Costa's Minimal Surface via Mathematica

Helaman Ferguson, Alfred Gray and Steen Markvorsen

1. Introduction

One of the most interesting minimal surfaces was defined by Costa (see [Costa1]). Hoffman and Meeks (see [HoMe]) showed that Costa's minimal surface is a complete embedded surface. We shall adapt the treatment of Costa's minimal surface given in [BaCo] to find a parametrization that can be used with *Mathematica*. For this we need the Weierstrass Zeta function ζ. First we explain the relevant theory of this function and also the Weierstrass function $\wp$.

2. The Weierstrass $\wp$ and ζ Functions

Let ω_1 and ω_2 be two complex numbers, both different from zero, and assume that $\mathfrak{Im}(\omega_2/\omega_1)$ is positive. We define a lattice $\mathcal{L} \subset \mathbb{C}$ by

$$\mathcal{L} = \{\, m\,\omega_1 + n\,\omega_2 \mid m, n \in \mathbb{Z} \,\}.$$

Put

$$g_2 = 60 \sum_{\substack{\omega\in\mathcal{L}\\ \omega\neq 0}} \frac{1}{\omega^4} \qquad \text{and} \qquad g_3 = 140 \sum_{\substack{\omega\in\mathcal{L}\\ \omega\neq 0}} \frac{1}{\omega^6}. \tag{1}$$

The **Weierstrass $\wp$ function** is defined by

$$\wp(z, g_2, g_3) = \frac{1}{z^2} + \sum_{\substack{\omega\in\mathcal{L}\\ \omega\neq 0}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right). \tag{2}$$

Then $z \longmapsto \wp(z, g_2, g_3)$ is a doubly periodic function with periods ω_1 and ω_2. The set

$$\mathbf{FPP}(g_2, g_3) = \{\, z \in \mathbb{C} \mid 0 \le \mathfrak{Re}(z) < \omega_1 \text{ and } 0 \le \mathfrak{Im}(z) < \omega_2 \,\}$$

is called the **fundamental period parallelogram** of $\wp(z, g_2, g_3)$; its importance lies in the fact that the image of $\mathbf{FPP}(g_2, g_3)$ under the Weierstrass $\wp$ function is the same as any translate $\mathbf{FPP}(g_2, g_3) + m\,\omega_1 + n\,\omega_2$ (where m and n are integers). The Weierstrass $\wp$ function satisfies an addition formula:

$$\wp(z_1 + z_2) = \frac{1}{4} \left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)} \right)^2 - \wp(z_1) - \wp(z_2). \tag{3}$$

We shall also need the **Weierstrass ζ function**, which is defined by

$$\zeta(z, g_2, g_3) = \frac{1}{z} + \sum_{\substack{\omega\in\mathcal{L}\\ \omega\neq 0}} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right). \tag{4}$$

When g_2 and g_3 are understood, we write

$$\wp(z) = \wp(z, g_2, g_3) \qquad \text{and} \qquad \zeta(z) = \zeta(z, g_2, g_3).$$

It is well-known that $\wp$ satisfies the differential equation

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3. \tag{5}$$

(See, for example, [Chand, page 29].) Furthermore, it is easy to see from (2) and (4) that ζ is related to $\wp$ by

$$\zeta'(z) = -\wp(z). \tag{6}$$

Since $\wp'(z)$ is an odd function of z with period ω_1, we have

$$\wp'\left(\frac{\omega_1}{2}\right) = -\wp'\left(-\frac{\omega_1}{2}\right) = -\wp'\left(-\frac{\omega_1}{2} + \omega_1\right) = -\wp'\left(\frac{\omega_1}{2}\right)$$

Hence

$$\wp'\left(\frac{\omega_1}{2}\right) = 0.$$

Similarly

$$\wp'\left(\frac{\omega_2}{2}\right) = \wp'\left(\frac{\omega_1 + \omega_2}{2}\right) = 0.$$

Moreover, $\wp'$ has no other zeros in the fundamental period parallelogram besides $\omega_1/2$, $\omega_2/2$ and $(\omega_1+\omega_2)/2$. It follows from (5) that the three roots of the cubic polynomial

$$4\wp(z)^3 - g_2\wp(z) - g_3 \tag{7}$$

are precisely $\omega_1/2$, $\omega_2/2$ and $(\omega_1 + \omega_2)/2$. If we put

$$e_1 = \wp\left(\frac{\omega_1}{2}\right), \qquad e_2 = \wp\left(\frac{\omega_2}{2}\right), \qquad e_3 = \wp\left(\frac{\omega_1 + \omega_2}{2}\right),$$

then we can rewrite (5) in the form

$$\wp'(z)^2 = 4\big(\wp(z) - e_1\big)\big(\wp(z) - e_2\big)\big(\wp(z) - e_3\big), \tag{8}$$

where

$$\begin{cases} e_1 + e_2 + e_3 = 0, \\ e_1e_2 + e_2e_3 + e_3e_1 = -\dfrac{g_2}{4}, \\ e_1e_2e_3 = \dfrac{g_3}{4}. \end{cases} \tag{9}$$

Moreover, e_1, e_2 and e_3 are all distinct; for details see [Chand, page 33].

The function ζ is not periodic. However,

$$\zeta(z + m\,\omega_1 + n\,\omega_2) = \zeta(z) + 2m\,\zeta\left(\frac{\omega_1}{2}\right) + 2n\,\zeta\left(\frac{\omega_2}{2}\right). \tag{10}$$

Then (10) implies

$$\zeta\left(\frac{\omega_1 + \omega_2}{2}\right) = \zeta\left(\frac{\omega_1}{2}\right) + \zeta\left(\frac{\omega_2}{2}\right) \tag{11}$$

for integers m and n. We shall need Legendre's relation

$$\zeta\left(\frac{\omega_1}{2}\right)\omega_2 - \zeta\left(\frac{\omega_2}{2}\right)\omega_1 = \pi. \tag{12}$$

For a proof of (12) see [Chand, page 50].

3. The Weierstrass $\wp$ and ζ Functions for a Square

The most symmetric fundamental period parallelogram is a square. Let us assume that the square is given by

$$\{\, z \in \mathbb{C} \mid 0 \le \mathfrak{Re}(z) < 1 \text{ and } 0 \le \mathfrak{Im}(z) < 1 \,\}. \tag{13}$$

The Weierstrass $\wp$ function corresponding to this square has $\omega_1 = 1$ and $\omega_2 = i$, and is explicitly given by

$$\wp(z, g_2, g_3) = \frac{1}{z^2} + \sum_{\substack{(m,n)\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \left(\frac{1}{(z - m - n\,i)^2} - \frac{1}{(m + n\,i)^2} \right). \tag{14}$$

Furthermore, the Weierstrass ζ function for the square (13) is given by

$$\zeta(z, g_2, g_3) = \frac{1}{z} + \sum_{\substack{(m,n)\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \left(\frac{1}{z - m - n\,i} + \frac{1}{m + n\,i} + \frac{z}{(m + n\,i)^2} \right). \tag{15}$$

To determine g_2 and g_3 in (14) and (15), we use (14) to calculate

$$e_1 + e_2 = \wp\left(\frac{1}{2}, g_2, g_3\right) + \wp\left(\frac{i}{2}, g_2, g_3\right) = 0. \tag{16}$$

Then it follows from (9) that

$$\begin{cases} e_3 = g_3 = 0, \\ g_2 = -4e_1e_2 = 4e_1^2. \end{cases} \tag{17}$$

In the case of the square given by (13), we use (17) to reduce the differential equation (5) to

$$\wp'(z)^2 = 4\wp(z)\big(\wp(z) - e_1^2\big) \tag{18}$$

In version 2.3 of *Mathematica*:

$\zeta(z, g_2, g_3)$	is implemented by	**WeierstrassZeta[z,g2,g3]**,
$\wp(z, g_2, g_3)$	is implemented by	**WeierstrassP[z,g2,g3]**,
$\wp'(z, g_2, g_3)$	is implemented by	**WeierstrassPPrime[z,g2,g3]**.

Furthermore, the command **WeierstrassInvariants** can be used to compute g_2 and g_3 from ω_1 and ω_2. For example, the command

```
N[WeierstrassInvariants[1/2,I/2],10]
```

elicits the response

```
                               -13
{189.0727201, 2.071757156 10      }
```

telling us that the g_2 that corresponds to the periods 1 and i is approximately given by

$$g_2 \approx 189.07272. \tag{19}$$

(We already know that $g_3 = 0$.) Therefore Weierstrass functions $\wp$ and ζ corresponding to the square (13) are $\wp(z, c, 0)$ and $\zeta(z, c, 0)$, where $c \approx 189.07272$. Furthermore,

```
Sqrt[N[WeierstrassInvariants[1/2,I/2],10][[1]]]/2
```

```
                    -7
{6.87519, 2.27583 10  }
```

tells us that $e_1 \approx 6.87519$.

We now introduce special notation for the Weierstrass $\wp$ and ζ functions corresponding to the square (13):

$$\begin{cases} \boldsymbol{P}(z) &= \wp(z, c, 0), \\ \boldsymbol{Z}(z) &= \zeta(z, c, 0). \end{cases} \tag{20}$$

Lemma 1. *For all z we have*

$$\boldsymbol{P}\left(z - \frac{1}{2}\right) - \boldsymbol{P}\left(z - \frac{i}{2}\right) - 2e_1 = \frac{16e_1^3 \boldsymbol{P}(z)}{\boldsymbol{P}'(z)^2}, \tag{21}$$

$$i\boldsymbol{Z}(i\,z) = \boldsymbol{Z}(z), \tag{22}$$

$$\boldsymbol{Z}\left(\frac{1}{2}\right) = i\boldsymbol{Z}\left(\frac{i}{2}\right) = \frac{\pi}{2}, \tag{23}$$

$$\boldsymbol{Z}\left(\frac{1+i}{2}\right) = \frac{(1-i)\pi}{2}. \tag{24}$$

Proof. To prove (21), we use (3) to compute

$$\boldsymbol{P}\left(z-\frac{1}{2}\right)=\frac{\boldsymbol{P}'(z)^2}{4\big(\boldsymbol{P}(z)-e_1\big)^2}-\boldsymbol{P}(z)-e_1. \tag{25}$$

Then (5) and (25) imply that

$$\begin{aligned}
\boldsymbol{P}\left(z-\frac{1}{2}\right) &= \frac{\boldsymbol{P}(z)\big(\boldsymbol{P}(z)^2-e_1^2\big)}{\big(\boldsymbol{P}(z)-e_1\big)^2}-\boldsymbol{P}(z)-e_1 \\
&= \frac{\boldsymbol{P}(z)\big(\boldsymbol{P}(z)+e_1\big)}{\boldsymbol{P}(z)-e_1}-\boldsymbol{P}(z)-e_1 \\
&= \frac{\boldsymbol{P}(z)\big(\boldsymbol{P}(z)+e_1\big)-\boldsymbol{P}(z)^2+e_1^2}{\boldsymbol{P}(z)-e_1} \\
&= \frac{e_1\boldsymbol{P}(z)+e_1^2}{\boldsymbol{P}(z)-e_1} \\
&= e_1+\frac{2e_1^2}{\boldsymbol{P}(z)-e_1}.
\end{aligned} \tag{26}$$

Similarly,

$$\boldsymbol{P}\left(z-\frac{i}{2}\right)=-e_1+\frac{2e_1^2}{\boldsymbol{P}(z)+e_1}. \tag{27}$$

From (26) and (27) we get

$$\begin{aligned}
\boldsymbol{P}\left(z-\frac{1}{2}\right)-\boldsymbol{P}\left(z-\frac{i}{2}\right)-2e_1 &= \frac{2e_1^2}{\boldsymbol{P}(z)-e_1}-\frac{2e_1^2}{\boldsymbol{P}(z)+e_1} \\
&= \frac{4e_1^3}{\boldsymbol{P}(z)^2-e_1^2}.
\end{aligned} \tag{28}$$

Now (21) follows from (28) and (5).

Equation (22) is evident from (15). Then (23) follows from (22) and (12). Finally (23) is a special case of (11). ▌

4. The Defintion of Costa's Minimal Surface

A description of Costa's minimal surface using the Weierstrass function $\boldsymbol{P}$ has been given by Barbosa and A. G. Colares (see [BaCo, pages 84–90], and also [HoMe]). We will adapt this description to *Mathematica*, but we shall use instead the Weierstrass Zeta function $\boldsymbol{Z}$ (given by (20)) to avoid one numerical integration. The use of the Weierstrass Zeta function $\boldsymbol{Z}$ instead of the Weierstrass

$\boldsymbol{P}$ function speeds up the plotting of Costa's minimal surface by a factor of at least 50.

Costa's minimal surface can be defined as a Weierstrass patch using the functions f and g defined by

$$f(z) = \boldsymbol{P}(z) \qquad \text{and} \qquad g(z) = \frac{A}{\boldsymbol{P}'(z)}. \tag{29}$$

In order that Costa's minimal surface have no self-intersections, we need to take

$$A = \sqrt{2\pi\, g_2} = 2\sqrt{2\pi}\, e_1 \approx 34.46707. \tag{30}$$

(See [BaCo, page 89]).

Definition. *The* **Costa minimal curve** *is the meromorphic minimal curve* **costamincurve**: $\mathbb{C} \longrightarrow \mathbb{C}^3$ *is the minimal curve defined as follows. First we put*

$$\textbf{costamincurve}'(z) = \frac{1}{2}\left(\frac{f(z)}{2}(1-g(z)^2), i\frac{f(z)}{2}(1+g(z)^2), f(z)g(z)\right), \tag{31}$$

where f and g are defined by (29). *We take* **costamincurve** *to be the antiderivative of* **costamincurve**$'$ *with the normalization*

$$\textbf{costamincurve}\left(\frac{1+i}{2}\right) = (0,0,0).$$

Next we show how to use the Weierstrass function $\boldsymbol{Z}$ to express **costamincurve** without integrals.

Theorem 2. *The* **Costa minimal curve** *is given by*

$$\textbf{costamincurve}(z) = \big(\text{costamincurve}_1(z), \text{costamincurve}_2(z), \text{costamincurve}_3(z)\big)$$

where

$$\begin{cases} \text{costamincurve}_1(z) = \dfrac{1}{2}\Bigg(-\boldsymbol{Z}(z) + \pi z - i\pi + \dfrac{\pi^2(1+i)}{4e_1} \\ \qquad\qquad + \dfrac{\pi}{2e_1}\left(\boldsymbol{Z}\left(z-\dfrac{1}{2}\right) - \boldsymbol{Z}\left(z-\dfrac{i}{2}\right)\right)\Bigg), \\ \text{costamincurve}_2(z) = \dfrac{i}{2}\Bigg(-\boldsymbol{Z}(z) - \pi z + \pi - \dfrac{\pi^2(1+i)}{4e_1} \\ \qquad\qquad - \dfrac{\pi}{2e_1}\left(\boldsymbol{Z}\left(z-\dfrac{1}{2}\right) - \boldsymbol{Z}\left(z-\dfrac{i}{2}\right)\right)\Bigg), \\ \text{costamincurve}_3(z) = \dfrac{\sqrt{2\pi}}{4}\left(\log\left(\dfrac{\boldsymbol{P}(z)-e_1}{\boldsymbol{P}(z)+e_1}\right) - \pi i\right). \end{cases}$$

Proof. Since costamincurve$_3(z)$ is the simplest, we tackle it first. Using (5), we compute

$$f(w)g(w) = \frac{A\,\boldsymbol{P}(w)}{\boldsymbol{P}'(w)} = \frac{A\,\boldsymbol{P}'(w)\boldsymbol{P}(w)}{\boldsymbol{P}'(w)^2} = \frac{A\,\boldsymbol{P}'(w)}{4\big(\boldsymbol{P}'(w)^2 - e_1^2\big)}. \tag{32}$$

We expand the right hand side of (32) by partial fractions and use (30) to obtain

$$\begin{aligned} f(w)g(w) \;&=\; \frac{A}{8e_1}\boldsymbol{P}'(w)\left(\frac{1}{\boldsymbol{P}(w)-e_1} - \frac{1}{\boldsymbol{P}(w)+e_1}\right) \\ &=\; \frac{\sqrt{2\pi}}{4}\left(\frac{\boldsymbol{P}'(w)}{\boldsymbol{P}(w)-e_1} - \frac{\boldsymbol{P}'(w)}{\boldsymbol{P}(w)+e_1}\right). \end{aligned} \tag{33}$$

We integrate both sides of (33), obtaining

$$\begin{aligned} \text{costamincurve}_3(z) = \int_{(1+i)/2}^{z} f(w)g(w)dw \;&=\; \frac{\sqrt{2\pi}}{4}\log\left(\frac{\boldsymbol{P}(w)-e_1}{\boldsymbol{P}(w)+e_1}\right)\Bigg|_{(1+i)/2}^{z} \\ &=\; \frac{\sqrt{2\pi}}{4}\left(\log\left(\frac{\boldsymbol{P}(z)-e_1}{\boldsymbol{P}(z)+e_1}\right) - \pi\, i\right). \end{aligned}$$

Next

$$\frac{1}{2}f(w)\big(1-g(w)^2\big) = \frac{1}{2}\left(\boldsymbol{P}(w) - \frac{A^2\boldsymbol{P}(w)}{\boldsymbol{P}'(w)^2}\right). \tag{34}$$

We use (21) and (30) to rewrite (34) as

$$\begin{aligned} \frac{1}{2}f(w)\big(1-g(w)^2\big) \;&=\; \frac{1}{2}\Bigg(\boldsymbol{P}(w) - \frac{A^2}{16e_1^3}\Bigg(\boldsymbol{P}\left(w-\frac{1}{2}\right) \\ &\qquad -\boldsymbol{P}\left(w-\frac{i}{2}\right) - 2e_1\Bigg)\Bigg) \\ &= \frac{1}{2}\left(\boldsymbol{P}(w) - \frac{\pi}{2e_1}\left(\boldsymbol{P}\left(w-\frac{1}{2}\right) - \boldsymbol{P}\left(w-\frac{i}{2}\right) - 2e_1\right)\right) \\ &= \frac{1}{2}\left(\boldsymbol{P}(w) + \pi - \frac{\pi}{2e_1}\left(\boldsymbol{P}\left(w-\frac{1}{2}\right) - \boldsymbol{P}\left(w-\frac{i}{2}\right)\right)\right). \end{aligned} \tag{35}$$

We integrate both sides of (35) from $(1+i)/2$ to z and use (23) and (24) to get

$$\int_{(1+i)/2}^{z} \frac{1}{2} f(w)\big(1-g(w)^2\big)\,dw \tag{36}$$

$$= \frac{1}{2}\left(-\boldsymbol{Z}(w) + \pi\, w + \frac{\pi}{2e_1}\left(\boldsymbol{Z}\left(w-\frac{1}{2}\right) - \boldsymbol{Z}\left(w-\frac{i}{2}\right)\right)\right)\Bigg|_{(1+i)/2}^{z}$$

$$= \frac{1}{2}\Bigg(- \boldsymbol{Z}(z) + \pi\, z + \frac{\pi}{2e_1}\left(\boldsymbol{Z}\left(z-\frac{1}{2}\right) - \boldsymbol{Z}\left(z-\frac{i}{2}\right)\right)$$

$$+Z\left(\frac{1+i}{2}\right) - \frac{\pi(1+i)}{2} - \frac{\pi}{2e_1}\left(Z\left(\frac{i}{2}\right) - Z\left(\frac{1}{2}\right)\right)\Bigg)$$

$$= \frac{1}{2}\left(- \boldsymbol{Z}(z) + \pi\, z + \frac{\pi}{2e_1}\left(\boldsymbol{Z}\left(z-\frac{1}{2}\right) - \boldsymbol{Z}\left(z-\frac{i}{2}\right)\right) - i\,\pi + \frac{\pi^2(1+i)}{4e_1}\right).$$

Similarly,

$$\int_{(1+i)/2}^{z} \frac{i}{2} f(w)\big(1+g(w)^2\big)\,dw \tag{37}$$

$$= \frac{i}{2}\left(- \boldsymbol{Z}(z) - \pi\, z - \frac{\pi}{2e_1}\left(\boldsymbol{Z}\left(z-\frac{1}{2}\right) - \boldsymbol{Z}\left(z-\frac{i}{2}\right)\right) + \pi - \frac{\pi^2(1+i)}{4e_1}\right). \blacksquare$$

Corollary 3. **Costa's minimal surface** *is given by*

$$\mathbf{costa}(u,v) = \big(\mathrm{costa}_1(u,v), \mathrm{costa}_2(u,v), \mathrm{costa}_3(u,v)\big)$$

where

$$\begin{cases} \mathrm{costa}_1(u,v) &= \dfrac{1}{2}\mathfrak{Re}\Bigg(-\boldsymbol{Z}(u+i\,v) + \pi\, u + \dfrac{\pi^2}{4e_1} \\ & \qquad + \dfrac{\pi}{2e_1}\left(\boldsymbol{Z}\left(u+i\,v-\dfrac{1}{2}\right) - \boldsymbol{Z}\left(u+i\,v-\dfrac{i}{2}\right)\right)\Bigg), \\ \mathrm{costa}_2(u,v) &= \dfrac{1}{2}\mathfrak{Re}\Bigg(-i\boldsymbol{Z}(u+i\,v) + \pi\, v + \dfrac{\pi^2}{4e_1} \\ & \qquad - \dfrac{\pi\, i}{2e_1}\left(\boldsymbol{Z}\left(u+i\,v-\dfrac{1}{2}\right) - \boldsymbol{Z}\left(u+i\,v-\dfrac{i}{2}\right)\right)\Bigg), \\ \mathrm{costa}_3(u,v) &= \dfrac{\sqrt{2\pi}}{4}\log\left|\dfrac{\boldsymbol{P}(u+i\,v)-e_1}{\boldsymbol{P}(u+i\,v)+e_1}\right|. \end{cases}$$

5. The **Mathematica** Implementation of Costa's Minimal Surface

We must use version 2.3 of *Mathematica* so that the Weierstrass Zeta function is available. First we obtain numerical approximations to the constants c, e_1 and A. The command

```
c=N[WeierstrassInvariants[1/2,I/2],10][[1]]
```

tells us that $c \approx 189.07272$. Therefore we enter the commands

```
e1=WeierstrassP[0.5,189.07272,0]
```

```
A=N[Sqrt[8Pi WeierstrassP[0.5,189.07272,0]^2],5]
```

telling us that $e_1 \approx 6.87519$ and $A \approx 34.467$.

Therefore we define Costa's minimal surface in *Mathematica* by

```
costa[u_,v_]:=
     {(1/2)Re[-WeierstrassZeta[u + I v,189.07272,0] +
     3.14159 u + 0.35889 +
     0.228473(WeierstrassZeta[u + I v - 1/2,189.07272,0] -

          WeierstrassZeta[u + I v - I/2,189.07272,0])],
     (1/2)Re[-I WeierstrassZeta[u + I v,189.07272,0] +
     3.14159 v + 0.35889 -
     0.228473(I WeierstrassZeta[u + I v - 1/2,189.07272,0] -

          I WeierstrassZeta[u + I v - I/2,189.07272,0])],
     0.626657Log[Abs[(WeierstrassP[u + I v,189.07272,0] -

          6.87519)/
          (WeierstrassP[u + I v,189.07272,0] + 6.87519)]]}
```

Our first attempt to plot Costa's minimal surface with *Mathematica* is

```
costaplot80=ParametricPlot3D[costa[u,v],
    {u,0.001,1.001},{v,0.001,1.001},
    PlotPoints->80]
```

This plot is incorrect, so we do not display it. The difficulty is that *Mathematica* has included some polygons whose vertices are too far from the origin. To exclude them we use the command **selectgraphics3d**.

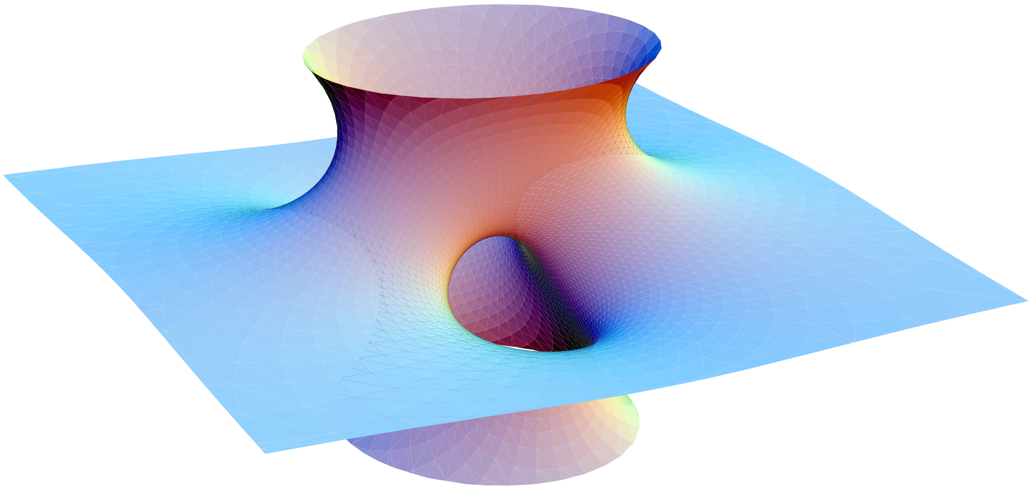
```
selectgraphics3d[graphics3dobj_,bound_,opts___]:=
    Show[Graphics3D[Select[graphics3dobj,
    (Abs[#[[1,1,1]]]<bound && Abs[#[[1,1,2]]]<bound &&
     Abs[#[[1,1,3]]]<bound && Abs[#[[1,2,1]]]<bound &&
     Abs[#[[1,2,2]]]<bound && Abs[#[[1,2,3]]]<bound &&
     Abs[#[[1,3,1]]]<bound && Abs[#[[1,3,2]]]<bound &&
     Abs[#[[1,3,3]]]<bound && Abs[#[[1,4,1]]]<bound &&
     Abs[#[[1,4,2]]]<bound && Abs[#[[1,4,2]]]<bound
     )&]],opts]
```

We also need the command **dip**

```
dip[ins_][g_]:=$DisplayFunction[Insert[g,ins,{1,1}]]
```

to eliminate the edges in the polygonal approximation of the surface. Here is the plotting command.

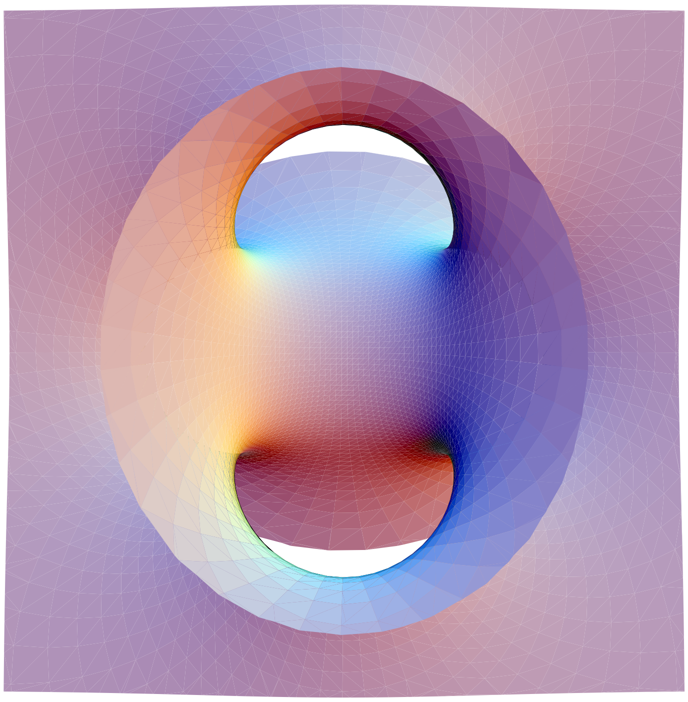
```
selectgraphics3d[costaplot80[[1]],8,
    Boxed->False,ViewPoint->{2.9, -1.4, 1.2},
    PlotRange->{{-4, 4}, {-4, 4}, {-2, 2}},
    DisplayFunction->dip[EdgeForm[]]]
```



Costa's minimal surface

To get a top view of Costa's minimal surface we use

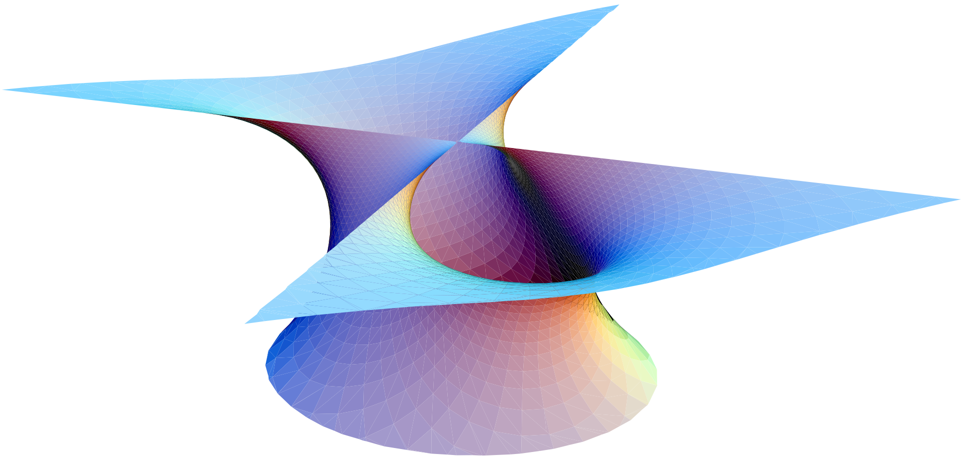
```
selectgraphics3d[costaplot80[[1]],8,
    Boxed->False,ViewPoint->{0,0,4},
    PlotRange->{{-2.5, 2.5}, {-2.5, 2.5}, {-2, 2}},
    DisplayFunction->dip[EdgeForm[]]]
```



Costa's minimal surface viewed from above

Costa's minimal surface contains two straight lines, as we can see from the following view:

```
selectgraphics3d[costaplot80[[1]],8,
    Boxed->False,ViewPoint->{2.9, -1.4, 1.2},
    PlotRange->{{-2.5, 2.5}, {-2.5, 2.5}, {-2, 2}},
    DisplayFunction->dip[EdgeForm[]]]
```



The lower half of Costa's minimal surface

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Department of Mathematics
University of Maryland
College Park, Maryland 20742
USA

Mathematics Institute
Technical University of Denmark
Lyngby
DENMARK