ON SEIBERG–WITTEN EQUATIONS ON SYMPLECTIC 4–MANIFOLDS

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Abstract

We discuss Taubes' idea to perturb the monopole equations on symplectic manifolds to compute the Seiberg–Witten invariants in the light of Witten's symmetry trick in the Kähler case. The article is thought as an supplement to a series of lectures given in Lindow and Berlin on that subject.

1 Introduction

In 1994 a new field equation on 4–manifolds came up simplifying lots of questions in low–dimensional topology, generally referred to as Donaldson theory. The main advantage of the new theory is, that it is an *abelean* field theory coupled to the Riemannian metric. The more or less direct consequence is that the moduli space of solutions of the field equations is compact (unlike in the case of anti-selfdual $SU(2)$–connections there is no bubbling–off phenomenon). With a few (but important) exceptions one can give a sequence of easier proofs of facts derived from Donaldson theory (some of them with even stronger consequences). See [Friedrich] for a more detailed discussion. Based on the vanishing of the Seiberg–Witten invariants there are new theorems in Riemannian geometry of spaces with positive scalar curvature (see [LeBrun]).

Most importantly, the computation of the new invariants started a development at the end of which we will probably have a much better understanding of the differential topology of symplectic 4–manifolds.

Let us recall the state of the art before the new invariants came up. A symplectic structure on an (oriented) 4–manifold is a closed 2–form ω such that $\omega \wedge \omega$ gives an orientation class. There exists a calibrating almost complex structure J or, equivalently, a Riemannian metric g such that ω is a self–dual and harmonic 2–form. So, we already have the basic two homotopy obstructions against the existence of a symplectic structure on a 4–manifold: $b_2^+ \geq 1$ and the (anti)–canonical bundle $K(K^{-1})$ satisfies

$$c_1(K)^2[M] = c_1(K^{-1})^2[M] = (2\chi(M) + 3\sigma(M))$$

where the existence of such a class $c \in H^2(M; Z)$ satisfying this equation implies the existence of an almost complex structure. With that description at hand one could prove various results like

Proposition 1.1 (Audin) *(i) If M,N are closed, compact, almost complex 4–manifolds then there is no almost complex structure on $M \sharp N$.*
(ii) There exists an almost complex structure on $kCP^2 \sharp l\overline{CP^2}$ inducing the given orientation iff $k \equiv 1 mod\, 2$.

But is was an open problem whether $kCP^2 \sharp l\overline{CP^2}$ admits a symplectic structure or not. This was settled by Taubes with the following

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Theorem 1.2 ([Taubes]) *Let M be a closed, compact, symplectic 4–manifold with $b_2^+ \geq 2$. Then the $Spin^C$–structure associated to the calibrating almost complex structure has Seiberg–Witten invariant ± 1.*

On the other hand there exist a Riemannian metric with positive scalar curvature on $kCP^2 \sharp l\overline{CP^2}$. The invariants vanish on such manifolds (see [Kr/Mr]) so he concludes

Corollary 1.3 *For $k > 1$ there is no symplectic structure on $kCP^2 \sharp l\overline{CP^2}$.*

Another circle of problems is posed by the question of rigidity of symplectic structures. So far, with the powerful techniques of Gromov's pseudo holomorphic curves, there have been rigidity results only for noncompact 4–manifolds:

Theorem 1.4 (Gromov,McDuff) *Suppose M is a (noncompact) symplectic 4–manifold with one end which is standard at infinity, then M is the blowing up of C^2 in a finite collection of points.*

The problem to let Gromov's tool work is usually the existence of just one such pseudo holomorphic curve.

By showing the existence of a pseudo holomorphic curve homologous to the hyperplane Taubes claims to prove the following conjecture of Gromov

Theorem 1.5 ([Taubes2]) *Every symplectic structure on CP^2 is diffeomorphic to the standard one.*

The paper is organized as follows. In Chapter 2 we discuss $Spin^C$–structures and study the monopole equations on almost complex and symplectic 4–manifolds, in Chapter 3 we compute the Lagrangian in this situation adapt the equations to the situation of a non integrable almost complex structure and discuss the failure of the corresponding Lagrangian to catch the symmetry of the more special Kähler case. Finally, we derive the family of equations Taubes considers to circumvent this problem. In Chapter 4 we proof ellipticity of the linearization and the basic a priori estimates for solutions of this family of equations (from which the compactness of the moduli space of monopoles and its smoothness in a nondegenerate solution follows). In Chapter 5 we prove the uniqueness and nondegeneracy of solutions having the special form proposed in Proposition 3.1 (which was left out by Taubes). Chapter 6 is basically extracted from Taubes paper, explaining *less* detailedly all necessary steps in the proof of Proposition 3.1. In the odds and ends of Chapter 7 we will discuss the limits of the application of Taubes' technic to keep the fortunate reader from applying it to enthusiastically and give some more corollaries and an overview of the important research announcements of Taubes in [Taubes2] in Chapter 8 .

2 Spinor bundles and Dirac operators on almost complex manifolds

Let (M^{2n}, g) be an oriented Riemannian manifold.

Definition 2.1 *A spinor bundle $S \longrightarrow M$ is a complex vector bundle which is an irreducible representation of the Clifford bundle, i.e.*

$$Cliff^C(M) \cong End(S)$$

as bundles of algebras.

Remark 2.2 *Such a spinor bundle is unique up to twisting by line bundles. This can be easily deduced from the uniqueness of the irreducible representation of the Clifford algebra $Cliff^C(V)$ of a $2n$–dimensional euclidean vector space V as algebra of endomorphisms(see [BGV]) $End(S)$ of a complex vector space S.*

Now we discuss the question of the existence of such spinor bundles.

Proposition 2.3 *An oriented $2n$–dimensional Riemannian manifold (M, g) admits a spinor bundle iff it admits a $Spin^C$–structure.*

Proof. The statement is rather elementary and probably well-known. In [BGV] the spinor module S is constructed locally using a maximal isotropic subspace of the complexified euclidean vector space $P \subset T^C M$. Having chosen an orienting orthonormal frame $\{e_j\}$ of TM, P may be set $P = \mathcal{L}_C(\{e_{2k-1} + ie_{2k}\})$. S is then defined to be the exterior algebra $S = \Lambda P$ and the Clifford action is given by exterior multiplication and contraction. (Equivalently, given locally an almost complex structure J, e.g. via $J(e_{2k-1}) = e_{2k}$, then P is the eigenspace to the eigenvalue $-\mathrm{i}$ of J). The algebra of matrices is simple, so given any other spinor module S' there is up to scalar multiplication a unique isomorphism of S and S' as Clifford modules. Now assume we have chosen a covering $\{U_\alpha\}_\alpha$ of M such that the open sets and pairwise intersections are contractible together with trivializations of (TM, g) and S. Trivialising the tangent bundle and the spinor bundle compatibly, we obtain transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow SO(2n)$ and $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow Gl(2^n; C)$ with

$$\phi_{\alpha\beta}(cs) = g_{\alpha\beta}(c)\phi_{\alpha\beta}(s)$$

for all $c \in Cliff(2n)$ and $s \in S$. On the other hand take liftings $\tilde{g}_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow Spin(2n)$. These do not define a cocycle in general. But via Clifford multiplication they do define transition functions for S commuting with the Clifford action. Denote these by $\tilde{g}_{\alpha\beta}$, too. By irreducibility of the representation of the Clifford module the difference between $\tilde{g}_{\alpha\beta}$ and $\phi_{\alpha\beta}$ is a scalar $\lambda_{\alpha\beta}$, i.e. cohomology

$$\tilde{g}_{\alpha\beta}\lambda_{\alpha\beta} = \phi_{\alpha\beta}$$

and so $\tilde{g}_{\alpha\beta} \times_{Z_2} \lambda_{\alpha\beta}$ defines a $Spin^C$–structure.

To begin with consider an almost complex 4–manifold (X, J) equipped with a hermitian metric g. Denote by ω the corresponding skew–symmetric 2–form. Clifford multiplication by ω splits any $Spin^C$–bundle into its eigenspaces. Assume a scaling of ω (by a constant) such that the corresponding eigenvalues are $\pm\mathrm{i}$, i.e.

$$\omega = \frac{\mathrm{i}}{4}(v^1 \wedge \bar{v^1} + v^2 \wedge \bar{v^2}),$$

for $v^1 = e^1 + \mathrm{i}e^2$ and $v^2 = e^3 + \mathrm{i}e^4$ with a compatible orthonormal frame $\{e^i\}$. Note that ω is self dual. J defines a maximal isotropic subspace of $T^C M$ globally. The line bundle is $L = K^{-1}$ for the corresponding $Spin^C$–bundle

$$\begin{aligned} S &\cong S^+ \oplus S^- \\ S^+ &\cong \Lambda^0 \oplus \Lambda^{0,2} \\ S^- &\cong \Lambda^{0,1}, \end{aligned}$$

(see [BGV] p.110) where ω acts as $-\mathrm{i}$ on the first and as $+\mathrm{i}$ on the second summand and trivially on S^-. At first let us compute the unique spinor connection $\nabla^S = \nabla^+ \oplus \nabla^-$ induced by the Levi–Civita connection (which will be denoted by ∇). By this we mean the unique $Spin^C$–connection associated to the metric and the connection on the characteristic line bundle $\det(S^+) = K^{-1}$ of the $Spin^C$–structure. Restricted and projected to the eigenbundles of the ω–action the Levi–Civita connection ∇ and ∇^S agree. Let us compute ∇^+ now. Unfortunately, ∇ is not compatible with the Clifford multiplication. We suppose that ∇ and ∇^+ induce the same connection on the summands which is equivalent to the assumption that $\det(\nabla^+)$ coincides with the covariant derivative ∇_{A_0} of the connection A_0 on $K^{-1} \cong \Lambda^{0,2}$ induced by the Levi–Civita connection. One necessary condition is

$$\nabla^+(\omega \cdot (f, \phi)) = \nabla^+(-\mathrm{i}f, \mathrm{i}\phi) = \nabla\omega \cdot (f, \phi) + \omega \cdot \nabla^+(f, \phi),$$

from which one deduces

$$\begin{aligned} \nabla^+ &= \nabla + b - b^* \\ b &\in \Omega^1(Hom(\Lambda^0; \Lambda^{0,2})) \\ &\cong \Omega^1(K^{-1}) \\ b &= \mathrm{i}(\nabla\omega)^{0,2}, \end{aligned} \tag{1}$$

where $(\nabla\omega)^{0,2}$ is the projection of the covariant derivative on the $(0,2)$–part. One easily deduces that this is indeed the desired spinor connection. The following lemma will be useful

Lemma 2.4 *Suppose $d\omega = d * \omega = 0$. Then*

$$\nabla^*_{A_0} b = -F^{0,2}_{A_0},$$

for the curvature of the connection A_0 on K^{-1}.

Proof. First note that $\nabla\omega$ consists of an $(0,2)$– and $(2,0)$– part only. On the other hand

$$\nabla^*\nabla\omega = \Delta\omega + \sum R_{ijkl}\epsilon^k\iota^l\epsilon^i\iota^j\omega$$

where ϵ denotes exterior multiplication and ι contraction via g (see [BGV]). Now choose geodesic coordinates around $x \in X$ such that

$$\begin{aligned} J\left(\frac{\partial}{\partial x_1}\right) &= \frac{\partial}{\partial x_2} \\ J\left(\frac{\partial}{\partial x_3}\right) &= \frac{\partial}{\partial x_4} \end{aligned} \tag{2}$$

in x. By assumption $\Delta\omega = 0$ and we compute in $x \in X$

$$
\begin{aligned}
\nabla^*\nabla\omega \;=\; & \frac{1}{2}(-R_{3112}e^1\wedge e^3 + R_{3142}e^3\wedge e^4 + R_{3113}e^1\wedge e^2 - R_{3143}e^2\wedge e^4 \\
& -R_{4112}e^1\wedge e^4 - R_{4132}e^3\wedge e^4 + R_{4114}e^1\wedge e^2 - R_{4134}e^2\wedge e^3 \\
& +R_{1334}e^1\wedge e^3 + R_{1324}e^1\wedge e^2 + R_{1331}e^3\wedge e^4 + R_{1321}e^2\wedge e^4 \\
& +R_{2334}e^2\wedge e^3 - R_{2314}e^1\wedge e^2 + R_{2332}e^3\wedge e^4 + R_{2312}e^1\wedge e^4 \\
& +R_{3221}e^2\wedge e^3 - R_{3241}e^3\wedge e^4 + R_{3223}e^1\wedge e^2 + R_{3243}e^1\wedge e^4 \\
& +R_{4221}e^2\wedge e^4 + R_{4231}e^3\wedge e^4 + R_{4224}e^1\wedge e^2 + R_{4234}e^1\wedge e^3 \\
& -R_{2443}e^2\wedge e^4 + R_{2413}e^1\wedge e^2 + R_{2442}e^3\wedge e^4 - R_{2412}e^1\wedge e^3 \\
& -R_{1443}e^1\wedge e^4 - R_{1423}e^1\wedge e^2 + R_{1441}e^3\wedge e^4 - R_{1421}e^2\wedge e^3).
\end{aligned}
$$

Denote the complex tensors by

$$
\begin{aligned}
v_a &= \frac{1}{2}(e_1 - \mathrm{i}e_2) \\
v^a &= e^1 + \mathrm{i}e^2 \\
v_b &= \frac{1}{2}(e_3 - \mathrm{i}e_4) \\
v^b &= e^3 + \mathrm{i}e^4.
\end{aligned}
$$

Inserting $(\overline{v_a}, \overline{v_b})$ we obtain the $(0,2)$–part of $\nabla^*\nabla\omega$

$$
\begin{aligned}
(\nabla^*\nabla\omega)^{(0,2)} &= \frac{1}{8}(-R_{3112} + R_{3143} - \mathrm{i}R_{4112} - \mathrm{i}R_{4134} \\
&\quad +R_{1334} - R_{1321} + \mathrm{i}R_{2334} + \mathrm{i}R_{2312} \\
&\quad +R_{4234} - R_{4221} + \mathrm{i}R_{3221} + \mathrm{i}R_{3243} \\
&\quad -R_{2412} + R_{2443} - \mathrm{i}R_{1421} - \mathrm{i}R_{1443})\overline{v^a}\wedge\overline{v^b} \\
&= \frac{1}{4}[(-R_{\bar b 112} - R_{\bar b 134} + R_{\bar a 334} + R_{\bar a 312}) \\
&\quad +\mathrm{i}(-R_{\bar b 212} - R_{\bar b 234} + R_{\bar a 434} + R_{\bar a 412})]\overline{v^a}\wedge\overline{v^b} \\
&= \frac{1}{2}(-R_{\bar b\bar a 34} - R_{\bar b\bar a 12} + R_{\bar a\bar b 12} + R_{\bar a\bar b 34})\overline{v^a}\wedge\overline{v^b}.
\end{aligned}
$$

Using the identity $e_1\wedge e_2 + e_3\wedge e_4 = -2\mathrm{i}(v_a\wedge\overline{v_a} + v_b\wedge\overline{v_b})$ on the last two indices we see that this is

$$
(\nabla^*\nabla\omega)^{(0,2)} = -2\mathrm{i}(R_{\bar a\bar b a\bar a} + R_{\bar a\bar b b\bar b})\overline{v^a}\wedge\overline{v^b}
$$

But $(\nabla\omega)^{(1,1)} = 0$, ∇ does not mix the $(2,0)$ and $(0,2)$–components and the claim follows with

$$
\begin{aligned}
R_{a\bar a\bar a\bar b} + R_{b\bar b\bar a\bar b} &= -\frac{1}{2}TrR^{0,2}_{T^{1,0}} \\
&= -\frac{1}{2}F^{0,2}_{A_0}.
\end{aligned}
$$

(The curvature of $\nabla^{1,0}$ is a combination of the Riemannian tensor as in the formula above plus a tensor incorporating the torsion ∇J of the form

$$const.((J+\mathrm{i})[\nabla J, \nabla J](J+\mathrm{i})).$$

But $\nabla J = -\omega^{-1} \circ \nabla\omega \circ J$ and by the closedness of ω

$$\nabla J \in \Omega^{1,0}(End(T^{1,0}, T^{0,1})) \oplus \Omega^{0,1}(End(T^{0,1}, T^{1,0})).$$

so that the torsion part of the curvature is purely of type $(1,1))$.

Now we write down a formula for the Dirac operator in the case where $d\omega = 0$. Note first that in general for a spinor $\psi = (f, \phi)$

$$D\psi = \sqrt{2}(\bar\partial f + \bar\partial^*\phi + \frac{1}{2}(((\partial + \partial^*)\omega)f + ((\bar\partial + \bar\partial^*)\omega)\phi)) \in \Lambda^{0,1}.$$

But if ω is closed then $\bar\partial\omega = \partial\omega = 0$ ($d = \partial + \bar\partial$ and $d^* = \partial^* + \bar\partial^*$ on $\Lambda^{1,1}$). So, if ω is closed the Dirac operator takes the form

$$D_A\psi = \sqrt{2}(\bar\partial_A f + \bar\partial_A^*\phi)$$

Here A is a connection on K^{-1} which induces a spinor connection and D_A is the corresponding Dirac operator. Note that $\bar\partial_{A_0} = \bar\partial$.

In general, for the Dirac operator we have the Weitzenböck formula

$$D_A^2\psi = (\nabla_A^S)^*\nabla_A^S\psi + \frac{R}{4}\psi + \frac{1}{2}F_A \cdot \psi.$$

Weitzenböck formula applied to $D_{A_0}^2 1 = 0$ with $1 = (1,0)$ gives

$$2\langle(\nabla^*(\nabla\omega)^{0,2}), \omega\rangle - |(\nabla\omega)^{0,2}|^2 = -\frac{F_{A_0}^\omega}{2} - \frac{R}{4}$$

where $F_{A_0}^+ = \mathrm{i}F_{A_0}^\omega\omega + F_{A_0}^{0,2} + F_{A_0}^{2,0}$.

Note that $\langle(\nabla^*(\nabla\omega)^{0,2}), \omega\rangle = |(\nabla\omega)^{0,2}|^2$ pointwise (just pair both sides with a cut–off and use $\langle(\nabla\omega)^{0,2}, \omega\rangle = 0 \in \Omega^1(M)$.

It follows that

$$|(\nabla\omega)^{0,2}| = -\frac{F_{A_0}^\omega}{2} - \frac{R}{4}$$

Another identity which could be derived exploiting $D(\alpha \cdot 1)$ for a one form α is, the following: the $(0,2)$–part of its exterior derivative is expressed by

$$(d\alpha)^{0,2} = \bar\partial(\alpha^{0,1}) + b \circ (\alpha^{1,0})$$

where "$\circ$" denotes the C–linear contraction of the common T^*M–component. Accordingly one obtains on functions

$$\bar\partial_A^2 + b \circ \partial_A = (F_A - F_{A_0})^{0,2}.$$

We conclude this collection of useful formulas with an identity resembling a similar one in Kähler geometry but which holds in the symplectic case on $(0,1)$–forms only:

Denote by Λ the operator on forms dual to the exterior multiplication by ω. Let $\alpha \in \Omega^{0,1}$. Then

$$\Lambda\partial\alpha = \mathrm{i}\bar\partial^*\alpha.$$

3 The monopole equations on symplectic 4–manifolds

$\psi = (f, \phi) \in \Gamma(S^+) = \Omega^0(M) \oplus \Omega^{0,2}(M)$. Using the hermitian product on S^+ (see [Kr/Mr]) we write

$$\sigma(\psi) = \mathrm{i}(\psi\bar{\psi}^T)_0 \in \underline{su}(S^+)$$

The action of the matrix entries is, of course, also defined with the help of the hermitian product. This yields

$$\sigma(\psi) = \mathrm{i}\begin{pmatrix} \frac{|f|^2-|\phi|^2}{2} & f\bar{\phi} \\ \bar{f}\phi & -\frac{|f|^2-|\phi|^2}{2} \end{pmatrix}.$$

Via Clifford multiplication one obtains

$$\underline{su}(S^+) \cong \Omega^+(M).$$

Then $\sigma(\psi)$ corresponds to

$$\sigma(\psi) = -\frac{1}{2}((|f|^2 - |\phi|^2)\omega + \frac{\mathrm{i}}{2}(\bar{f}\phi - f\bar{\phi}).$$

Note that for the $(2,0)$–form $f\bar{\phi}$ the Clifford action is $f\bar{\phi}\cdot\Psi = -2\langle\Psi, \bar{f}\phi\rangle$.

The second monopole equation is

$$F_A^+ = -2\mathrm{i}\sigma(\psi) = \mathrm{i}(|f|^2 - |\phi|^2)\omega + (\bar{f}\phi - f\bar{\phi}).$$

For the Lagrangian we compute

$$\begin{aligned}
& \int_M (2|D_A\psi|^2 + |F_A^+ + 2\mathrm{i}\sigma(\psi)|^2)dM \\
= & \int_M (2|\nabla_A^+\psi|^2 + \frac{R}{2}|\psi|^2 + \langle F_A^+\cdot\psi, \psi\rangle + |F_A^+|^2 + 2\langle F_A^+, \sigma(\psi)\rangle + 4|\sigma(\psi)|^2)dM \\
= & \int_M (2|\nabla_A^+\psi|^2 + \frac{R}{2}|\psi|^2 + F_A^\omega(|f|^2 - |\phi|^2) + 2\langle F_A^{0,2}f, \phi\rangle - 2\langle\phi, F_A^{\bar{2},0}f\rangle + |F_A^+|^2 \\
& -2\langle \mathrm{i}F_A^\omega\omega + F_A^{0,2} + F_A^{2,0}, \mathrm{i}(|f|^2 - |\phi|^2)\omega + (\bar{f}\phi - f\bar{\phi})\rangle + 4|\sigma(\psi)|^2)dM \\
= & \int_M (2|\nabla_A^+\psi|^2 + \frac{R}{2}|\psi|^2 - F_A^\omega(|f|^2 - |\phi|^2) + 4\mathrm{Re}(f\langle F_A^{0,2}, \phi\rangle) + |F_A^+|^2 \\
& +F_A^\omega(|f|^2 - |\phi|^2) - 4\mathrm{Re}(f\langle F_A^{0,2}, \phi\rangle) + \frac{1}{2}(|f|^2 + |\psi|^2)^2)dM \\
= & \int_M (2|\nabla_A^+\psi|^2 + \frac{R}{2}(|f|^2 + |\psi|^2) + \frac{1}{2}(|f|^2 + |\psi|^2)^2)dM.
\end{aligned}$$

Now

$$|\nabla_A^+\psi|^2 = |\nabla_A f|^2 + |\nabla_A\phi|^2 + 2\mathrm{Re}(\langle bf, \nabla_A\phi\rangle - \langle\langle\phi, b\rangle, \nabla_A f\rangle) + (|bf|^2 + |b^*\phi|^2),$$

so in the non-Kähler case (where $b = (\nabla\omega)^{0,2} \neq 0$) we have terms in the integrand of the Lagrangian mixing f and ϕ. In the Kähler case ($b \equiv 0$) there are no such terms and we obtain an additional symmetry

$$\begin{array}{rcl} A & \mapsto & A \\ f & \mapsto & -f \\ \phi & \mapsto & \phi. \end{array}$$

(see [Witten] for a discussion). We would like to adopt this idea for the general symplectic 4–manifold. Observe that

$$d^*(\mathrm{Re}\langle bf, \phi\rangle) = \mathrm{Re}(-\langle bf, \nabla_A\phi\rangle - \langle\langle\phi, b\rangle, \nabla_A f\rangle + \langle\nabla^* b, \bar{f}\phi\rangle).$$

which vanishes after integration. We change the second equation to

$$F_A^+ = \mathrm{i}(|f|^2 - |\phi|^2 + \delta)\omega + (\bar{f}\phi - f\bar{\phi}) - (\nabla^* b - \overline{\nabla^* b}),$$

where $\delta \in C^\infty(M)$. Then the Lagrangian changes to

$$\int_M (2(|\nabla_A f|^2 + |\nabla_A\phi|^2) + \frac{R}{2}(|f|^2 + |\phi|^2) + (|bf|^2 + |b^*\phi|^2) - 8\mathrm{Re}\langle\langle\phi, b\rangle, \nabla_A f\rangle$$
$$+\langle\delta, (|f|^2 - |\phi|^2)\rangle + |F_A^+ + (\nabla^* b - \overline{\nabla^* b}) - \mathrm{i}\delta\omega|^2 + \frac{1}{2}(|f|^2 + |\phi|^2)^2)dM$$

Now, if one adds, hypothetically, the unfortunately singular term

$$\frac{2}{|f|^2}(\bar{f}b \circ \nabla_A f - f\overline{b \circ \nabla_A f})$$

to the second equation the last mixed term vanishes in the corresponding Lagrangian. So, we have a good heuristic to change the second equation to

$$F_A^+ = \mathrm{i}(|f|^2 - |\phi|^2 + \delta)\omega + (\bar{f}\phi - f\bar{\phi}) - (\nabla^* b - \overline{\nabla^* b}) + \frac{2r}{1 + r|f|^2}(\bar{f}b \circ \nabla_A f - f\overline{b \circ \nabla_A f}),$$

where we obtain the singular term in the process $r \to \infty$. Now we choose the parameters $\delta = F_{A_0}^\omega - 1$ and use the above computed identity $\nabla^* b = -F_{A_0}^{0,2}$ and obtain the final version of the equations

$$\bar{\partial}_A f + \bar{\partial}_A^* \phi = 0$$
$$F_A^+ - F_{A_0}^+ = \mathrm{i}(|f|^2 - |\phi|^2 - 1)\omega + (\bar{f}\phi - f\bar{\phi}) + \frac{2r}{1 + r|f|^2}(\bar{f}b \circ \nabla_A f - f\overline{b \circ \nabla_A f}).$$

For each r the tupel $(A_0, 1)$ is a solution. The main result of Taubes' paper is to prove the following

Proposition 3.1 *For r sufficiently large this is up to gauge transformation the only solution of the above monopole equations. Moreover, this solution is nondegenerate for all values of $r > 0$.*

4 Ellipticity and compactness

First one should note that the monopole equations are still a (nonlinear) elliptic problem, i.e. the linearization of the equations modulo gauge equivalence is an elliptic operator (of first order). We just compute it to be

$$\begin{pmatrix} D_A & 0 \\ \frac{2r}{1+r|f|^2}(\bar{f}b \circ \nabla_A|_{\Lambda^0} - f\overline{b \circ \nabla_A|_{\Lambda^0}}) & d^* \oplus d^+ \end{pmatrix} + \text{terms of zeroth order.}$$

The diagonal part is known to be elliptic and an off–diagonal smooth first order differential operator cannot destroy that property.

The second important issue is the a priori estimate of solutions: if we loose the compactness of the moduli space we did not gain much, introducing the r–dependent term. But this is as easily established as the former: in addition to that we obtain some a priori estimates independent of r.

Proposition 4.1 *For arbritrary r a solution (A, ψ) of the monopole equations satisfies*

$$|\psi| \le \max(0, -R + 2\|F^+_{A_0}\|_\infty + 2\|b\|^2_\infty).$$

Consequently, the moduli space $\mathcal{M}_r$ of solutions is compact for all r. Moreover, the right hand side does not depend on r. In addition, we have an r–independent bound

$$\int_M |\nabla_A\psi|^2 \le \max(0, -\frac{R}{2} + \|b\|^2_\infty + \|F_{A_0}\|_\infty) \int_M |\psi|^2.$$

Proof. The proof goes along the line of [Kr/Mr]. Suppress the issue of smoothness (just suppose the solution is smooth). We compute

$$\begin{aligned} d^*d|\psi|^2 &= 2\langle(\nabla^+_A)^*\nabla^+_A\psi, \psi\rangle - 2\langle\nabla^+_A\psi, \nabla^+_A\psi\rangle \\ &= -\frac{1}{2}\langle(\mathrm{i}(|f|^2 - |\phi|^2 - 1)\omega + (\bar{f}\phi - f\bar{\phi}) + \frac{4r}{1+r|f|^2}\mathrm{Im}(\bar{f}b \circ \nabla_A f) + F^+_{A_0}) \cdot \psi, \psi\rangle \\ &\quad -\frac{R}{2}|\psi|^2 - 2|\nabla^+_A\psi|^2, \end{aligned}$$

using Weitzenböck formula.

We end up with

$$d^*d|\psi|^2 + 2|\nabla^+_A\psi|^2 + \frac{1}{2}|\psi|^4 = -\frac{R}{2}|\psi|^2 + \langle F^+_{A_0} \cdot \psi, \psi\rangle + \frac{4r|f|^2}{1+2r|f|^2}\mathrm{Re}\langle b \circ \nabla_A f, \phi\rangle.$$

for solutions of the monopole equations. The proposition is now derived as in the before mentioned paper: considering the equality at a maximal point of $|\psi|^2$, where $d^*d|\psi|^2 \ge 0$ for the first, and integrating over M for the second statement. We only have to take care of the r–dependent part:

$$\left|\frac{4r|f|^2}{1+2r|f|^2}\mathrm{Re}\langle b \circ \nabla_A f, \phi\rangle\right| \le \epsilon\|b\|^2_\infty|\nabla^S_A\psi|^2 + \frac{1}{\epsilon}|\psi|^2$$

Choose $\epsilon\|b\|^2_\infty = 1$ to obtain the result.

5 Uniqueness and nondegeneracy

We establish the uniqueness of the solution $(A_0, 1)$ up to gauge. In the next section we prove the vanishing of the $(0,2)$–component for any solution of the equations for large r. Suppose now we are given a solution of the form $(A, (f, 0))$. We construct a gauge transformation such that A is transformed into A_0 and f into the constant function 1. Note that from the first equation follows

$$\bar{\partial}_A^* \bar{\partial}_A f = 0,$$

i.e. f is the zero of a Schrödinger type operator on functions. Minimax principle of such operators shows that f vanishes nowhere. We are thus able to transform the solution by $g = \frac{\bar{f}}{|f|} \in \mathcal{G}$. That means w.l.o.g. we may suppose f to be a (positive) real function. The conjugate of the first equation then gives

$$\partial_A f = 0.$$

It follows that f is a parallel section having constant length. For topological reasons (integrating the second equation $\wedge\omega$ over M) $f \equiv 1$. Remembering that

$$\bar{\partial}_A f = \bar{\partial} f + (A - A_0)^{(0,1)} f$$

we finally obtain $A = A_0$.

It remains to verify the nondegeneracy of the functional at $(A_0, 1)$. That means to verify that the second cohomology of the deformation complex vanishes which in the case the virtual dimension of the moduli space is zero is equivalent to the vanishing of the first cohomology at irreducible solutions. We use the observation of LeBrun (see[LeBrun]) which can be carried out for the irreducible solution $(A_0, 1)$.

Assume $(a, g, \varphi) \in \Omega^1(\mathrm{i}R) \times (\Lambda^0 \oplus \Lambda^{0,2})$ is a 1–cocycle. Write $a = \alpha - \bar{\alpha}$ with $\alpha \in \Omega^{0,1}$.

This is equivalent to lying in the kernel of the Hessian of the corresponding Lagrangian (which has semi–definite Hessian at solutions of the equations). With Weitzenböck the Lagrangian can be transformed to

$$\int_M (2(|\nabla_A f|^2 + |\nabla_A \phi|^2) + \frac{R}{2}(|f|^2 + |\phi|^2) + (|bf|^2 + |b^*\phi|^2) - \frac{8}{1 + r|f|^2}\mathrm{Re}\langle\langle\phi, b\rangle, \nabla_A f\rangle$$

$$+\langle F^\omega_{A_0} - 1, (|f|^2 - |\phi|^2)\rangle + |F^+_A - F^+_{A_0} + \mathrm{i}\omega|^2 + \frac{1}{2}(|f|^2 + |\phi|^2)^2) dM =$$

$$\int_M (4(|\bar{\partial}_A f|^2 + |\bar{\partial}^*_A \phi|^2) + \frac{1}{2}|F^\omega_A - F^\omega_{A_0} + 1 - (|f|^2 - |\phi|^2)|^2 + 2|F^{0,2}_A - F^{0,2}_{A_0}|^2 + 2|f|^2|\phi|^2$$

$$-\frac{8}{1 + r|f|^2}\mathrm{Re}\langle\langle\phi, b\rangle, \nabla_A f\rangle) dM$$

Computing the Hessian at $(A_0, 1)$ we obtain

$$\int_M 8(|\bar{\partial} g + \frac{\alpha}{2}|^2 + |\bar{\partial}^* \varphi|^2) + |\Lambda(da) - 2\mathrm{iRe}g|^2 + 4|(da)^{0,2}|^2 + 4|\varphi|^2$$

$$-\frac{16}{1 + r}\mathrm{Re}\langle\varphi, b \circ \nabla g + b \circ \frac{a}{2}\rangle) dM.$$

Now using $\bar{\partial}(\beta^{0,1}) + b \circ (\beta^{1,0}) = (d\beta)^{0,2}$ for one forms β one obtains

$$\int_M \left(\frac{8}{1+r}|\bar\partial g + \frac{\alpha}{2} + \bar\partial^*\phi|^2\right) + \frac{8r}{1+r}\left(|\bar\partial g + \frac{\alpha}{2}|^2 + |\bar\partial^*\varphi|^2\right) + |\Lambda(da) - 2\mathrm{iRe}g|^2$$
$$+\frac{4}{1+r}\left(|(da)^{0,2} - \varphi|^2\right) + \frac{4r}{1+r}\left(|(da)^{0,2}|^2 + |\varphi|^2\right)\right)dM$$

from which we conclude for $r > 0$ that $\varphi \equiv 0$ and g purely imaginary and constant:

$$\bar\partial g = -\frac{\alpha}{2}$$

and

$$\Lambda(da) = 2\mathrm{iRe}g$$

imply

$$\bar\partial^*\bar\partial g + \partial^*\partial\bar g + \mathrm{Re}g = 0.$$

On account of $\partial\bar\partial + \bar\partial\partial = 0$ one has $2\bar\partial^*\bar\partial = 2\partial^*\partial = \Delta$ the Laplace–Beltrami operator on functions. But then the tripel $(a, g, 0)$ was a coboundary and the first cohomology vanishes.

6 The behaviour for large r

We now turn to the proof of Taubes' main result. The idea goes as follows: Instead of having the symmetry of the Kähler case the Lagrangian is "nearly symmetric" where the failure of having the symmetry decreases with growing r. To be more precise: Denote by $\mathcal{L}(A,\psi)$ the Lagrangian described above. Then

$$\mathcal{L}(A,(f,\phi)) - \mathcal{L}(A,(-f,\phi)) = \int_M \frac{-16}{1+r|f|^2}\mathrm{Re}\langle b\circ\nabla_A f, \phi\rangle. \tag{3}$$

Remember that we have r–independent a priori estimates on the solutions for a parameter r. So the only serious contribution on the right hand side comes from the set where f is small compared to r. The left hand side will be expressed differently to get L^2_1 estimates on ψ.

Let us first turn to this latter issue. Having a solution (A,ψ) to the monopole equations of parameter r ($\psi = (f,\phi)$) we compute

$$\mathcal{L}(A,(f,\phi)) - \mathcal{L}(A,(-f,\phi)) =$$
$$\int_M \left(16\mathrm{Re}\langle\bar\partial_A f, \bar\partial_A^*\phi\rangle - 4\mathrm{Re}\langle \bar f\phi - f\bar\phi, F_A^+ - F_{A_0}^+ - \frac{2r(\bar f b\circ\nabla_A f - f\overline{b\circ\nabla_A f})}{1+r|f|^2}\rangle\right)dM$$
$$= -\int_M (16|\bar\partial_A f|^2 + 8|f|^2|\phi|^2)dM = -\int_M (16|\bar\partial_A^*\phi|^2 + 8|f|^2|\phi|^2)dM. \tag{4}$$

We need another expression for $2|\bar\partial_A f|^2 = |D_A(f,0)|^2$, namely

$$\int_M |D_A(f,0)|^2 dM = \int_M \left(|\nabla_A^+(f,0)|^2 + \frac{R}{4}|f|^2 + \frac{F_A^\omega}{2}|f|^2\right)dM,$$

via Weitzenböck which is with one of the final identities of the second chapter easily seen to be

$$\int_M (|\nabla_A f|^2 + |(\nabla\omega)^{0,2}|^2|f|^2 + \frac{R}{4}|f|^2 + \frac{F_A^\omega}{2}|f|^2)dM = \int_M (|\nabla_A f|^2 + \frac{(F_A^\omega - F_{A_0}^\omega)}{2}|f|^2)dM.$$

Proof of Proposition 3.1. Assume we have an unbounded increasing sequence $\{r_i\}$ and solutions $\{(A_i, \psi_i)\}$ such that $\phi_i \neq 0$ for all i and $\psi_i = (f_i, \phi_i)$. By Proposition 4.1 and some standard arguments we have a subsequence such that $|\psi_i|$ converges strongly in L^p for any $p \in [2, \infty)$ and $|\nabla^+_{A_i}\psi_i|$ converges strongly in L^2.

Denote by $V_i = \{x \in M || f_i| < \frac{1}{2}\}$. Taubes proves successively

1. For the above subsequence the measure

$$\lim_{i\to\infty} \mu(V_i) = 0.$$

2. For the same subsequence

$$\int_M (|\nabla_{A_i}\phi_i|^2 + |\phi_i|^2)dM \leq const. \int_{V_i} |\phi_i|^2 dM$$

independently of the parameter r_i.

3.

$$\begin{aligned} \int_{V_i} |\phi_i|^2 dM &\leq vol(V_i)^{\frac{1}{2}}(\int_{V_i} |\phi_i|^4 dM)^{\frac{1}{2}} \\ &\leq const.vol(V_i)^{\frac{1}{2}}(\int_M (|d|\phi_i||^2 + |\phi|^2)dM) \\ &\leq const.vol(V_i)^{\frac{1}{2}} \int_M (|\nabla_{A_i}\phi_i|^2 + |\phi_i|^2)dM \\ &\leq const.vol(V_i)^{\frac{1}{2}} \int_{V_i} |\phi_i|^2 dM \end{aligned}$$

by Kato's inequality $|d|\phi|| \leq |\nabla_{A_i}\phi_i|$. This contradicts the assumption of nonvanishing ϕ_i if $vol(V_i)$ tends to zero and proves the main theorem.

It remains to explain the first two steps in the proof. Note that the parameter r in (3) is in the denominator. So, whereever $|f_i|$ is big the contribution to the integral will be small (if you like only $r_i^\epsilon |f_i|^2$ has to be big for $\epsilon < 1$, because we derived L^2–bounds on $|\nabla_{A_i} f_i|$ and L^∞–bounds on ϕ_i). So we will just divide the domain of integration in the part where $|f_i|$ is big and where it is small. First we integrate the right hand side of (3) by parts to obtain

$$\int_M (2|\bar{\partial}_A^*\phi_i|^2 + |f_i|^2|\phi_i|^2)dM \leq \int_M \frac{2}{1 + r_i|f_i|^2}(|f_i||\nabla_A\phi_i| + |f_i||\phi_i| + |\phi_i||d|f_i||)dM.$$

Now using the convergence assumption on the subsequence, observing $\frac{s}{1+rs^2} \leq \frac{\sqrt{2}}{3\sqrt{r}}$ and dividing the domain in $V^i = \{x \in M | r_i^{\frac{1}{4}}|f_i| \leq 1\}$ and its complement we have

$$\int_M (2|\bar{\partial}^*\phi_i|^2 + |f_i|^2|\phi_i|^2)dM \leq const.(\frac{1}{\sqrt{r_i}} + \int_{V^i} |\phi_i| \frac{|d|f_i||}{1 + r_i|f_i|^2}) \tag{5}$$

The key observation at this point is that the right hand side of the inequality has to converge to zero under the assumption made on the subsequence. This is proved by standard techniques. Remember that the $|f_i|$ converge to some positive function f in L_1^2. So it is sufficient to prove that

$$\int_M \frac{|df|}{1+r_i|f_i|^2}dM$$

converges to zero. We first show that for

$$\int_M \frac{|df|}{1+r_i f^2}dM$$

and then for the difference. We divide the domain in the part where $f^2 > \frac{1}{n}$ and its complement. Over the first domain the integral is bounded by

$$const.\frac{n}{r_i}\|df\|_{L^2}$$

which goes to zero for fixed n as r tends to infinity. For the integral over the latter region Taubes' uses

Lemma 6.1 $0 \leq f \in L_1^2(M)$. *Then*

$$\lim_{\epsilon\to 0}\int_{\{x\in M\,|\,f<\epsilon\}}|df| = 0.$$

For the difference compute the integrand

$$|df|r_i^2\frac{|f_i|^2-|f|^2}{(1+r_i|f|^2)(1+r_i|f_i|^2)}.$$

Integrate over $\{f^2 > \frac{1}{n}\}$ and its complement. By appeal to Lemma 6.1 the latter is bounded by a sequence which r–independently tends to zero as n tends to infinity where the former tends to zero for fixed n as r tends to infinity because the measure of the set $\{||f_i|^2 - |f|^2| > \frac{1}{2n}\}$ tends to zero.

Now go back to (4). We obtain

$$\int_M(|\nabla_{A_i}f_i|^2+\frac{(|f_i|^2-|\phi_i|^2-1)}{2}(|f_i|^2-1)+|f_i|^2|\phi_i|^2))dM = \int_M\frac{2}{1+r_i|f_i|^2}\langle b\circ\nabla_{A_i}f_i,\phi_i\rangle dM$$

using

$$\int_M(|f_i|^2-|\phi_i|^2-1)dM = \int_M(F_A-F_{A_0})\wedge\omega = 0.$$

With $w_i = (1-|f_i|^2)$ we obtain

$$\int_M(|\nabla_{A_i}f_i|^2+\frac{w_i^2}{2}+\frac{|f_i|^2|\phi_i|^2}{2}+|\phi_i|^2)dM = \int_M\frac{1}{1+r_i|f_i|^2}\langle b\circ\nabla_{A_i}f_i,\phi_i\rangle dM.$$

But we have seen that the right hand side of this equation tends to zero forcing the L^2–norm of w_i to vanish at infinity, so the measure for *any* $\delta > 0$ of $\{x \in M | |f_i|^2 < \delta\}$ converges to zero, completing the first step.

On the other hand taking absolute values and integrating the right hand side over $V_i = \{|f_i|^2 < \frac{1}{2}\}$ and its complement gives

$$\int_M (|\nabla_{A_i} f_i|^2 + w_i^2 + |\phi_i|^2 + \frac{|f_i|^2|\phi_i|^2}{2})dM \leq \frac{2\|b\|_\infty}{r_i} \int_M |\nabla_{A_i} f_i||\phi_i| dM + \|b\|_\infty \int_{V_i} |\nabla_{A_i} f_i||\phi_i| dM.$$

Use inequalities like $2xy \leq \frac{1}{\epsilon}x^2 + \epsilon y^2$ and suppose r_i to be large enough to get

$$\int_M (2|\bar{\partial}_{A_i} f_i|^2 + |f_i|^2|\phi_i|^2)dM =$$
$$\int_M (|\nabla_{A_i} f_i|^2 + w_i^2 + |\phi_i|^2 + \frac{|f_i|^2|\phi_i|^2}{2})dM \leq \frac{\|b\|_\infty^2}{2\epsilon(1 - \epsilon - \frac{\|b\|_\infty}{r_i})} \int_{V_i} |\phi_i|^2 dM.$$

Then with (3) - (4) we have an estimate

$$\int_M (2|\bar{\partial}^*_{A_i} \phi_i|^2 + |f_i|^2|\phi_i|^2)dM \leq const. \int_{V_i} |\phi_i|^2 dM$$

for i sufficiently large. Again Weitzenböck and the formula for $|(\nabla\omega)^{0,2}|^2$ gives

$$\int_M |\bar{\partial}^*_{A_i} \phi_i|^2 dM = \int_M (|\nabla_{A_i} \phi_i|^2 - \frac{|f_i|^2 - |\phi_i|^2 - 1}{2}|\phi_i|^2 - F^\omega_{A_0}|\phi_i|^2)dM$$

or

$$\int_M |\nabla_{A_i} \phi_i|^2 dM \leq |\int_M (|\bar{\partial}^*_{A_i} \phi_i|^2 + \frac{|f_i|^2|\phi_i|^2}{2} + (F^\omega_{A_0} - 1)|\phi_i|^2)dM| \leq const. \int_{V_i} |\phi_i|^2 dM$$

proving the second statement and completing the proof of the theorem.

7 Further consequences and limits

From the computations of the previous section one obtains some more immediate obstructions against symplectic structures and vanishing results for Seiberg–Witten invariants summarized in [Taubes1]. Proving the first step in the verification of Proposition 3.1 we made use of the fact that

$$\int_M (|f_i|^2 - |\phi_i|^2 - 1)dM = \int_M -\mathrm{i}(F_A - F_{A_0}) \wedge \omega = 0.$$

Note that if we have a general $Spin^C$–structure with $\det S^+(L) = K^{-1} \otimes L^2$, or more precisely

$$S^+(L) = L \oplus K^{-1} \otimes L$$

then everything goes as in the case where L was the trivial bundle, but

$$\int_M -\mathrm{i}(F_A - F_{A_0}) \wedge \omega = -4\pi c_1(L) \cup [\omega][M]$$

So, carrying out the remaining estimates one concludes the vanishing of the Seiberg–Witten invariant for $S^+(L)$ if $c_1(L) \cup [\omega][M] < 0$. The case when this number vanishes is more subtle: One still obtains the vanishing of the form-part of the spinor, i.e. $0 = \phi_i \in \Omega^{0,2}(L)$. For generic almost complex structure the zero set has to be a smooth pseudo-holomorphic curve transversally cut out which is homologous to $P.D.(c_1(L))$. But ω integrated over this curve is zero, so the zero set is empty. That means that L has to be trivial. One should also note that the monopole equations have a symmetry in the two components of the spinor. If we play with the parameter δ and consider the monopole equation

$$\bar{\partial}_A f + \bar{\partial}_A^* \phi = 0$$

$$F_A^+ - F_{A_0}^+ = \mathrm{i}(|f|^2 - |\phi|^2 + 1)\omega + (\bar{f}\phi - f\bar{\phi}) + \frac{2r}{1 + r|f|^2}(\bar{f}b \circ \nabla_A f - f\overline{b \circ \nabla_A f})$$

instead, exchanging in all the estimates the f's and ϕ's we obtain a similar result for the Chern class $c_1(K \otimes L^{-1})$. We summarize

Theorem 7.1 ([Taubes1]) *Suppose M is a compact, closed, symplectic 4–manifold with $b_2^+ > 1$. Then the Seiberg-Witten invariants for the $Spin^C$–structure with*

$$S^+(L) = L \oplus K^{-1} \otimes L$$

vanishes if
(i) $c_1(L) \cup [\omega][M] < 0$
(ii) $c_1(K \otimes L^{-1}) \cup [\omega] < 0$ or
(iii) $c_1(L) \cup [\omega][M] = 0$ and L is a nontrivial line bundle or
(iv) $c_1(K \otimes L^{-1}) \cup [\omega][M] = 0$ and $K \otimes L^{-1}$ is a nontrivial line bundle.

As an immediate consequence we obtain that $c_1(K) \cup [\omega][M]$ has to be positive or K has to be the trivial line bundle. A somewhat different observation can be made in the case of CP^2:

Theorem 7.2 ([Taubes1]) *There is no symplectic structure on CP^2 with $c_1(K) \cup [\omega][M] > 0$.*

Proof. One observes that the original monopole equation

$$D_A\psi = 0$$

$$F_A^+ = \mathrm{i}(|f|^2 - |\phi|^2)\omega + (\bar{f}\phi - f\bar{\phi})$$

gives zero Seiberg–Witten invariant on account of the existence of a metric with positive scalar curvature, while the perturbed version

$$D_A\psi = 0$$

$$F_A^+ - F_{A_0}^+ = \mathrm{i}(|f|^2 - |\phi|^2 - 1)\omega + (\bar{f}\phi - f\bar{\phi})$$

gives nonzero invariant. The only reason for that can be a "wall crossing" for the family of equations

$$D_A\psi = 0$$

$$F_A^+ - sF_{A_0}^+ = \mathrm{i}(|f|^2 - |\phi|^2 - s)\omega + (\bar{f}\phi - f\bar{\phi})$$

at some parameter $s \in (0,1)$. At this parameter exists a reducible solution ($\psi \equiv 0$) and we end up with

$$(1-s)c_1(K^{-1}) \cup [\omega][M] = s \cdot vol_\omega(M) > 0.$$

The considerations above also show to what extend these techniques of Taubes can be used: If $c_1(L) \cup [\omega][M] > 0$ and $c_1(K \otimes L^{-1}) \cup [\omega][M] > 0$ the line of arguments stops at the point described in this chapter. One has to use different methods which Taubes proposes to do in [Taubes2].

8 Monopoles and pseudo holomorphic curves

Being so powerful in detecting holomorphic sections of line bundles in the Kähler case there was strong hope that the Seiberg–Witten invariants could compute the Gromov invariants counting pseudo holomorphic curves homologous to a certain homology class. In a big research announcement [Taubes2] Taubes claims to be able to settle this issue. We shortly discuss the results.

Denote by $\mathcal{M}_c(M,\omega)$ the space of smooth pseudo holomorphic curves in a closed, compact symplectic 4–manifold (M,ω) equipped with a compatible almost complex structure J homologous to $c \in H_2(M;Z)$. Note that only in the case of a torus there are topologically different parametrizations with smooth immersions, producing smooth pseudo holomorphic tori with multiplicities. For a generic almost complex structure this is a smooth, oriented manifold of even dimension

$$c \cdot c - c \cdot c_1(K) \equiv 0 \mathrm{mod} 2.$$

Pick a set of $(c \cdot c - c \cdot c_1(K))/2$ points $\{x_i\}$ in M. For a generic choice of the latter

$$\mathcal{M}_{c,\{x_i\}} = \{C \in \mathcal{M}_c | \{x_i\} \subset C\}$$

is a discrete set of points with orientation. The Gromov invariant as an invariant of the symplectic manifold (M,ω) is defined as the oriented number of points

$$G_c(M,\omega) = \hat{\sharp}\mathcal{M}_{c,\{x_i\}}$$

Now the main theorem Taubes announces to prove is

Theorem 8.1 *Let (M,ω) be a compact, closed, symplectic 4–manifold with $b_+^2 > 1$. Then the Seiberg–Witten invariant for the $Spin^C$–structure with $S^+(L) = L \oplus K^{-1} \otimes L$ is $\pm G_{c_1(L)}$. In other words the Gromov invariants are diffeomorphism invariants of the 4–manifold.*

From the theorem and this observation one concludes a series of remarkable statements: Suppose $b_2^+ > 1$ and J is a generic, compatible almost complex structure.

1. $c_1(K)$ is represented by a smooth pseudo holomorphic curve consisting of several components C of genus $g(C) = C \cdot C + 1$ (possibly with multiplicities only if the component is a torus). If M is minimal (i.e. admits no embedded pseudo holomorphic 2–spheres of self intersection -1) then $c_1(K) \cdot c_1(K) \geq 0$.

   Assume the Seiberg–Witten invariant is nonzero for $S^+(L) = L \oplus K^{-1} \otimes L$. Then $c_1(L)$ and $c_1(K \otimes L^{-1})$ are represented by smooth pseudo holomorphic curves. Their components are components of the representative of $c_1(K)$ with smaller multiplicities. The converse statement is true.

   An immediate consequence is the vanishing of the Seiberg-Witten invariants in the case of a positive dimensional moduli space of monopoles if $b_2^+ > 1$.

2. If M is minimal, then

$$-\frac{4}{3}(1 - b_1) - \frac{2}{3}b_2 \leq \sigma(M).$$

3. Suppose M admits an embedded sphere S with $S \cdot S = -1$ and $c_1(K) \cdot S \neq 0$. Then M admits a pseudo holomorphic 2–sphere in the same homology class.

4. Suppose M is minimal and $c_1(K) \cdot c_1(K) = 0$. Then $c_1(K)$ is Poincaré dual to a union of disjoint, embedded, pseudo holomorphic tori (possibly with multiplicities) with , of course, zero self intersection (by adjunction formula). Moreover, for any $S^+(L)$ with nontrivial Seiberg–Witten invariant, $c_1(L)$ is Poincaré dual to a subset of these tori (with smaller multiplicities).

Remark 8.2 *The statements are (although reminiscent) slightly weaker than those obtained in Kähler geometry. There, in the case of $c_1(K) \cdot c_1(K) > 0$, one obtains that the holomorphic curve representing $c_1(K)$ is connected obtained by purely algebraic means using the Hodge index theorem (see [Friedm/Morg]). It would be interesting to know whether there are symplectic manifolds with non-connected pseudo holomorphic representatives of $c_1(K)$ allowing more $Spin^C$–structures $S^+(L)$ with nontrivial Seiberg–Witten invariant than $L = \underline{C}$ or $L = K$ as in the Kähler case under this assumption.*

Proof. The first statement is a direct consequence of the two main Theorems. Recall that the dimension of the moduli space of monopoles for $S^+(L)$ is

$$d(L) = c_1(L) \cdot c_1(L) - c_1(K) \cdot c_1(L)$$

but $P.D.(c_1(L))$ consists of components of $P.D.(c_1(K))$ with smaller multiplicities by the second part of the statement. So, the dimension obstruction follows. The second statement plays with the index formula for the virtual dimension of the moduli space using $c_1(K) \cdot c_1(K) \geq 0$. The third statement observes that there is a diffeomorphism acting on $H_2(M; Z)$ as reflection on the hyperplane orthogonal to the class $[S]$. But then $c_1(K)$ changes to $c_1(K) - (2c_1(K) \cdot S)(P.D.([S]))$ which has to have nontrivial Seiberg–Witten invariant. That means that a multiple of $[S]$ has to be represented by a multiple of pseudo holomorphic embedded spheres. But the only way to achieve this is to represent $[S]$ by a pseudoholmorphic sphere of self intersection -1. The last statement is the special case of the first.

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