# Fuchsian groups of the second kind and representations carried by the limit set

*Dedicated to Rolf Sulanke on the occasion of his 65'th birthday.*

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## Contents




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# 1 Introduction

Let $\Gamma \subset SL(2,\mathbf{R}) =: G$ be a discrete subgroup acting freely on the real hyperbolic plane $H^2 = SL(2,\mathbf{R})/S^1 =: X$, i.e., the upper half space $\mathbf{R}^2_+$ with the metric $ds^2 = \frac{1}{y^2}(dx^2+dy^2)$. We assume this action to be geometrically finite and convex co-compact. Under these circumstances $\Gamma$ will be called Fuchsian of the second kind. Another model for the hyperbolic plane is the Poincaré disc $\{|z|<1\} \subset \mathbf{C}$. The corresponding group of isometries is $SU(1,1)$ which is, of course, ismorphic to $SL(2,\mathbf{R})$.

The group $G$ acts on the circle $S^1$ which can be identified with the boundary $\partial X$ of $X$ using the Poincaré disc model. The set of accumulation points (in the Euclidean topology) of an arbitrary orbit $\Gamma x$, $x \in X$, is called the limit set $\Lambda \subset \partial X$. The complement $\Omega := \partial X \setminus \Lambda$ is called the ordinary set. $\Gamma$ acts freely and co-compactly on $\Omega$ with the compact quotient $B$. As a manifold $B$ is a finite union of circles.

Let $T \to S^1$ be the complexified tangent bundle of $S^1$. It is $G$-homogeneous and we can form complex powers $T^\lambda \to S^1$, $\lambda \in \mathbf{C}$. Let $\partial$ be the fundamental vector field of the action of $U(1) \subset G = SU(1,1)$. It provides a natural trivialization of $T$. Frequently we will identify a section $\phi\partial^\lambda \in \Gamma(T^\lambda)$ with the function $\phi$. The number $\lambda$ parametrizes a principal series representation $H^\lambda$ of $G$ on the Hilbert space $L^2(S^1, T^{\lambda-1/2})$. By $H^\lambda_{-\omega}$ we denote the space of its hyperfunction vectors. As a topological vector space $H^\lambda_{-\omega}$ can be identified with the space of hyperfunction sections of $T^{\lambda-1/2}$. Thus $f \in H^\lambda_{-\omega}$ is a continuous functional on $H^{-\lambda}_\omega := C^\omega(S^1, T^{-\lambda-1/2})$.

**Definition 1.1** *By $H^\lambda_{-\omega,\Lambda}$ we denote the subspace of $H^\lambda_{-\omega}$ of hyperfunction sections with support in the limit set $\Lambda$.*

Since $\Lambda$ is a $\Gamma$-invariant closed subset $H^\lambda_{-\omega,\Lambda}$ is a closed $\Gamma$-invariant subspace of $H^\lambda_{-\omega}$.

The goal of the present paper is to study the cohomology groups $H^*(\Gamma, H^\lambda_{-\omega,\Lambda})$. Our interest in these cohomology groups is motivated by a conjecture of S. Patterson [19] (see also [2]) relating their dimensions with the order of the singularities of the Selberg zeta function associated to $\Gamma$. As a by-product we describe an easy way to obtain a meromorphic continuation of the scattering matrix and the Eisenstein series. Here we recover results of Patterson [15], [16], [17].

With the exception of Section 5 the methods extend to spherical principal series representations in higher dimensions. To obtain the results of Section 5 and to cover also non-spherical principal series representations is still a challenging project.

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# 2 The case $\mathrm{Re}(\lambda) > 0$

Let $A_\lambda := \Delta - 1/4 + \lambda^2$, where $\Delta$ is the Laplace-Beltrami operator of $X$. Let $\mathcal{E} = C^\infty(X)$ and $\mathcal{E}(A_\lambda) = \ker(A_\lambda) \subset \mathcal{E}$. Helgason ([6], [7] Introduction Thm. 4.3) proved that for $\mathrm{Re}(\lambda) > 0$ the Poisson transform $P_\lambda : H^\lambda_{-\omega} \to \mathcal{E}(A_\lambda)$ is a $G$-isomorphism. We characterize the subspace $P_\lambda(H^\lambda_{-\omega,\Lambda}) \subset \mathcal{E}(A_\lambda)$ using certain semi-norms. For all $W \subset X$ contained in finitely many translates of a fundamental domain of $\Gamma$ and $k \in \mathbf{N}_0$ we define

$$q_{W,k}(f) := \|A^k_\lambda f\|_{L^2(W)} \ .$$

By $\mathcal{E}_\Lambda$ we denote the Frechet space of all functions $f \in \mathcal{E}$ with $q_{W,k}(f) < \infty$ for all such $W$ and $k \geq 0$. Let $\mathcal{E}_\Lambda(A_\lambda) = \mathcal{E}_\Lambda \cap \ker(A_\lambda)$.

**Lemma 2.1**

$$P_\lambda(H^\lambda_{-\omega,\Lambda}) = \mathcal{E}_\Lambda(A_\lambda) \ .$$

*Proof.* We first employ the upper half-plane model and assume that $\infty \in \Lambda$. The Poisson transform $P_\lambda$ is given by the kernel $P_\lambda((x,y),b) := (\frac{y}{y^2+(x-b)^2})^{\lambda+1/2}$, where $b \in \mathbf{R}$. The closures of the sets $W$ in the Euclidean topology are well separated from $\Lambda$. It is easy to see that for all $W$ the kernel $b \to P_\lambda(.,b)$ defines an analytic function from a neighbourhood of $\Lambda$ to $L^2(W)$. It also satisfies $A_\lambda P_\lambda(.,b) = 0$. If we pair the Poisson kernel with a hyperfunction supported on $\Lambda$ we end up with a smooth eigenfunction of $A_\lambda$ which is in $L^2(W)$ for all $W$. It follows that

$$P_\lambda(H^\lambda_{-\omega,\Lambda}) \subset \mathcal{E}_\Lambda(A_\lambda) \ .$$

It remains to show that the inverse (up to a scalar) of $P_\lambda$, the boundary value map $\beta_\lambda$, satisfies

$$\beta_\lambda(\mathcal{E}_\Lambda(A_\lambda)) \subset H^\lambda_{-\omega,\Lambda} \ .$$

For this argument it is more convenient to employ the ball model. We use polar coordinates $(r,\alpha)$, $r \in [0,1)$, $\alpha \in (0,2\pi]$ of $X$. Let $f \in \mathcal{E}_\Lambda(A_\lambda)$ and $\phi \in H^{-\lambda}_\omega$. Then

$$\langle \beta_\lambda(f), \phi\rangle = \lim_{r\to 1}(1-r^2)^{\lambda-1/2}\int_{S^1} f(r\mathrm{e}^{i\alpha})\phi(\mathrm{e}^{i\alpha})\frac{d\alpha}{2\pi} \ .$$

Now let $(a,b) \subset \Omega$. The $L^2$-condition satisfied by $f$ along $\Omega$ allows us to restrict the integration to the complement of $(a,b)$, i.e.,

$$\langle \beta_\lambda(f), \phi\rangle = \lim_{r\to 1}(1-r^2)^{\lambda-1/2}\int_{S^1\setminus(a,b)} f(r\mathrm{e}^{i\alpha})\phi(\mathrm{e}^{i\alpha})\frac{d\alpha}{2\pi} \ .$$

Thus $\beta_\lambda(f)$ is in fact a continuous functional on germs of analytic sections of $T^{-\lambda-1/2}$ on $S^1 \setminus (a,b)$, i.e, $\beta_\lambda(f) \in \mathcal{A}'(S^1\setminus(a,b), T^{\lambda-1/2})$. The hyperfunction sections of $T^{\lambda-1/2}$ on $(a,b)$ are $\mathcal{B}((a,b),T^{\lambda-1/2}) \cong \mathcal{A}'(S^1,T^{\lambda-1/2})/\mathcal{A}'(S^1\setminus(a,b),T^{\lambda-1/2})$. Hence the restriction to $(a,b)$ of the hyperfunction $\beta_\lambda(f)$ vanishes. Since this holds for all intervals $(a,b) \subset \Omega$ we conclude $\mathrm{supp}\,\beta_\lambda(f) \subset \Lambda$. The lemma follows. $\square$

The proof of the following lemma is completely analogous to that of [2], Lemma 2.4.

**Lemma 2.2** *$\mathcal{E}_\Lambda$ is $\Gamma$-acyclic.*

The following lemma shows that

$$0 \longrightarrow H^\lambda_{-\omega,\Lambda} \xrightarrow{P_\lambda} \mathcal{E}_\Lambda \xrightarrow{A_\lambda} \mathcal{E}_\Lambda \longrightarrow 0 \tag{1}$$

is a $\Gamma$-acyclic resolution of $H^\lambda_{-\omega,\Lambda}$.

**Lemma 2.3** *$A_\lambda : \mathcal{E}_\Lambda \to \mathcal{E}_\Lambda$ is surjective.*

*Proof.* Let ${}^tA_\lambda : \mathcal{E}'_\Lambda \to \mathcal{E}'_\Lambda$ be the dual operator. It is enough to show that ${}^tA_\lambda$ is injective and has closed range (see Treves [24] for basic techniques to deal with surjectivity questions).

Since $C_c^\infty(X)$ is dense in $\mathcal{E}_\Lambda$ we can embed $\mathcal{E}'_\Lambda$ into the distributions on $X$. If $f \in \mathcal{E}'_\Lambda$, then there exist $k, W$ such that $|\langle f, \phi\rangle| \le C q_{W,k}(\phi)$ for all $\phi \in C_c^\infty(X)$. Hence $f$ is a distribution of order at most $2k$ with support on $W$. The bound of the order follows from the fact that $q_{W,k}(\phi)$ estimates the local $H^{2k}$-norm of $\phi$.

If $f \in \mathcal{E}'_\Lambda$, then it is supported on a finite number of fundamental domains. If in addition ${}^tA_\lambda f = 0$, then $f = 0$, since $f$ is real analytic and vanishes on an open set. This shows the injectivity of ${}^tA_\lambda$.

The range of ${}^tA_\lambda$ is closed, if $\mathrm{im}({}^tA_\lambda) \cap B'$ is closed for any bounded set $B' \subset \mathcal{E}'_\Lambda$. Let $B'$ be bounded. Since $\mathcal{E}_\Lambda$ is Fréchet, by a theorem of Schwartz [22], p.90, there are $W$, $k$, $\epsilon > 0$ such that $X \setminus W$ is connected and $|\langle f, \phi\rangle| \le 1$ for all $f \in B'$, $\phi \in C_c^\infty(X)$ with $q_{W,k}(\phi) \le \epsilon$. It follows that $\mathrm{supp}(f) \subset W$, $\forall f \in B'$.

Let ${}^tA_\lambda h_i = f_i \in B'$ converge to $f$. Unique continuation for ${}^tA_\lambda$ implies that $\mathrm{supp}(h_i) \subset W$. We extend $\Delta - 1/4$ to a non-negative self-adjoint unbounded operator on $L^2(X)$. Since either $\mathrm{Im}(\lambda^2) \neq 0$ or $\lambda^2 > 0$ the operator ${}^tA_\lambda = \Delta - 1/4 + \lambda^2$ has a bounded inverse on $L^2(X)$. We use ${}^tA_\lambda$ in order to define the scale of Sobolev spaces $H^l(X)$, $l \in 2\mathbf{Z}$. Here $A_\lambda : H^l(X) \to H^{l-2}(X)$ is an isometry and $H^0(X) = L^2(X)$. It is easy to see that $f_i$ converges in the direct limit $\lim_{l\to\infty} H^{-l}(X)$. In fact it already converges in $H^{-2k}(X)$. Hence $h_i \to h = {}^tA_\lambda^{-1}(f)$. Since $\mathrm{supp}(h_i) \subset W$ we also have $\mathrm{supp}(h) \subset W$. It follows that $h \in B'$ and ${}^tA_\lambda h = f$. This shows the closedness of the range of ${}^tA_\lambda$. The lemma follows. □

In order to calculate the cohomology of $\Gamma$ with coefficients in $H^\lambda_{-\omega,\Lambda}$ we take the subcomplex of $\Gamma$-invariants of the complex (1). Let $Y = \Gamma\backslash X$ be the Riemann surface corresponding to $\Gamma$. Let $A_{\lambda,Y} = \Delta_Y - 1/4 + \lambda^2$, where $\Delta_Y$ is the Laplacian of $Y$. Let $\mathcal{E}_Y := \{f \in C^\infty(Y) \mid A^k_{\lambda,Y} f \in L^2(Y)\ \forall k \in \mathbf{N}_0\}$. The cohomology $H^*(\Gamma, H^\lambda_{-\omega,\Lambda})$ is the cohomology of

$$0 \longrightarrow \mathcal{E}_Y \xrightarrow{A_{\lambda,Y}} \mathcal{E}_Y \longrightarrow 0.$$

Since $-\lambda^2$ is separated from the essential spectrum $[0,\infty)$ of $\Delta_Y - 1/4$ (see [15]) we obtain

**Proposition 2.4** *If $\mathrm{Re}(\lambda) > 0$, then $H^p(\Gamma, H^\lambda_{-\omega,\Lambda}) = 0$ for all $p \ge 2$ and $H^0(\Gamma, H^\lambda_{-\omega,\Lambda}) = H^1(\Gamma, H^\lambda_{-\omega,\Lambda}) = \ker_{L^2}(A_{\lambda,Y})$. There is a finite set $\Sigma \subset (0,\infty)$ such that $\ker_{L^2}(A_{\lambda,Y})$ is non-trivial iff $\lambda \in \Sigma \subset \mathbf{C}$. Here $\ker_{L^2}(A_\lambda)$ is always finite-dimensional.*

The finiteness of the point spectrum of $\Delta_Y$ was shown e.g. in Lax/Phillips [10].

# 3 Intertwining operators and the scattering matrix

We introduce the spaces $H^\lambda_\infty(\Omega)$ and $H^\lambda_{-\omega,(\Lambda)}$ of smooth functions on $\Omega$ and hyperfunctions on $S^1$ with singular support in $\Lambda$, respectively. The index $\lambda$ indicates the way $\Gamma$ acts on these spaces. Since the sheaf of hyperfunctions is flabby we have an exact sequence

$$0 \longrightarrow H^\lambda_{-\omega,\Lambda} \longrightarrow H^\lambda_{-\omega,(\Lambda)} \stackrel{res}{\longrightarrow} H^\lambda_\infty(\Omega) \longrightarrow 0$$

of $\Gamma$-modules. The long exact cohomology sequence gives

$$\begin{array}{l} 0 \longrightarrow {}^\Gamma H^\lambda_{-\omega,\Lambda} \longrightarrow {}^\Gamma H^\lambda_{-\omega,(\Lambda)} \stackrel{res}{\longrightarrow} {}^\Gamma H^\lambda_\infty(\Omega) \longrightarrow \\ \longrightarrow H^1(\Gamma, H^\lambda_{-\omega,\Lambda}) \longrightarrow \dots\ . \end{array}$$

Recall the definition of the finite set $\Sigma := \{\lambda \in \mathbf{C} \mid 1/2 > \mathrm{Re}(\lambda) > 0, H^0(\Gamma, H^\lambda_{-\omega,\Lambda}) \neq 0\}$. For all $\lambda \in \mathbf{C} \setminus \Sigma$, $\mathrm{Re}(\lambda) > 0$, by Proposition 2.4 an invariant function $\phi_\lambda \in {}^\Gamma H^\lambda_\infty(\Omega)$ has a unique invariant extension as a hyperfunction $\Phi_\lambda \in {}^\Gamma H^\lambda_{-\omega,(\Lambda)}$, i.e., $res(\Phi_\lambda) = \phi_\lambda$.

We fix a smooth positive invariant function $\varrho \in {}^\Gamma H^{3/2}_\infty(\Omega)$ (for existence see Perry [21]). Multiplication by

$$\varrho^s : H^\lambda_\infty(\Omega) \to H^{\lambda+s}_\infty(\Omega)$$

is an isomorphism of $\Gamma$-modules. Let $\phi \in {}^\Gamma H^{1/2}_\infty(\Omega)$ and set $\phi_\lambda := \varrho^{\lambda-1/2}\phi \in {}^\Gamma H^\lambda_\infty(\Omega)$. Then $\phi_\lambda$ is a holomorphic family of smooth functions on $\Omega$. For $\lambda \in \mathbf{C} \setminus \Sigma$, $\mathrm{Re}(\lambda) > 0$, let $\Phi_\lambda \in {}^\Gamma H^\lambda_{-\omega,(\Lambda)}$ be the unique extension of $\phi_\lambda$.

**Lemma 3.1** *For $\lambda \in \mathbf{C} \setminus \Sigma$, $\mathrm{Re}(\lambda) > 0$, the family $\Phi_\lambda$ is a holomorphic family of hyperfunctions.*

*Proof.* Consider the Poincaré series

$$\sum_{g\in\Gamma} g'(b)^{\lambda+1/2}$$

for some $b \in \Omega$. Here $g'(b)$ is the conformal dilatation of the map $b \mapsto g(b)$ with respect to the Euclidean metric on $S^1 = \partial X$ in the Poincaré disc model. Let $\delta_\Gamma \in \mathbf{R}$ be the smallest number such that the Poincaré series converges for all $\lambda \in \mathbf{C}$ with $\mathrm{Re}(\lambda) > \delta_\Gamma$. For $\mathrm{Re}(\lambda) > \delta_\Gamma$ the convergence is independent of the choice of $b \in \Omega$ and uniform in compact subsets of $\Omega$. It is known that $\delta_\Gamma < 1/2$ and that $\delta_\Gamma + 1/2$ is the Hausdorff dimension of $\Lambda$ (Patterson [18], Sullivan [23]). We claim that for $\mathrm{Re}(\lambda) > \delta_\Gamma$ the hyperfunction extension $\Phi_\lambda$ of $\phi_\lambda$ is given by integration against $\phi_\lambda$. This extension is in fact a measure. Let $f \in C(S^1)$ and $F \subset \Omega$ be a compact fundamental domain for $\Gamma$. We have

$$\begin{aligned} < \Phi_\lambda, f\partial^{-\lambda-1/2} > &:= \int_\Omega \varrho^{\lambda-1/2}(b)\phi(b)f(b)db \qquad (2)\\ &= \sum_{g\in\Gamma} \int_{g^{-1}F} \varrho^{\lambda-1/2}(b)\phi(b)f(b)db \end{aligned}$$

$$\begin{aligned}
&= \sum_{g\in\Gamma}\int_F \varrho^{\lambda-1/2}(gb)\phi(gb)f(gb)d(gb)\\
&= \sum_{g\in\Gamma}\int_F g'(b)^{\lambda+1/2}\varrho^{\lambda-1/2}(b)\phi(b)f(gb)db\\
&= \int_F(\sum_{g\in\Gamma} g'(b)^{\lambda+1/2}f(gb))\varrho^{\lambda-1/2}(b)\phi(b)db\ .
\end{aligned}$$

Since $f$ is uniformly bounded on $S^1$ the right-hand side converges for $\mathrm{Re}(\lambda) > \delta_\Gamma$. It is easy to see that $\Phi_\lambda$ is a holomorphic family of measures with respect to the parameter $\lambda$. Hence it is a holomorphic family of hyperfunctions for $\mathrm{Re}(\lambda) > \delta_\Gamma$.

It remains to show that $\Phi_\lambda$ is holomorphic in the strip $0 < \mathrm{Re}(\lambda) < 1/2$, $\lambda \notin \Sigma$. This requires analysis on the symmetric space $X$ and its quotient $Y = \Gamma\backslash X$. Let $T : (0,\epsilon)\times\Omega \to X$, $\epsilon > 0$, be the coordinates of a $\Gamma$- invariant neighbourhood of $\Omega$ introduced by Perry [20]. These coordinates depend on the choice of $\varrho$ and are defined as follows:

$$\begin{aligned}
y(r,b) &= \frac{r\varrho(b)}{1+\frac{r^2}{4}\varrho'(b)^2}\\
x(r,b) &= b-\frac{1}{2}\frac{r^2\varrho(b)\varrho'(b)}{1+\frac{r^2}{4}\varrho'(b)^2}\ .
\end{aligned}$$

Here $(x,y)$, $y > 0$, are coordinates of the upper half-plane model. The map $T$ satisfies $gT(r,b) = T(r,gb)$ for $g\in\Gamma$, $r\in(0,\epsilon)$, $b\in\Omega$. Hence we obtain induced coordinates of a collar neighbourhood $E\subset Y$ at infinity:

$$T_Y : (0,\epsilon)\times B \to Y\ . \tag{3}$$

In these coordinates the Laplacian $\Delta_Y$ has the form

$$\Delta_Y = -(r\frac{\partial}{\partial r})^2 + r\frac{\partial}{\partial r} + r^2\Delta_B + rP(r\frac{\partial}{\partial r}, r\frac{\partial}{\partial b})\ ,$$

where $\Delta_B$ is the Laplacian on $B$ with respect to the metric induced by $\varrho^{-2}(db)^2$ and $P$ is of second order with bounded coefficients. Let $A_{\lambda,Y} = \Delta_Y - 1/4 + \lambda^2$. Asymptotically the volume form is $r^{-2}$-times a bounded form. Fix a cut-off function $\chi(r)$ being zero near $r=\epsilon$ and one near $r=0$. Form $f_\lambda(r,b) = \chi(r)r^{-\lambda+1/2}\phi(b)$. One can check that $g_\lambda := A_{\lambda,Y}f_\lambda$ is a holomorphic family in $L^2(Y)$ for $\mathrm{Re}(\lambda) < 1$. For $\mathrm{Re}(\lambda) > 0$, $\lambda\notin\Sigma$, we can solve

$$A_{\lambda,Y}h_\lambda = -g_\lambda$$

in $L^2(Y)$ and obtain a holomorphic family of eigenfunctions $F_\lambda := f_\lambda + h_\lambda \in \ker(A_{\lambda,Y})$. Lifting $F_\lambda$ to $X$ we find a holomorphic family of invariant eigenfunctions $\tilde{F}_\lambda$. The boundary value $\beta_\lambda(\tilde{F}_\lambda)$ is a hyperfunction and the desired extension $\Phi_\lambda$ of $\phi_\lambda$ for $\mathrm{Re}(\lambda) > 0$, $\lambda\notin\Sigma$. In fact on sets $W$ we have $\tilde{F}_\lambda(x,y) \sim y^{-\lambda+1/2}\varrho^{\lambda-1/2}(x)\phi(x)\ (\mathrm{mod}\ L^2(W))$ and thus $res\ \beta_\lambda(\tilde{F}_\lambda) = \phi_\lambda$. For $\delta_\Gamma < \mathrm{Re}(\lambda) < 1$ the extension just constructed coincides with the extension defined by (2) by uniqueness. Since the boundary value map depends holomorphically on $\lambda$ for $\mathrm{Re}(\lambda) > 0$ the lemma follows. □

**Definition 3.2** *We define the Eisenstein series associated to* $\phi \in {}^\Gamma H^{1/2}_\infty(\Omega)$ *and* $\lambda \in \mathbf{C}$ *with* $\mathrm{Re}(\lambda) > 0$*,* $\lambda \notin \Sigma$*, by*

$$E(\phi, \lambda) = P_\lambda(\Phi_\lambda) \ .$$

The term 'series' stems from the fact that for $\mathrm{Re}(\lambda) > \delta_\Gamma$ it can be written as

$$E(\phi,\lambda)(p) = \sum_{g\in\Gamma} \int_F P_\lambda(gp, b)\phi_\lambda(b) db$$

(compare [15], [12], [21], [14]), since the Lebesgue measure of the limit set is zero and we can alternatively integrate over all of $S^1$:

$$E(\phi,\lambda)(p) = \int_{S^1} P_\lambda(p, b)\Phi_\lambda(b) db \ .$$

We now discuss the singularities of the extension $\Phi_\lambda$ at $\lambda_0 \in \Sigma$. Let $\{F_i\}$, $i = 1, \dots, k$, be an orthonormal base of $\ker_{L^2}(A_{\lambda_0,Y})$. Then near $\lambda_0$ the family $F_\lambda$ has the form

$$F_\lambda = \frac{1}{\lambda_0^2 - \lambda^2} \sum_{i=1}^k \langle F_i, g_\lambda\rangle_{L^2(Y)} F_i + \hat{F}_\lambda \ ,$$

where $\hat{F}_\lambda$ is holomorphic and orthogonal to $\ker_{L^2}(A_{\lambda_0,Y})$. Applying the boundary value map $\beta_\lambda$ we obtain

$$\Phi_\lambda = \frac{1}{\lambda_0^2 - \lambda^2} \sum_{i=1}^k \langle F_i, g_\lambda\rangle_{L^2(Y)} \beta_\lambda(\tilde{F}_i) + \hat{\Phi}_\lambda \ , \tag{4}$$

where $\hat{\Phi}_\lambda$ is holomorphic. Note that $F_i(r,b) \sim r^{\lambda+1/2}\beta_Y(F_i)(b) + o(r^{\lambda+1/2})$, where $\beta_Y$ is the boundary value on $Y$. In more detail, $\beta_Y(F_i)(.) = \lim_{r\to 0} r^{-\lambda-1/2}F_i(r,.)$, where the limit exists in the $L^2(B)$-sense. Applying Green's formula we obtain

$$\begin{aligned}
\langle F_i, g_{\lambda_0}\rangle_{L^2(Y)} &= \langle F_i, A_{\lambda_0,Y} f_{\lambda_0}\rangle_{L^2(Y)} - \langle A_{\lambda_0,Y} F_i, f_{\lambda_0}\rangle_{L^2(Y)} \\
&= \lim_{r\to 0}\langle F_i, r\frac{d}{dr} f_{\lambda_0}\rangle_{L^2(\{r\}\times B)} - \lim_{r\to 0}\langle r\frac{d}{dr}F_i, f_{\lambda_0}\rangle_{L^2(\{r\}\times B)} \\
&= -2\lambda_0 \langle \beta_Y(F_i), \phi\rangle_{L^2(B)} \ .
\end{aligned}$$

Taking the Poisson transform of (4) and using

$$P_\lambda \circ \beta_\lambda = \frac{1}{\sqrt{2}} \frac{\Gamma(\lambda)}{\Gamma(1/2+\lambda)} \mathrm{id}$$

we obtain

**Proposition 3.3** *The Eisenstein series* $E(\phi,\lambda)$ *is meromorphic on* $\mathrm{Re}(\lambda) > 0$ *with singularities in* $\Sigma$ *and residues*

$$\frac{1}{\sqrt{2}} \frac{\Gamma(\lambda_0)}{\Gamma(1/2+\lambda_0)} \sum_{i=1}^k \langle \beta_Y(F_i), \phi\rangle_{L^2(B)} F_i$$

*at* $\lambda = \lambda_0 \in \Sigma$*.*

Recall the Knapp-Stein intertwining operator $J_\lambda : H^\lambda_{-\omega} \to H^{-\lambda}_{-\omega}$. For $\mathrm{Re}(\lambda) < 0$ it is given by the integral operator

$$(J_\lambda \phi)(b) = \int_{\mathbf{R}} |b-b'|^{-1-2\lambda} \phi(b') db', \quad \phi \in C^\infty_c(R) \ .$$

For $\mathrm{Re}(\lambda) \geq 0$ it is defined by analytic continuation. $J_\lambda$ is a meromorphic family of elliptic pseudodifferential operators with singularities at $\lambda \in \mathbf{N}_0$. The restriction of $J_\lambda$ to $H^\lambda_{-\omega,(\Lambda)}$ defines a meromorphic family of $\Gamma$-intertwining operators $H^\lambda_{-\omega,(\Lambda)} \to H^{-\lambda}_{-\omega,(\Lambda)}$.

**Definition 3.4** *For $\mathrm{Re}(\lambda) > 0$, $\lambda \not\in \Sigma$, the scattering matrix $S_\lambda$ is the operator*

$$S_\lambda : {}^\Gamma H^\lambda_\infty(\Omega) \to {}^\Gamma H^{-\lambda}_\infty(\Omega)$$

*given by*

$${}^\Gamma H^\lambda_\infty \ni \phi \mapsto \Phi \in {}^\Gamma H^\lambda_{-\omega,(\Lambda)} \mapsto res \circ J_\lambda(\Phi) \in {}^\Gamma H^{-\lambda}_\infty(\Omega) \ .$$

For $\mathrm{Re}(\lambda) > \delta_\Gamma$, $\lambda \not\in \mathbf{N}$, and $\phi \in {}^\Gamma C^\infty(\Omega)$ we have

$$\begin{aligned} S_\lambda \phi_\lambda(b) &= \int_\Omega |b-b'|^{-1-2\lambda} \varrho^{\lambda-1/2} \phi(b') db' \\ &= \sum_{g\in\Gamma} \int_{g^{-1}F} |b-b'|^{-1-2\lambda} \varrho^{\lambda-1/2}(b') \phi(b') db' \\ &= \sum_{g\in\Gamma} \int_F |b-gb'|^{-1-2\lambda} \varrho^{\lambda-1/2}(gb') \phi(gb') d(gb') \\ &= \sum_{g\in\Gamma} \int_F |b-gb'|^{-1-2\lambda} g'(b')^{\lambda+1/2} \varrho^{\lambda-1/2}(b') \phi(b') db' \ , \end{aligned}$$

where the singularity of the kernel near $b = b'$ has to be interpreted in the regularized sense. Thus for $\mathrm{Re}(\lambda) > \delta_\Gamma$, $\lambda \not\in \mathbf{N}$, the scattering matrix is given by the integral kernel

$$\sum_{g\in\Gamma} |b-gb'|^{-1-2\lambda} g'(b')^{\lambda+1/2}$$

and this coincides with other definitions in the literature ([15], [14], [21]). The scattering matrix is a meromorphic family of elliptic pseudodifferential operators on $B$ in the following sense. Let $\phi \in C^\infty(B) = {}^\Gamma H^{1/2}_\infty(\Omega)$. Define $S(\lambda) : C^\infty(B) \to C^\infty(B)$ by

$$S(\lambda)\phi := \varrho^{1/2+\lambda} S_\lambda \varrho^{\lambda-1/2} \phi \ .$$

**Proposition 3.5** *The family*

$$S(\lambda)\phi := \varrho^{1/2+\lambda} S_\lambda \varrho^{\lambda-1/2} \phi$$

*is meromorphic for $\mathrm{Re}(\lambda) > 0$ with poles at $\Sigma$ and residues*

$$\phi \mapsto \frac{1}{\sqrt{2}} \frac{\Gamma(\lambda_0)}{\Gamma(1/2+\lambda_0)} \sum_{i=1} \langle \beta_Y(F_i), \phi \rangle_{L^2(B)} \beta_Y(F_i) \tag{5}$$

*at $\lambda = \lambda_0 \in \Sigma$. Moreover, it has poles at $\lambda \in \mathbf{N}$ (induced by the singularities of the on-diagonal regularization of the intertwining operator) and $S(k+1/2) = \varrho^{k+1} d^{2k+1} \varrho^k$, $k \in \mathbf{N}_0$.*

Equation (5) is obtained from (4) and

$$J_\lambda \circ \beta_\lambda = \frac{1}{\sqrt{2}} \frac{\Gamma(\lambda)}{\Gamma(1/2+\lambda)} \beta_{-\lambda} \ .$$

In Section 7 we obtain a meromorphic continuation of the scattering matrix and the Eisenstein series to all of $\mathbf{C}$.

# 4 The case $\mathrm{Re}(\lambda) < 0$, $\lambda \neq -1/2, -3/2, \dots$

For $2\lambda \notin \mathbf{Z}$ the intertwining operator $J_\lambda$ is an isomorphism. By restriction we obtain an isomorphism of $\Gamma$-modules $J_\lambda : H^\lambda_{-\omega,(\Lambda)} \to H^{-\lambda}_{-\omega,(\Lambda)}$. The operator $J_\lambda$ has first order poles at $\lambda = k \in \mathbf{N}$. At these points we consider its renormalization $\tilde{J}_k := \lim_{\lambda \to k} (\lambda - k) J_\lambda$. It satisfies $\tilde{J}_k \circ J_{-k} = -\frac{\pi}{2k} \mathrm{id}$ and provides an isomorphism $\tilde{J}_k : H^k_{-\omega,(\Lambda)} \to H^{-k}_{-\omega,(\Lambda)}$ .

**Lemma 4.1** *$H^\lambda_{-\omega,(\Lambda)}$ is $\Gamma$-acyclic.*

*Proof.* It is enough to show that $H^{-\lambda}_{-\omega,(\Lambda)}$ is $\Gamma$-acyclic. Since $\Gamma$ acts freely and properly on $\Omega$ the modules $H^\lambda_\infty(\Omega)$ are $\Gamma$-acyclic by [2], Lemma 2.4, for all $\lambda \in \mathbf{C}$. Consider the exact sequence of $\Gamma$-modules

$$0 \longrightarrow H^{-\lambda}_{-\omega,\Lambda} \longrightarrow H^{-\lambda}_{-\omega,(\Lambda)} \stackrel{res}{\longrightarrow} H^{-\lambda}_\infty(\Omega) \longrightarrow 0 \ .$$

The associated long exact sequence is

$$\begin{aligned} 0 \longrightarrow H^0(\Gamma, H^{-\lambda}_{-\omega,\Lambda}) \longrightarrow H^0(\Gamma, H^{-\lambda}_{-\omega,(\Lambda)}) \stackrel{res}{\longrightarrow} H^0(\Gamma, H^{-\lambda}_\infty(\Omega)) \stackrel{\delta}{\longrightarrow} \\ \longrightarrow H^1(\Gamma, H^{-\lambda}_{-\omega,\Lambda}) \longrightarrow H^1(\Gamma, H^{-\lambda}_{-\omega,(\Lambda)}) \longrightarrow 0 \ . \end{aligned}$$

In case that $-\lambda \notin \Sigma$ we immediately deduce $H^1(\Gamma, H^{-\lambda}_{-\omega,(\Lambda)}) = 0$. Thus assume that $-\lambda \in \Sigma$. We have to show that $\delta$ is surjective. Let $d := \dim {}^\Gamma H^{-\lambda}_{-\omega,\Lambda} = \dim H^1(\Gamma, H^{-\lambda}_{-\omega,\Lambda})$. It is suffices to show that

$$\dim \operatorname{coker}(res : {}^\Gamma H^{-\lambda}_{-\omega,(\Lambda)} \to {}^\Gamma H^{-\lambda}_\infty(\Omega)) \geq d \ .$$

Let $\tilde{W} \subset {}^\Gamma H^\lambda_\infty(\Omega)$ be the subspace spanned by the boundary values $\varrho^{\lambda-1/2}\beta_Y(F)$ for all $F \in \ker_{L^2}(A_{-\lambda,Y})$ and $V \subset {}^\Gamma H^{-\lambda}_\infty(\Omega)$ be the orthogonal complement of $\tilde{W}$. Then $\operatorname{codim} V = d$. It is sufficient to show that $res({}^\Gamma H^{-\lambda}_{-\omega,(\Lambda)}) \subset V$. Let $\Phi \in {}^\Gamma H^{-\lambda}_{-\omega,(\Lambda)}$. We must verify that $\langle \beta_Y(F), \varrho^{\lambda+1/2} res(\Phi) \rangle_{L^2(B)} = 0$ for all $F \in \ker_{L^2}(A_{-\lambda,Y})$. Thus let $E = P_{-\lambda}\Phi \in C^\infty(Y)$. Then by Green's formula

$$\begin{aligned} & -2\lambda \langle \beta_Y(F), \varrho^{\lambda+1/2} res(\Phi) \rangle_{L^2(B)} \\ = \ & \lim_{r\to 0} \langle F, r\frac{d}{dr}E \rangle_{L^2(\{r\}\times B)} - \lim_{r \to 0} \langle r\frac{d}{dr}F, E\rangle_{L^2(\{r\}\times B)} \\ = \ & \langle F, A_{\lambda,Y}E \rangle_{L^2(Y)} - \langle A_{\lambda,Y}F, E\rangle_{L^2(Y)} \\ = \ & 0 \ . \end{aligned}$$

This finishes the proof of the Lemma for $-\lambda \in \Sigma$. □
If $\lambda \notin -\mathbf{N}_0$, then $J_{-\lambda} \circ J_\lambda = -\frac{\cot(\pi\lambda)}{2\lambda}\mathrm{id}$. By Lemma 4.1 we obtain the following $\Gamma$-acyclic resolution of $H^\lambda_{-\omega,\Lambda}$:

$$0 \longrightarrow H^\lambda_{-\omega,\Lambda} \stackrel{J_\lambda}{\longrightarrow} H^{-\lambda}_{-\omega,(\Lambda)} \stackrel{res\circ J_{-\lambda}}{\longrightarrow} H^\lambda_\infty(\Omega) \longrightarrow 0 \ .$$

Restricting to the $\Gamma$-invariants and taking Definition 3.4 of the scattering matrix into account we obtain

**Proposition 4.2** *For* $\mathrm{Re}(\lambda) < 0$, $\lambda \notin -\frac{1}{2}\mathbf{N}_0$, $-\lambda \notin \Sigma$, *we have*

$$\begin{aligned} H^p(\Gamma, H^\lambda_{-\omega,\Lambda}) &= 0, \quad p \geq 2 \ , \\ H^0(\Gamma, H^\lambda_{-\omega,\Lambda}) &= \ker(S_{-\lambda}) \ , \\ H^1(\Gamma, H^\lambda_{-\omega,\Lambda}) &= \mathrm{coker}(S_{-\lambda}) \ . \end{aligned}$$

*Since* $S_{-\lambda}$ *is elliptic and* $\mathrm{index}(S_{-\lambda}) = 0$ *we have*

$$\dim H^0(\Gamma, H^\lambda_{-\omega,\Lambda}) = \dim H^1(\Gamma, H^\lambda_{-\omega,\Lambda}) < \infty \ .$$

Now we discuss the points $\lambda$, where the intertwining operator or the scattering matrix have singularities. First we consider the points $k \in \mathbf{N}$. We obtain the $\Gamma$-acyclic resolution

$$0 \longrightarrow H^k_{-\omega,\Lambda} \stackrel{J_{-k}}{\longrightarrow} H^{-k}_{-\omega,(\Lambda)} \stackrel{res\circ \tilde{J}_k}{\longrightarrow} H^k_\infty(\Omega) \longrightarrow 0 \ .$$

Let $\tilde{S}_k := \lim_{\lambda \to k}(\lambda - k)S_\lambda$ be the renormalized scattering matrix defined in a similar way as $\tilde{J}_k$. Then we have

**Proposition 4.3** *For* $k \in \mathbf{N}$, $k \neq 0$, *we have*

$$\begin{aligned} H^p(\Gamma, H^{-k}_{-\omega,\Lambda}) &= 0, \quad p \geq 2 \ , \\ H^0(\Gamma, H^{-k}_{-\omega,\Lambda}) &= \ker(\tilde{S}_k) \ , \\ H^1(\Gamma, H^{-k}_{-\omega,\Lambda}) &= \mathrm{coker}(\tilde{S}_k) \ . \end{aligned}$$

*Since* $\tilde{S}_{-\lambda}$ *is elliptic and* $\mathrm{index}(\tilde{S}_k) = 0$ *we have*

$$\dim H^0(\Gamma, H^{-k}_{-\omega,\Lambda}) = \dim H^1(\Gamma, H^{-k}_{-\omega,\Lambda}) < \infty \ .$$

We now consider the case $-\lambda \in \Sigma$. By Lemma 4.1 we have $\dim \mathrm{coker}(res : {}^\Gamma H^{-\lambda}_{-\omega,(\Lambda)} \to {}^\Gamma H^{-\lambda}_\infty(\Omega)) = d$. In fact $res({}^\Gamma H^{-\lambda}_{-\omega,(\Lambda)}) = V$ is the orthogonal complement of the space spanned by the boundary values of the elements of $\ker_{L^2}(A_{-\lambda,Y})$ and thus $V$ is closed. We choose a complement $W \subset {}^\Gamma H^{-\lambda}_\infty(\Omega)$ to $V$. By Proposition 3.5 the singular part of the scattering matrix $S_{-\lambda}$ vanishes on $V$. We fix an identification $i : W \stackrel{\sim}{\to} H^{-\lambda}_{-\omega,\Lambda}$ and define a regularized scattering matrix by $(S^{reg}_{-\lambda})_{|V} = S_{-\lambda}$ and $(S^{reg}_{-\lambda})_{|W} = res \circ J_{-\lambda} \circ i$. Then $S^{reg}_{-\lambda}$ is still an elliptic pseudodifferential operator of index zero. For any $\phi \in V$ by (4) we can construct the extension $\Phi \in {}^\Gamma H^{-\lambda}_{-\omega,(\Lambda)}$ thus obtaining a map $ext : V \to {}^\Gamma H^{-\lambda}_{-\omega,(\Lambda)}$ which is right inverse to $res$. It induces a decomposition ${}^\Gamma H^{-\lambda}_{-\omega,(\Lambda)} = \mathrm{im}(ext) \oplus {}^\Gamma H^{-\lambda}_{-\omega,\Lambda}$. Then

$$\tilde{res} = res_{|\mathrm{im}(ext)} \oplus i^{-1} : \mathrm{im}(ext) \oplus H^{-\lambda}_{-\omega,\Lambda} \stackrel{\sim}{\to} V \oplus W$$

satisfies $S^{reg}_{-\lambda} \circ \tilde{res} = res \circ J_{-\lambda}$. Hence we can identify $H^i(\Gamma, H^\lambda_{-\omega,\Lambda})$ for $i = 0, 1$ with the kernel and cokernel of $S^{reg}_{-\lambda}$, respectively. We claim that $S^{reg}_{-\lambda}(V)$ is transverse to $res \circ J_{-\lambda}({}^\Gamma H^{-\lambda}_{-\omega,\Lambda})$. We employ the meromorphic continuation of the scattering matrix to all of $\mathbf{C}$ and its functional equation obtained in Section 7. Using Lemma 7.2 we find $res \circ J_{-\lambda}({}^\Gamma H^{-\lambda}_{-\omega,\Lambda}) \subset \ker(S_\lambda)$ and $S_\lambda \circ (S^{reg}_{-\lambda})_{|V} = -\frac{\cot(\pi\lambda)}{2\lambda}\mathrm{id}_V$. Also, $res \circ J_\lambda \circ i^{-1}$ is injective. In fact a nontrivial element in the kernel of this composition would correspond to a non-trivial $L^2$-eigenfunction on $Y$ with both leading exponents vanishing. Since this is impossible we have shown the claim.

We conclude

**Proposition 4.4**

$$\begin{aligned} H^p(\Gamma, H^\lambda_{-\omega,\Lambda}) &= 0, \quad p \ge 2\ , \\ H^0(\Gamma, H^\lambda_{-\omega,\Lambda}) &= \ker(S^{reg}_{-\lambda})\ , \\ H^1(\Gamma, H^\lambda_{-\omega,\Lambda}) &= \mathrm{coker}(S^{reg}_{-\lambda})\ . \end{aligned}$$

*Moreover,* $\dim H^0(\Gamma, H^\lambda_{-\omega,\Lambda}) = \dim H^1(\Gamma, H^\lambda_{-\omega,\Lambda}) = \dim\ker\,(S^{reg}_{-\lambda})_{|V} < \infty$.

# 5 The case $\lambda = -1/2, -3/2, \ldots$

Let $t$ be the parameter of $S^1$ and $d := d/dt$. Recall the exact sequences

$$0 \to F_k \to H^{k/2} \stackrel{d^k}{\to} H^{-k/2} \to F_k \to 0\ ,$$

$k \in 2\mathbf{N}_0 + 1$, of $G$-modules, where $F_k$ is the finite-dimensional representation of $G$ of dimension $k$. Define the $\Gamma$-modules $M_k$, $N_k$ by

$$\begin{aligned} &0 \to H^{k/2}_{-\omega,\Lambda} \stackrel{d^k}{\to} H^{-k/2}_{-\omega,\Lambda} \to M_k \to 0 \\ &0 \to N_k \to H^{k/2}_{-\omega}(\Omega) \stackrel{d^k}{\to} H^{-k/2}_{-\omega}(\Omega) \to 0\ , \end{aligned}$$

where $H^\lambda_{-\omega}(\Omega)$ denotes the hyperfunctions on $\Omega$ with the $\Gamma$-module structure given by $\lambda$. The short exact sequence of complexes

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & H^{k/2}_{-\omega,\Lambda} & \to & H^{k/2}_{-\omega} & \to & H^{k/2}_{-\omega}(\Omega) & \to & 0 \\ & & d^k\downarrow & & d^k\downarrow & & d^k\downarrow & & \\ 0 & \to & H^{-k/2}_{-\omega,\Lambda} & \to & H^{-k/2}_{-\omega} & \to & H^{-k/2}_{-\omega}(\Omega) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

induces the exact sequence

$$0 \to F_k \to N_k \to M_k \to F_k \to 0 \ . \tag{6}$$

Since $H^{k/2}_{-\omega,\Lambda}$ has trivial $\Gamma$-cohomology by Proposition 2.4, we have

$$H^*(\Gamma, M_k) \cong H^*(\Gamma, H^{-k/2}_{-\omega,\Lambda})$$

leading to the problem of computing the cohomology of $M_k$.

We assume that $\Omega \neq \emptyset$ and that $\Gamma$ is non-trivial. Then $\Gamma$ is a free group

$$\Gamma = \langle a_1, \ldots, a_g, b_1, \ldots, b_g, \sigma_1, \ldots, \sigma_{t-1} \rangle \ ,$$

where $g$ is the genus of the surface $Y$ and $t$ is the number of boundary components, i.e., $\dim H^0(B)$. The generators $\sigma_i$ correspond to the boundary circles. The missing circle $\sigma_t$ can be expressed in terms of the other generators. Let $\Sigma_i \cong \mathbf{Z}$ be the group generated by $\sigma_i$.

**Lemma 5.1** *As a representation of* $\Gamma$ *we have* $N_k = \oplus_{i=1}^t \mathrm{Ind}_{\Sigma_i}^\Gamma F_{k|\Sigma_i}$. *Moreover, the embedding* $F_k \to N_k$ *is given by* $u : F_k \ni v \mapsto \oplus_{i=1}^t (\gamma \mapsto \gamma^{-1} v)$.

*Proof.* As a vector space, $N_k = \oplus_{\text{components of }\Omega} F_k$, where we identify $F_k$ with the kernel of $d^k$, i.e., with the polynomials of order at most $k-1$. The $\Gamma$-action is easy to check. □

The cohomology of $N_k$ can be computed using the Shapiro Lemma [1]:

$$H^*(\Gamma, N_k) = H^*(\Gamma, \oplus_{i=1}^t \mathrm{Ind}_{\Sigma_i}^\Gamma F_{k|\Sigma_i}) = \oplus_{i=1}^t H^*(\Sigma_i, F_{k|\Sigma_i}) \ .$$

Now $\sigma_i$ is hyperbolic and thus has a unique fixed line in $F_k$. It follows that $H^p(\Sigma_i, F_{k|\Sigma_i}) = \mathbf{C}$ for $p = 0, 1$ and $H^p(\Sigma_i, F_{k|\Sigma_i}) = 0$ for $p \geq 2$. We finally obtain

$$H^*(\Gamma, N_k) = H^*(B) \ .$$

From the exact sequence (6) we see that $H^*(\Gamma, M_k)$ is finite-dimensional and $\chi(\Gamma, M_k) = \chi(\Gamma, N_k) = 0$. We now split the sequence (6) into two short exact sequences

$$\begin{aligned} 0 \to F_k \to M_k \to R_k \to 0 \ , \\ 0 \to R_k \to N_k \to F_k \to 0 \ , \end{aligned}$$

where $R_k$ is a certain $\Gamma$-module. We obtain two long exact sequences

$$\begin{aligned} &0 \to H^0(\Gamma, F_k) \to H^0(B) \to H^0(\Gamma, R_k) \to \\ &\to H^1(\Gamma, F_k) \to H^1(B) \to H^1(\Gamma, R_k) \to 0 \end{aligned} \tag{7}$$

$$\begin{aligned} &0 \to H^0(\Gamma, R_k) \to H^0(\Gamma, M_k) \to H^0(\Gamma, F_k) \to \\ &\to H^1(\Gamma, R_k) \to H^1(\Gamma, M_k) \to H^1(\Gamma, F_k) \to 0 \ . \end{aligned} \tag{8}$$

In the following discussion we will assume that $\Gamma$ is non-abelian. Then for $k \geq 2$ we have

$$H^0(\Gamma, F_k) = 0 \quad \text{and} \quad \dim H^1(\Gamma, F_k) = (2g-2+t)k \ ,$$

while in case $k = 1$ we have

$$\dim H^0(\Gamma, F_1) = 1 \text{ and } \dim H^1(\Gamma, F_1) = 2g+t-1 \ .$$

Let $h^1 := \dim H^1(\Gamma, M_k)$ and $q$ be the dimension of the image of $u_* : H^1(\Gamma, F_k) \to H^1(B)$. Then we can read off from (7) that $\dim H^1(\Gamma, R_k) = t - q$. In case $k \geq 2$ from (8) we obtain

$$h^1 = t - q + (2g-2+t)k \ . \tag{9}$$

In case $k = 1$ we claim

$$h^1 = t - q + 2g + t - 1 \ . \tag{10}$$

We must show that $H^0(\Gamma, M_1) \to H^0(\Gamma, F_1)$ is surjective. This amounts to construct an invariant hyperfunction one-form $\omega$ on $S^1$ with non-trivial integral and support in $\Lambda$. In fact, $H_{-\omega}^{-1/2}$ can be identified with the hyperfunction one-forms and the map $H_{-\omega}^{-1/2} \to F_1 = \mathbf{C}$ is the integral of the one-form over $S^1$. Let $\chi$ be a locally constant function on $B$. Lift it to a locally constant function $\tilde{\chi}$ on $\Omega$. Let $\mathbf{R}^1 \to S^1$ be the universal cover and $\hat{\chi}$ be the lift of $\tilde{\chi}$ to $\mathbf{R}^1$. Assume that $0 \in \mathbf{R}^1$ projects to a point of $\Lambda$. Consider $[0,1]$ as a fundamental domain of that cover. We define the function $\chi_1(t) := \hat{\chi}(t) + k$ for all $t \in \mathbf{R}^1$, $t - k \in (0,1)$, projecting to a point of $\Omega$. Then $d\chi_1$ is a distribution on $\mathbf{R}^1$ carried by the lift of the limit set. It projects down to $S^1$ to give the desired $\Gamma$-invariant hyperfunction (in fact distribution) one-form with integral one. Thus (10) holds for $k = 1$.

In order to compute $h^1$ it remains to compute $t - q$.

**Lemma 5.2** *We have $t - q = 1$ for $k = 1$ or $g = 0$ and $t - q = 0$ for $k \geq 2$, $g \neq 0$.*

*Proof.* Let $V$ be a $\Gamma$-module. Recall the space of group cochains

$$C^i(\Gamma, V) = \{\phi : \underbrace{\Gamma \times \ldots \times \Gamma}_{i\times} \to V\}$$

and the boundary operators $\partial^0 : C^0(\Gamma, V) \to C^1(\Gamma, V)$, $(\partial v)(g) = gv - v$, $v \in C^0(\Gamma, V) = V$ and $\partial^1 : C^1(\Gamma, V) \to C^2(\Gamma, V)$, $(\partial^1\phi)(g_1, g_2) = g_1(\phi(g_2) - \phi(g_1^{-1})) - \phi(g_1 g_2)$. A one-cocycle is uniquely determined by its values on the generators. Since $\Gamma$ is free, the values on the generators can be prescribed arbitrarily. Hence

$$Z^1(\Gamma, V) = \mathbf{C}^{2g+t-1} \otimes V \ .$$

Let $\phi \in Z^1(\Gamma, F_k)$ be given by its values on the generators $\phi(a_i), \phi(b_i), \phi(\sigma_i)$. Then $u\phi \in Z^1(\Gamma, N_k)$ is given by $(u\phi)(g) = \oplus_{i=1}^t (h \mapsto h^{-1}\phi(g))$. This cocycle is a boundary iff the following equations have a solution $\oplus_{i=1}^t \psi_i \in N_k$:

$$\begin{aligned} h^{-1}\phi(g) &= \psi_i(g^{-1}h) - \psi_i(h), \quad i = 1, \ldots, t, \quad \forall g, h \in \Gamma & (11) \\ (1 - \sigma_i^{-1})\psi_i(e) &= \phi(\sigma_i), \quad i = 1, \ldots, t \ . & (12) \end{aligned}$$

Here the first equation (11) encodes the boundary map while the second (12) ensures that $\oplus_{i=1}^t \psi_i \in N_k$. Since $\phi$ was a cocycle there are always functions $\psi_i : \Gamma \to F_k$ solving (11) and the solutions are determined by $\psi_i(e)$. Thus $u\phi$ is a boundary iff the equations

$$(1 - \sigma_i^{-1})v = \phi(\sigma_i), \quad i = 1, \ldots, t \ , \tag{13}$$

are solvable. Let us first discuss the cases $k = 1$ or $g = 0$. $F_1$ is the trivial representation and (13) is not solvable if $\phi(\sigma_i) \neq 0$. We claim that for $k = 1$ or $g = 0$ the value $\phi(\sigma_t)$ depends on $\phi(\sigma_i)$, $i = 1, \ldots, t-1$.

The case $g = 0$ is obvious. Consider $g \geq 1$. Then $\sigma_t^{-1} = [a_1, b_1] \ldots [a_g, b_g]\sigma_1 \ldots \sigma_{t-1}$. A group one-cocycle $\phi$ in the trivial representation vanishes on commutators. Thus $\phi([a_1, b_1] \ldots [a_g, b_g]) = 0$. The value of $\phi(\sigma_1 \ldots \sigma_{t-1})$ only depends on $\phi(\sigma_i)$, $i = 1, \ldots, t-1$. Thus $\phi(\sigma_t)$ depends on $\phi(\sigma_i)$, $i = 1, \ldots, t-1$, too. This proves the claim. Since the $\sigma_i$ are hyperbolic we have

$$\dim(\mathrm{coker}(1 - \sigma_i^{-1}) : F_k \to F_k) = 1 \ .$$

By choosing non-zero $\phi(\sigma_i)$, $i = 1, \ldots, t-1$, appropriately we can produce a $t-1$-dimensional subspace in $H^1(\Gamma, N_1)$. Thus the claim implies $t - q = 1$.

We now discuss the case $k \geq 2$, $g \neq 0$. By chosing non-zero $\phi(\sigma_i)$, $i = 1, \ldots, t-1$, as above we can produce a $t-1$-dimensional subspace in $H^1(\Gamma, N_k)$. In order to find a one-dimensional complement, we set $\phi(\sigma_i) = 0$ for $i = 1, \ldots, t-1$, $\phi(a_i) = \phi(b_i) = 0$, $i = 2, \ldots, g$, $\phi(b_1) = 0$ and choose $\phi(a_1)$ appropriately. In fact by the cocycle equation

$$\begin{aligned} \phi(\sigma_t^{-1}) &= [a_1, b_1]\phi([a_2, b_2] \ldots [a_g, b_g]\sigma_1 \ldots \sigma_{t-1}) + \phi([a_1, b_1]) \\ &= \phi([a_1, b_1]) \\ &= (1 - a_1 b_1 a_1^{-1})\phi(a_1) \ . \end{aligned}$$

Since $[a_1 b_1 a_1^{-1}, \sigma_t] \neq 0$ we can choose $\phi(a_1)$ such that $(1 - a_1 b_1 a_1^{-1})\phi(a_1)$ represents a non-trivial element in the cokernel of $(1 - \sigma_t)$. Then the equation (13) has no solution for $i = t$. It follows that $q = t$. $\square$

**Proposition 5.3** *Let $\Gamma$ be non-abelian. Then we have*

$$\dim H^0(\Gamma, H^{-k/2}_{-\omega,\Lambda}) = \dim H^1(\Gamma, H^{-k/2}_{-\omega,\Lambda}) = \left\{ \begin{array}{cc} (2g - 2 + t)k, & k \geq 2, g \neq 0 \\ (2g - 2 + t)k + 1, & k \geq 2, g = 0 \\ 2g + t, & k = 1 \end{array} \right. .$$

If $\Gamma$ is abelian, then the limit set consists of two points. $H^*(\Gamma, H^{\lambda}_{-\omega,\Lambda})$ is represented by derivatives of delta distributions located at $\Lambda$. The following result is an easy exercise.

**Corollary 5.4** *If $\Gamma$ is non-trivial and abelian, then*

$$\dim H^0(\Gamma, H^{-k/2}_{-\omega,\Lambda}) = \dim H^1(\Gamma, H^{-k/2}_{-\omega,\Lambda}) = 2 \ .$$

# 6 The case $\mathrm{Re}(\lambda)=0$, $\lambda\neq 0$

Let $\lambda=i\mu\neq 0$ and $A_{i\mu}:=\Delta-1/4-\mu^2$. By the theorem of Helgason the Poisson transform provides an embedding $P_{i\mu}:H^{i\mu}_{-\omega,\Lambda}\hookrightarrow\mathcal{E}$. In order to characterize the range of $P_{i\mu}$ we introduce the following semi-norms: $q_{W,k}(f):=\|A^k_{i\mu}f\|_{B(W)}$, $p_W(f):=\|f\|_{B^*(W)}$, $s_W(f)=p_W(D_\mu f)$, and $r_{K,i}(f):=\|A^i_{i\mu}f\|_{L^2(K)}$. The indices run over the domains $k\geq 1$, $i\geq 0$. $K$ runs over the compact subsets of $X$ and $W$ over the sets of the form $W=M\times(0,a)\subset\mathbf{R}^2_+$, $a>0$, for compact $M\subset\Omega$, using the coordinates $(x,y)$, $y>0$, of the upper half-plane model. The operator $D_\mu$ is given in these coordinates by $D_\mu:=y\frac{d}{dy}-(i\mu+1/2)$. The $B$- and $B^*$-norms are defined by $\|f\|_{B(W)}:=\sum_{j\geq 0}2^{j/2}\|f\|_{L^2(W\cap\Omega_j)}$, $\|f\|_{B^*(W)}:=\sup_{j\geq 0}2^{-j/2}\|f\|_{L^2(W\cap\Omega_j)}$, $\Omega_j:=\{(x,y)\in\mathbf{R}^2_+|-\ln(y)\in[2^{j-1},2^j)\}$, $j\geq 1$, and $\Omega_0:=\{(x,y)\in\mathbf{R}^2_+|y\geq \mathrm{e}\}$. By $B(W)$ and $B^*(W)$ we denote the Banach spaces of functions on $W$ with finite $B$- or $B^*$-norm. We also introduce the closed subspace ${}^\circ B^*(W)=\{f\in B^*(W)|\limsup_R\frac{1}{R}\|f\|^2_{L^2(M\times(a,\mathrm{e}^{-R}))}=0\}$. These norms are natural in the framework of scattering theory [8], Ch.14.

Let $\mathcal{E}^\mu_\Lambda$ be the Frechet space of all $f\in\mathcal{E}$ such that for all $W$ and $K$ described above $q_{W,k}(f)<\infty$, $\forall k\geq 1$, $p_W(f)<\infty$, $s_W(f)<\infty$, $r_{K,i}(f)<\infty$, $\forall i\geq 0$ and $D_\mu f\in{}^\circ B^*(W)$. Set $\mathcal{E}^\mu_\Lambda(A_{i\mu})=\mathcal{E}^\mu_\Lambda\cap\ker(A_{i\mu})$.

**Lemma 6.1**

$$P_{i\mu}(H^{i\mu}_{-\omega,\Lambda})=\mathcal{E}^\mu_\Lambda(A_{i\mu})$$

*Proof.* Let $P_{i\mu}((x,y),b)=(\frac{y}{y^2+(x-b)^2})^{i\mu+1/2}$ be the kernel defining the Poisson transform. Then $A_{i\mu}P_{i\mu}(.,b)=0$, $P_{i\mu}(.,b)\in B^*(W)$ and $D_\mu P_{i\mu}(.,b)\in{}^\circ B^*(W)$, $W=M\times(0,a)$, for $b\notin M$. Hence $b\to P_{i\mu}(.,b)$ defines an analytic function from a neighbourhood of $\Lambda$ to $\mathcal{E}^\mu_{\Lambda|W}$ (with the obvious definition) for all $W$. It follows $P_{i\mu}(H^{i\mu}_{-\omega,\Lambda})\subset\mathcal{E}^\mu_\Lambda(A_{i\mu})$.

We now show that $f\in\mathcal{E}^\mu_\Lambda(A_{i\mu})$ implies that the boundary value $\beta_{i\mu}(f)$ vanishes on $\Omega\cap\bar W$ and hence $\beta_{i\mu}(\mathcal{E}^\mu_\Lambda(A))\subset H^{i\mu}_{-\omega,\Lambda}$. Here $\beta_{i\mu}$ is the boundary value corresponding to the asymptotic $y^{-i\mu+1/2}$.

The argument is similar to the corresponding argument in the proof of Lemma 2.1. We again employ the coordinates of the disk model. Let $\psi\in H^{-i\mu}_\omega$. Then

$$\langle\beta_{i\mu}(f),\psi\rangle=\lim_{R\to\infty}\frac{1}{R}\int_0^R f_\psi(t)\mathrm{e}^{t(1/2-i\mu)}dt\ ,$$

where

$$f_\psi(t)=\int_{S^1}f(t,\alpha)\psi(\mathrm{e}^{i\alpha})\frac{d\alpha}{2\pi}\ ,$$

and the $t$-coordinate is the hyperbolic distance from the origin. Let $(a,b)\subset\Omega$. Using $(\frac{d}{dt}+i\mu+1/2)f=-D_\mu f+o(t^{-1})$ and $D_\mu f\in{}^\circ B^*$ along $\Omega$ we obtain

$$\begin{aligned}&\frac{1}{R}\int_0^R\int_{(a,b)}(\frac{d}{dt}+i\mu+1/2)f(t,\alpha)\mathrm{e}^{t(\mp i\mu-1/2)}\psi(\mathrm{e}^{i\alpha})\frac{d\alpha}{2\pi}\mathrm{e}^t dt\\ &\quad\leq\quad (R^{-1/2}\|D_\mu f\|_{L^2((0,R)\times(a,b))}+o(R^{-3/2}))\|\psi\|_{L^2((a,b))}\\ &\quad\stackrel{R\to\infty}{\to}\quad 0\ .\end{aligned}$$

By partial integration this implies on the one hand

$$\lim_{R\to\infty} \frac{1}{R} \mathrm{e}^{R(1/2+i\mu)} \int_{(a,b)} f(R,\alpha)\psi(\mathrm{e}^{i\alpha}) \frac{d\alpha}{2\pi} = 0$$

and on the other hand

$$\begin{aligned} 2i\mu \lim_{R\to\infty} \frac{1}{R} \int_0^R \int_{(a,b)} f(t,\alpha)\psi(\mathrm{e}^{i\alpha}) \frac{d\alpha}{2\pi} \mathrm{e}^{t(1/2-i\mu)} dt \\ = \ - \lim_{R\to\infty} \frac{1}{R} \mathrm{e}^{R(1/2-i\mu)} \int_{(a,b)} f(R,\alpha)\psi(\mathrm{e}^{i\alpha}) \frac{d\alpha}{2\pi} \ . \end{aligned}$$

Thus

$$\lim_{R\to\infty} \frac{1}{R} \int_0^R \int_{(a,b)} f(t,\alpha)\psi(\mathrm{e}^{i\alpha}) \frac{d\alpha}{2\pi} \mathrm{e}^{t(1/2-i\mu)} dt = 0 \ .$$

We see that $\beta_{i\mu}(f)$ defines a continuous functional on the germs of analytic sections of $T^{-i\mu-1/2}$ on $S^1 \setminus (a,b)$. The argument can now be completed as in Lemma 2.1. $\square$

We introduce another Frechet space of functions

$$\mathcal{E}_\Lambda := \{ f \in C^\infty(X) \mid q_{W,k}(f) < \infty, \forall W, k \geq 0, r_{K,i}(f) < \infty, \forall K, i \geq 0 \} \ .$$

The next lemma is proved in analogy with [2], Lemma 2.4.

**Lemma 6.2** *$\mathcal{E}^\mu_\Lambda$, $\mathcal{E}_\Lambda$ are $\Gamma$-acyclic.*

The following Lemma implies that

$$0 \longrightarrow H^{i\mu}_{-\omega,\Lambda} \xrightarrow{P} \mathcal{E}^\mu_\Lambda \xrightarrow{A_{i\mu}} \mathcal{E}_\Lambda \longrightarrow 0 \qquad (14)$$

is a $\Gamma$-acyclic resolution of $H^{i\mu}_{-\omega,\Lambda}$.

**Lemma 6.3** *$A_{i\mu} : \mathcal{E}^\mu_\Lambda \to \mathcal{E}_\Lambda$ is surjective.*

*Proof.* We consider the adjoint operator ${}^tA_{i\mu} : \mathcal{E}'_\Lambda \to (\mathcal{E}^\mu_\Lambda)'$. We must show that ${}^tA_{i\mu}$ is injective and has closed range. Since $C_c^\infty(X)$ is dense in $\mathcal{E}_\Lambda$, we can embed $\mathcal{E}'_\Lambda$ into the distributions on $X$. In fact if $f \in \mathcal{E}'_\Lambda$, then it is a distribution with support in a finite union of fundamental domains of $\Gamma$ (see the proof of Lemma 2.3 for a similar argument). If ${}^tA_{i\mu}f = 0$, then $f$ is real analytic. Since it vanishes on a non-empty open subset of $X$, it vanishes identically. Thus ${}^tA_{i\mu}$ is injective.

We must show that ${}^tA_{i\mu}$ has closed range. As in Lemma 2.3 we can restrict the consideration to bounded sets $B' \subset (\mathcal{E}^\mu_\Lambda)'$. There is a subset $U \subset X$ being the union of a compact set $K \subset X$ and $W = (0,b) \times M$, $M$ compact, $M \subset \Omega$, such that $X \setminus U$ is connected and $\mathrm{supp}(h) \subset U$ for all $h \in B'$. Let ${}^tA_{i\mu}f_i =: h_i \in B'$ such that $h_i \to h$ in $(\mathcal{E}^\mu_\Lambda)'$. We have $\mathrm{supp}(f_i) \subset U$. We must find a $f \in \mathcal{E}'_\Lambda$ with ${}^tA_{i\mu}f = h$.

Let $\mathcal{E}^\mu_\emptyset, \mathcal{E}_\emptyset$ be defined like $\mathcal{E}^\mu_\Lambda, \mathcal{E}_\Lambda$ but for the trivial limit set. Then $f_i \in \mathcal{E}'_\emptyset$ and ${}^tA_{i\mu}f_i \to h$ holds in $(\mathcal{E}^\mu_\emptyset)'$. Using a result of Perry [20] (essentially a corollary to Lemma

7.1) we conclude that $A_{i\mu} : \mathcal{E}^\mu_\emptyset \to \mathcal{E}_\emptyset$ is a topological isomorphism. Thus $f_i \to f = {}^t A^{-1}_{i\mu} h$ in $(\mathcal{E}_\emptyset)'$. Since $\mathrm{supp}(f_i) \subset U$ we also have $\mathrm{supp}(f) \subset U$ and hence $f \in \mathcal{E}'_\Lambda$. Thus the range of ${}^t A_{i\mu} : \mathcal{E}'_\Lambda \to (\mathcal{E}^\mu_\Lambda)'$ is closed. This finishes the proof of surjectivity of $A_{i\mu}$ and of the lemma. □

We now consider the $\Gamma$-invariants in the resolution (14). Using the coordinates (3) of a collar neighbourhood $E$ of infinity $T_Y : (0, \epsilon) \times B \to E \subset Y$ we can define the spaces $B(Y)$, $B^*(E)$ and ${}^\circ B^*(E)$ as in [20]. Let $A_{i\mu,Y} = \Delta_Y - 1/4 - \mu^2$, where $\Delta_Y$ is the Laplacian on $Y$ and $D_\mu = \frac{d}{dr} - i\mu$. Then ${}^\Gamma\mathcal{E}^\mu_\Lambda = \mathcal{E}^\mu_Y, {}^\Gamma\mathcal{E}_\Lambda = \mathcal{E}_Y$ with

$$\begin{aligned}
\mathcal{E}^\mu_Y &:= \{f \in C^\infty(Y) \| \|A^k_{i\mu,Y} f\|_{B(Y)} < \infty, k \geq 1, \|f\|_{B^*(E)} < \infty, D_\mu f \in {}^\circ B^*(E)\} \\
\mathcal{E}_Y &:= \{f \in C^\infty(Y) \| \|A^k_{i\mu,Y} f\|_{B(Y)} < \infty, k \geq 0\}
\end{aligned}$$

It again follows from the results of [20] that $A_{i\mu,Y} : \mathcal{E}^\mu_Y \to \mathcal{E}_Y$ is a topological isomorphism. Thus $\ker(A_{i\mu,Y}) = \mathrm{coker}(A_{i\mu,Y}) = 0$.

**Proposition 6.4** *For $0 \neq \lambda$, $\mathrm{Re}(\lambda) = 0$, we have*

$$H^*(\Gamma, H^\lambda_{-\omega,\Lambda}) = 0 \ .$$

# 7 The scattering matrix near $\mathrm{Re}(\lambda) = 0$

In Section 3 we constructed a continuation of the scattering matrix up to the imaginary axis. In this section we first show that it is continuous at the imaginary line. Then we employ the functional equation in order to provide the meromorphic continuation to all of $\mathbf{C}$. Consider the operator $A_{\lambda,Y} = \Delta_Y - 1/4 + \lambda^2$. For $\mathrm{Re}(\lambda) > 0$ it is invertible on $L^2(Y)$. If $\lambda$ approaches the imaginary axis we consider its inverse on the slightly smaller space $B(Y)$.

**Lemma 7.1 (Perry, [20], Prop. 5.5)** *The inverse $A^{-1}_{\lambda,Y} : B(Y) \to B^*(Y)$ is a weakly continuous family of bounded operators uniformly bounded on compact subsets of $\lambda \in \mathbf{C} \setminus (\{0\} \cup \Sigma)$, $\mathrm{Re}(\lambda) \geq 0$.*

We now employ the constructions and notations introduced in Section 3. Let $\phi \in C^\infty(B) = {}^\Gamma H^{1/2}_\infty(\Omega)$. We can consider $\phi_\lambda := \varrho^{\lambda - 1/2}\phi \in {}^\Gamma H^\lambda_\infty(\Omega)$. We want to construct a hyperfunction extension $\Phi_\lambda \in H^\lambda_{-\omega}$. The exact sequence

$$0 \longrightarrow H^\lambda_{-\omega,\Lambda} \longrightarrow H^\lambda_{-\omega,(\Lambda)} \stackrel{res}{\longrightarrow} H^\lambda_\infty(\Omega) \longrightarrow 0$$

implies the long exact sequence

$$\begin{aligned}
&0 \longrightarrow {}^\Gamma H^\lambda_{-\omega,\Lambda} \longrightarrow {}^\Gamma H^\lambda_{-\omega,(\Lambda)} \stackrel{res}{\longrightarrow} {}^\Gamma H^\lambda_\infty(\Omega) \longrightarrow \\
&\longrightarrow H^1(\Gamma, H^\lambda_{-\omega,\Lambda}) \longrightarrow \dots \ .
\end{aligned}$$

From Proposition 6.4 it follows that $\Phi_\lambda$ exists for $\mathrm{Re}(\lambda) \geq 0$, $\lambda \notin \Sigma$, $\lambda \neq 0$. As we have seen in Section 3 $\Phi_\lambda$ is holomorphic near the imaginary axis. We show that $\Phi_\lambda$ is continuous up to $\mathrm{Re}(\lambda)=0$, $\lambda\neq 0$. We construct the family of approximate eigenfunctions $f_\lambda$ and note that $g_\lambda := A_{\lambda,Y} f_\lambda \in B(Y)$. Lemma 7.1 shows that $F_\lambda = f_\lambda - A_{\lambda,Y}^{-1} g_\lambda$ extends continuously to $\mathrm{Re}(\lambda)=0$, $\lambda\neq 0$, as a family of eigenfunctions in $C^\infty(Y)$ (weakly continuous in $B^*(Y)$). Lifting to the universal cover and taking the boundary value we see that the family of hyperfunctions $\Phi_\lambda$ is continuous up to the imaginary axis, too. We define the right limit $S_{\lambda+0}$ of the scattering matrix for $\lambda\neq 0$, $\mathrm{Re}(\lambda)=0$, by

$$S_{\lambda+0}\phi_\lambda := (res\circ J_\lambda)\Phi_{\lambda+0}\ .$$

Then by definition $S(\lambda) = \varrho^{1/2+\lambda} S_\lambda \varrho^{\lambda-1/2}$ is continuous in the strong topology as an operator on $C^\infty(B)$ for $\mathrm{Re}(\lambda)\geq 0$ and $\lambda\neq 0$, $\lambda\notin\Sigma$ and $\lambda\notin\mathbf{N}_0$.

The Knapp-Stein intertwining operators satisfy the functional equation

$$J_\lambda\circ J_{-\lambda} = -\frac{\cot(\pi\lambda)}{2\lambda}\ .$$

It follows for $\mathrm{Re}(\lambda)=0$, $\lambda\neq 0$,

$$\begin{aligned} S_{\lambda+0}\circ S_{-\lambda+0}\phi_\lambda &= \lim_{\epsilon\to 0}(res\circ J_{\lambda+\epsilon})(res\circ J_{-\lambda+\epsilon}\Phi_{-\lambda+\epsilon})_{\mathrm{extended}} \\ &= res\circ J_\lambda\circ J_{-\lambda}\Phi_{\lambda+0} \\ &= -\frac{\cot(\pi\lambda)}{2\lambda}\phi_\lambda\ . \end{aligned}$$

We define $S_\lambda$ for $\mathrm{Re}(\lambda)<0$ by

$$S_\lambda := -\frac{\cot(\pi\lambda)}{2\lambda} S_{-\lambda}^{-1}\ .$$

A standard application of the Fredholm theory for Fréchet spaces [5] similar as in [16] shows that $S_\lambda$ is meromorphic with at most finite-dimensional singularities at $\lambda\notin\frac{1}{2}\mathbf{N}_0$. It remains to show continuity of $S_\lambda$ at $\mathrm{Re}(\lambda)=0$. In fact we have

$$S_{\lambda+0} - S_{\lambda-0} = S_{\lambda+0} + \frac{\cot(\pi\lambda)}{2\lambda} S_{-\lambda+0}^{-1} = 0\ .$$

Thus we have shown

**Lemma 7.2** *$S(\lambda)$ is a meromorphic family of elliptic pseudodifferential operators on $C^\infty(B)$. It has at most finite-dimensional poles at the spectral points $\lambda\in\Sigma$ and in the set of resonances $\mathrm{Re}(\lambda)<0$ with $H^0(\Gamma, H^\lambda_{-\omega,\Lambda})\neq 0$. There are further singularities at $\lambda\in\mathbf{N}_0$.*

**Corollary 7.3** *The set of $\lambda\in\mathbf{C}$ with $H^*(\Gamma, H^\lambda_{-\omega,\Lambda})\neq 0$ is discrete.*

We now can find an analytic continuation of $\Phi_\lambda$. For $\mathrm{Re}(\lambda)<0$, $\lambda$ not a resonance or in $\frac{1}{2}\mathbf{Z}$, define

$$\Phi_\lambda := -2\lambda\tan(\pi\lambda) J_{-\lambda}\circ res^{-1}\circ S_\lambda\phi_\lambda\ .$$

Then for $\mathrm{Re}(\lambda) = 0$, $\lambda \neq 0$, we have

$$\begin{aligned}
\Phi_{\lambda-0} &= -2\lambda\tan(\pi\lambda)J_{-\lambda+0} \circ res^{-1} \circ S_{\lambda-0}\phi_{\lambda-0} \\
&= -2\lambda\tan(\pi\lambda)J_{-\lambda} \circ J_\lambda \circ res^{-1}\phi_\lambda \\
&= res^{-1}\phi_\lambda \\
&= \Phi_{\lambda+0} \ .
\end{aligned}$$

It follows that $\Phi_\lambda$ is a meromorphic family of hyperfunctions. We obtain the Eisenstein series by applying the Poisson transform: $E(\phi, \lambda) = P_\lambda \Phi_\lambda$.

**Proposition 7.4** *The Eisenstein series has a meromorphic continuation to all of* $\mathbf{C} \setminus \{0\}$ *with poles in* $\Sigma$*, the set of resonances, and the negative half-integers. The Eisenstein series satisfies the functional equation*

$$E(\phi, \lambda) = \sqrt{2}\frac{\Gamma(1/2-\lambda)}{\Gamma(-\lambda)} E(S_\lambda\phi, -\lambda) \ .$$

The functional equation follows easily from the corresponding functional equation of the Poisson transform [9]. The meromorphic continuation of the scattering matrix and the Eisenstein series for surfaces was first obtained by Patterson [15],[17],[17]. Generalizations to higher dimensions can be found in [14], [21], [12], [11]. Since a generalization with more technical details can be found in [3] in the present paper we kept the arguments concerning the scattering matrix and the Eisenstein series sketchy.

# 8 Summary

Let $\Gamma$ be a torsion-free Fuchsian group of the second kind without parabolic elements. By $H^\lambda_{-\omega,\Lambda}$ we denote the hyperfunction vectors of the principal series representation $H^\lambda_{-\omega}$ with support in the limit set of $\Gamma$. For $\lambda \neq 0$ the cohomology $H^*(\Gamma, H^\lambda_{-\omega,\Lambda})$ is finite-dimensional and has vanishing Euler characteristic. In greater detail:

- For $\mathrm{Re}(\lambda) > 0$, we have

$$H^0(\Gamma, H^\lambda_{-\omega,\Lambda}) = H^1(\Gamma, H^\lambda_{-\omega,\Lambda}) = \ker_{L^2}(\Delta_Y - 1/4 + \lambda^2) \ ,$$

  where $\Delta_Y$ is the Laplace-Beltrami operator on the hyperbolic surface $Y := \Gamma \backslash H^2$. Let $\Sigma$ be the set of $\lambda$, $\mathrm{Re}(\lambda) > 0$, with $H^0(\Gamma, H^\lambda_{-\omega,\Lambda})$ non-trivial.

- For $\mathrm{Re}(\lambda) = 0$, $\lambda \neq 0$, we have

$$H^*(\Gamma, H^\lambda_{-\omega,\Lambda}) = 0 \ .$$

- The case $\lambda = 0$ is still unknown.

- For $\mathrm{Re}(\lambda) < 0$, $\lambda \notin -\frac{1}{2}\mathbf{N}$, $-\lambda \notin \Sigma$, we have

$$\begin{aligned} H^0(\Gamma, H^\lambda_{-\omega,\Lambda}) &= \ker(S_{-\lambda}) \\ H^1(\Gamma, H^\lambda_{-\omega,\Lambda}) &= \mathrm{coker}(S_{-\lambda}) \ , \end{aligned} \tag{15}$$

  where $S_\lambda$ is the scattering matrix. We have $\dim H^0(\Gamma, H^\lambda_{-\omega,\Lambda}) = \dim H^1(\Gamma, H^\lambda_{-\omega,\Lambda})$.

- For $\mathrm{Re}(\lambda) < 0$, $-\lambda \in \Sigma$, (15) holds with $S_{-\lambda}$ replaced by $S^{reg}_{-\lambda}$ defined in Section 4.

- For $\mathrm{Re}(\lambda) < 0$, $-\lambda \in \mathbf{N}$, (15) holds with $S_{-\lambda}$ replaced by the renormalized $\tilde{S}_{-\lambda}$.

- For $\lambda = -k/2$, $k$ odd, we have for non-abelian $\Gamma$

$$\dim H^0(\Gamma, H^{-k/2}_{-\omega,\Lambda}) = \dim H^1(\Gamma, H^{-k/2}_{-\omega,\Lambda}) = \begin{cases} (2g-2+t)k, & k \geq 2, g \neq 0 \\ (2g-2+t)k+1, & k \geq 2, g = 0 \\ 2g+t, & k = 1 \end{cases}$$

  and for abelian $\Gamma$

$$\dim H^0(\Gamma, H^{-k/2}_{-\omega,\Lambda}) = \dim H^1(\Gamma, H^{-k/2}_{-\omega,\Lambda}) = 2 \ .$$

  Here $g$ is the genus and $t$ is the number of boundary components of $Y$.

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