necessary.

Theorem 4.1 *Let Γ be a smooth algebraic curve of genus g with fixed local coordinates $w_{j,\pm}(P)$ in neighbourhoods of $2l$ points $P_j^{\pm}$, $w_{j,\pm}(P_j^{\pm}) = 0$, $j = 1, \ldots, l$ and with fixed cuts between the points $P_j^{\pm}$. Then for each set of $g + l - 1$ points $\gamma_1, \ldots, \gamma_{g+l-1}$ in general position there exists a unique function $\psi_\alpha(x, T, P)$, $\alpha = 1 \ldots, l$, $T = \{t_{i,j;\pm}, \ i = 1, \ldots, \infty; \ j = 1, \ldots, l\}$ such that*

1^0. *The function ψ_α of the variable $P \in \Gamma$ is meromorphic outside the cuts and has at most simple poles at the points γ_s (if all of them are distinct);*

2^0. *The boundary values $\psi_\alpha^{(\pm)}$ of this function at opposite sides of the cuts satisfy the relation*

$$\psi_\alpha^{(+)}(x, T, P) = \psi_\alpha^{(-)}(x, T, P) e^{2\pi i x/\eta}. \tag{4.1}$$

3^0. *In a neighbourhood of the point $P_j^{\pm}$ it has the form*

$$\psi_\alpha(x, T, P) = w_{j,\pm}^{\mp x/\eta} \left(\sum_{s=0}^{\infty} \xi_s^{\alpha j;\pm}(x, T) w_{j,\pm}^s \right) \exp \left(\sum_{i=1}^{\infty} w_{j,\pm}^{-i} t_{i,j;\pm} \right), \quad w_{j,\pm} = w_{j,\pm}(P), \tag{4.2}$$

$$\xi_0^{\alpha j;+}(x, T) \equiv \delta_{\alpha j}. \tag{4.3}$$

The proof of theorems of this kind, as well as the explicit formula for ψ_α in terms of Riemann theta-functions, are standard in the finite-gap integration theory. We use the notation of the paper [19].

It follows from the Riemann-Roch theorem that for each divisor $D = \gamma_1 + \cdots + \gamma_{g+l-1}$ in general position there exists a unique meromorphic function $h_\alpha(P)$ such that the divisor of its poles coincides with D and such that

$$h_\alpha(P_j^+) = \delta_{\alpha j}. \tag{4.4}$$

The basis of cycles a_i^0, b_i^0 on Γ with canonical intersection matrix being fixed, this function may be written as follows:

$$h_\alpha(P) = \frac{f_\alpha(P)}{f_\alpha(P_\alpha^+)}; \quad f_\alpha(P) = \theta(A(P) + Z_\alpha)\frac{\prod_{j\neq\alpha}\theta(A(P)+R_j)}{\prod_{i=1}^l \theta(A(P)+S_i)}, \tag{4.5}$$

where the Riemann theta-function $\theta(z_1,\ldots,z_g) = \theta(z_1,\ldots,z_g|B)$ is defined by the matrix $B = (B_{ik})$ of periods of holomorphic differentials on Γ; $A(P)$ is the Abel map: $A : P \in \Gamma \to J(\Gamma)$;

$$R_j = -\mathcal{K} - A(P_j^+) - \sum_{s=1}^{g-1} A(\gamma_s), \quad j = 1,\ldots,l \tag{4.6}$$

$$S_i = -\mathcal{K} - A(\gamma_{g-1+i}) - \sum_{s=1}^{g-1} A(\gamma_s), \tag{4.7}$$

and

$$Z_\alpha = Z_0 - A(P_\alpha^+), \quad Z_0 = -\mathcal{K} - \sum_{i=1}^{g+l-1} A(\gamma_i) + \sum_{j=1}^{l} A(P_j^+), \tag{4.8}$$

where $\mathcal{K}$ is the vector of Riemann's constants (the proofs can be found in [19]).

Let $d\Omega^{(i,j;\pm)}$ be the unique meromorphic differential holomorphic on Γ outside the point $P_j^\pm$, $j = 1,\ldots,l$, which has the form

$$d\Omega^{(i,j;\pm)} = d(w_{j,\pm}^{-i} + O(w_{j;\pm})), \tag{4.9}$$

near the point $P_j^\pm$ and normalized by the conditions

$$\oint_{a_k^0} d\Omega^{(i,j;\pm)} = 0. \tag{4.10}$$

It defines a vector $U^{(i,j;\pm)}$ with coordinates

$$U_k^{(i,j;\pm)} = \frac{1}{2\pi i}\oint_{b_k^0} d\Omega^{(i,j;\pm)}. \tag{4.11}$$

Further, define a differential $d\Omega^{(0)}$, which is holomorphic outside the points $P_j^\pm$, has the form

$$d\Omega^{(0)} = \pm\frac{dw_{j,\pm}}{\eta w_{j,\pm}} + O(1)dw_{j,\pm}, \tag{4.12}$$

near these points, and has zero a-periods. It defines a vector $U^{(0)}$ with coordinates

$$U_k^{(0)} = \frac{1}{2\pi i}\oint_{b_k^0} d\Omega^{(0)} \tag{4.13}$$

It follows from Riemann's bilinear relations that

$$U^{(0)} = \eta^{-1}\sum_{j=1}^{l}(A(P_j^-) - A(P_j^+)). \tag{4.14}$$

Theorem 4.2 *The components ψ_α of the Baker-Akhiezer function $\psi(x,T,P)$ are given by the formula*

$$\psi_\alpha = h_\alpha(P)\frac{\theta(A(P)+U^{(0)}x+\sum_A U^{(A)}t_A+Z_\alpha)\theta(Z_0)}{\theta(A(P)+Z_\alpha)\theta(U^{(0)}x+\sum_A U^{(A)}t_A+Z_0)}e^{(x\Omega^{(0)}(P)+\sum_A t_A\Omega^{(A)}(P))}, \quad (4.15)$$

$$\Omega^{(A)}(P)=\int_{q_0}^{P} d\Omega^{(A)}, \quad A=(i,j;\pm). \quad (4.16)$$

Remark. Note that since the abelian integral $\Omega^{(0)}$ has logarithmic singularities at the marked points, one may define a single-valued branch of ψ only after cutting the curve Γ between the points $P_j^\pm$.

Let us now define the dual Baker-Akhiezer function. For each set of $g+l-1$ points in general position there exists a unique differential $d\Omega$ holomorphic outside the points $P_j^\pm$ such that it has simple poles at these points with residues ± 1, i.e.,

$$d\Omega = \pm\frac{dw_{j,\pm}}{w_{j,\pm}}+O(1)dw_{j,\pm} \quad (4.17)$$

and such that it equals zero at the points γ_s:

$$d\Omega(\gamma_s)=0. \quad (4.18)$$

Becides γ_s this differential has $g+l-1$ other zeros which we denote γ_s^+.

The dual Baker-Akhiezer function is the unique function $\psi^+(x,T,P)$ with components $\psi^{+,\alpha}(x,t,P)$ such that

1^0. The function $\psi^{+,\alpha}$ of the variable $P\in\Gamma$ is meromorphic outside the cuts and has at most simple poles at the points γ_s^+ (if all of them are distinct);

2^0. The boundary values $\psi^{+;\alpha,(\pm)}$ of this function at opposite sides of the cuts satisfy the relation

$$\psi^{+;\alpha,(+)}(x,T,P)=\psi^{+;\alpha,(-)}(x,T,P)e^{-2\pi ix/\eta}; \quad (4.19)$$

3^0. In a neighbourhood of $P_j^\pm$ it has the form

$$\psi^{+,\alpha}(x,T,P)=w_{j,\pm}^{\pm x/\eta}\left(\sum_{s=0}^{\infty}\xi_s^{+;\alpha j;\pm}(x,T)w_{j,\pm}^s\right)\exp\left(-\sum_{i=1}^{\infty}w_{j,\pm}^{-i}t_{i,j;\pm}\right), \quad (4.20)$$

$$\xi_0^{+;\alpha j;+}(x,T)\equiv\delta_{\alpha j}. \quad (4.21)$$

Let $h_\alpha^+(P)$ be the function that has poles at the points of the dual divisor $\gamma_1^+,\ldots,\gamma_{g+l-1}^+$ and normalized by $h_\alpha^+(P_j^+)=\delta_{\alpha j}$. It can be written in the form (4.5) in which γ_s are replaced by γ_s^+. It follows from the definition of dual divisors that

$$\sum_{s=1}^{g+l-1}A(\gamma_s)+\sum_{s=1}^{g+l-1}A(\gamma_s^+)=K_0+\sum_{j=1}^{l}(A(P_j^+)+A(P_j^-)), \quad (4.22)$$

where K_0 is the canonical class (i.e. the equivalence class of the divisor of zeros of a holomorphic differential). Thus the vector Z_0^+ in the formulas for h_α^+ is connected to Z_0 by the relation

$$Z_0+Z_0^+=-2\mathcal{K}-K_0+\sum_{j=1}^{l}(A(P_j^+)-A(P_j^-))=-2\mathcal{K}-K_0-U^{(0)}\eta. \quad (4.23)$$

Theorem 4.3 *The components $\psi^+(x,T,P)$ of the dual Baker-Akhiezer function are given by*

$$\psi^{+;\alpha} = h^+_\alpha(P)\frac{\theta(A(P) - U^{(0)}x - \sum_A U^{(A)}t_A + Z^+_\alpha)\theta(Z^+_0)}{\theta(A(P) + Z^+_\alpha)\theta(U^{(0)}x + \sum_A U^{(A)}t_A - Z^+_0)} e^{-(x\Omega^{(0)}(P) + \sum_A t_A\Omega^{(A)}(P))}, \quad (4.24)$$

where

$$Z^+_0 = -Z_0 - 2\mathcal{K} - K_0 - U^{(0)}\eta, \quad Z^+_\alpha = Z^+_0 - A(P^+_\alpha). \quad (4.25)$$

The above results are valid for any algebraic curve with two sets of marked points. Consider now the class of curves corresponding to the spin generalizations of the Ruijsenaars-Schneider model.

Theorem 4.4 *Let $\tilde\Gamma$ be a smooth algebraic curve defined by an equation of the form*

$$\hat R(K,z) = K^N + \sum_{i=1}^N R_i(z)K^{N-i} = 0, \quad (4.26)$$

where $R_j(z)$ are meromorphic functions of z such that

$$R_j(z + 2\omega_\alpha) = R_j(z)e^{-2j\zeta(\omega_\alpha)\eta}. \quad (4.27)$$

and holomorphic in the fundamental domain of the lattice with periods $2\omega_\alpha$ outside the point $z = -\eta$. Let us assume that in a neighbourhood of $z = -\eta$ the polynomial $\hat R$ has the following factorization:

$$\hat R(K,z) = \prod_{i=1}^l (K + (z+\eta)^{-1}H_i(z+\eta)) \prod_{i=l+1}^N (K + (z+\eta)H_i(z+\eta)), \quad (4.28)$$

where $H_i(z)$ have no singularity at the point $z = -\eta$.

Then the Baker-Akhiezer function ψ corresponding to: (i) the curve Γ, which is the factor of $\tilde\Gamma$ with respect to the transformation group

$$z \longmapsto z + 2\omega_\alpha, \quad K \longmapsto Ke^{-2\zeta(\omega_\alpha)\eta}, \quad (4.29)$$

(ii) local coordinates $w_{j,+} = (z+\eta)H_j^{-1}(0)$ near the poles $P^+_j, j = 1,\ldots,l$ of the multi-valued function $K = K(P)$ and arbitrary local coordinates $w_{j,-}$ near the zeros P^-_j of this function obeys the relation:

$$\psi(x + 2\omega_\alpha, T, P) = \varphi_\alpha(P)\psi(x,T,P). \quad (4.30)$$

where

$$\varphi_\alpha(P) = K(P)^{2\omega_\alpha/\eta}e^{\zeta(\omega_\alpha)z}. \quad (4.31)$$

The proof is easy. It follows from the monodromy properties (4.29) that the values of $\varphi_\alpha(P)$ do not change under shifts of z by periods of the elliptic curve, i.e., it yields a well-defined function on Γ. Eq. (4.30) follows from the fact that its left and right hand sides have the same analitical properties. ∎

Corollary 4.1 *The Baker-Akhiezer function $\psi(x,T,P)$ with components $\psi_\alpha(x,T,P)$ corresponding to the data of Theorem 4.4 can be written in the form*

$$\psi(x,T,P)=\sum_{i=1}^{m} s_i(T,P)\Phi(x-x_i(T)),z)k^{x/\eta},\quad k=K\left[\frac{\sigma(z+\eta)}{\sigma(z-\eta)}\right]^{1/2}. \tag{4.32}$$

Under these very conditions the dual Baker-Akhiezer function has the form

$$\psi^+(x,T,P)=\sum_{i=1}^{m} s_i^+(T,P)\Phi(-x+x_i(T)-\eta),z)k^{-x/\eta}. \tag{4.33}$$

Proof of the lemma is identical to the proof of the corresponding lemma from the paper [19] and it was already presented at the beginning of Sect. 2, because Theorem 4.4 implies that ψ and ψ^+ are double-Bloch functions. (Note that from (4.23) and (4.24) it follows that ψ^+ has poles at the points $x_i-\eta$). ■

So far $t_{i,\alpha;\pm}$ were arbitrary parameters, entering ψ through the form of the essential singularity at $P_\alpha^\pm$. Fix now some values of these parameters for $i>1$, i.e., $t_{i,\alpha;\pm}=t^0_{i,\alpha;\pm}$, whereas for $i=1$ we put

$$t_{1,\alpha;\pm}=t_\pm+t^0_{1,\alpha;\pm}. \tag{4.34}$$

The Baker-Akhiezer function now depends on the variables (x,t_+,t_-). For the sake of brevity, we denote it $\psi(x,t_+,t_-,P)$, skipping the dependence on the constants T^0.

Theorem 4.5 *For any choice of the constants T^0 the Baker-Akhiezer function $\psi(x,t_+,t_-,P)$ satisfies the equations*

$$\partial_+\psi(x,t_+,t_-,P)=\psi(x+\eta,t_+,t_-,P)+v(x,t_+,t_-)\psi(x,t_+,t_-,P), \tag{4.35}$$

$$\partial_-\psi(x,t_+,t_-,P)=c(x,t_+,t_-)\psi(x-\eta,t_+,t_-,P),\quad \partial_\pm=\partial/\partial t_\pm, \tag{4.36}$$

where

$$v(x,t_+,t_-)=\partial_+g(x,t_+,t_-)g^{-1}(x,t_+,t_-)=\xi_1^+(x,t_+,t_-)-\xi_1^+(x+\eta,t_+,t_-), \tag{4.37}$$

$$c(x,t_+,t_-)=g(x,t_+,t_-)g^{-1}(x-\eta,t_+,t_-)=\partial_-\xi_1^+(x,t_+,t_-), \tag{4.38}$$

and the matrices g and ξ_1^+ are defined by the coefficients of the expansion (4.2):

$$g^{\alpha,j}(x,t_+,t_-)=\xi_0^{\alpha,j;-}(x,t_+,t_-),\quad \xi_1^+(x,t_+,t_-)=\{\xi_1^{\alpha j;+}\} \tag{4.39}$$

Proof. Eq. (4.35) follows from the fact that the function

$$\partial_+\psi_\alpha(x,t_+,t_-,P)-\psi_\alpha(x+\eta,t_+,t_-,P)$$

has the same analitic properties as ψ except the normalization 4.3). Thus we can write it as a linear combination of the components ψ_β with coefficients $v^{\alpha\beta}$. The explicit form of $v^{\alpha\beta}$ follows from comparing the coefficients in the left and right hand sides of (4.35) at the points P_j^-. Hence we get the first equality in (4.37). On the other hand, we may expand ψ near the points P_j^+. Then we get the second equality in (4.37). Eqs. (4.36) and (4.38) are proved in a similar way. ■

Corollary 4.2 *The matrix function $g_n(t_+,t_-) = g(n\eta + x_0, t_+, t_-)$, corresponding (according to the definitions of the Baker-Akhiezer functions) to the Riemann surface Γ with fixed local coordinates near the marked points $P_j^\pm$ and to the set of points $\gamma_1, \ldots, g+l-1$ is a solution of the 2D Toda chain equations (2.1).*

Remark. The dependence of g_n on the variables $t_{i,j}$, $i = 1, \ldots, \infty$, $j = 1, \ldots, l$, corresponds to higher flows of the 2D Toda chain hierarchy.

Theorem 4.6 *The dual Baker-Akhiezer function satisfies the equations*

$$-\partial_+\psi^+(x,t_+,t_-,P) = \psi^+(x-\eta,t_+,t_-,P) + \psi^+(x,t_+,t_-,P)v(x,t_+,t_-), \tag{4.40}$$

$$-\partial_-\psi^+(x,t_+,t_-,P) = \psi^+(x+\eta,t_+,t_-,P)c(x+\eta,t_+,t_-), \tag{4.41}$$

where $c(x,t_+,t_-)$, $v(x,t_+,t_-)$ are the same as in (4.35, 4.36).

Proof. The same arguments as in the proof of Theorem 4.5 show that ψ^+ obeys eqs. (4.40), (4.41) with coefficients v^+ and c^+ given by

$$c^+(x,t_+,t_-) = [\xi_0^{+;-}(x+\eta,t_+,t_-)]^{-1}\xi_0^{+;-}(x,t_+,t_-), \tag{4.42}$$

$$v^+(x,t_+,t_-) = -[\xi_0^{+;-}(x,t_+,t_-)]^{-1}\partial_+\xi_0^{+;-}(x,t_+,t_-), \tag{4.43}$$

where the matrix elements $\xi_0^{+;-} = \{\xi_0^{+;\alpha j;-}\}$ are determined from (4.20). The coincidence of the coefficients for ψ and ψ^+ is easily seen from the relation

$$[\xi_0^{+;-}]^{-1} = \xi_0^-(x,t_+,t_-) = g(x,t_+,t_-), \tag{4.44}$$

which follows from the definition of the dual Baker-Akhiezer function. To prove (4.44), consider the differential $\psi_\alpha\psi^{+\beta}d\Omega$, where $d\Omega$ is the same as in the definition of the dual divisor γ_s^+. It is a meromorphic differential on Γ with the only poles at $P_j^\pm$. Its residue at P_j^+ is

$$\mathrm{res}_{P_j^+}\psi_\alpha\psi^{+\beta}d\Omega = \delta_{\alpha,j}\delta_{\beta,j}. \tag{4.45}$$

Further, the residue of this differential at P_j^- is

$$\mathrm{res}_{P_j^+}\psi_\alpha\psi^{+\beta}d\Omega = -\xi_0^{\alpha,j;-}\xi_0^{+;\beta,j;-}. \tag{4.46}$$

Since the sum of the residues should be zero, then (4.44) follows from (4.45) and (4.46). ■

Theorem 4.7 *Let the curve Γ, the marked points $P_j^\pm$ and local coordinates in their neighbourhoods be the same as in Theorem 4.4, then the corresponding algebro-geometric "potentials" v and c in eqs. (4.35, 4.36) are elliptic functions in x. In general position they have the form*

$$v(x,T) = \sum_{i=1}^N a_i(T)b_i^+(T)V(x-x_i(T)), \tag{4.47}$$

$$c(x,T) = \partial_-\left(S_0(T) + \sum_{i=1}^N a_i(T)b_i^+(T)\zeta(x-x_i(T)\right), \tag{4.48}$$

where a_i, b_i^+ are some vectors, and S_0 is a matrix functions that does not depend on x.

Proof. The potentials in eqs. (4.35) and (4.36) are elliptic functions due to (4.30). It follows from (4.15) that the poles $x = x_i(T)$ of the Baker-Akhiezer function correspond to solutions of the equation

$$\theta(U^{(0)}x + \sum_A U^{(A)}t_A + Z_0) = 0. \tag{4.49}$$

It is clear from (4.8) that for corresponding solutions $(x_i(T),\ T)$ the first factor in the numerator of (4.15) is zero at $P = P_\alpha^+$. At the points P_β^+, $\beta \neq \alpha$ the function $h_\alpha(P)$ vanishes. Therefore, the residue $\psi_{\alpha,i}^0(T, P)$ of the function $\psi_\alpha(x, T, P)$ at $x = x_i(T)$ (as a function of P) has the following analitic properties:

1^0. It is a meromorphic function on Γ outside the cuts between the points $P_j^\pm$ and has the same poles as ψ;

2^0. Its boundary values $\psi_{\alpha,i}^{0;(\pm)}(T, P)$ at opposite sides of the cuts satisfy the relation

$$\psi_{\alpha,j}^{0;(+)}(T, P) = \psi_{\alpha,j}^{0;(-)}(T, P)e^{2\pi i x_j(T)/\eta}; \tag{4.50}$$

3^0. In a neighbourhood of $P_j^\pm$ we have

$$\psi_{\alpha,i}^0(T, P) = w_{j,\pm}^{\mp x_i(T)/\eta} \exp(\sum_{s=1}^{\infty} w_{j,\pm}^{-s} t_{s,j;\pm}) F_{i,j,\alpha}^\pm(w_{j,\pm}), \tag{4.51}$$

where $F_{i,j,\alpha}^\pm$ are regular in these neighbourhoods, and

$$F_{i,j,\alpha}^+(0) = 0. \tag{4.52}$$

Hence we see that $\psi_{\alpha,i}^0$ has the same analitic properties as the Baker-Akhiezer function except the following. The regular factor in the expansions of this function near *all* the points P_j^+ has the vanishing leading term. For general x, t_A there is no such function. For the special values $(x = x_i(T), T)$ such function $\psi_{i0}(T, P)$ does exist and is *unique* up to a constant (with respect to P) factor (it is uniquely defined in general case when $x_i(T)$ is a simple root of the equation (4.49)). So we can represent ψ_α in the form

$$\psi_\alpha(x, T, P) = \frac{\phi_\alpha(T)\,\psi_{i0}(T, P)}{x - x_i(T)} + O((x - x_i(T))^0). \tag{4.53}$$

It follows from the last equality that the residues $\rho_i(T)$ of the matrix $\xi_1^+(x, T)$ with matrix elements $\xi_1^{\alpha j;+}(x, T)$

$$\xi_1^+(x, T) = \frac{\rho_i(T)}{x - x_i(T)} + O((x - x_i(t))^0) \tag{4.54}$$

have rank 1. This means that there exist vectors $a_i(T)$ and covectors $b_i^+(T)$ such that $\rho_i = a_i(T)b_i^+(T)$. We see from (4.30) that the matrix ξ_1^+ obeys the following monodromy properties:

$$\xi_1^+(x + 2\omega_l) = \xi_1^+(x, T) + 2\zeta(\omega_l)r, \tag{4.55}$$

where r is a constant. It follows from these equations and (4.54) that ξ_1^+ can be written in the form

$$\xi_1^+ = S_0(T) + \sum_{i=1}^{N} a_i(T)b_i^+(T)\zeta(x - x_i(T)). \tag{4.56}$$

Combining this with the second equalities in (4.37) and (4.38), we get the desired result. ∎

Remark. In the abelian case ($l = 1$) there is an equivalent expression for $c(x)$ as a product of pole factors:

$$c(x,T) = \prod_{i=1}^{N} \frac{\sigma(x - x_i(T) + \eta)\sigma(x - x_i(T) - \eta)}{\sigma^2(x - x_i(T))} \tag{4.57}$$

(see (1.17)). Comparing the pole terms in (4.48) and (4.57), we get the relations

$$\partial_+\partial_- x_i(T) = -\sigma^2(\eta) \prod_{k\neq i} \frac{\sigma(x_i - x_k + \eta)\sigma(x_i - x_k - \eta)}{\sigma^2(x_i - x_k)}, \tag{4.58}$$

$$\partial_+\partial_- x_i(T) = -\partial_+ x_i(T)\partial_- x_i(T) \sum_{k\neq i} (V(x_i - x_k) - V(x_k - x_i)). \tag{4.59}$$

It is easy to check that if the dynamics with respect to $t_\pm$ is given by the hamiltonians $\sigma(\pm\eta)H_\pm$ (1.20), then these relations follow from equations of motion.

The connection between algebraic-geometrical potentials in eqs. (4.35), (4.36) corresponding to equivalent divisors is well known in the theory of finite-gap integration. Let $D = \gamma_1 + \cdots + \gamma_{g+l-1}$ and $D^{(1)} = \gamma_1^{(1)} + \cdots + \gamma_{g+l-1}^{(1)}$ be two equivalent divisors. This means that there exist a meromorphic function $h(P)$ on Γ such that D is the divisor of its poles and $D^{(1)}$ is the divisor of its zeros.

Corollary 4.3 *The algebraic-geometrical potentials $v(x,T)$, $c(x,T)$ and $v^{(1)}(x,T)$, $c^{(1)}(x,T)$ corresponding to $\Gamma, P_j^\pm, w_{j,\pm}(P)$ and equivalent divisors D and $D^{(1)}$ are gauge equivalent:*

$$v^{(1)}(x,T) = Hv(x,T)H^{-1}, \quad c^{(1)}(x,T) = Hc(x,T)H^{-1}, \quad H^{\alpha j} = h(P_j^+)\delta^{\alpha j}. \tag{4.60}$$

Corollary 4.4 *In general position the curve Γ satisfies the conditions of Theorem 4.4 if and only if it is the spectral curve (3.13) of the Lax matrix L defined by eq. (2.32), in which $x_i, \dot{x}_i$ are arbitrary constants and the vectors a_i, b_i^+ satisfy the conditions (1.25).*

It follows from Theorem 3.2 that the Baker-Akhiezer function $\Psi_\alpha(x,t,P)/\Psi_1(0,0,P)$ is related to the normalized Baker-Akhiezer function $\psi_\alpha(x,t,P)$ by the formula

$$\frac{\Psi_\alpha(x,t,P)}{\Psi_1(0,0,P)} = \sum_\beta \chi_0^{\alpha\beta} \psi_\beta(x,t,P), \tag{4.61}$$

where $\psi(x,t,P)$ is the Baker-Akhiezer function defined in the beginning of this section. It corresponds to the following values of the parameters $T = \{t_{i,j;\pm}\}$:

$$t_{1,j;+} = t, \quad t_{i,j,\pm} = 0, \ (i,j,\pm) \neq (1,j,+), \tag{4.62}$$

The equality (4.61) yields the corollary:

Corollary 4.5 *Let $a_i(t), b_i(t), x_i(t)$ be solutions to the equations of motion (1.21)-(1.23), then*

$$\sum_{i=1}^{N} a_i(t) b_i^+(t) V(x - x_i(t)) = \chi_0 v(x,t) \chi_0^{-1}, \tag{4.63}$$

where $v(x,t) = v(x, t_+ = t, t_- = 0)$ is the potential corresponding (according to Theorem 4.5) to the normalized Baker-Akhiezer function $\psi(x,t,P)$ constructed from the data (a curve with marked points) obeying the conditions of Theorem 4.4.

Corollary 4.6 *The map*

$$a_i(t),\ b_i^+(t), x_i(t) \longmapsto \{\Gamma,\ [D]\}, \tag{4.64}$$

where $[D]$ is the equivalence class of the divisor D is an isomorphism up to the transformations (3.50).

The curve Γ does not depend on time, while $[D]$ depends on the choice of the initial point $t_0 = 0$. The follwoing theorem shows that this dependence $[D(t_0)]$ is linearized on the Jacobian.

Theorem 4.8 *Let Γ be a curve that is defined by the equation (4.28) and $D = \gamma_1, \ldots, \gamma_{g+l-1}$ be a set of points in general position. Then the formulas*

$$\theta(U^{(0)} x_i(t) + U^{(+)} t + Z_0) = 0, \quad U^{(+)} = \sum_j U^{(1,j,+)}, \tag{4.65}$$

$$a_{i,\alpha}(t) = Q_i^{-1}(t) h_\alpha(q_0) \frac{\theta(U^{(0)} x_i(t) + U^{(+)} t + Z_\alpha)}{\theta(Z_\alpha)}, \tag{4.66}$$

$$b_i^\alpha(t) = Q_i^{-1}(t) h_\alpha^+(q_0) \frac{\theta(U^{(0)} x_i(t) + U^{(+)} t - Z_\alpha^+)}{\theta(Z_\alpha^+)}, \tag{4.67}$$

where

$$Q_i^2(t) = \frac{1}{2} \sum_{\alpha=1}^{l} h_\alpha^+(q_0) h_\alpha(q_0) \frac{\theta(U^{(0)} x_i(t) + U^{(+)} t - Z_\alpha) \theta(U^{(0)} x_i(t) + U^{(+)} t - Z_\alpha^+)}{\theta(Z_\alpha) \theta(Z_\alpha^+)}, \tag{4.68}$$

(q_0 is an arbitrary point of the curve Γ) define solutions of the system (1.21), (2.23), (2.24). Any solution of this system in general position may be obtained from (4.65)-(4.67) by means of the symmetries (3.50).

Remark. If in eqs. (4.65)-(4.67) the vector Z_0 is substituted by

$$Z_0 \longmapsto Z_0 + \sum_A U^{(A)} t_A, \tag{4.69}$$

then the corresponding quantities $x_i(T),\ a_i(T),\ b_i(T),\ T = \{t_A\}$, depende on t_A according to the higher commuting flows of the system (1.21)-(1.23). We would like to emphasize that the points $P_j^{\pm}$ enter symmetrically. This means that the dependence of $x_i(T),\ a_i(T),\ b_i(T)$ on the variable $t_- = l^{-1} \sum_{j=1}^{l} t_{1,j;-}$ is described by the same equations as that for $t = t_+$.

5 Difference analogs of Lame operators

Consider the operator S_0 given by eq. (1.9) for integer ℓ. First of all we note that due to the obvious symmetry $-\ell \leftrightarrow \ell - 1$ it is enough to consider only positive values of ℓ. In what follows we imply that $\ell \in \mathbf{Z}_+$. The finite-gap property of the operator S_0 means that the Bloch solutions to the equation

$$(S_0 f)(x) = \varepsilon f(x) \tag{5.1}$$

are parametrized by points of a hyperelliptic curve of genus 2ℓ.

Any solution $f(x)$ of eq. (5.1) may be represented in the form

$$f(x) = \Psi(x)\,(\theta_1(\eta/2))^{-x/\eta} \prod_{j=1}^{\ell} \theta_1(x - j\eta), \tag{5.2}$$

where $\Psi(x)$ satisfies the equation

$$(\tilde{S}_0 \Psi)(x) \equiv \Psi(x+\eta) + c_\ell(x)\Psi(x-\eta) = \varepsilon \Psi(x), \tag{5.3}$$

$$c_\ell(x) = \theta_1^2(\eta/2) \frac{\theta_1(x+\ell\eta)\theta_1(x-(\ell+1)\eta)}{\theta_1(x)\theta_1(x-\eta)}. \tag{5.4}$$

This transformation sends Bloch solutions of the first equation to Bloch solutions of the second one. To construct them explicitly, we use the ansatz similar to the one for the linear equation (2.2):

$$\Psi = \sum_{j=1}^{\ell} s_j(z,k,\varepsilon)\Phi_\ell(x - j\eta, z) k^{x/\eta}, \tag{5.5}$$

where

$$\Phi_\ell(x,z) = \frac{\theta_1(z+x+N\eta)}{\theta_1(z+N\eta)\theta_1(x)} \left[\frac{\theta_1(z-\eta)}{\theta_1(z+\eta)}\right]^{x/2\eta}, \quad N = \frac{\ell(\ell+1)}{2}. \tag{5.6}$$

(note that Φ_1 coincides with $\Phi(x,z)$ that is given by (2.6) and $c_1(x)$ coincides with $c(x-\eta)$ from (2.8) up to a constant factor).

The function $\Phi_\ell(x,z)$ is double-periodic in z:

$$\Phi_\ell(x, z+2\omega_\alpha) = \Phi_\ell(x,z) \tag{5.7}$$

(In this section we take the periods to be $2\omega_1 = 1$, $2\omega_2 = \tau$.) For values of x such that $x/2\eta$ is a half-integer number the function Φ_ℓ is single-valued on the Riemann surface $\hat{\Gamma}_0$ of the function $E(z)$ that is defined in (2.9). For general values of x one can define a single-valued branch of $\Phi_\ell(x,z)$ by cutting the elliptic curve Γ_0 between the points $z = \pm\eta$.

As a function of x, $\Phi_\ell(x,z)$ is a double-Bloch function:

$$\Phi_\ell(x+2\omega_\alpha, z) = T_\alpha^{(\ell)}(z)\Phi_\ell(x,z), \tag{5.8}$$

where

$$T_1^{(\ell)}(z) = \left(\frac{\theta_1(z-\eta)}{\theta_1(z+\eta)}\right)^{1/2\eta}. \tag{5.9}$$

$$T_2^{(\ell)}(z)=\exp(-2\pi i(z+N\eta))\left(\frac{\theta_1(z-\eta)}{\theta_1(z+\eta)}\right)^{\tau/2\eta}. \tag{5.10}$$

In the fundamental domain of the lattice defined by $2\omega_\alpha$ the function $\Phi_\ell(x,z)$ has a unique pole at the point $x=0$:

$$\Phi_\ell(x,z)=\frac{1}{\theta_1'(0)x}+O(1). \tag{5.11}$$

Substituting (5.5) into (5.3) and computing the residues at the points $z=j\eta,\ j=0,\ldots,\ell$, we get $\ell+1$ linear equations

$$\sum_{j=1}^{\ell} L_{i,j}s_j=0,\quad i=0,\ldots,\ell, \tag{5.12}$$

for ℓ unknown parameters $s_j=s_j(z,k,\varepsilon)$. Matrix elements $L_{i,j}$ of this system are:

$$L_{0,1}=k+h\Phi_\ell(-2\eta,z)k^{-1},\quad L_{0,j}=h\Phi_\ell(-(j+1)\eta,z)k^{-1},\quad j=2,\ldots,\ell; \tag{5.13}$$

$$L_{1,1}=-\varepsilon-h\Phi_\ell(-\eta,z)k^{-1},\ L_{1,2}=k-h\Phi_\ell(-2\eta,z)k^{-1},\ L_{1,j}=-h\Phi_\ell(-j\eta,z)k^{-1},\ \ j>2; \tag{5.14}$$

$$L_{i,j}=\delta_{i,j+1}c_jk^{-1}-\varepsilon\delta_{i,j}+\delta_{i,j-1}k,\quad i>1, \tag{5.15}$$

where

$$h=\theta_1'(0)\mathrm{res}_{x=0}c_\ell(x)=\frac{\theta_1^2(\eta/2)}{\theta_1(\eta)}\theta_1(\ell\eta)\theta_1((\ell+1)\eta), \tag{5.16}$$

$$c_j=c_\ell(j\eta)=\theta_1^2(\eta/2)\frac{\theta_1((j+\ell)\eta)\theta_1((j-\ell-1)\eta)}{\theta_1(j\eta)\theta_1((j-1)\eta)}. \tag{5.17}$$

The overdetermined system (5.12) has non-trivial solutions if and only if rank of the rectangular matrix $L_{i,j}$ is less than ℓ. By $L^{(0)}$ and $L^{(1)}$ denote $l\times l$ matrices obtained from L by deleting the rows with $i=0$ and $i=1$, respectively. Then the set of parameters z,k,ε for which eq. (5.3) has solutions of the form (5.12) is defined by the system of two equations:

$$\det L^{(i)}\equiv R^{(i)}(z,k,\varepsilon)=0,\quad i=0,1. \tag{5.18}$$

Expanding the determinants with respect to the upper row, we get that $R^{(i)}$ have the form:

$$R^{(0)}(z,k,\varepsilon)=r_\ell^{(0)}(\varepsilon)+\sum_{j=1}^{\ell}k^{-j}\Phi_\ell(-j\eta,z)r_{\ell-j}^{(0)}(\varepsilon), \tag{5.19}$$

$$R^{(1)}(z,k,\varepsilon)=k\tilde r_{\ell-1}(\varepsilon)+\sum_{j=1}^{\ell}k^{-j}\Phi_\ell(-(j+1)\eta,z)r_{\ell-j}^{(1)}(\varepsilon), \tag{5.20}$$

where $\tilde r_{\ell-1},\ r_{\ell-j}^{(0)},\ r_{\ell-j}^{(1)}$ are polynomials in ε of degrees $\ell-1$ and $\ell-j$ respectively.

These equations define an algebraic curve $\hat\Gamma$ realized as $\ell(\ell+1)/2$-fold ramified covering of the genus 2 curve $\hat\Gamma_0$ on which the functions $\Phi_\ell(-j\eta,z)$ are single-valued. This curve enjoys an obvious symmetry

$$(z,\ k,\ \varepsilon)\longmapsto(z,-k,-\varepsilon). \tag{5.21}$$

This symmetry is a direct corollary of the following general property of the equation (5.3). If Ψ is a solution of (1.4.3) then $\Psi(x)\exp(\pi i x/\eta)$ is a solution of the same equation but with a different sign of ε. This transformation corresponds to the change of sign for k.

Note that the simultanious change of signs of the variable k in (5.5) and change of sheets of $E(z)$ does not change a function $\Psi(x,z,k)$ at all. Therefore, $\hat{\Gamma}$ may be considered as a $l(l+1)$-fold ramified covering of Γ^0.

Let us show that this curve is invariant with respect to another involution,

$$(z,\ k,\ \varepsilon) \longmapsto (-z,\ k^{-1}\theta_1^2(\eta/2),\ \varepsilon), \tag{5.22}$$

as well.

Let $\Psi(x)$ be a solution of the equation (5.3). Then the function

$$\tilde{\Psi}(x) = \Psi(-x)A(x), \tag{5.23}$$

where

$$A(x) = \theta_1(\eta/2)^{2x/\eta} \prod_{j=1}^{\ell} \frac{\theta_1(x+j\eta)}{\theta_1(x-j\eta)} \tag{5.24}$$

is also a solution of the same equation. Moreover, if Ψ is a double-Bloch solution then $\tilde{\Psi}$ is also a double-Bloch solution. It is easy to check that if Bloch multipliers of Ψ are parametrized by the pair z,k, then the Bloch mltipliers of $\tilde{\Psi}$ correspond to the pair $-z,\ k^{-1}\theta_1^2(\eta/2)$.

The variable ε being considered as a function $\varepsilon(P)$ on the curve $P \in \hat{\Gamma}$ is a meromorphic function. It can not take any value more than twice because for given value of ε the second order difference equation (5.3) has at most two different Bloch solutions. The involution (5.22) is not trivial. Therefore, the function ε does take generic value two times. Therefore, the algebraic curve $\hat{\Gamma}$ is a hyperelliptic curve of finite genus g. Due to the symmetry (5.21) it can be represented in the form

$$y^2 = \prod_{i=1}^{g+1} (\varepsilon^2 - \varepsilon_i^2). \tag{5.25}$$

The involution (5.22) is the hyperelliptic involution corresponding to the interchanging of the sheets of the ramified covering (5.25).

Now we are going to prove that $g = 2\ell$. From (5.25) it follows that there are $2g+2$ fixed points of the hyperelliptic involution. On the other hand, the number of fixed points of the hyperelliptic involution (5.22) is equal to the number of preimages of the second-order points $\omega_a \in \Gamma_0,\ a = 0,\ldots,3$, (that are fixed points of the involution $z \to -z$ on Γ_0) on $\hat{\Gamma}$ such that the corresponding value of k is equal to $\pm\theta_1(\eta/2)$. Explicitly, from now on we adopt the following notation [1]:

$$\omega_0 = 0,\ \omega_1 = 1/2,\ \omega_2 = (1+\tau)/2,\ \omega_3 = \tau/2 \tag{5.26}$$

[1]this notation differs from that adopted in Sect.2.

For values ε corresponding to the fixed points there is only one Bloch solution. This implies that corresponding solution has a definite parity with respect to the involution (5.23), i.e.

$$\Psi(x) = \nu\Psi(-x)A(x), \quad \nu = \pm 1. \tag{5.27}$$

We are going to prove that $\nu = (-1)^\ell$.

The equality (5.27) for $x = \eta$ implies that

$$s_1 = \nu k^{-1}\Psi(-\eta)(-1)^{\ell-1}\frac{\theta_1^2(\eta/2)}{\theta_1(\eta)}\theta_1(\ell\eta)\theta_1((\ell+1)\eta), \tag{5.28}$$

Comparing this equality with (5.3) (taken at $x = 0$), we see that if $s_1 \neq 0$, then $\nu = (-1)^\ell$. Otherwise, $s_1 = 0$ and $\Psi(-\eta) = 0$.

From (5.3) it follows that coefficients s_1 and s_2 in the representation (5.5) of a double-Bloch solution can not equal zero simulteniously. Indeed, let j be a minimal index such that $s_j \neq 0$. If $j > 2$ then the left hand side of (5.3) has a pole at the point $z = (j-1)\eta$ but the right hand side has no such pole.

Therefore, if $s_1 = 0$, then $s_2 \neq 0$. In that case eq. (5.3) implies that $\Psi(0) \neq 0$. At the same time from (5.27) (taken at $x = 0$) it follows that

$$\Psi(0) = (-1)^\ell \nu \Psi(0). \tag{5.29}$$

Therefore, $\nu = (-1)^\ell$.

For $z = \omega_a$, $k = (-1)^{a+1}\theta_1(\eta/2)$ the representation (5.5) of the double-Bloch functions is equivalent to

$$\Psi(x) = (\theta_1(\eta/2))^{x/\eta}\exp(\pi i(\delta_{2,a} + \delta_{3,a})x)\prod_{j=1}^{\ell}\frac{\theta_1(x + x_j)}{\theta_1(x - j\eta)}, \tag{5.30}$$

where

$$\sum_{j=1}^{\ell} x_j = \omega_a. \tag{5.31}$$

Lemma 5.1 *The hyperelliptic involution of $\hat{\Gamma}$ has $2d$ fixed points, where d equals sum of dimensions of the functional subspaces consisting of functions that have the form (5.30) and satisfy the relation (5.27) with $\nu = (-1)^\ell$.*

Proof. As it was shown above, for each pair of fixed points of the hyperelliptic involution of $\hat{\Gamma}$ that are invariant with respect to the involution (5.21) (corresponding to the change of sign for k) there exists the unique solution to eq. (5.3) having the form (5.30) and satisfing (5.27) with $\nu = (-1)^\ell$. On the other hand, the space of such functions is invariant with respect to the operator $\tilde{S}_0$. Indeed, the equality (5.28) with $\nu = (-1)^\ell$ (that is a corrolary of (5.27)) implies that $\tilde{S}_0\Psi$ has no pole at $z = 0$. At the same time $\tilde{S}_0$ commutes with the linear operator (5.23). Therefore, the number of solutions of (5.3) having the form (5.30) and satisfing (5.27) is equal to the number of eigenvalues ε_i, $i = 1, \ldots, d$ of $\tilde{S}_0$ on these finite-dimensional spaces, i.e., is equal to sum of their dimensions. ∎

It is easy to see that dimension of the space defined in the last lemma for $\omega_0 = 0$ is equal to $(\ell - 1)/2$ if ℓ is odd and $\ell/2 + 1$ if ℓ is even. For the other three points of second order corresponding dimensions are equal to $(\ell + 1)/2$ if ℓ is odd and $\ell/2$ if ℓ is even. Therefore, the number of fixed points is $4\ell + 2$ and we do prove that $g = 2\ell$.

Lemma 5.2 *The direct sum of spaces of functions having the form (5.30) and satisfying (5.27) with $\nu = (-1)^\ell$ is invariant with respect to the operators $\tilde{S}_a$,*

$$(\tilde{S}_a\Psi)(x) = H_a\left[\frac{\theta_{a+1}(x-\ell\eta)}{\theta_1(x-\ell\eta)}\Psi(x+\eta) - \theta_1^2(\eta/2)\frac{\theta_1(x-(\ell+1)\eta)\theta_{a+1}(-x-\ell\eta)}{\theta_1(x)\theta_1(x-\eta)}\Psi(x-\eta)\right], \tag{5.32}$$

$$H_a = (i)^{\delta_{a,2}}\frac{\theta_{a+1}(\eta/2)}{\theta_1(\eta/2)} \tag{5.33}$$

which are gauge equivalent to the operators (1.9).

The proof is straightforward. Again, eq. (5.28) implies that $\tilde{S}_a\Psi$ has no pole at $z = 0$. At the same time these operators commute with the transformaton (5.23) and keep invariant the set of Bloch multipliers corresponding to spaces of functions having the form (5.30). Note that coefficients of $\tilde{S}_a$, $a \neq 0$ are not elliptic; that's why they do not preserve each space but only their direct sum.

Remark. The invariant space for the operators $\tilde{S}_a$ described above coincides (after the gauge transformation (5.2)) with the finite-dimensional representation space of the Sklyanin algebra found in [8]. More information on invariant functional subspaces for Sklyanin's operators is contained in the next section.

Equations (5.18) represent k and ε as multi-valued functions of z. Let us consider now analytical properties of the double-Bloch solutions $\Psi(x, Q)$, $Q = (z,\ k,\ \varepsilon) \in \hat{\Gamma}$ on the hyperelliptic curve $\hat{\Gamma}$.

Theorem 5.1 *The Bloch solution $\psi(x, P) = \Psi(x, P)\Psi^{-1}(0, P)$ of eq. (5.3) is a meromorphic function on the genus 2ℓ hyperelliptic curve $\hat{\Gamma}$ defined by the equation*

$$y^2 = \prod_{i=1}^{2l+1}(\varepsilon^2 - \varepsilon_i^2), \tag{5.34}$$

(where ε_i are eigenvalues of $\tilde{S}_0$ on finite-dimensional invariant space of function having the form (5.30), (5.31) outside a cut between the infinity points $P^\pm$, that are preimages of $\varepsilon = \infty$ on $\hat{\Gamma}$. Outside the cut $\psi(x, P)$ has 2ℓ poles independent of x invariant with respect to involution of $\hat{\Gamma}$ that covers the involution $\varepsilon \to -varepsilon$ (5.21). The boundary values of $\psi^{(\pm)}(x, P)$ at opposite sides of the cut are connected by the relation

$$\psi^{(+)} = \psi^{(-)}e^{2\pi i/\eta}. \tag{5.35}$$

In a neighbourhood of $P^\pm$ the function $\psi(x, P)$ has the form:

$$\psi = \varepsilon^{\pm x/\eta}\left(\sum_{s=0}^{\infty}\xi_s^\pm(x)\varepsilon^{-s}\right), \quad \xi_0^+ \equiv 1. \tag{5.36}$$

Proof. The coefficients s_j in (5.5) are solutions of linear system (5.12). Let us normalize them by the condition $s_1 = 1$. Then all other s_j are meromorphic functions on $\hat{\Gamma}$. Therefore, we may conclude that Ψ as a function of Q is well-defined on $\hat{\Gamma}$ with a cuts between zeros and poles of

$$K = \left(\frac{\theta_1(z-\eta)}{\theta_1(z+\eta)}\right)^{1/2} k(z). \tag{5.37}$$

At the edges of these cuts Ψ has a singularity of the form:

$$\Psi = K^{x/\eta} O(1). \tag{5.38}$$

From (5.3) it directly follows that a function Ψ with this type of singularity might be a solution of this equation if at the point of singularity $\varepsilon = \infty$. At the same time we have that in neighbourhoods of these points: $K = \varepsilon^{\pm 1} O(1)$. That proves (5.36). The proof that the number of poles of ψ is equal to 2ℓ is standart. First of all note that poles of ψ do not depend on x because poles of s_j do not depend on x. After that we may consider a meromorphic function of ε that is equal to

$$F(\varepsilon) = (\psi(\eta, P_1(\varepsilon)) - \psi(\eta, P_2(\varepsilon)))^2, \tag{5.39}$$

where $P_i(\varepsilon)$ are two preimages of ε on $\hat{\Gamma}$. Due to square this function does not depend on the choice of orders of this points, i.e. F is a meromorphic function of ε. It has double poles at projections of poles of ψ, has pole of order 2 at infinity and has simple zeros at the branch points. The numbers of zeros and poles of a meromorphic function are equal to each other. That implies that ψ has g poles. ∎

We would like to mention that the theorem might be proved by a direct consideration of analytical behaviour of Ψ on $\hat{\Gamma}$ represented in the form that is defined by equations (5.18), as well. The corresponding proof is very similar to the proof of the theorem 3.2. We skip it, but present only arguments that explain why the multi-valued function K has only one pole and one zero.

For $\ell > 1$ the function $\Phi_\ell(-j\eta, z)$ has a pole and zero of order j at the points $z = \pm\eta$, respectively. Therefore, k has zeros and poles at all preimages on $\hat{\Gamma}$ of $z = -\eta$ and $z = \eta$, respectively. Hence, K is regular at all these points. The function Φ_ℓ has a simple pole at the point $z = -N\eta$. It turns out that at one of the preimages of this point the function ε as well as k has pole. The corresponding point is one of the infinities of $\hat{\Gamma}$ in representation it in the form (5.34). The other infinity is one of the primages of the point $z = N\eta$.

Remark. Eqs. (1.9) define one particular series of representations of the Sklyanin algebra found in [8] (called series a) there). In this series the operator S_0 has the finite-gap property. It should be noted that in each of the other series there is again an operator having the finite-gap property (see (6.31) below). The other three operators in each series in general do not have this property. Trigonometric degenerations of these operators were considered in [32]. They correspond to solitonic solutions of the Toda chain.

6 Representations of the Sklyanin algebra

In this section we show how to construct representations of the Sklyanin algebra from "vacuum vectors" of the L-operator (1.4). The notions of vacuum vectors and the vacuum curve of an L-operator were introduced by one of the authors [33] in analysis of the Yang-Baxter equation by methods of algebraic geometry. Let us recall the main definitions.

To begin with, consider an *arbitrary* L-operator $\mathcal{L}$ with two-dimensional auxiliary space $\mathbf{C}^2$, i.e., an arbitrary $2n \times 2n$ matrix represented as 2×2 matrix whose matrix elements are $n \times n$ matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$:

$$\mathcal{L} = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}. \tag{6.1}$$

The operators $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ act in a linear space $\mathcal{H} \cong \mathbf{C}^n$ which is called the *quantum space* of the L-operator. We emphasize that so far no conditions on $\mathcal{L}$ are implied. In particular, we do not impose the relation (1.5) and do not imply any specific parametrization of the matrix elements.

Let us consider a vector $X \otimes U \in \mathcal{H} \otimes \mathbf{C}^2$ ($X \in \mathcal{H}$, $U \in \mathbf{C}^2$) such that

$$\mathcal{L}(X \otimes U) = Y \otimes V, \tag{6.2}$$

where $Y \in \mathcal{H}$, $V \in \mathbf{C}^2$ are some vectors. The relation (6.2) means that the undecomposable tensor $X \otimes U$ is transformed by $\mathcal{L}$ into another undecomposable tensor. Being written in components, this relation has the form

$$\mathcal{L}^{i\alpha}_{j\beta} X_j U_\beta = Y_i V_\alpha, \tag{6.3}$$

where indices α, β (resp., i, j) correspond to $\mathbf{C}^2$ (resp., $\mathcal{H}$) and summation over repeated indices is implied.

Suppose (6.2) holds; then the vector X is called a *vacuum vector* of the L-operator $\mathcal{L}$. Multiplying (6.2) from the left by the covector $\tilde{V} = (V_2, -V_1)$, orthogonal to V, we get

$$(\tilde{V}\mathcal{L}U)X = 0\,. \tag{6.4}$$

Here $\tilde{V}\mathcal{L}U$ is an operator in $\mathcal{H}$ with matrix elements $\tilde{V}_\alpha \mathcal{L}^{i\alpha}_{j\beta} U_\beta$. Conversely, suppose (6.4) holds. Then we have (6.2) with some vector Y which is uniquely determined by U, V and X.

The relation (6.2) (in the particular case $\mathcal{H} \cong \mathbf{C}^2$) was the starting point for Baxter in his solution of the eight-vertex model [14]. In the papers on integrable lattice models of statistical physics this relation is called "pair propagation through a vertex". In the context of quantum inverse scattering method [2] the equivalent condition (6.4) is more customary. It defines the local vacuum of the gauge-transformed L-operator (this explains the terminology introduced above). In general form this relation appeared for the first time in [33].

The necessary and sufficient condition for the existence of vacuum vectors is

$$\det(\tilde{V}\mathcal{L}U) = 0\,. \tag{6.5}$$

Putting for simplicity $U_2 = V_2 = 1$ and using (6.1), one may represent (6.5) in a more explicit form:

$$\det(U_1\mathcal{A} + \mathcal{B} - U_1V_1\mathcal{C} - V_1\mathcal{D}) = 0, \quad U_2 = V_2 = 1. \tag{6.6}$$

This equation defines an algebraic curve in $\mathbf{C}^2$ which is called the *vacuum curve* of the L-operator $\mathcal{L}$. So vacuum vectors are parametrized by points of the vacuum curve (i.e. by pairs (U_1, V_1) satisfying (6.6)). In general position the space of vacuum vectors corresponding to each point of the curve is one-dimensional.

Suppose now that $\mathcal{H} \cong \mathbf{C}^2$ and $\mathcal{L}$ satisfies the equation (1.5) with some matrix R. In this case the vacuum curve has genus 1, i.e., it is an elliptic curve $\mathcal{E}_0$. It is parametrized by points z of one-dimensional complex torus with periods 1 and τ.

Fixing a suitable normalization (for example, putting second components of all the vectors equal to 1), we may consider components of the vectors $U(z)$, $V(z)$, $X(z)$, $Y(z)$ as meromorphic functions on $\mathcal{E}_0$ having at most 2 simple poles. With this normalization, the right hand side of (6.2) must be multiplied by a scalar meromorphic function $h(z)$. It follows [33] from the Yang-Baxter equation that

$$Y(z) = X(z + \frac{\eta}{2}), \quad V(z) = U(z - \frac{\eta'}{2}), \tag{6.7}$$

where η and η' are some constants. Therefore, we can write the basic relation (6.2) in the form [33]

$$\mathcal{L}(X(z) \otimes U(z)) = h(z)Y(z) \otimes V(z) = h(z)X(z + \frac{\eta}{2}) \otimes U(z - \frac{\eta'}{2}). \tag{6.8}$$

Let D_X (resp., D_U) be the divisor of poles of the meromorphic vector $X(z)$ (resp., $U(z)$). Let $\mathcal{M}(D)$ be the space of functions associated to an effective divisor D, i.e., functions having poles at points of D of order not higher than the multiplicity of the corresponding point in D. For divisors of degree 2 on elliptic curves this space is two-dimensional in general position, so $\dim \mathcal{M}(D_X) = \dim \mathcal{M}(D_U) = 2$, and components of the vectors X and U form bases in these spaces. Further, the functions $X_i(z)U_\alpha(z),\ i, \alpha = 1, 2$, form a basis in the space $\mathcal{M}(D_X + D_U)$. According to (6.8), the functions $h(z)X_i(z + \frac{\eta}{2})U_\alpha(z - \frac{\eta'}{2})$ form another basis in this space and the matrix $\mathcal{L}$ connects the two bases. The divisors of poles of the left and right hand sides of (6.8) must be equivalent, i.e., must be equal modulo periods of the lattice. Since under the shift by $\eta'/2$ the divisor of poles of the function having 2 poles is shifted by η', this means that

$$\eta' - \eta = M + N\tau\,, \qquad M, N \in \mathbf{Z}. \tag{6.9}$$

The vectors $X(z)$, $U(z)$ are double-periodic, hence there are 4 different cases:

$$\eta' = \eta + 2\omega_a\,, \tag{6.10}$$

where $\omega_0 = 0$, $\omega_1 = 1/2$, $\omega_2 = (\tau + 1)/2$, $\omega_3 = \tau/2$.

Baxter's parametrization of the L-operator follows from eq. (6.8). Indeed, the equivalence class of the pole divisor of $X(z)$ may differ from that of $U(z)$ by only a shift on $\mathcal{E}_0$. By means of a "gauge" transformation one may represent $\mathcal{L}$ in the form (1.4). The value of this shift is then identified with the spectral parameter of the L-operator. In this parametrization, it is natural to write (6.8) explicitly in terms of θ-functions.

To do this, it is convenient to use another normalization, specifically, the one in which the vectors are entire functions in z (in this case they are sections of certain line bundles on $\mathcal{E}_0$).

Let us introduce the vector

$$\Theta(z) = \begin{pmatrix} \theta_4(z|\frac{\tau}{2}) \\ \\ \theta_3(z|\frac{\tau}{2}) \end{pmatrix}. \tag{6.11}$$

Its components form a basis in the space of θ-functions $\theta(z)$ of second order with monodromy propertties $\theta(z+1) = \theta(z)$, $\theta(z+\tau) = \exp(-2\pi i\tau - 4\pi i z)\theta(z)$. They have 2 zeros in the fundamental domain of the lattice with periods $1\,,\tau$. Then, putting

$$X(z) = \Theta(z)\,, \tag{6.12}$$

$$U(z) = U^{\pm}(z) = \Theta\left(z \pm \frac{1}{2}(u + \frac{\eta}{2})\right) \tag{6.13}$$

(for each choice of the sign), we may rewrite (6.2) in the form [14], [2]:

$$L^{(a)}(u)\Theta(z)\otimes\Theta(z\pm\frac{1}{2}(u+\frac{\eta}{2})) = 2g_a^{\pm}\frac{\theta_1(u+\frac{\eta}{2}|\tau)}{\theta_1(\eta|\tau)}\Theta(z+\frac{\eta}{2})\otimes\Theta(z\pm\frac{1}{2}(u-\frac{\eta}{2})\pm\omega_a), \tag{6.14}$$

where

$$g_0^{\pm} = g_1^{\pm} = 1, \quad -ig_2^{\pm} = g_3^{\pm} = -\exp(\pm 2\pi i z + \pi i(u + \frac{\tau-\eta}{2})), \tag{6.15}$$

$$L^{(a)}(u) = \sum_{b=0}^{3}\frac{\theta_{b+1}(u|\tau)}{\theta_{b+1}(\frac{\eta}{2}|\tau)}\sigma_b\otimes(\sigma_a\sigma_b) \tag{6.16}$$

(see (1.4) and (1.7)). We note that $L^{(a)}(u) = \sigma_a L(u)$ ($L(u)$ is given by (1.4) with $S_a = \sigma_a$ and the matrix product is performed in the auxiliary space) satisfies the "$RLL = LLR$" relation (1.5) with *the same* R-matrix (1.7) for each $a = 0, \ldots, 3$. The scalar factor in the right hand side of (6.14) is determined from the condition that $L(\frac{\eta}{2})$ is proportional to the permutation operator in $\mathbf{C}^2\otimes\mathbf{C}^2$. One may verify (6.14) directly using identities for θ-functions (see the Appendix).

Remark. Given an elliptic curve, we can always choose the vector $X(z)$ to be an even function, $X(-z) = X(z)$. We introduce the even function $X(z)$ (6.11) from the very beginning. Then the equality corresponding to the minus sign in (6.14) follows from the similar equality with the plus (it is enough to change $z \to -z$). However, in Baxter's approach it is useful to deal with both equalities.

Let us turn to the case of arbitrary spin. Consider an L-operator of the form (1.4):

$$L(u) = \begin{pmatrix} W_0(u)S_0 + W_3(u)S_3 & W_1(u)S_1 - iW_2(u)S_2 \\ W_1(u)S_1 + iW_2(u)S_2 & W_0(u)S_0 - W_3(u)S_3 \end{pmatrix}, \tag{6.17}$$

where

$$W_a(u) = \frac{\theta_{a+1}(u|\tau)}{\theta_{a+1}(\frac{\eta}{2}|\tau)}\,, \tag{6.18}$$

and S_a are generators of some algebra (at this stage the commutation relations (1.1), (1.2) are not imposed). We are going to obtain explicit formulas for representations of this algebra from a simple generalization of (6.8).

Before presenting the main result of this section we need some more preliminaries.

Since we are interested in the general form of representations in terms of difference operators, in what follows we take for the quantum space of the L-operator (6.17) the space of *meromorphic functions of one complex variable* z. The generators S_a act on elements of this space (i.e. functions $X(z)$):

$$S_a : X(z) \longmapsto (S_a X)(z). \tag{6.19}$$

Consider the following generalization of (6.8) and (6.14):

$$U^{\pm}(z)^T(\sigma_a L(u))X(z) = g_a(u)U^{\pm}(z \mp l\eta \mp \omega_a)^T X(z \pm \frac{\eta}{2}), \quad a = 0, \ldots, 3. \tag{6.20}$$

Here $z \in \mathcal{E}_0$, $U^T = (U_1, U_2)$, l is a parameter, $g_a(u)$ are scalar functions independent of z, and

$$U^{\pm}(z) = \Theta\left(z \pm \frac{u + l\eta}{2}\right) \tag{6.21}$$

(for $l = 1/2$ it coincides with (6.13)). As before, $L(u)$ acts on $U^{\pm}$ as 2×2-matrix, while each matrix element of $L(u)$ acts on $X(z)$ according to (6.19). We have written (6.20) in terms of the *covector* U^T for the following reason: since the operator (6.19) acts on a function *from the left*, this is equivalent to the *right* action of the corresponding matrix (representing this operator in a fixed basis) on the covector formed by components of the function with respect to the basis. As it is clear from what follows, at $l = 1/2$ (6.20) coincides with the *conjugated* equality (6.14).

It should be noted that at present time we can not suggest any explicit description of the vacuum curve of the L-operator (6.17). Moreover, we do not know any direct argument establishing the equivalence between (6.20) and the "intertwining" relation (1.5) for $L(u)$ (taken together with the Yang-Baxter equation for $R(u)$). Theorem 6.1 (see below) states that Sklyanin's commutation relations (1.1, 1.2) for S_a follow from (6.20), hence $L(u)$ should satisfy (1.5). Let us stress once again that our arguments are in a sense inverse to original Sklyanin's approach (see also [34], [35], where some formulas for vacuum vectors in the higher spin XYZ model were obtained). Our starting point is the relation (6.20), where no conditions on S_a are implied. It turns out that the Sklyanin algebra for S_a (together with its functional realization) follows from (6.20).

The main result of this section is the following

Theorem 6.1 *Let $L(u)$ be given by (6.17), where S_a are some operators in the space of meromorphic functions $X(z)$ of one complex variable z. Suppose the relation (6.20) holds for $a = 0$, i.e.,*

$$U^{\pm}(z)^T L(u)X(z) = g_0(u)U^{\pm}(z \mp l\eta)^T X(z \pm \frac{\eta}{2}), \tag{6.22}$$

where $U^{\pm}(z)$ is defined in (6.21), l is a parameter and $g_0(u)$ is a scalar function independent of z. Then S_a are difference operators of the following form:

$$(S_aX)(z) = \lambda \frac{(i)^{\delta_{a,2}}\theta_{a+1}(\frac{\eta}{2}|\tau)}{\theta_1(2z|\tau)} \left(\theta_{a+1}(2z - l\eta|\tau)X(z + \frac{\eta}{2}) - \theta_{a+1}(-2z - l\eta|\tau)X(z - \frac{\eta}{2}) \right), \tag{6.23}$$

where λ is an arbitrary constant. Conversely, if S_a are defined by (6.23), then (6.22) holds, and

$$g_0(u) = 2\lambda\theta_1(u + l\eta|\tau). \tag{6.24}$$

Remark. a) Using transformation properties of the vector $\Theta(z)$ under shifts by half-periods ω_a it is readily seen that the relations (6.20) for $a \neq 0$ follow from (6.22); b) For even functions $X(z)$ the two relations (6.22) are equivalent.

Proof. The relations (6.22) form a system of 4 linear equations for 4 functions $(S_aX)(z)$ entering the left hand side. More explicitly, we have

$$U_1^{\pm}(z)(L_{11}(u)X)(z) + U_2^{\pm}(z)(L_{21}(u)X)(z) = g_0(u)U_1^{\pm}(z \mp l\eta)X(z \pm \frac{\eta}{2}), \tag{6.25}$$

$$U_1^{\pm}(z)(L_{12}(u)X)(z) + U_2^{\pm}(z)(L_{22}(u)X)(z) = g_0(u)U_2^{\pm}(z \mp l\eta)X(z \pm \frac{\eta}{2}), \tag{6.26}$$

where $(L_{\alpha\beta}(u)X)(z)$ are expressed through $(S_aX)(z)$ according to (6.17). Fix $g_0(u)$ to be given by (6.24). Solving this system, we get (6.23) (all necessary identities for θ-functions are presented in the Appendix to this section). Therefore, (6.23) is equivalent to (6.22, 6.24). ∎

Corollary 6.1 *Suppose for some $L(u)$ of the form (6.17) the equation (6.22) holds. Then $L(u)$ satisfies the "intertwining" relation (1.5) with the R-matrix (1.7).*

The proof follows from the identification of (6.23) with formulas (1.9) for representations of the Sklyanin algebra by putting $2z \equiv x$, $X(x/2) \equiv f(x)$. The constant λ is not essential since the commutations relations (1.1, 1.2) are homogeneous.

Remark. From the technical point of view, the derivation of formulas (1.9) by solving the system (6.22) is much simpler than the direct verification of the commutation relations. The amount of computations in the former case is comparable with that in the latter one if we identify the coefficients only in front of $f(x \pm 2\eta)$.

Let us consider now the equality (6.20) for $a = 1, 2, 3$.

Lemma 6.1 *Let us define the transformation of the generators $S_b \longmapsto \mathcal{Y}_a(S_b)$, $a, b = 0, \ldots, 3$, by the relation*

$$\sigma_a L(u + \omega_a) = h_a(u) \sum_{b=0}^{3} \frac{\theta_{b+1}(u|\tau)}{\theta_{b+1}(\frac{\eta}{2}|\tau)} \mathcal{Y}_a(S_b) \otimes \sigma_b, \tag{6.27}$$

where $h_0(u) = h_1(u) = 1$, $h_2(u) = -ih_3(u) = \exp(-\frac{i\pi\tau}{4} - i\pi u)$. Then $\mathcal{Y}_a$, $a = 0, \ldots, 3$, is an automorphism of the algebra generated by S_b.

The proof follows from the fact that $\sigma_a L(u+\omega_a)$, $a=0,\dots,3$ satisfies (1.5) with R-matrix (1.7). (This is because the matrices σ_a are c-number solutions of (1.5).) From (6.27) we find the explicit form of $\mathcal{Y}_a$ (here $\theta_a(\frac{\eta}{2})\equiv\theta_a(\frac{\eta}{2}|\tau)$):

$$\mathcal{Y}_1:(S_0,S_1,S_2,S_3)\longmapsto\left(-\frac{\theta_1(\frac{\eta}{2})}{\theta_2(\frac{\eta}{2})}S_1,\ \frac{\theta_2(\frac{\eta}{2})}{\theta_1(\frac{\eta}{2})}S_0,\ -\frac{i\theta_3(\frac{\eta}{2})}{\theta_4(\frac{\eta}{2})}S_3,\ \frac{i\theta_4(\frac{\eta}{2})}{\theta_3(\frac{\eta}{2})}S_2\right), \tag{6.28}$$

$$\mathcal{Y}_2:(S_0,S_1,S_2,S_3)\longmapsto\left(\frac{\theta_1(\frac{\eta}{2})}{\theta_3(\frac{\eta}{2})}S_2,\ \frac{i\theta_2(\frac{\eta}{2})}{\theta_4(\frac{\eta}{2})}S_3,\ \frac{\theta_3(\frac{\eta}{2})}{\theta_1(\frac{\eta}{2})}S_0,\ -\frac{\theta_4(\frac{\eta}{2})}{\theta_2(\frac{\eta}{2})}S_1\right), \tag{6.29}$$

$$\mathcal{Y}_3:(S_0,S_1,S_2,S_3)\longmapsto\left(\frac{\theta_1(\frac{\eta}{2})}{\theta_4(\frac{\eta}{2})}S_3,\ -\frac{\theta_2(\frac{\eta}{2})}{\theta_3(\frac{\eta}{2})}S_2,\ \frac{\theta_3(\frac{\eta}{2})}{\theta_2(\frac{\eta}{2})}S_1,\ \frac{\theta_4(\frac{\eta}{2})}{\theta_1(\frac{\eta}{2})}S_0\right), \tag{6.30}$$

and $\mathcal{Y}_0$ is the identity transformation. These automorphisms were considered by Sklyanin in [8].

Shifting $u\to u+\omega_a$ in (6.20) and applying these automorphisms to (6.23), we get for each $b=0,\dots,3$ the following representations:

$$(S_aX)(z)=\frac{(i)^{\delta_{a,2}}\theta_{a+1}(\frac{\eta}{2})}{\theta_1(2z)}\left(\theta_{a+1}(2z-l\eta-\omega_b)X(z+\frac{\eta}{2})-\theta_{a+1}(-2z-l\eta-\omega_b)X(z-\frac{\eta}{2})\right). \tag{6.31}$$

We remark that (6.31) can be obtained from (6.23) by the formal change $l\to l+\omega_b/\eta$ provided l is considered to be an arbitrary complex parameter. However, we will see soon that the operators (6.31) restricted to invariant subspaces yield non-equivalent finite-dimensional representations of the Sklyanin algebra.

Suppose $l\in\frac{1}{2}\mathbf{Z}_+$; then one may identify $X(z)$ with a section of some linear bundle on the initial elliptic curve $\mathcal{E}_0$. Repeating the arguments presented after eq. (6.8), we conclude that degree of this bundle equals $4l$. ¿From the Riemann-Roch theorem for elliptic curves it follows that in general position the space of holomorphic sections of this bundle is $4l$-dimensional. It is convenient to identify these sections with θ-functions of order $4l$. Consider the space $\mathcal{T}_{4l}^{+}$ of *even* θ-functions of order $4l$, i.e., the space of entire functions $F(z)$, $z\in\mathbf{C}$, such that $F(-z)=F(z)$ and

$$\begin{aligned}&F(z+1)=F(z)\,,\\&F(z+\tau)=\exp(-4l\pi i\tau-8l\pi iz)F(z)\,.\end{aligned} \tag{6.32}$$

In complete analogy with the paper [8], one can see that the space $\mathcal{T}_{4l}^{+}$ is invariant under action of the operators (6.31) for $b=0$ and $b=1$, while for $b=2$ and $b=3$ the invariant space is $\exp(-\pi iz^2/\eta)\mathcal{T}_{4l}^{+}$. It is known that $\dim\mathcal{T}_{4l}^{+}=2l+1$ provided $l\in\frac{1}{2}\mathbf{Z}_+$. Restricting the difference operators (6.31) to these invariant subspaces, we get 4 series of finite-dimensional representations.

Generally speaking, these representations are mutually non-equivalent. This follows from analysis of values of the central elements. The Sklyanin algebra has two independent central elements:

$$\mathbf{K}_0=\sum_{a=0}^{3}S_a^2\,,\quad \mathbf{K}_2=\sum_{i=1}^{3}J_iS_i^2 \tag{6.33}$$

(the constants J_i are defined by (1.3) and (1.10)). Their values for the representations (6.31) at $b=0,\dots,3$ are:

$$\mathbf{K}_0=4\theta_1^2\left((l+\frac{1}{2})\eta+\omega_b|\tau\right), \tag{6.34}$$

$$\mathbf{K}_2 = 4\theta_1\left((l+1)\eta + \omega_b|\tau\right)\theta_1\left(l\eta + \omega_b|\tau\right). \tag{6.35}$$

The arguments given above lead to the following statement.

Theorem 6.2 *The Sklyanin algebra (1.1, 1.2) has 4 different series of finite-dimensional representations indexed by* $b = 0, 1, 2, 3$. *Representations of each series are indexed by the discrete parameter* $l \in \frac{1}{2}\mathbf{Z}_+$ *("spin"). They are obtained by restriction of the operators (6.31) to the invariant* $(2l+1)$*-dimensional functional subspaces* $\mathcal{T}_{4l}^+$ *for* $b = 0, 1$ *and* $\exp(-\pi i z^2/\eta)\mathcal{T}_{4l}^+$ *for* $b = 2, 3$. *For general values of parameters these representations are mutually non-equivalent.*

■

Let us identify these representations with those obtained by Sklyanin in [8]. For $b = 0$ and $b = 3$ we get series a) and c) respectively. These representations are self-adjoint with respect to the real form of the algebra studied in [8]. The other two series (corresponding to $b = 1$ and $b = 2$) in general are not self-adjoint. Consider first the case $b = 1$. For rational values of η, $\eta = p/q$, and special values of l, $l = (q-1)/2 \bmod q$, these representations are self-adjoint and are equivalent to some subset of representations of series b) [2]. To the best of our knowledge, the series corresponding to $b = 2$ was never mentioned in the literature (though, in a sense, it is implicitly contained in Sklyanin's paper). Another outcome of our approach is the natural correspondence between the different series of representations and points of order 2 on the elliptic curve.

It is natural to surmise that representations of the last two series become self-adjoint with respect to other real forms of the algebra. A real form is defined by an anti-involution ($*$-operation) on the algebra. It should be noted that classification of non-equivalent real forms of the Sklyanin algebra and its generalizations is an interesting open problem.

Concluding this section, we would like to remark that the variable z in (6.20) and (6.31) may be identified with the statistical variable ("height") in IRF-type models [36] (after a suitable discretization of the former). This is readily seen from the well known vertex-IRF correspondence recalling that the vertex-IRF transformation is explicitly performed by means of the vacuum vectors.

Finally, it seems to be instructive to carry out a detailed analysis of the trigonometric and rational limits of the constructions presented in this section. Some particular related problems have been already discussed in the literature. In the recent paper [37], vacuum vectors for the higher spin XXZ-type quantum spin chains are constructed. Vacuum curves of trigonometric L-operators have been described in [38]. In the simplest case they are collections of rational curves intersecting at 2 points. Trigonometric degenerations of the Sklyanin algebra that are in a sense "intermediate" between the initial algebra and the standard quantum deformation of gl_2 are studied in [32].

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[2] The whole family of representations of series b) found in [8] has 3 continuous parameters. These representations are self-adjoint and exist only if $\eta = p/q$. In this case all of them have dimension q. They are obtained by restriction of the operator (6.31) to a finite discrete uniform lattice.

Appendix to Section 6

We use the following definition of the θ-functions:

$$\theta_1(z|\tau) = \sum_{k\in\mathbf{Z}} \exp\left(\pi i\tau(k+\frac{1}{2})^2 + 2\pi i(z+\frac{1}{2})(k+\frac{1}{2})\right), \tag{6.36}$$

$$\theta_2(z|\tau) = \sum_{k\in\mathbf{Z}} \exp\left(\pi i\tau(k+\frac{1}{2})^2 + 2\pi iz(k+\frac{1}{2})\right), \tag{6.37}$$

$$\theta_3(z|\tau) = \sum_{k\in\mathbf{Z}} \exp\left(\pi i\tau k^2 + 2\pi izk\right), \tag{6.38}$$

$$\theta_4(z|\tau) = \sum_{k\in\mathbf{Z}} \exp\left(\pi i\tau k^2 + 2\pi i(z+\frac{1}{2})\right). \tag{6.39}$$

Here is the list of identities used in the computations.

The first group of identities (addition theorems):

$$\theta_4(x|\tau)\theta_3(y|\tau) = \theta_4(x+y|2\tau)\theta_4(x-y|2\tau) + \theta_1(x+y|2\tau)\theta_1(x-y|2\tau), \tag{6.40}$$

$$\theta_4(x|\tau)\theta_4(y|\tau) = \theta_3(x+y|2\tau)\theta_3(x-y|2\tau) - \theta_2(x+y|2\tau)\theta_2(x-y|2\tau), \tag{6.41}$$

$$\theta_3(x|\tau)\theta_3(y|\tau) = \theta_3(x+y|2\tau)\theta_3(x-y|2\tau) + \theta_2(x+y|2\tau)\theta_2(x-y|2\tau), \tag{6.42}$$

$$\theta_2(x|\tau)\theta_2(y|\tau) = \theta_3(x+y|2\tau)\theta_2(x-y|2\tau) + \theta_2(x+y|2\tau)\theta_3(x-y|2\tau), \tag{6.43}$$

$$\theta_1(x|\tau)\theta_1(y|\tau) = \theta_3(x+y|2\tau)\theta_2(x-y|2\tau) - \theta_2(x+y|2\tau)\theta_3(x-y|2\tau), \tag{6.44}$$

Here are simple consequences of them convenient in the computations:

$$\theta_4(x|\tau)\theta_3(y|\tau) + \theta_4(y|\tau)\theta_3(x|\tau) = 2\theta_4(x+y|2\tau)\theta_4(x-y|2\tau), \tag{6.45}$$

$$\theta_4(x|\tau)\theta_3(y|\tau) - \theta_4(y|\tau)\theta_3(x|\tau) = 2\theta_1(x+y|2\tau)\theta_1(x-y|2\tau), \tag{6.46}$$

$$\theta_3(x|\tau)\theta_3(y|\tau) + \theta_4(y|\tau)\theta_4(x|\tau) = 2\theta_3(x+y|2\tau)\theta_3(x-y|2\tau), \tag{6.47}$$

$$\theta_3(x|\tau)\theta_3(y|\tau) - \theta_4(y|\tau)\theta_4(x|\tau) = 2\theta_2(x+y|2\tau)\theta_2(x-y|2\tau). \tag{6.48}$$

The second group of identities:

$$2\theta_1(x|2\tau)\theta_4(y|2\tau) = \theta_1(\frac{x+y}{2}|\tau)\theta_2(\frac{x-y}{2}|\tau) + \theta_2(\frac{x+y}{2}|\tau)\theta_1(\frac{x-y}{2}|\tau), \tag{6.49}$$

$$2\theta_3(x|2\tau)\theta_2(y|2\tau) = \theta_1(\frac{x+y}{2}|\tau)\theta_1(\frac{x-y}{2}|\tau) + \theta_2(\frac{x+y}{2}|\tau)\theta_2(\frac{x-y}{2}|\tau), \tag{6.50}$$

$$2\theta_3(x|2\tau)\theta_3(y|2\tau) = \theta_3(\frac{x+y}{2}|\tau)\theta_3(\frac{x-y}{2}|\tau) + \theta_4(\frac{x+y}{2}|\tau)\theta_4(\frac{x-y}{2}|\tau), \tag{6.51}$$

$$2\theta_2(x|2\tau)\theta_2(y|2\tau) = \theta_3(\frac{x+y}{2}|\tau)\theta_3(\frac{x-y}{2}|\tau) - \theta_4(\frac{x+y}{2}|\tau)\theta_4(\frac{x-y}{2}|\tau). \tag{6.52}$$

Particular cases of them:

$$2\theta_1(z|\tau)\theta_4(z|\tau) = \theta_2(0|\frac{\tau}{2})\theta_1(z|\frac{\tau}{2}), \tag{6.53}$$

$$2\theta_2(z|\tau)\theta_3(z|\tau) = \theta_2(0|\frac{\tau}{2})\theta_2(z|\frac{\tau}{2}). \tag{6.54}$$

Two more identities:

$$\theta_1(z|\frac{\tau}{2})\theta_2(z|\frac{\tau}{2}) = \theta_4(0|\tau)\theta_1(2z|\tau), \tag{6.55}$$

$$\theta_4(z|\frac{\tau}{2})\theta_3(z|\frac{\tau}{2}) = \theta_4(0|\tau)\theta_4(2z|\tau). \tag{6.56}$$

7 Concluding remarks

This work elaborates upon the following three subjects:

I) Dynamics of poles for elliptic solutions to the 2D non-abelian Toda chain;

II) Difference analogs of Lame operators;

III) Representations of the Sklyanin algebra in terms of difference operators.

Let us outline the results:

- The poles move according to equations of motion for spin generalizations of the Ruijsenaars-Schneider model; the action-angle variables for the latter are constructed in terms of some algebraic-geometrical data;

- One of the generators of the Sklyanin algebra, represented as a difference operator with elliptic coefficients, has the "finite-gap" property that is a motivation for the analogy with Lame operators;

- Starting from the notion of vacuum vectors of an L-operator, a general simple scheme for constructing functional realizations of the Sklyanin algebra is suggested.

Here we would like to explain why the three themes are to be intimately connected.

To each problem I) - III), a distinguished class of algebraic curves has been associated. In case I), these are spectral curves Γ for the L-operators of the Ruijsenaars-Schneider-type models; in II), we deal with the spectral curve $\hat{\Gamma}$ for the difference Lame operator S_0 (a generator of the Sklyanin algebra); in III), the representations are defined on sections of certain line bundles on a vacuum curve $\mathcal{E}$ of the elliptic higher spin L-operator (1.4) (though it is implicit in Section 6). It has been shown that Γ and $\hat{\Gamma}$ are ramified coverings of the initial elliptic curve. The characteristic property (6.20) of the vacuum vectors suggests that the same should be true for $\mathcal{E}$, i.e., $\mathcal{E}$ is a ramified covering of the initial elliptic curve $\mathcal{E}_0$ (the vacuum curve of the spin-1/2 L-operator).

The connection between I) and II) is similar to the relation between elliptic solutions of KP and KdV equations. Specifically, the elliptic solutions of the abelian 2D Toda chain, which are *stationary with respect to the time flow* $t_+ + t_-$ correspond to *isospectral deformations* of the difference Lame operator S_0 (considered as a Lax operator for 1D Toda chain). In other words, the hyperelliptic curves $\hat{\Gamma}$ form a specific subclass of the curves Γ. A similar reduction in the non-abelian case yields spin generalizations of difference Lame operators. Their properties and a possible relation to the Sklyanin-type quadratic algebras are to be figured out.

Apart from the apparent result that the construction of Sect. 6 provides a natural source of difference Lame-like operators, we expect a more deep connection between II) and III). Specifically, the spectral curves $\hat{\Gamma}$ are expected to be very close to the vacuum curves $\mathcal{E}$. Conjecturally, they may even coincide, at least in some particular cases. At the moment we can not present any more arguments and leave this as a further problem.

At last, we would like to note an intriguing similarity between the basic ansatz (2.26) used in this work and the functional Bethe ansatz [39]. Indeed, in the latter case wave functions are sought in the form of an "elliptic polynomial" $\prod \sigma(z - z_j)$, where the roots z_j are subject to Bethe equations. Similarly, in the former case we deal with *a ratio* of two "elliptic polynomials" (see, for example (5.30)). However, this function is parametrized by residues at the poles rather than zeros of the numerator. This may indicate a non-trivial interplay[3] between Calogero-Moser-type models (and more general Hithin's systems) and quantum integrable models solved by means of Bethe ansatz.

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