On Conformally flat hypersurfaces, Curved flats and Cyclic systems

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Abstract

It will be shown that suitable "Gauß maps" associated to a conformally flat hypersurface in S^{n+1} $(n \geq 3)$ yield normal congruences of circles having a whole one parameter family of conformally flat orthogonal hypersurfaces. However such a "cyclic system" is not uniquely associated to a conformally flat hypersurface. The key idea is to show that these Gauß maps are "curved flats" in a pseudo Riemannian symmetric space. Additionally in this context some characterizations of three dimensional conformally flat hypersurfaces arise with a new flavour. Moreover, the curved flat approach allows us to handle conformally flat hypersurfaces in the context of integrable system theory.

A Riemannian manifold (M, g) is called CONFORMALLY FLAT if around each point there exists a function u such that the conformally equivalent metric $e^{2u}g$ is flat. For hypersurfaces in the sphere[1] S^{n+1} we may ask the following

Question: *How can we characterize hypersurfaces being conformally flat with respect to the induced metric by means of their extrinisic geometry?*

In case of surfaces in the three sphere this question is completely answered by the well known theorem on the existence of isothermic coordinates for every twodimensional Riemannian manifold: *every surface in the three sphere is conformally flat.* For higher dimensional hypersurfaces this question was first posed by É. Cartan [4] who gave a complete answer in case $n > 3$ (see also [3]): *a hypersurface of dimension $n > 3$ in the sphere is conformally flat if and only if it is a (branched) channel hypersurface, i.e. if it envelopes a (perhaps stationary) one parameter family of hyperspheres.*

In case of three dimensional hypersurfaces these branched channel hypersurfaces are conformally flat as well — but there are "more" conformally flat hypersurfaces: for example stereographic projections of

cones over surfaces of constant Gauß curvature in $S^3 \subset \mathbb{R}^4$,

cylinders over surfaces of constant Gauß curvature in $\mathbb{R}^3 \subset \mathbb{R}^4$ or

rotational hypersurfaces over surfaces of constant Gauß curvature in H^3 considered as the Poincaré half space in $\mathbb{R}^4$ (the infinity plane of H^3 being the "axis" of rotation)

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[1]As we will see in the next section 1 all spaces of constant curvature may be identified in (local) conformal geometry by means of stereographic projections.

yield conformally flat hypersurfaces in S^4 with three distinct principal curvatures[2]. Nevertheless, contrary to the two dimensional case, there are also hypersurfaces which are not conformally flat: appropriate examples are tubes over the Veronese surface in S^4.

In his paper Cartan gave a quite curious characterization of the conformally flat hypersurfaces in S^4: *a hypersurface with three distinct principal curvatures in S^4 is conformally flat if and only if the six umbilic distributions (the distributions of those (complex) planes where the second fundamental form is a multiple of the first fundamental form) are integrable.* In my previous paper [9] I gave an extension of this theorem in showing that *a three dimensional hypersurface in S^4 is conformally flat if and only if the "conformal fundamental forms" are integrable.* Having three distinct principal curvatures, this gives the connection to "Guichard's nets" in $\mathbb{R}^3$ and hence a possibility to "look at" such hypersurfaces — but the flavour of this condition is not less mystical.

Recently D. Ferus and F. Pedit [6] developed the concept of CURVED FLATS: these are curvature isotropic submanifolds of a (pseudo) Riemannian symmetric space, i.e. submanifolds on which the ambient space's curvature tensor vanishes[3]. Here this curvature tensor is considered as a 2 - form on the submanifold with values in the space of (1,1) - tensors of the symmetric space. The (generalized) Gauß maps of some classes of submanifolds turn out to be curved flats — and since curved flats can be handled in the context of completely integrable systems so do these classes of submanifolds. One example are the isothermic surfaces in three space [2] which we related to curved flats in $\frac{O_1(5)}{O(3)\times O_1(2)}$. In a quite similar (but nevertheless different) way, a kind of Gauß map for the conformally flat hypersurfaces will be shown to fit into this approach. Moreover regular curved flats (having nondegenerate induced metrics) come with special coordinates: for three dimensional conformally flat hypersurfaces this might yield a "demystification" of the two theorems cited above (see page 8).

A curved flat associated to a conformally flat hypersurface in S^{n+1} — which actually is not unique — gives a whole one parameter family[4] of conformally flat hypersurfaces orthogonal to an n - parameter family of circles. Thus such a curved flat yields a circle congruence with a one parameter family of orthogonal hypersurfaces, i.e. a CYCLIC SYSTEM — the orthogonal hypersurfaces being all conformally flat. Here we are led to our research's third topic.

We should remark, that according to the results of J. Moore [11], in general we cannot *globally* associate a curved flat to a conformally flat hypersurface. Consequently also the corresponding cyclic system — although it is a Möbius geometric construction in contrary to that of the associated curved flat — is generally *not globally* defined. Having this in mind we will — for reasons of simplicity of the notations — assume the function u making the metric of the hypersurface flat to be defined globally: so in truth we are examining *globally conformally flat hypersurfaces.* At some points we will come back to this problem.

In conformal geometry — and especially in the context of conformal flatness — it seems to be convenient to look at objects from various points of view. We will recall the corresponding models[5] in a sketchy introduction on

[2] Branched channel surfaces may be characterized as "quasiumbilic" hypersurfaces, consequently they have at most two distinct principal curvatures.

[3] Thus a curved flat may be considered as the envelope of a congruence of flats.

[4] This is *not* the one parameter family of curved flats: that one yields a one parameter family of one parameter families.

[5] A very good introduction to sphere geometry will be found in Blaschke's famous book [1], an intro-

1 Möbius geometry

The spatial elements considered in the Möbius geometry of S^{n+1} are points and hyperspheres — hyperspheres are considered as intersections of S^{n+1} with affine hyperplanes of $\mathbb{R}^{n+2}$: Möbius transformations are exactly those point transformations mapping (hyper -) spheres into (hyper -) spheres. Embedding S^{n+1} as a quadric in projective space $\mathbb{R}P^{n+2}$ we may identify a hypersphere in S^{n+1} and the pole S of its hyperplane. In this way the outer[6] space of $S^{n+1} \subset \mathbb{R}P^{n+2}$ becomes the space of spheres in S^{n+1}. In an analogous way we may identify circles (or more generally m - spheres) and projective planes of dimension $n-1$ (resp. $n-m$) not meeting S^{n+1}. In this model the group of Möbius transformations becomes the group of those projective transformations leaving the quadric S^{n+1} invariant — therefore Möbius geometry is nothing but "projective geometry with absolute quadric[7]".

Now let's turn to the coordinate space of our projective space $\mathbb{R}P^{n+2}$. Up to scaling there is exactly one scalar product $\langle .,. \rangle$ (of Lorentz type) such that the quadric S^{n+1} is the space of light like lines — so we may think of this coordinate space as the Minkowski space $\mathbb{R}^{n+3}_1$. According to the above discussions the space of m - spheres in S^{n+1} equals the Grassmanian $\frac{O_1(n+3)}{O(n+1-m)\times O_1(m+2)}$ of spacelike $(n+1-m)$ - planes in $\mathbb{R}^{n+3}_1$. In case of hyperspheres $(m=n)$ this is just the space of diametrical point pairs of the unit Lorentz sphere S^{n+2}_1. Choosing representatives $S, \tilde{S} \in S^{n+2}_1$ for two spheres having at least one common point, their angle of intersection is given by the scalar product[8]: $\cos^2 \angle(S,\tilde{S}) = \langle S, \tilde{S}\rangle^2$. In particular the two spheres do touch if $\langle \tilde{S}, S\rangle = \pm 1$ and they do intersect orthogonally[9] if $\langle \tilde{S}, S\rangle = 0$. This notion of orthogonal intersection generalizes to m - spheres: in this case the corresponding subspaces have to be orthogonal.

Now consider an immersion $f: M^n \to S^{n+1}$. Choosing a representative in the light cone L^{n+2} (which we also denote by f) its induced metric $g = \langle df, df \rangle$ on M^n will be a Riemannian one[10]. A conformal change $g \rightsquigarrow e^{2u} g$ of this metric ($u: M^n \to \mathbb{R}$) may be realized by the passage $f \rightsquigarrow e^u f$, i.e. by the passage to another light cone representative. A sphere S and the hypersurface $f(M)$ will have first order contact at a point $f(p)$ if[11]

$$\langle f(p), S \rangle = 0 \quad \text{and} \quad \langle d_p f, S \rangle = 0 : \tag{1}$$

the first condition is just the incidence of S and $f(p)$ while the second one encodes additional incidence of S and the points of $f(M)$ in an infinitesimal neighbourhood to $f(p)$ — thus the hypersurface is said to (be) *envelope* (of) a sphere congruence $S: M^n \to S^{n+2}_1$

duction to the theory of cyclic systems in three space will be found in Coolidge's book [5], a generalization to higher dimensions is given in the first part of our recent paper [10].

[6] The *inner* space of $S^{n+1} \subset \mathbb{R}P^{n+2}$ may be characterized as the set of those points containing no real tangent to S^{n+1}; the *outer* space is its complement in $\mathbb{R}P^{n+2} \setminus S^{n+1}$.

[7] We may also think of the inner space of S^{n+1} as a Klein model of the hyperbolic $(n+2)$ - space. Hence Möbius geometry is the "boundary geometry" of hyperbolic geometry.

[8] Since a Möbius transformation in this linearized model is just a pseudo orthogonal transformation of the Minkowski space this intersection angle is clearly a conformal invariant.

[9] Projectively speaking: the poles representing these spheres lie in the polar hyperplanes one of each other.

[10] This is exactly the condition on f to yield a projective immersion.

[11] Remark that the condition (1) says that S is a unit normal vector to $f(M) \subset L^{n+2}$ at $f(p)$. We will use this interpretation at some points.

if it touches each sphere $S(p)$ at the point $f(p)$. Performing a change $S \rightsquigarrow S + a\cdot f$ (where $a : M^n \to \mathbb{R}$) of a sphere congruence S enveloped by f results in another sphere congruence enveloped by f: obviously the conditions (1) are preserved by such a change of the sphere congruence. These remarks lead to the following

Definition: A transformation $(f, S) \rightsquigarrow (e^u f, S + a\, f)$ will be called CONFORMAL DEFORMATION of the STRIP[12] (f, S).

The idea of conformal deformation is very similar to that of stereographic projection: we know that given a fixed vector $n_k \in \mathbb{R}^{n+3}_1$ the quadrics

$$M^{n+1}_k \;=\; \{y \in L^{n+2} |\, \langle y, n_k\rangle = -1\} \tag{2}$$

are spaces of constant curvature[13] $k = -\langle n_k, n_k\rangle$. In this model a stereographic projection is just the central projection from the lightcone's origin of one quadric (2) onto another one. Hence a stereographic projection acts on a hypersurface $f : M^n \to M^{n+1}_k$ with unit normal field $S : M^n \to TM^{n+1}_k$ as a conformal deformation of the strip (f, S).

At this point, I think, we are prepared to start our investigations.

2 Conformally flat hypersurfaces

Let $f : M^n \to L^{n+2}$ represent an immersion into the sphere S^{n+1} and $S : M^n \to S^{n+2}_1$ be a sphere congruence enveloped by f. We choose a framing

$$\begin{array}{rcl} F &=& (S_1, \dots, S_n, S, f, \hat{f}) : M^n \to O_1(n+3) \quad \text{with} \\ \mathrm{span}\{S_1, \dots, S_n\}|_p &=& d_p f(T_p M^n) \end{array} \tag{3}$$

for the strip[14] (f, S) and calculate its connection form $\Phi = F^{-1}dF : TM \to \mathfrak{o}_1(n+3)$:

$$\begin{array}{rcl} \Phi &=& \begin{pmatrix} \Omega & \eta \\ -\eta^\star & \mathcal{V} \end{pmatrix} \quad \text{where} \quad \Omega = (-\omega_{ij})_{ij} \;: TM \to \mathfrak{o}(n), \\ \eta &=& \begin{pmatrix} -\psi_1 & \omega_1 & \chi_1 \\ \vdots & \vdots & \vdots \\ -\psi_n & \omega_n & \chi_n \end{pmatrix} \;: TM \to M(n\times 3) \quad \text{and} \\ \mathcal{V} &=& \begin{pmatrix} 0 & 0 & \nu \\ -\nu & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;: TM \to \mathfrak{o}_1(3). \end{array} \tag{4}$$

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[12]We may think of a *strip* (f, S) as a fixing of the metric of S^{n+1} in an infinitesimal neighbourhood of $f(M)$. Hence a conformal deformation may be considered as a conformal change of the ambient space's metric "along f".

[13]The isometries of a space M^{n+1}_k are exactly those Möbius transformations leaving the vector n_k invariant (cf. [8]).

[14]The relative scaling of the lightlike vectors f and $\hat{f}$ is assumed to be $\langle \hat{f}, f\rangle = 1$. Because of the condition $\mathrm{span}\{S_1, \dots, S_n\}|_p = d_p f(T_p M^n)$ we will refer to such a framing as a "first order adapted framing for f".

The Maurer Cartan equation[15] $d\Phi + \frac{1}{2}[\Phi \wedge \Phi] = 0$ yields the Codazzi equation $0 = d\mathcal{H} + \Omega \wedge \mathcal{H} + \mathcal{H} \wedge \mathcal{V}$:

$$\begin{array}{rcl} 0 & = & d\psi_i - \Sigma\omega_{ij} \wedge \psi_j + \omega_i \wedge \nu \ , \\ 0 & = & d\omega_i - \Sigma\omega_{ij} \wedge \omega_j \ , \\ 0 & = & d\chi_i - \Sigma\omega_{ij} \wedge \chi_j - \psi_i \wedge \nu \ ; \end{array} \tag{5}$$

the Ricci equation $d\mathcal{V} = \mathcal{H}^\star \wedge \mathcal{H}$:

$$\begin{array}{rcl} 0 & = & \Sigma\psi_j \wedge \omega_j \ , \\ 0 & = & \Sigma\chi_j \wedge \omega_j \ , \\ d\nu & = & \Sigma\chi_j \wedge \psi_j \end{array} \tag{6}$$

and the Gauß equation $\mathcal{R} = \mathcal{H} \wedge \mathcal{H}^\star$ with the curvature form $\mathcal{R} = d\Omega + \Omega \wedge \Omega$:

$$\begin{array}{rcl} \varrho_{ij} & = & -d\omega_{ij} + \Sigma\omega_{ik} \wedge \omega_{kj} \\ & = & \psi_i \wedge \psi_j + \omega_i \wedge \chi_j + \chi_i \wedge \omega_j \ . \end{array} \tag{7}$$

From the Ricci equations we learn that the second fundamental form[16] $II_S = \Sigma\omega_j\psi_j$ is symmetric and hence the corresponding Weingarten tensor is diagonalizable. So we may assume $\psi_i = a_i\omega_i$: the functions a_i will be called *the principal curvatures* of f and we will refer to the framing F as a *principal curvature framing*[17].

Now let's turn to conformally flat hypersurfaces in S^{n+1} — here we may assume the induced metric $\Sigma\omega_j^2$ of the light cone representative f to be flat[18]. Consequently the curvature forms ϱ_{ij} in the Gauß equation (7) vanish. This fact enables us to calculate the second fundamental form of f with respect to the lightlike normal $\hat{f}$: with the ansatz

$$\psi_i = a_i\omega_i \quad \text{and} \quad \chi_i = \Sigma b_{ik}\omega_k \tag{8}$$

— and assuming $n \geq 3$ — the Gauß equations (7) become

$$0 = a_ia_j + b_{ii} + b_{jj} \quad \text{and} \quad 0 = b_{ij} \ \text{ for } i \neq j \ . \tag{9}$$

Hence the immersion's second fundamental form with respect to $\hat{f}$ also becomes diagonal — and therefore the Ricci equations (6) give us the following

[15]As usual, for Lie algebra valued 1 - forms Φ and Ψ the symmetric product $[\Phi \wedge \Psi]$ is defined by

$$[\Phi \wedge \Psi](v,w) = [\Phi(v), \Psi(w)] - [\Phi(w), \Psi(v)] \ .$$

For matrix valued 1 - forms we may also write

$$\Phi \wedge \Psi = (\Sigma\varphi_{ik} \wedge \psi_{kj})_{ij}$$

to obtain $[\Phi \wedge \Psi] = \Phi \wedge \Psi + \Psi \wedge \Phi$ in case the matrix space forms a Lie algebra.

[16]Here S is considered as a unit normal field to the immersion $f : M^n \to I\!R^{n+3}_1$.

[17]In points where the multiplicities of the principal curvatures are not locally constant, problems may occur: the fields of principal curvature directions may become singular. So we may additionally need a certain "genericity assumption".

[18]However, by this assumption, the light cone representative of the hypersurface is not uniquely determined at all: a conformal deformation $(f,S) \rightsquigarrow (e^uf,S)$ with a function $u : M^n \to I\!R$ satisfying $0 = d(e_iu) - \Sigma(e_ju)\omega_{ij} - (e_iu)du + \frac{1}{2}(\Sigma(e_ju)^2)\omega_i$ ($(e_1,\ldots,e_n)$ denoting the dual basis to $(\omega_1,\ldots,\omega_n)$) (cf. [9, eq. 26]) might be performed. Since such a function u in general is not a constant (nevertheless each constant function fulfills these equations) the framing of the strip will change totally in this case: in particular the spheres S_i do change (cf. (16)).

Lemma (flatness of the normal bundle): *If the induced metric $\langle df, df\rangle$ of a light cone representative $f : M^n \to \mathbb{R}^{n+3}_1$ of a hypersurface in S^{n+1} is flat then its normal bundle is also flat.*

This Lemma leads directly to an associated curved flat — but before adressing this topic we might see how Cartan proved his characterization of conformally flat hypersurfaces: the equations (9) enable us to express the forms χ_i in terms of the principal curvatures a_i of f:

$$\chi_i \;=\; -\tfrac{1}{2}(a_i a_j + a_i a_k - a_j a_k)\,\omega_i$$

for pairwise distinct indices i, j, k. In case the dimension of the hypersurface $n = 3$ these are three equations for the three forms χ_i but in case of higher dimension we get "too many equations". Hence in case of dimension $n \geq 4$ we also derive relations for the principal curvatures:

$$0 \;=\; (a_i - a_j)(a_k - a_l)$$

for pairwise distinct indices i, j, k, l. Consequently $n-1$ principal curvatures have to coincide — this was Cartan's original proof in [4] to show that *a conformally flat hypersurface of S^{n+1} is a channel hypersurface.*

Now let's turn back to our lemma and see how it drives us to the theory of

3 Curved flats

To an immersion $f : M^n \to L^{n+2}$ we associate its Gauß map $c_f : p \mapsto d_p f(T_p M) \subset \mathbb{R}^{n+3}_1$. In case f induces a Riemannian metric (positive definite) on M^n, the Gauß map will take values in the Grassmannian of spacelike n - planes in $\mathbb{R}^{n+3}_1$:

$$c_f : M^n \to \frac{O_1(n+3)}{O(n) \times O_1(3)}\,. \tag{10}$$

Endowed with the Killing metric, this Grassmannian becomes a $3n$ - dimensional pseudo Riemannian symmetric space of signature $(2n, n)$. A framing c_k (a lift into the isometry group $O_1(n+3)$) will be given by our above framing (3) — the normal field (sphere congruence) S not yet being specified. According to the symmetric decomposition[19]

$$\mathfrak{o}_1(n+3) \;=\; \mathfrak{k} \oplus \mathfrak{p}\ ,\quad \mathfrak{k} \;=\; \mathfrak{o}(n) \times \mathfrak{o}_1(3)\ ,$$

of the Lie algebra $\mathfrak{o}_1(n+3)$ the connection form of our framing splits into a $\mathfrak{k}$ - part and a $\mathfrak{p}$ - part:

$$\begin{aligned} \Phi &= \Phi_{\mathfrak{k}} + \Phi_{\mathfrak{p}} \quad \text{with} \\ \Phi_{\mathfrak{k}} &= \begin{pmatrix} \Omega & 0 \\ 0 & \mathcal{V} \end{pmatrix} : TM \to \mathfrak{k}\ , \\ \Phi_{\mathfrak{p}} &= \begin{pmatrix} 0 & \eta \\ -\eta^{\star} & 0 \end{pmatrix} : TM \to \mathfrak{p}\ . \end{aligned} \tag{11}$$

Note that the $\mathfrak{p}$ - part $\Phi_{\mathfrak{p}}$ of the connection form describes the differential dc_f of c_f.

From the Lemma 2 in the previous section we know that if the immersion f induces a flat metric on M^n then the induced connection on the normal bundle is also flat. Hence

[19] A "symmetric decomposition" has to satisfy the relations $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

the 2 - forms on both sides of the Gauß and Ricci equations (7,6) has to vanish. In a quite elegant way this may be written

$$0 \quad = \quad [\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}] \ , \tag{12}$$

or $0 \ = \ d\Phi_{\mathfrak{k}} + \frac{1}{2}[\Phi_{\mathfrak{k}} \wedge \Phi_{\mathfrak{k}}]$ respectively — but this is exactly the condition on c_f to be a curved flat[20] [6, Lemma 1]. Summarizing we obtain the following

Theorem: *The light cone representative $f : M^n \to L^{n+2}$ of a hypersurface in S^{n+1} induces a flat metric on M^n if and only if its Gauß map $c_f : M^n \to \frac{O_1(n+3)}{O(n)\times O_1(3)}$ is a curved flat.*

Corollary: *To every conformally flat hypersurface in S^{n+1} we may associate a curved flat in the Grassmannian $\frac{O_1(n+3)}{O(n)\times O_1(3)}$ of spacelike n - planes in $\mathbb{R}^{n+3}_1$ — however this curved flat is not unique*[21].

The Killing metric on $\frac{O_1(n+3)}{O(n)\times O_1(3)}$ is just a multiple of the trace form [7] — and hence the metric induced by the Gauß map c_f is a multiple of

$$\operatorname{tr} \Phi_{\mathfrak{p}}^2 \quad = \quad -2\textstyle\sum_{j=1}^{n}(\psi_j^2 + \chi_j \cdot \omega_j + \omega_j \cdot \chi_j) \ .$$

Denoting, as above, the principal curvatures of f (with respect to S) by a_i we see that this metric highly degenerates for channel hypersurfaces: with $a_2 = \ldots = a_n$ we get $\operatorname{tr}\Phi_{\mathfrak{p}}^2 \ = \ -2(a_1 - a_2)^2\omega_1^2$. In case of a three dimensional hypersurface in S^4 this metric is given by

$$\begin{array}{rcl} \operatorname{tr}\Phi_{\mathfrak{p}}^2 & = & 2\Sigma\gamma_j^2 \quad \text{where} \\ \gamma_1 & = & \sqrt{a_3 - a_1}\sqrt{a_1 - a_2}\cdot\omega_1 \ , \\ \gamma_2 & = & \sqrt{a_1 - a_2}\sqrt{a_2 - a_3}\cdot\omega_2 \ , \\ \gamma_3 & = & \sqrt{a_2 - a_3}\sqrt{a_3 - a_1}\cdot\omega_3 \end{array} \tag{13}$$

denote the *conformal fundamental forms*[22] of the hypersurface [9]. So we see that in the case of a three dimensional hypersurface in S^4 with three distinct principal curvatures, the metric induced by c_f is nondegenerate. In this case c_f happens to be a *regular* curved flat. Moreover the conformal fundamental forms γ_i of a conformally flat hypersurface are closely related to the *roots* of the abelian form $\Phi_{\mathfrak{p}}$, i.e. the eigenvalue forms of $\operatorname{ad}(\Phi_{\mathfrak{p}}) : TM \to \operatorname{End}(\mathfrak{o}_1(6))$. We perform a Gauge transformation (together with a transform to an orthonormal basis of $\mathbb{C}^6$) —

$$\begin{array}{rcl} \operatorname{Ad}(\begin{pmatrix} I & 0 \\ 0 & H \end{pmatrix})\Phi_{\mathfrak{p}} & = & i\begin{pmatrix} 0 & \operatorname{diag}(\gamma_1,\gamma_2,\gamma_3) \\ -\operatorname{diag}(\gamma_1,\gamma_2,\gamma_3) & 0 \end{pmatrix} \qquad \text{where} \\ H & = & \begin{pmatrix} \sqrt{a_1^2 + 2b_1} & 0 & 0 \\ 0 & \sqrt{a_2^2 + 2b_2} & 0 \\ 0 & 0 & \sqrt{a_3^2 + 2b_3} \end{pmatrix}^{-1} \begin{pmatrix} -a_1 & 1 & b_1 \\ -a_2 & 1 & b_2 \\ -a_3 & 1 & b_3 \end{pmatrix} \end{array} \tag{14}$$

[20]The pullback of the symmetric space's curvature tensor R considered as a 2 - form on M is given by $R(dc_f, dc_f) \ = \ -\frac{1}{2}[[\Phi_{\mathfrak{p}} \wedge \Phi_{\mathfrak{p}}], .]$. Since $\mathfrak{o}_1(n+3)$ is semisimple, $0 \ = \ R(dc_f, dc_f) : \mathfrak{p} \to \mathfrak{p}$ is equivalent to (12).

[21]Cf. footnote 18 on page 5.

[22]The forms γ_i are conformal invariants, i.e. invariant under conformal deformations of the strip (f, S). Consequently the metric induced by c_f is also conformally invariant! Note that one of the forms γ_i is real while the two others take values in $i\mathbb{R}$.

— to find $\gamma_i \pm \gamma_j$ and $-\gamma_i \pm \gamma_j$ $(i < j)$ as the roots of $\Phi_\mathfrak{p}$.

Now integration of the (closed) forms γ_i yield those canonical principal curvature[23] coordinates which always come with regular curved flats [6]; on the other hand this yields just the canonical Guichard net existing on every conformally flat hypersurface [9]. This fact puts my characterization of conformally flat hypersurfaces into a wider context. But Cartan's characterization of conformally flat hypersurfaces in S^4 with three distinct principal curvatures also arises in a new flavour: Cartan's "umbilical distributions" are exactly the "root distributions" $\gamma_i \pm \gamma_j = 0$ of $\Phi_\mathfrak{p}$. Thus, perhaps these two theorems (cited on page 2) characterizing conformally flat hypersurfaces seem a bit less "mystical" in the context of regular curved flats[24].

In this preceeding section we saw how we can associate a curved flat to a conformally flat hypersurface. It seems natural to be question whether we can recover a conformally flat hypersurface from a given curved flat. This is the theme of the following section:

4 Cyclic systems

By means of the Gauß map of a light cone representative of a hypersurface in S^{n+1} we associated to each point of the hypersurface a spacelike n - plane of the Minkowski space $\mathbb{R}^{n+3}_1$. Recalling our introduction, we realize that in this way we associate to each point a circle intersecting the hypersurface orthogonally at this point — that's why the Gauß map was denoted by "c_f". Actually these circles depend on the choice of the light cone representative: Performing a conformal deformation $(f, S) \rightsquigarrow (e^u f, S)$ the circles will change since the spheres S_i change[25]: $S_i \rightsquigarrow S_i + (e_i u) \cdot f$.

If moreover we choose a flat light cone representative for a conformally flat hypersurface in S^{n+1}, this circle congruence will be a *cyclic system*: From our Lemma on page 6 we saw that the bundle given by $p \mapsto c_f^\perp$ is flat with respect to the induced connection and hence the circle congruence allows a one parameter family of orthogonal hypersurfaces [10, part I] (see also [5]). Now we may choose S to be a *parallel* normal field to f, i.e. $d_p S(T_p M) \subset \text{span}\{S_1, \ldots, S_n\}|_p$ for all $p \in M$ — this fixes S up to a conformal deformation $(f, S) \rightsquigarrow (f, S + a\, f)$ where a is a constant. In the context of Möbius geometry this means that $\hat{f}$ will be the second envelope of the sphere congruence S. Moreover the Ricci equations (6) show that the curvature lines on the two envelopes[26] do correspond: the sphere congruence is *Ribaucour*. Now the one parameter family of orthogonal hypersurfaces may be parametrized[27] by $t \mapsto f_t := t\, S + f - \frac{1}{2} t^2\, \hat{f}$. Since the tangential frame $(S_1, \ldots, S_n)$ is the same for all these orthogonal hypersurfaces — for t

[23]These coordinates also arise from the integration of a Lax system yielding a conformally flat hypersurface if the initial loop is choosen in an appropriate way (cf. [8]).

[24]However the theory of curved flats gives us only one half of each theorem for free: starting just with the integrability of the conformal fundamental forms γ_i (or of the umbilical distributions $\gamma_i \pm \gamma_j = 0$) a priori there is no curved flat to be associated with the hypersurface in S^4.

[25]*All* congruences of circles orthogonal to the hypersurface might be obtained by performing transformations $S_i \rightsquigarrow S_i + u_i \cdot f$ where u_i are n independent functions on M^n.

[26]This is in the generic case of pairwise distinct principal curvatures on both envelopes. Nevertheless the sphere congruence $S : M^n \to S^{n+2}_1$ will be called "Ribaucour" if its normal bundle is flat — also in less generic cases (for example in "the worst case" where one envelope collapses to a single point).

[27]Remark that there are numerous possibilities to parametrize this one parameter family — for example the most general "arc length parametrization" is given by $\frac{1}{g'}(g\, S + f - \frac{1}{2} g^2\, \hat{f})$ where $g : \mathbb{R} \to \mathbb{R}$ denotes an arbitrary function.

small enough these hypersurfaces will be nondegenerate too — also the curvature forms do not change: we get the following

Theorem: *To a conformally flat hypersurface in S^{n+1} there exists a normal congruence of circles having a whole one parameter family of conformally flat hypersurfaces*[28] *intersecting the circles of the congruence orthogonally. This cyclic system is not uniquely associated to the hypersurface.*

In case of a channel hypersurface this theorem is not too surprising: for example, thinking of a channel hypersurface in Euclidean ambient space $\mathbb{R}^{n+1}$ all the parallel hypersurfaces will also be channel hypersurfaces and hence they will be conformally flat. Indeed, this construction would be too special in Möbius geometry to expect its occurance in all cases of curved flats. So let's analyze the situation carefully: in the case of a channel hypersurface (assuming $a_2 = \ldots = a_n$) the Codazzi equations (5) for the normal field S yield $0 = da_2 \wedge \omega_k - (a_1 - a_2)\omega_1 \wedge \omega_{1k}$, for $k = 2, \ldots, n$, and therefore the constancy of the principal curvature a_2 on the $(n-1)$ - spheres $\omega_1 = 0$ foliating our channel hypersurface[29] is obtained:

$$0 = da_2 \wedge \omega_1 . \tag{15}$$

Now we conclude that the vectors[30]

$$p_0 = a_2 S + \tfrac{1}{2}a_2^2 f - \hat{f} \quad \text{and} \quad S_0 = \tfrac{1}{2}a_2^2 f + \hat{f}$$

are also constant along these $(n-1)$ - spheres. So all the circles orthogonally intersecting the channel surface in a fixed $(n-1)$ - sphere $\omega_1 = 0$ will contain the point p_0 which, moreover, lies on the sphere $S + a_2 f$ of the one parameter congruence enveloped by the channel hypersurface[31]: $0 = \langle S + a_2 f, p_0\rangle$. If the sphere S does *not* belong to the enveloped one parameter congruence, i.e. if $a_2 \neq 0$, the vector S_0 will give us a sphere since $\langle S_0, S_0\rangle = a_2^2$. Then we have $\langle p_0, S_0\rangle = 0$ and $\frac{\langle S_0, S + a_2 f\rangle^2}{\langle S_0, S_0\rangle} = 1$ which implies that the two spheres S_0 and $S + a_2 f$ have first order contact in the point p_0. Consequently all the circles orthogonally intersecting our channel hypersurface in a fixed $(n-1)$ - sphere belong to the parabolic circle bundle orthogonal to the parabolic pencil of spheres spanned by S_0 and the principal curvature sphere $S + a_2 f$. Reflecting the foliating $(n-1)$ - sphere on the hypersurface f at the sphere S_0 yields the corresponding $(n-1)$ - sphere on the second envelope[32] $\hat{f}$ of S: $f - 2\frac{\langle f, S_0\rangle}{\langle S_0, S_0\rangle} S_0 = -\frac{2}{a_2^2}\hat{f}$.

If the principal curvature a_2 is constant over the entire hypersurface — not only on the foliating $(n-1)$ - spheres $\omega_1 = 0$ — then we have $dp_0 = 0$ — instead of $dp_0 \wedge \omega_1 = 0$. So we may interprete p_0 as the point at infinity of a Euclidean ambient

[28]Moreover it is easy to see that these conformally flat hypersurfaces are in general *not* conformally equivalent: using the above parametrization for the family of orthogonal hypersurfaces their metrics are given by $\langle df_t, df_t\rangle = \Sigma(1 - ta_i - \frac{1}{2}t^2 b_{ii})^2\omega_i^2$ — so, if they were conformally equivalent we should have $(1 - ta_i - \frac{1}{2}t^2 b_{ii})^2 = (1 - ta_j - \frac{1}{2}t^2 b_{jj})^2$ for all $i, j = 1, \ldots, n$; comparing coefficients we see that conformal equivalence implies $a_i = a_j$.

[29]Here, for simplicity, we assume that we have a "real" channel hypersurface, i.e. $a_1 \neq a_2$.

[30]Remark that S_0 changes when $S \rightsquigarrow S + a f$, while p_0 does not.

[31]From the Codazzi equations (5) and (15) we deduce $d[(a_1 - a_2)\omega_1] = 0$ which gives us a natural parameter for the one parameter congruence enveloped by our channel hypersurface.

[32]This second envelope may become singular if $a_1 = 0$; otherwise it is a regular channel hypersurface.

space M_0^{n+1}. Consequently our curved flat c_f yields a normal line congruence in M_0^{n+1} the family of orthogonal hypersurfaces being parallel flat hypersurfaces: the enveloped one parameter family $S + a_2 f$ consists of planes — spheres containing p_0. This is exactly the case where the family of orthogonal hypersurfaces of c_f becomes a family of parallel hypersurfaces in Euclidean space.

At this point we are actually able to construct an example showing that a cyclic system associated to a conformally flat hypersurface in S^{n+1} hardly depends on the choice of a corresponding flat light cone immersion (cf. footnote 18 on page 5): let's consider the cylinder $f : M^n \to I\!R^{n+1} \cong M_0^{n+1}$ over a planar curve together with its tangent hyperplane congruence $S : M^n \to S_1^{n+2}$. In this situation we may choose a *parallel* principal curvature framing $F : M^n \to O_1(n+3)$ for the strip (f, S), i.e. the metric fundamental forms integrate to coordinate functions and consequently the corresponding connection form takes a very simple form:

$$F^{-1}dF \;=\; \left(\begin{array}{cccc|ccc} 0 & 0 & \dots & 0 & -a_1 dx_1 & dx_1 & 0 \\ 0 & 0 & \dots & 0 & 0 & dx_2 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & dx_n & 0 \\ \hline a_1 dx_1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -dx_1 & -dx_2 & \dots & -dx_n & 0 & 0 & 0 \end{array}\right)$$

where $a_1 = a_1(x_1)$ denotes the curvature of the planar curve. According to the above discussions, the corresponding cyclic system is a normal line congruence in $I\!R^{n+1}$. Now let's perform a conformal deformation

$$(f, S) \;\rightsquigarrow\; (\tfrac{1}{\|x\|^2} f, S + \tfrac{a}{\|x\|^2} f) \quad \text{where} \quad da \;=\; 2a_1 x_1 dx_1 \;. \tag{16}$$

It is not too hard to see that $d(\frac{\partial u}{\partial x_i}) - \frac{\partial u}{\partial x_i} du + \frac{1}{2}\Sigma(\frac{\partial u}{\partial x_j})^2 \cdot dx_i \;=\; 0$ which implies the induced metric of $\frac{1}{\|x\|^2} f$ be flat (cf. [8]). Further, we have

$$d(S + \tfrac{a}{\|x\|^2} f) \;=\; -a_1 \|x\|^2 \cdot \tfrac{\partial}{\partial x_1}(\tfrac{1}{\|x\|^2} f) dx_1 + a \cdot d(\tfrac{1}{\|x\|^2} f)$$

which shows that $S + \frac{a}{\|x\|^2} f$ is a parallel unit normal field to $\tilde{f} \;=\; \frac{1}{\|x\|^2} f : M^n \to L^{n+2}$ and that the principal curvatures are given by $\tilde{a}_1 \;=\; a_1 \|x\|^2 - a$ and $\tilde{a}_i \;=\; -a$ where $i \;=\; 2, \dots, n$. Obviously $\tilde{a}_2 \neq$ *const* and consequently the cyclic system $c_{\tilde{f}}$ corresponding to the flat light cone immersion $\tilde{f}$ is *not* a normal line congruence in some Euclidean space. Hence we also see that we can only *locally* associate our cyclic systems to a given conformally flat hypersurface — on domains where the induced metric is "globally conformally flat" (cf. [11]).

From the preceding discussions we see that a curved flat $c : M^n \to \frac{O_1(n+3)}{O(n) \times O_1(3)}$ in the space of spacelike n - planes in Minkowski space $I\!R_1^{n+3}$ — now the curved flat is not assumed to be the Gauß map of an immersion — may be interpreted as a cyclic system having a family of conformally flat orthogonal hypersurfaces: in fact every parallel lightlike section $f_t : M^n \to L^{n+2}$ of the vector bundle $c^\perp$ which induces a nondegenerate metric on

M^n yields a flat immersion of M^n into the light cone and consequently a conformally flat immersion into S^{n+1}. Assuming the curved flat to be immersed assures that generically[33] every lightlike section of $c^\perp$ will induce a nondegenerate metric. Summarizing we formulate the following

Theorem: *Every immersed curved flat in the space of spacelike n - planes in Minkowski space $\mathbb{R}^{n+3}_1$ gives us a cyclic system in S^{n+1} with a one parameter family of conformally flat orthogonal hypersurfaces.*

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[33] Taking any lift $F : M^n \to O_1(n+3)$ of the curved flat c with vanishing $\mathfrak{o}_1(3)$ - part of $\Phi_\mathfrak{k}$ the section $Fv : M^n \to L^{n+2}$ will (locally) induce a Riemannian metric on M^n for almost all isotropic initial vectors $v \in L^{n+2}$, $F|_{p_0}v \in c(p_0)^\perp$ — note that $L^{n+2} \cap F^{-1}c^\perp|_{p_0} \cong L^2 \subset \mathbb{R}^3_1$.