Cohomological properties of the smooth globalization of a Harish-Chandra module

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Contents

1 Introduction

Let G be a connected semisimple Lie group with finite center. Let $\mathbf{g}$ be the Lie algebra of G and K be its maximal compact subgroup. By $\mathcal{HC}(\mathbf{g}, K)$ we denote the category of Harish-Chandra modules of G.

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The Harish-Chandra module $(\pi, V_{\pi,K}) \in \mathcal{HC}(\mathbf{g}, K)$ has natural globalizations (analytic, smooth, distribution, and hyperfunction vectors)

$$V_{\pi,\omega} \subset V_{\pi,\infty} \subset V_{\pi,-\infty} \subset V_{\pi,-\omega} \ .$$

Here $V_{\pi,*}$ is a complete locally convex vector space admitting a continuous G-action such that the underlying $(\mathbf{g}, K)$-module is $V_{\pi,K}$.

Let $\Gamma \subset G$ be a torsion-free discrete subgroup of finite covolume. Then a natural problem is to study the cohomology of Γ with coefficients in the different globalizations $V_{\pi,*}$. The cohomology groups again become locally convex topological vector spaces. Our work in [3] led us to the following conjecture.

Conjecture 1.1 *If Γ is cocompact, then for all $p \geq 0$ there are isomorphisms*

$$\begin{aligned} H^p(\Gamma, V_{\pi,\omega}) &= H^p(\Gamma, V_{\pi,\infty}) && (1)\\ H^p(\Gamma, V_{\pi,-\omega}) &= H^p(\Gamma, V_{\pi,-\infty}) && (2)\\ H^p(\Gamma, V_{\pi,*})^* &= H^{\dim(G/K)-p}(\Gamma, V_{\tilde{\pi},-*}),\ \ * = \omega, \infty && (3)\end{aligned}$$

and all vector spaces are Hausdorff and finite-dimensional. If Γ has finite covolume, then $H^*(\Gamma, V_{\pi,-\infty})$ is finite-dimensional and Hausdorff.

In [3] we settled this conjecture in the rank-one case for $* = \omega$ and cocompact Γ. In the present paper we prove the conjecture in the case $\mathrm{rank}_{\mathbf{R}}(G) = 1$ for $* = \infty$ admitting non-cocompact Γ of finite covolume (Section 8). As indicated in [3], the main result needed to extend the method of [3] to the case $* = \infty$ was the surjectivity of the $B := \Omega - \mu$ ($\mu \in \mathbf{C}$, Ω - Casimir of G) on the space of sections of moderate growth of homogeneous bundles over $X := G/K$. This is proved in Theorem 2.5 of the present paper.

We also obtain a more concrete description of the cohomology groups in terms of automorphic and cusp forms.

Let now G be of general rank and Γ be cocompact. Employing the recent result of Kashiwara-Schmid [13], Thm. 6.13, one can show (3) for $* = \omega$ and dim $H^*(\Gamma, V_{\pi,\pm\omega}) < \infty$. Using Theorem 1.3 below for $* = \infty$ it is possible to prove (1) and (2). We will explain this in a forthcoming paper. The case of non-cocompact Γ remains open for $\mathrm{rank}_R(G) > 1$.

Our main motivation for considering the Γ-cohomology of globalizations of Harish-Chandra modules was to prove a conjecture of Patterson about the singularities of the Selberg zeta functions associated to Γ.

Let G be a semisimple Lie group of real rank one and and $P = MAN$ be a minimal parabolic subgroup of G. Let $\mathbf{a}, \mathbf{n}$ be the Lie-algebras of A, N. Let $(\sigma, V_\sigma) \in \hat{M}$ be an irreducible representation of M and $\Gamma \subset G$ be a discrete cocompact torsion-free subgroup. Then the Selberg zeta function $Z_\Gamma(s, \sigma)$, $s \in \mathbf{a}^*_{\mathbf{C}}$, is defined as the analytic continuation of the infinite product

$$Z_\Gamma(s,\sigma) = \prod_{[g]\in C\Gamma, n_\Gamma(g)=1} \prod_{k=0}^{\infty} \det\left(1 - a_g^{-\rho-s} S^k(Ad(m_g a_g)_{\mathbf{n}}^{-1}) \otimes \sigma(m_g)\right) .$$

Here $C\Gamma$ is the set of hyperbolic conjugacy classes in Γ, $n_\Gamma(g)$ is the maximal number $n \in \mathbf{N}$ such that $g = h^n$ for some $h \in \Gamma$ and $m_g \in M$, $a_g \in A^+$ are such that g is conjugated in G to $m_g a_g$. $S^k(\mathrm{Ad}(m_g a_g)_{\mathbf{n}}^{-1})$ stands for the k'th symmetric power of $\mathrm{Ad}(m_g a_g)^{-1}$ restricted to $\mathbf{n}$ and $\rho \in \mathbf{a}^*$ is defined by $\rho(H) := \frac{1}{2}\mathrm{tr}(\mathrm{ad}(H)_{\mathbf{n}})$. The infinite product converges for $\mathrm{Re}(s) > \rho$. In this generality the Selberg zeta function was introduced by Fried [7]. He applied Ruelle's techniques of hyperbolic dynamics and gave a meromorphic continuation of $Z_\Gamma(s,\sigma)$.

The parameters $(\sigma,\lambda) \in \hat{M} \times \mathbf{a}^*_{\mathbf{C}}$ also define a principal series representation of G on the Hilbert space

$$H^{\sigma,\lambda} = \{f : G \to V_\sigma \mid f(gman) = a^{\lambda-\rho}\sigma(m)^{-1}f(g),\ \forall man \in MAN, f_{|K} \in L^2\},$$

where G acts by the left regular representation. By $H^{\sigma,\lambda}_{-\infty}$ we denote the space of its distribution vectors.

S. Patterson [17] conjectured the relationship of the singularities of Selberg zeta functions with the cohomology of Γ with coefficients in the distribution vectors of principal series representations. In [3] we have shown an analog of Pattersons conjecture for the hyperfunction vectors of principal series representations. Pattersons original conjecture now follows from (2).

Theorem 1.2 *The cohomology $H^p(\Gamma, H^{\sigma,\lambda}_{-\infty})$ is finite dimensional for all $p \geq 0$,*

$$\chi(\Gamma, H^{\sigma,\lambda}_{-\infty}) = \sum_{p=0}^{\infty}(-1)^p \dim H^p(\Gamma, H^{\sigma,\lambda}_{-\infty}) \quad = \quad 0, \tag{4}$$

$$\chi(\Gamma, \hat{H}^{\sigma,\lambda}_{-\infty}) = \sum_{p=0}^{\infty}(-1)^p \dim H^p(\Gamma, \hat{H}^{\sigma,\lambda}_{-\infty}) \quad = \quad 0, \tag{5}$$

and the order of $Z_\Gamma(s,\sigma)$ at $s \in \mathbf{a}^*_{\mathbf{C}}$ can be expressed in terms of the group cohomology of Γ with coefficients in $H^{\sigma,\lambda}_{-\infty}$ as follows :

$$\mathrm{ord}_{s=\lambda\neq 0} Z_\Gamma(s,\sigma) \quad = \quad -\sum_{p=0}^{\infty}(-1)^p p \dim H^p(\Gamma, H^{\sigma,\lambda}_{-\infty}), \tag{6}$$

$$\mathrm{ord}_{s=0} Z_\Gamma(s,\sigma) \quad = \quad -\sum_{p=0}^{\infty}(-1)^p p \dim H^p(\Gamma, \hat{H}^{\sigma,0}_{-\infty}), \tag{7}$$

where $\hat{H}^{\sigma,\lambda}_{-\infty}$ is a certain non-trivial extension of $H^{\sigma,\lambda}_{-\infty}$ with itself.

Our description of the cohomology for non-cocompact Γ of finite covolume implies that the analog of Pattersons conjecture is false in that case.

A very detailed analysis of the cohomology of Fuchsian groups of the first kind with coefficients in principal series representations is given in Section 9. We express the cohomology in terms of automorphic and cusp forms. For the non-irreducible principal series representations we find explicit formulas for the dimensions of the cohomology groups in terms of the topology of $Y = \Gamma\backslash X$.

The methods of this paper can also be used to study the $\mathbf{n}$-cohomology of globalizations of Harish-Chandra modules.

Let G be of general rank and $P \subset G$ be a parabolic subgroup with Langlands decomposition $P = M_P A_P N_P$. Let $\mathbf{n}_P$ be the Lie algebra of N_P.

If $(\pi, V_{\pi,K}) \in \mathcal{HC}(\mathbf{g}, K)$, then by a theorem of Hecht-Schmid [9] the Lie algebra cohomology groups $H^*(\mathbf{n}_P, V_{\pi,K})$ have natural structures of modules in $\mathcal{HC}(\mathbf{m}_P, K_P)$, where $K_P = K \cap M_P$ and $\mathbf{m}_P$ is the Lie algebra of M_P. Moreover, all elements of $H^*(\mathbf{n}_P, V_{\pi,K})$ are finite under $\mathbf{a}_P$, where $\mathbf{a}_P$ is the Lie algebra of A_P.

It is natural to ask whether globalization is compatible with $\mathbf{n}_P$-cohomology. The standard $\mathbf{n}_P$-cohomology complex induces a natural topology on $H^p(\mathbf{n}_P, V_{\pi,*})$ such that $H^p(\mathbf{n}_P, V_{\pi,*})$ becomes a continuous representation of $M_P A_P$. The natural theorem is

Theorem 1.3 *There are topological isomorphisms of $M_P A_P$ modules*

$$\begin{aligned} H^p(\mathbf{n}_P, V_{\pi,*}) &\cong H^p(\mathbf{n}_P, V_{\pi,K})_* \ , \\ H^p(\mathbf{n}_P, V_{\pi,-*})^* &\cong H^{\dim(\mathbf{n}_P)-p}(\mathbf{n}_P, V_{\tilde{\pi},K})_* \otimes \Lambda^{\dim(\mathbf{n}_P)}\mathbf{n}_P \ , \quad \forall p \geq 0 \end{aligned}$$

for $* = \infty, \omega$, where $(\tilde{\pi}, V_{\tilde{\pi},K})$ is the dual Harish-Chandra module.

For $* = \infty$ this was proved by Casselman (unpublished). In the rank-one case there were previous partial results by Osborne [16]. In the present paper we give a proof in the rank-one case using different methods (Theorem 5.2).

In [3] we were able to prove this theorem in the case $\mathrm{rank}_{\mathbf{R}}(G) = 1$ for $* = \omega$. For G of general rank and $* = \omega$ the theorem was announced by Bratten and Hecht-Taylor (compare [10]). It is now an easy consequence of the above-mentioned result [13], Thm. 6.13.

2 Surjectivity on weighted spaces

From now on let G be a connected semisimple Lie group with finite center of *real rank one* and K be its maximal compact subgroup. Let $G \times_K V_\gamma = E$ be a G-homogeneous bundle over the rank-one symmetric space $X = G/K$. Let $\mathcal{E} = C^\infty(X, E)$ be the space of smooth sections of E. We define the increasing sequence of subspaces $S_R\mathcal{E}$, $R \in \mathbf{R}$, by

$$S_R\mathcal{E} := \{f \in \mathcal{E} \mid p_{-R,L}(f) < \infty \ \forall L \in \mathcal{U}(\mathbf{g})\},$$

where the seminorms $p_{R,L}$ are defined by

$$p_{R,L}(f) := \sup_G \mathrm{e}^{R\mathrm{dist}(gK,\mathcal{O})}|f(Lg)| \ ,$$

$\mathcal{O} \in X$ is the class $[K] \in G/K$ considered as the base point of X, and we view f as a function on G with values in V_γ satisfying $f(gk) = \gamma(k)^{-1}f(g)$, $\forall k \in K$. $S_R\mathcal{E}$ is a Fréchet space and for $R' \geq R$ we have a continuous inclusion $S_R\mathcal{E} \hookrightarrow S_{R'}\mathcal{E}$. Let $S_R\mathcal{E}' := (S_{-R}\mathcal{E})^*$

be the topological conjugate dual of $S_{-R}\mathcal{E}$. Then for $R \geq R'$ there is a continuous projection $S_{R'}\mathcal{E}' \to S_R\mathcal{E}'$. We also consider

$$\begin{aligned} S_\infty\mathcal{E}' &:= \lim_{\overrightarrow{R}} S_R\mathcal{E}' \ , \\ S_\infty\mathcal{E} &:= \{f \in \mathcal{E} \mid \forall L \in \mathcal{U}(\mathbf{g})\, \exists R \in \mathbf{R}\; s.t.\, p_{R,L}(f) < \infty\} \ . \end{aligned}$$

The space $S_\infty\mathcal{E}$ is a limit of Fréchet spaces in the following way. Let $\mathcal{I}$ be the partially ordered set of all monotone functions $u : \mathbf{N}_0 \to \mathbf{R}$, where $u \geq u'$, iff $u(n) \geq u'(n)$, $\forall n \in \mathbf{N}$. For any $u \in \mathcal{I}$ we define the Fréchet space

$$S_u\mathcal{E} := \{f \in \mathcal{E} \mid p_{-u(\deg(L)),L}(f) < \infty \quad \forall L \in \mathcal{U}(\mathbf{g})\} \ ,$$

where $\deg(L)$ is the order of L as a differential operator on G. Then

$$S_\infty\mathcal{E} = \lim_{\overrightarrow{u \in \mathcal{I}}} S_u\mathcal{E} \ .$$

We equip $S_\infty\mathcal{E}$ with the topology of the direct limit. The symbol $S_\infty\mathcal{E}$ is an abuse of notation since

$$S_\infty\mathcal{E} \neq \lim_{\overrightarrow{R \in \mathbf{R}}} S_R\mathcal{E} \ .$$

$S_\infty\mathcal{E}$ is called the space of sections of E of moderate growth.

The space $S_\infty\mathcal{E}'$ is the topological conjugate dual of $\cap_R S_R\mathcal{E}$. Since $C_c^\infty(X, E)$ is dense in $\cap_R S_R\mathcal{E}$, we have an inclusion $S_\infty\mathcal{E}' \hookrightarrow C^{-\infty}(X, E)$.

Let Ω be the Casimir operator of G and for $\mu \in \mathbf{C}$ let $B := \Omega - \mu$. Let $\mathbf{C}[B]$ be the ring of all polynomials in B. We consider the functor Fin_μ on the category of $\mathbf{C}[B]$-modules defined by

$$\mathrm{Fin}_\mu(V) = \{v \in V \mid \exists l \geq 0 \quad B^l v = 0\} \ .$$

This functor is left exact and its higher derived functors are denoted by $\mathrm{Fin}^i_\mu(V)$. Since the inductive limit is an exact functor on the category of $\mathbf{C}[B]$-modules and

$$\mathrm{Fin}_\mu(V) \cong \lim_{\overrightarrow{j}} \mathrm{Hom}_{\mathbf{C}[B]}(\mathbf{C}[B]/(B)^j, V)$$

we have

$$\mathrm{Fin}^i_\mu(V) \cong \lim_{\overrightarrow{j}} \mathrm{Ext}^i_{\mathbf{C}[B]}(\mathbf{C}[B]/(B)^j, V) \ . \tag{8}$$

Since $\mathbf{C}[B]$ is a regular ring of dimension one we have $\mathrm{Fin}^i_\mu(V) = 0$ for $i \geq 2$. A $\mathbf{C}[B]$-module V is called Fin_μ-acyclic, iff $\mathrm{Fin}^1_\mu(V) = 0$.

Theorem 2.1 *$S_\infty\mathcal{E}'$ is Fin_μ-acyclic.*

Proof. We consider the Schwartz space

$$S_{-log}\mathcal{E} := \{f \in \mathcal{E} \mid q_{L,k}(f) < \infty \quad \forall k \in \mathbf{N}, L \in \mathcal{U}(\mathbf{g})\} \ ,$$

where

$$q_{L,k}(f)^2 = \int_G \mathrm{dist}(gK, \mathcal{O})^k |f(Lg)|^2 dg$$

and its topological conjugate dual space

$$S_{log}\mathcal{E}' = (S_{-log}\mathcal{E})^* \ .$$

Lemma 2.2 *The inclusion $S_{log}\mathcal{E}' \hookrightarrow S_\infty\mathcal{E}'$ induces a surjection*

$$Fin^1_\mu(S_{log}\mathcal{E}') \to Fin^1_\mu(S_\infty\mathcal{E}') \ .$$

Proof. Let V be a $\mathbf{C}[B]$-module. We can compute $\mathrm{Ext}^1_{\mathbf{C}[B]}(\mathbf{C}[B]/(B)^k, V)$ using the Koszul complex

$$0 \to \mathrm{Ext}^0_{\mathbf{C}[B]}(\mathbf{C}[B]/(B)^k, V) \to V \stackrel{B^k}{\to} V \to \mathrm{Ext}^1_{\mathbf{C}[B]}(\mathbf{C}[B]/(B)^k, V) \to 0 \ .$$

In order to perform the direct limit in (8) note that the map

$$\mathrm{Ext}^1_{\mathbf{C}[B]}(\mathbf{C}[B]/(B^k), V) \to \mathrm{Ext}^1_{\mathbf{C}[B]}(\mathbf{C}[B]/(B^{k+1}), V)$$

is induced by $B : V \to V$.

Hence Lemma 2.2 follows from the next result.

Lemma 2.3 *Let $k \geq 1$. For any $f \in S_\infty\mathcal{E}'$ there exists a $g \in S_\infty\mathcal{E}'$ with $f - B^k g \in S_{log}\mathcal{E}'$.*

In fact, Lemma 2.3 implies that any element $f \in S_\infty\mathcal{E}'$ representing some class in $\mathrm{Ext}^1_{\mathbf{C}[B]}(\mathbf{C}[B]/(B)^k, S_\infty\mathcal{E}')$ can be replaced by $f - B^k g \in S_{log}\mathcal{E}'$ representing the same class in $\mathrm{Ext}^1_{\mathbf{C}[B]}(\mathbf{C}[B]/(B)^k, S_\infty\mathcal{E}')$.

Proof. It is enough to show that for $f \in S_R\mathcal{E}'$ there exists $g \in S_R\mathcal{E}'$ such that $f - B^k g \in S_{R-1}\mathcal{E}'$. In fact, it is sufficient that this holds for all weights R in a dense subset of $\mathbf{R}$. Then the assertion of the lemma follows by a finite iteration.

We consider polar coordinates $\mathbf{a}^+ \times K/M$ of $X \setminus \{\mathcal{O}\}$. We identify the fibre of E over (a, kM) with the fibre over (a', kM) using the radial parallel transport.

There is a constant coefficient operator B^k_{rad} on $\mathbf{a}^+$ such that $B^k - B^k_{rad} = \mathrm{e}^{-a}Q$ (we identify $\mathbf{a}^+$ with $\mathbf{R}_+$ such that the short root has length one) and Q has bounded coefficients up to infinity. Note that Q also contains differentiations along $\mathbf{a}^+$ but with exponentially decreasing prefactors. There are an endomorphism $F(kM) \in \mathrm{End}(E_{(a,kM)})$ and $c \in \mathbf{C}$ such that

$$B_{rad} = -\frac{d^2}{da^2} + c\frac{d}{da} + F(kM) \ .$$

We decompose the bundle $E_{|X\setminus\mathcal{O}} = \sum_\sigma E(\sigma)$ according to the eigenvalues of $F(kM)$ (which are independent of kM). There are $x_\sigma, y_\sigma \in \mathbf{C}$ such that $B_{rad} = \sum_\sigma B_{rad}(\sigma)$ and $B_{rad}(\sigma) = -(\frac{d}{da} - \bar{x_\sigma})(\frac{d}{da} - \bar{y_\sigma})$. We exclude the weights $R = \mathrm{Re}(x_\sigma), \mathrm{Re}(y_\sigma)$ for all σ from the following consideration.

Let $S_R\mathcal{E}_0 \subset S_R\mathcal{E}$ be the subspace of all sections vanishing in a neighbourhood of the origin $\mathcal{O} \in X$. Then $S_R\mathcal{E}_0$ is a limit of Fréchet spaces. Let $S_R\mathcal{E}'_0$ be the topological conjugate dual of $S_R\mathcal{E}_0$. Then $S_R\mathcal{E}' \subset S_R\mathcal{E}'_0$. Choose a cut-off function $\chi \in C^\infty(\mathbf{a}^+)$ such that $\chi(a) = 0$ for $a < 1$ and $\chi(a) = 1$ for $a > 2$. Then the multiplication by χ defines a continuous operator $S_R\mathcal{E}'_0 \to S_R\mathcal{E}'$.

Below we will construct a continuous solution operator $H : S_R\mathcal{E}_0 \to S_R\mathcal{E}_0$ such that $H(B^*_{rad})^k - \mathrm{id} : S_R\mathcal{E}_0 \to C_c^\infty(X \setminus \mathcal{O}, E)$ is continuous and $\mathrm{supp}(H(B^*_{rad})^k - \mathrm{id})f \subset [1,2] \times K/M$, $\forall f \in S_R\mathcal{E}_0$. Here $(B^*_{rad})^k$ is the formal adjoint of B^k_{rad}. The adjoint $H^* : S_R\mathcal{E}'_0 \to S_R\mathcal{E}'_0$ then has the property that $B^k_{rad}H^*\phi - \phi \in \mathcal{E}'$ for $\phi \in S_R\mathcal{E}'_0$. Here $\mathcal{E}'$ is the space of distributions with compact support. The composition $\chi \circ H^*_{|S_R\mathcal{E}'} : S_R\mathcal{E}' \to S_R\mathcal{E}'$ satisfies $B^k_{rad}\chi \circ H^*\phi - \phi \in \mathcal{E}'$, $\phi \in S_R\mathcal{E}'$. Hence $B^kH^*\phi - \phi = e^{-a}QH^*\phi \,(\mathrm{mod}\, \mathcal{E}')$. But $e^{-a}QH^*\phi \in S_{R-1}\mathcal{E}'$. This proves the lemma assuming that we have already constructed H.

Note that $B^*_{rad}(\sigma)^k$ is a product of operators of first order $\frac{d}{da} + x_\sigma$, $\frac{d}{da} + y_\sigma$. It is enough to construct solution operators $H_{x_\sigma}, H_{y_\sigma} : S_R\mathcal{E}_0(\sigma) \to S_R\mathcal{E}_0(\sigma)$ such that $\mathrm{supp}(H_{x_\sigma}(\frac{d}{da} + x_\sigma)f - f) \subset [1,2] \times K/M$, $\forall f \in S_R\mathcal{E}_0(\sigma)$, and similarly for H_{y_σ}. Then $H := (-1)^k \sum_\sigma H^k_{x_\sigma} H^k_{y_\sigma}$ has the required properties.

For $R - \mathrm{Re}(x_\sigma) > 0$ we set (simplifying the notation by omitting the angular variable)

$$(H_{x_\sigma} f)(a) := -\chi(a)\mathrm{e}^{-x_\sigma a} \int_a^\infty \mathrm{e}^{x_\sigma b} f(b)db \ .$$

Since $f \in S_R\mathcal{E}_0(\sigma)$ we have for some $C < \infty$ and all $b \in \mathbf{a}^+$ that $|f(b)| \le C\mathrm{e}^{-Rb}$. Hence the integral converges. If $R - \mathrm{Re}(x_\sigma) < 0$, then we set

$$(H_{x_\sigma} f)(a) := \chi(a)\mathrm{e}^{-x_\sigma a} \int_0^a \mathrm{e}^{x_\sigma b} f(b)db \ .$$

It is easy to check that $\mathrm{supp}(H_{x_\sigma}(\frac{d}{da} + x_\sigma)f - f) \subset [1,2] \times K/M$. We must show that $H_{x_\sigma} : S_R\mathcal{E}_0(\sigma) \to S_R\mathcal{E}_0(\sigma)$ and that H_{x_σ} is continuous.

To define the topology of $S_R\mathcal{E}_0(\sigma)$ it is sufficient to consider the set of seminorms $\tilde{p}_{R,L}$ with $L = (L_1, L_2) \in \mathcal{U}(\mathbf{a}) \times \mathcal{U}(\mathbf{k})$. Set $(Lf)(a, kM) = f(L_1 a, L_2 kM)$. Then

$$\tilde{p}_{R,L} := \sup_{a\in\mathbf{a}^+, k\in K} \mathrm{e}^{Ra}|(Lf)(a, kM)| \ .$$

It is clear that $LH_x f = H_x Lf + W_L f$, where $W_L : S_R\mathcal{E}_0(\sigma) \to C_c^\infty(X \setminus \mathcal{O}, E(\sigma))$ is continuous. Thus in order to show that H_{x_σ} is continuous it is enough to verify the estimate

$$\tilde{p}_{R,1}(H_{x_\sigma} f) \le C\tilde{p}_{R,1}(f) \ .$$

We employ that $|f(a, kM)| \le \tilde{p}_{R,1}(f)\mathrm{e}^{-Ra}$. If $R - \mathrm{Re}(x_\sigma) > 0$, then

$$\begin{aligned} \tilde{p}_{R,1}(H_{x_\sigma} f) &= \sup_{a\in\mathbf{a}^+, k\in K} \mathrm{e}^{Ra}|(H_{x_\sigma} f)(a, kM)| \\ &\le \sup_{a\in\mathbf{a}^+, k\in K} \mathrm{e}^{Ra}\tilde{p}_{R,1}(f)\mathrm{e}^{-\mathrm{Re}(x_\sigma)a} \int_a^\infty \mathrm{e}^{(\mathrm{Re}(x_\sigma)-R)b}db \\ &\le C\tilde{p}_{R,1}(f) \ . \end{aligned}$$

If $R-\mathrm{Re}(x_\sigma)<0$, then

$$\begin{aligned}\tilde p_{R,1}(H_{x_\sigma}f) &= \sup_{a\in\mathbf{a}^+,k\in K} \mathrm{e}^{Ra}|(H_{x_\sigma}f)(a,kM)|\\ &\le \sup_{a\in\mathbf{a}^+,k\in K} \mathrm{e}^{Ra}\tilde p_{R,1}(f)\mathrm{e}^{-\mathrm{Re}(x_\sigma)a}\int_1^a \mathrm{e}^{(\mathrm{Re}(x_\sigma)-R)b}db\\ &\le C\tilde p_{R,1}(f)\ .\end{aligned}$$

This finishes the construction of H and hence the proof of Lemma 2.2. □

Lemma 2.4 *$S_{log}\mathcal{E}'$ is Fin_μ-acyclic.*

Proof. Using the generalization of the Helgason-Fourier transform to bundles (see [4], 1.4.3 , Branson-Olafsson-Schlichtkrull [2]) and a result of Arthur [1] we can identify the Schwartz space $S_{-log}\mathcal{E}$ with

$$[\mathcal{S}(\imath\mathbf{a}^*)\hat\otimes C^\infty(K\times_M V_\gamma)]^W\oplus\oplus_{i=1}^r V_{i,\infty}\ .$$

Here $V_{i,\infty}$ are the spaces of smooth vectors of certain irreducible discrete series representations, $\mathcal{S}(\imath\mathbf{a}^*)$ is the Schwartz space on $\imath\mathbf{a}^*$ and $W=\mathbf{Z}_2$ is the Weyl group. $\hat\otimes$ denotes the nuclear tensor product. The operation of W on $\mathcal{S}(\imath\mathbf{a}^*)\hat\otimes C^\infty(K\times_M V_\gamma)$ is implemented by a family of unitary operators

$$A_\lambda : C^\infty(K\times_M V_\gamma)\to C^\infty(K\times_M V_\gamma),\quad \lambda\in\imath\mathbf{a}^*\ ,$$

which are closely related to Knapp-Stein intertwining operators of principal series representations. The non-trivial element $w\in W$ acts by $(wf)(-\lambda,k)=(A_\lambda f(\lambda,.))(k)$. Dually, we can identify $S_{log}\mathcal{E}'$ with

$$[\mathcal{S}'(\imath\mathbf{a}^*)\hat\otimes C^{-\infty}(K\times_M V_\gamma)]^W\oplus\oplus_{i=1}^r V'_{i,-\infty}\ .$$

On the Fourier image the operator B acts as multiplication by the polynomial $P(\lambda):=\lambda^2+\rho^2-\gamma(\Omega_M)-\mu$, $\lambda\in\imath\mathbf{a}^*$, on the first summand and as a scalar on $V'_{i,-\infty}$.

It follows that $\mathrm{Fin}^1_\mu(V'_{i,-\infty})=0$. In order to see that

$$\mathrm{Fin}^1_\mu([\mathcal{S}'(\imath\mathbf{a}^*)\hat\otimes C^{-\infty}(K\times_M V_\gamma)]^W)=0$$

we show that the multiplication by P is surjective on $[\mathcal{S}'(\imath\mathbf{a}^*)\hat\otimes C^{-\infty}(K\times_M V_\gamma)]^W$. We consider the dual situation and show that the multiplication by P on $[\mathcal{S}(\imath\mathbf{a}^*)\hat\otimes C^\infty(K\times_M V_\gamma)]^W$ is injective and has a closed range. Injectivity is easy since $P(\lambda)$ is non-invertible at most on a set of codimension 1 in $\imath\mathbf{a}^*\times K/M$. Note that $C^\infty(K\times_M V_\gamma)=\oplus_{\sigma\in\hat M}\oplus_{h=1}^{[\gamma:\sigma]} C^\infty(K\times_M V_\sigma)$ and $P(\lambda)$ acts by the scalar polynomial $P_\sigma(\lambda)=\lambda^2+\rho^2-\sigma(\Omega_M)-\mu$ on each summand $\mathcal{S}'(\imath\mathbf{a}^*)\hat\otimes C^\infty(K\times_M V_\sigma)$. Let Z_σ be the set of zeros of $P_\sigma(\lambda)$. Let $n_\sigma(\lambda)$ be the order of the zero of P_σ at $\lambda\in Z_\sigma$. Then the range of the multiplication by P_σ on $\mathcal{S}(\imath\mathbf{a}^*)\hat\otimes C^\infty(K\times_M V_\sigma)$ consists of all sections $f\in\mathcal{S}(\imath\mathbf{a}^*)\hat\otimes C^\infty(K\times_M V_\sigma)$ such that $f(\lambda, kM)$ has a zero of order n_σ for all $\lambda \in Z_\sigma$ and for all $k \in K$. This vanishing condition is a closed condition. The range of P on $[\mathcal{S}(\imath\mathbf{a}^*)\hat{\otimes}C^\infty(K \times_M V_\gamma)]^W$ consists of all $f \in [\mathcal{S}(\imath\mathbf{a}^*)\hat{\otimes}C^\infty(K \times_M V_\gamma)]^W$ such that $f_\sigma(\lambda, kM)$ has a zero of order $n_\sigma(\lambda)$ for all $\lambda \in Z_\sigma$, $k \in K$ and σ. This condition is again closed.

This finishes the proof of Lemma 2.4 and hence of Theorem 2.1. $\square$

Theorem 2.5 *The operator $B : S_\infty\mathcal{E} \to S_\infty\mathcal{E}$ is surjective.*

Proof. Let $S_\infty\mathcal{E}'(B)$ and C be defined by the following exact sequence:

$$0 \to S_\infty\mathcal{E}'(B) \to S_\infty\mathcal{E}' \stackrel{B}{\to} S_\infty\mathcal{E}' \to C \to 0 \ . \tag{9}$$

Lemma 2.6 $C = 0$.

Proof. We have $BC = 0$ and $BS_\infty\mathcal{E}'(B) = 0$. Hence $\mathrm{Fin}^1_\mu(S_\infty\mathcal{E}'(B)) = 0$, $\mathrm{Fin}_\mu(S_\infty\mathcal{E}'(B)) = S_\infty\mathcal{E}'(B)$, $\mathrm{Fin}^1_\mu(C) = 0$, and $\mathrm{Fin}_\mu(C) = C$. Thus applying Fin_μ to the exact sequence of Fin_μ-acyclic spaces (9) we obtain the exact sequence

$$0 \to S_\infty\mathcal{E}'(B) \to \mathrm{Fin}_\mu(S_\infty\mathcal{E}') \stackrel{B}{\to} \mathrm{Fin}_\mu(S_\infty\mathcal{E}') \to C \to 0 \ .$$

Let $S_\infty\mathcal{E}'(B^k) := \ker(B^k : S_\infty\mathcal{E}' \to S_\infty\mathcal{E}')$. Then

$$\mathrm{Fin}_\mu(S_\infty\mathcal{E}') = \lim_{\overrightarrow{k}} S_\infty\mathcal{E}'(B^k)$$

and in order to show that $C = 0$ it is enough to show that

$$\lim_{\overrightarrow{k}} \mathrm{coker}(B : S_\infty\mathcal{E}'(B^k) \to S_\infty\mathcal{E}'(B^k)) = 0 \ . \tag{10}$$

The space of K-finite vectors $S_\infty\mathcal{E}'(B^k)_K$ is an admissible $(\mathbf{g}, K)$-module and $S_\infty\mathcal{E}'(B^k)$ is its canonical distribution vector globalization ([20], Ch. 11). Since the globalization functor is exact (Wallach [20], Ch. 11, Casselman [5]) to show (10) is suffices to verify that

$$\lim_{\overrightarrow{k}} \mathrm{coker}(B : S_\infty\mathcal{E}'(B^k)_K \to S_\infty\mathcal{E}'(B^k)_K) = 0 \ . \tag{11}$$

Since the multiplication by B on $(\mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{k})} V_{\tilde\gamma})_K$ is injective we obtain by K-type wise algebraic dualization

$$\mathrm{coker}(B : \mathrm{Hom}_{\mathcal{U}(\mathbf{k})}(\mathcal{U}(\mathbf{g}), V_\gamma)_K \to \mathrm{Hom}_{\mathcal{U}(\mathbf{k})}(\mathcal{U}(\mathbf{g}), V_\gamma)_K) = 0 \ .$$

Thus $\mathrm{Hom}_{\mathcal{U}(\mathbf{k})}(\mathcal{U}(\mathbf{g}), V_\gamma)_K$ is Fin_μ-acyclic. Applying Fin_μ to the exact sequence of Fin_μ-acyclic spaces

$$0 \to \mathrm{Hom}_{\mathcal{U}(\mathbf{k})}(\mathcal{U}(\mathbf{g}), V_\gamma)_K(B) \to \mathrm{Hom}_{\mathcal{U}(\mathbf{k})}(\mathcal{U}(\mathbf{g}), V_\gamma)_K \stackrel{B}{\to} \mathrm{Hom}_{\mathcal{U}(\mathbf{k})}(\mathcal{U}(\mathbf{g}), V_\gamma)_K \to 0$$

we obtain the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{U}(\mathbf{k})}(\mathcal{U}(\mathbf{g}), V_\gamma)_K(B) \to \mathrm{Fin}_\mu(\mathrm{Hom}_{\mathcal{U}(\mathbf{k})}(\mathcal{U}(\mathbf{g}), V_\gamma)_K) \stackrel{B}{\to} \mathrm{Fin}_\mu(\mathrm{Hom}_{\mathcal{U}(\mathbf{k})}(\mathcal{U}(\mathbf{g}), V_\gamma)_K) \to 0\ .$$

The exactness at the last place is equivalent to

$$\lim_{\substack{\longrightarrow \\ k}} \mathrm{coker}(B : \mathrm{Hom}_{\mathcal{U}(\mathbf{k})}(\mathcal{U}(\mathbf{g}), V_\gamma)_K(B^k) \to \mathrm{Hom}_{\mathcal{U}(\mathbf{k})}(\mathcal{U}(\mathbf{g}), V_\gamma)_K(B^k)) = 0\ .$$

The claim (11) now follows from $S_\infty\mathcal{E}'(B^k)_K \cong \mathrm{Hom}_{\mathcal{U}(\mathbf{k})}(\mathcal{U}(\mathbf{g}), V_\gamma)_K(B^k)$ (compare [3], Lemma 3.5). □

We have shown that $B : S_\infty\mathcal{E}' \to S_\infty\mathcal{E}'$ is surjective. The G-invariant Hermitian scalar product on E induces an inclusion $S_\infty\mathcal{E} \subset S_\infty\mathcal{E}'$. Theorem 2.5 is now a consequence of the following regularity result.

Lemma 2.7 *If $f \in S_\infty\mathcal{E}'$, $Bf \in S_\infty\mathcal{E}$, then $f \in S_\infty\mathcal{E}$.*

Proof. Let $\tilde{W}$ be a G-invariant fundamental solution of B, i.e., $B\tilde{W} = \tilde{W}B = \mathrm{id}$ on $C_c^\infty(X,E)$. Consider $X \times X$ with the diagonal G-action. Let $\chi \in {}^GC^\infty(X \times X)$ be a G-invariant cut-off function defined by $\chi(x,y) = \rho(\mathrm{dist}(x,y))$, where $\rho \in C^\infty(\mathbf{R})$, $\rho(r) \in [0,1]$, $\rho(r) = 1$ for $r < 1$ and $\rho = 0$ for $r > 2$.

Multiplying the distributional kernel of $\tilde{W}$ by χ we obtain a G-invariant parametrix W of B. It has finite propagation (the support of $W\phi$ is contained in a 1-neighbourhood of the support of ϕ for all $\phi \in C_c^\infty(X,E)$) and is applicable to arbitrary distributions in $C^{-\infty}(X,E)$. Let $f \in S_\infty\mathcal{E}'$ and $Bf = F \in S_\infty\mathcal{E}$. Then $f = WF + Sf$, where S is a G-invariant smoothing operator on E. We must show that $WF, Sf \in S_\infty\mathcal{E}$.

Let $L \in \mathcal{U}(\mathbf{g})$. Then

$$(Sf)(Lg) = \langle f, s(LgK,.)\rangle\ ,$$

where s is the integral kernel of S and $s(LgK,.) \in C_c^\infty(X,E)$. For any $L_1 \in \mathcal{U}(\mathbf{g})$ and $R \in \mathbf{R}$ there is a $P \in \mathbf{R}$ depending on L_1, L such that

$$p_{R,L_1}(s(LgK,.)) = \sup_{h\in G} \mathrm{e}^{R\mathrm{dist}(\mathcal{O},hK)}|s(LgK, L_1hK)| \le C\mathrm{e}^{(P+R)\mathrm{dist}(\mathcal{O},gK)}\ ,$$

where C may depend on L, L_1, R, P but not on g. In fact, for all $A, B \in \mathcal{U}(\mathbf{g})$ we have $\sup_{g,h\in G}|s(gA, hB)| < \infty$ since S is G-invariant. If $f \in S_R\mathcal{E}'$ for some R, then there is a finite set $\{L_1, \dots, L_r\} \subset \mathcal{U}(\mathbf{g})$ such that for all $L \in \mathcal{U}(\mathbf{g})$ there exists $P \in \mathbf{R}$ with

$$\begin{aligned} p_{-(R+P),L}(Sf) &= \sup_{g\in G} \mathrm{e}^{-(R+P)\mathrm{dist}(\mathcal{O},gK)}\langle f, s(LgK,.)\rangle \\ &\le C(f) \sup_{g\in G, i=1,\dots,r} \mathrm{e}^{-(R+P)\mathrm{dist}(\mathcal{O},gK)} p_{R,L_i}(s(LgK,.)) \\ &< \infty\ . \end{aligned}$$

This proves $Sf \in S_\infty\mathcal{E}$.

We now show that $WF \in S_\infty\mathcal{E}$. Let $L \in \mathcal{U}(\mathbf{g})$. We must show that for some $R \in \mathbf{R}$

$$p_{R,L}(WF) = \sup_{g\in G} \mathrm{e}^{R\mathrm{dist}(\mathcal{O},gK)}|(WF)(LgK)| < \infty \ .$$

Let $w(gK, hK)$ be the distributional kernel of W. The family $gK \mapsto w(LgK, .)$ is a smooth family of distributions such that $\mathrm{supp}(w(LgK, .))$ is contained in the unit ball in X with center at gK. Thus we can write

$$(WF)(LgK) = \langle w(LgK, .), F\rangle \ .$$

Since W is G-invariant there is a finite set $\{L_1, \dots, L_r\} \subset \mathcal{U}(\mathbf{g})$ such that we have

$$|(WF)(LgK)| \le \mathrm{e}^{P\mathrm{dist}(\mathcal{O},gK)} \sup_{h\in G, \mathrm{dist}(hK,gK)\le 1, i=1,\dots,r} |F(hL_iK)|$$

for some $P \in R$ depending on L. Hence there is another finite set $\{L'_1, \dots, L'_{r'}\} \subset \mathcal{U}(\mathbf{g})$ and an exponent $Q \in \mathbf{R}$ such that

$$\begin{aligned} |(WF)(LgK)| &\le \mathrm{e}^{Q\mathrm{dist}(\mathcal{O},gK)} \sup_{h\in G, \mathrm{dist}(hK,gK)\le 1, i=1,\dots,r'} |F(L'_i hK)| \\ &\le \mathrm{e}^{(Q-R)\mathrm{dist}(\mathcal{O},gK)} \sup_{i=1,\dots,r'} p_{R,L'_i}(F) \end{aligned}$$

Hence $p_{-(Q-R),L}(WF) < \infty$ if R is small enough.

We conclude that $WF \in S_\infty\mathcal{E}$. This finishes the proof of the Lemma and of Theorem 2.5. □

3 De Rham complexes

In this section we prove the local acyclicity of the weighted de Rham complex on X twisted with the functions of moderate growth on G, the global acyclicity of the weighted de Rham complex on X twisted with the functions of moderate growth on $\Gamma\backslash G$, and the $\mathbf{n}$-acyclicity of $S_\infty\mathcal{E}$ for a homogeneous vector bundle $E \to X$. A complex is called acyclic if it is exact in all non-zero degrees.

Lemma 3.1 *For any vector bundle $E \to X$ the space $S_\infty\mathcal{E}$ is $\mathbf{n}$-acyclic.*

Proof. Fix a basis $\{X_i\}_{i=1}^{\dim(\mathbf{n})}$ of $\mathbf{n}$. Let $\{X^i\}$ be the basis of $\mathbf{n}^*$ dual to $\{X_i\}$. Let $Xn \in T_nN$, $X \in \mathbf{n}$, $n \in N$, be the fundamental vector at n corresponding to $-X$.

Let I_p be the set of all multi-indices $\{1 \le i_1 < i_2 < \dots < i_p \le \dim(\mathbf{n})\}$. For $I \in I_p$ we define $X^I n \in \Lambda^p T_n^* N$ by $X^I n(X_J n) = \delta_{IJ}$, $\forall J \in I_p$. Here $X_J n = X_{j_1} n \wedge \dots \wedge X_{j_p} n$. Now identify $N \times A \stackrel{\sim}{\to} X$, $(n, a) \mapsto naK$. We also employ the $\mathbf{n}$-equivariant trivialization $G \times_K V_\gamma = E \stackrel{\sim}{\to} N \times A \times V_\gamma$, $[nak, v] \mapsto (n, a) \times \gamma^{-1}(k)v$. We identify $C^\infty(N, \Lambda^* T^* N \otimes C^\infty(A) \otimes V_\gamma)$ with $\Lambda^*\mathbf{n}^* \otimes \mathcal{E}$ such that $\omega \in C^\infty(N, \Lambda^p T^* N \otimes C^\infty(A) \otimes V_\gamma)$ corresponds to $(X_I, na) \to \omega(na)(X_I n)$, $I \in I_p$, $n \in N$, $a \in A$, $\omega(na)(X_I n) \in V_\gamma$. Under this identification the $\mathbf{n}$-cohomology complex of $\mathcal{E}$ becomes the de Rham complex of N twisted with $C^\infty(A)\otimes V_\gamma$.

Using the trivialization $\{X^I n\}_{I\in I_p}$ of the p-form bundle we write $\omega(na)=\sum_{I\in I_p}\omega_I(na)X^I n$. The subspace $\Lambda^p\mathbf{n}\otimes S_\infty\mathcal{E}$ is identified with

$$\begin{aligned}\Lambda^p\mathbf{n}\otimes S_\infty\mathcal{E} &= \{\omega\in C^\infty(N,\Lambda^pT^*N\otimes C^\infty(A)\otimes V_\gamma)\\ &\quad \mid \forall L_1\in\mathcal{U}(\mathbf{n}),L_2\in\mathcal{U}(\mathbf{a})\ \ \exists R\in\mathbf{R}\\ &\quad s.t.\ \sup_{na\in NA,I\in I_p}\mathrm{e}^{-R\mathrm{dist}(\mathcal{O},naK)}|\omega_I(L_1n,L_2a)|<\infty\}\ .\end{aligned}$$

We define the contraction Ψ_t, $t\in[0,1]$, of N by $\Psi_t(n):=\exp(t\log(n))$. Let $T_t:=\frac{d}{ds}_{|s=0}\Psi_{t+s}(n)\in T_{\Psi_t(n)}N$. We set $H_t\omega:=\Psi_t^*(i_{T_t}\omega)$ and $H=\int_0^1H_tdt$. Then

$$H:C^\infty(N,\Lambda^pT^*N\otimes C^\infty(A)\otimes V_\gamma)\to C^\infty(N,\Lambda^{p-1}T^*N\otimes C^\infty(A)\otimes V_\gamma)$$

is a zero homotopy of the de Rahm complex $(C^\infty(N,\Lambda^*T^*N\otimes C^\infty(A)\otimes V_\gamma),d)$. In order to prove the Lemma we have to show that this zero homotopy is compatible with the subspaces $\Lambda^*\otimes S_\infty\mathcal{E}$. It is enough to show that

$$t\to H_t\in\mathrm{Hom}^{cont}_{\mathbf{C}}(\Lambda^p\otimes S_\infty\mathcal{E},\Lambda^{p-1}\otimes S_\infty\mathcal{E})$$

is continuous. Here we equip Hom^{cont} with the strong topology (pointwise convergence). We call a function f on N a polynomial, if $f(\exp(n))$ is a polynomial on $\mathbf{n}$. Using the fact that N is nilpotent, one can easily show that

$$(H_t\omega)(n)=\sum_{J\in I_{p-1},I\in I_p}\Phi_{I,J}(t,n)\omega_I(\Psi_t(n))X^Jn\ ,$$

where $n\to\Phi_{I,J}(t,n)$ are polynomial functions on N (and in fact also polynomials in t). Let L_r be the set of tuples $\{i_1,\dots,i_{\dim(\mathbf{n})}\}$, $i_k\in\mathbf{N}_0$, with $\sum_{k=1}^{\dim(\mathbf{n})}i_k\le r$. Let $X(l):=X_1^{l_1}\dots X_{\dim(\mathbf{n})}^{l_{\dim(\mathbf{n})}}\in\mathcal{U}(\mathbf{n})$, $l\in L_r$. By $X(l)n$ we denote the corresponding differential operator $(X_1n)^{l_1}\dots(X_{\dim(\mathbf{n})}n)^{l_{\dim(\mathbf{n})}}$. If $L_1\in\mathcal{U}(\mathbf{n})$, $\deg(L_1)=r$, $L_2\in\mathcal{U}(\mathbf{a})$, then also

$$(H_t\omega)_J(L_1n,L_2a)=\sum_{l\in L_r,I\in I_p}\Phi_{l,I,J}(t,n)\omega_I(X(l)\Psi_t(n),L_2a)$$

with polynomial functions $\Phi_{l,I,J}(t,n)$. We obtain the estimate

$$\sup_{J\in I_{p-1}}|(H_t\omega)_J(L_1n,L_2a)|\le C(1+|\log(n)|)^P\sup_{l\in L_r,I\in I_p}|\omega_I(X(l)\Psi_t(n),L_2a)|\ ,$$

where P is sufficiently large. Note that $\mathrm{e}^{Q\mathrm{dist}(\mathcal{O},naK)}\ge c(1+|\log(n)|)^P$ for some $Q,c>0$ and all $na\in NA$. We also have $\mathrm{dist}(\mathcal{O},naK)\ge\mathrm{dist}(\mathcal{O},\Psi_t(n)aK)$ for all $t\in[0,1]$. Fix $R\in\mathbf{R}$. Then for all ω with

$$\sup_{l\in L_r,na\in NA,I\in I_p}\mathrm{e}^{-R\mathrm{dist}(\mathcal{O},naK)}|\omega_I(X(l)n,L_2a)|=:S(\omega)<\infty$$

we have

$$\sup_{na\in NA, J\in I_{p-1}} \mathrm{e}^{-(R+Q)\mathrm{dist}(\mathcal{O},naK)}|(H_t\omega)_J(L_1n, L_2a)| < CS(\omega)$$

with $C<\infty$ independent of ω. These estimates imply that if $\omega\in\Lambda^p\mathbf{n}\otimes S_u\mathcal{E}$, $u\in\mathcal{I}$, then $H_t\omega\in\Lambda^{p-1}\mathbf{n}\otimes S_{u+Q}\mathcal{E}$ and depends continuously on $t\in[0,1]$. This proves the required continuity of H_t. Thus H provides a zero homotopy of the $\mathbf{n}$-cohomology complex for $S_\infty\mathcal{E}$, and the lemma is proved. $\square$

Fix an Iwasawa decomposition $G=NAK$ and identify $X\cong NA$. Using these coordinates we attach a boundary $\partial X = N\times\{\infty\}$ to X obtaining $\bar X$. Note that this boundary is different from the geodesic boundary. The motivation for this definition is that small neighbourhoods of $x\in\partial X$ look like small neighbourhoods of points in the boundary of the Borel-Serre compactification of quotients $Y=\Gamma\backslash X$ of finite volume.

We want to twist the de Rham complex of X with the space of functions of moderate growth on G. Let $\pi: X\times G\to X$ denote the projection onto the first factor. Let $L^p:=\Lambda^pT^*X$. We define

$$\begin{aligned} S_\infty\mathcal{L}^p[G] \;:=\; & \{\omega\in C^\infty(X\times G,\pi^*L^*) \\ & \mid \forall L_1,L_2\in\mathcal{U}(\mathbf{g})\ \ \exists R,Q\in\mathbf{R} \\ & s.t.\ \sup_{g,h\in G} \mathrm{e}^{-R\mathrm{dist}(\mathcal{O},gK)}\mathrm{e}^{-Q\mathrm{dist}(\mathcal{O},hK)}|\omega(L_1g,L_2h)|<\infty\}\ , \end{aligned}$$

where we consider ω as a function from $G\times G$ with values in $\Lambda^pT^*_{\mathcal{O}}X$.

There is a natural sheafification $\underline{S_\infty\mathcal{L}^*[G]}$ on $\bar X$ such that for an open set $U\subset\bar X$ the vector space $\underline{S_\infty\mathcal{L}^*[G]}(U)$ consists of those forms $\omega\in C^\infty(U\times G,\pi^*L^*)$ which satisfy

$$\sup_{g\in U, h\in G}|\omega(L_1g,L_2h)|\mathrm{e}^{-R\mathrm{dist}(\mathcal{O},gK)}\mathrm{e}^{-Q\mathrm{dist}(\mathcal{O},hK)}<\infty$$

for any $L_1,L_2\in\mathcal{U}(\mathbf{g})$ and appropriate $Q,R\in\mathbf{R}$ which may depend on L_1,L_2. Let d be the differential of the de Rham complex acting trivially with respect to the second variable $g\in G$.

Lemma 3.2 *The complex of sheaves $(\underline{S_\infty\mathcal{L}^*[G]},d)$ is acyclic.*

Proof. Let $x\in\bar X$. Then we have to show that the complex of germs $(\underline{S_\infty\mathcal{L}^*[G]}_x,d)$ is acyclic. If $x\in X$, then we employ the standard homotopy formula associated to the radial contraction of small balls in X with center at x. We leave to the reader to verify that the zero homotopy is compatible with the growth conditions with respect to second variable. We discuss a similar problem in the proof of Lemma 3.3.

It remains to consider $x\in\partial X$. Without loss of generality we can assume $x=e\times\{\infty\}$, where $e\in N$ is the identity. Let $U_i\subset\mathbf{n}$, $i\in\mathbf{N}$, be a fundamental system of balls around 0. Then $W_i:=\exp(U_i)(i,\infty]\subset NA$ is a fundamental system of neighbourhoods of x. Here we identified $\mathbf{a}\cong\mathbf{R}$ such that $(i,\infty]=\exp((i,\infty))\cup\{\infty\}$. Let da be the one-form dual to the fundamental vector field H^* on A corresponding to $H\in\mathbf{a}^+$ with $|H|=1$. We decompose forms ω in $\mathcal{L}^p$ as $\omega = \omega_1 \oplus da \wedge \omega_2$, where $i_{H^*}\omega_j = 0$, $j = 1, 2$, and i_{H^*} is the insertion of H^*. One can check that

$$\underline{S_\infty \mathcal{L}^p[G]}(W_i) = V^p(W_i) \oplus da \wedge V^{p-1}(W_i) \ ,$$

where

$$\begin{aligned} V^p(W_i) \ := \ & \{\omega \in C^\infty(U_i \times (i,\infty) \times G, \pi_1^* \Lambda^p T^* U_i) | \\ & \forall L_1 \in \mathcal{U}(\mathbf{n}), L_2 \in \mathcal{U}(\mathbf{a}), L_3 \in \mathcal{U}(\mathbf{g}) \ \ \exists R, Q \in \mathbf{R} \\ & s.t. \quad \sup_{a \in (i,\infty), h \in G} a^{-R} \mathrm{e}^{-Q \mathrm{dist}(\mathcal{O}, hK)} |\omega(L_1 n, L_2 a, L_3 h)| < \infty\} \ , \end{aligned}$$

and $\pi_1 : U_i \times (i, \infty) \times G \to U_i$ is the projection onto the first factor. The complex $(\underline{S_\infty \mathcal{L}^*[G]}(W_i), d)$ is the total complex of a double complex. The latter consists of two rows equal $(V^*(W_i), d)$, where d is the differential of the de Rham complex of U_i. The vertical differential of the double complex is the differentiation along the A-direction given by $\pm H$. The balls U_i can be contracted radially. The associated zero homotopy of the de Rham complex of U_i extends to $V^*(W_i)$. Again we leave the verification to the reader (see also the proof of Lemma 3.1). Thus the rows $(V^*(W_i), d)$ of the double complex are acyclic. The zeroth horizontal cohomology is equal to ${}^{\mathbf{n}}V^0(W_i)$, the functions in $V^0(W_i)$ that do not depend on $n \in U_i$. To finish the proof of the Lemma we must show that $H : {}^{\mathbf{n}}V^0(W_i) \to {}^{\mathbf{n}}V^0(W_i)$ is surjective. In fact the equation $Hf = g$, $g \in {}^{\mathbf{n}}V^0(W_i)$, can explicitly be solved by integration such that $f \in {}^{\mathbf{n}}V^0(W_i)$. Set

$$f(n, a, h) = \int_i^a g(n, b, h) db \ .$$

□

Let $\pi : X \times \Gamma \backslash G$ be the projection onto the first factor. We define

$$\begin{aligned} S_\infty \mathcal{L}^p[\Gamma \backslash G] \ := \ & \{\omega \in C^\infty(X \times \Gamma \backslash G, \pi^* L^*) \\ & | \ \forall L_1, L_2 \in \mathcal{U}(\mathbf{g}) \ \ \exists R, Q \in \mathbf{R} \\ & s.t. \ \sup_{g,h \in G} \mathrm{e}^{-R \mathrm{dist}(\mathcal{O}, gK)} \mathrm{e}^{-Q \mathrm{dist}_Y(\Gamma\mathcal{O}, \Gamma hK)} |\omega(L_1 g, L_2 h)| < \infty\} \ , \end{aligned}$$

where $\mathrm{dist}_Y(\Gamma\mathcal{O}, \Gamma hK)$ is the distance of ΓhK from $\Gamma\mathcal{O}$ in $Y := \Gamma \backslash X$.

Lemma 3.3 *The de Rham complex $(S_\infty \mathcal{L}^*[\Gamma \backslash G], d)$ is acyclic.*

Proof. As in the proof of Lemma 3.2 the de Rham complex $(S_\infty \mathcal{L}^*[\Gamma \backslash G], d)$ is the total complex of a double complex consisting of two rows each equal to the $\mathbf{n}$-cohomology complex $(\Lambda^* \mathbf{n} \otimes S_\infty \mathcal{L}^0[\Gamma \backslash G], d)$. Let $\pi_1 : N \times A \times \Gamma \backslash G \to N$ be the projection onto the first factor. We identify $\Lambda^p \mathbf{n} \otimes S_\infty \mathcal{L}^0[\Gamma \backslash G]$ with

$$\begin{aligned} & \{\omega \in C^\infty(N \times A \times \Gamma \backslash G, \pi_1^* \Lambda^p T^* N) \\ & | \ \forall L_1 \in \mathcal{U}(\mathbf{n}), L_2 \in \mathcal{U}(\mathbf{a}), L_3 \in \mathcal{U}(\mathbf{g}) \ \ \exists R, Q \in \mathbf{R} \\ & s.t. \quad \sup_{na \in NA, g \in G, I \in I_p} \mathrm{e}^{-R \mathrm{dist}(\mathcal{O}, naK)} \mathrm{e}^{-Q \mathrm{dist}_Y(\Gamma\mathcal{O}, \Gamma gK)} |\omega_I(L_1 n, L_2 a, L_3 g)| < \infty\} \ . \end{aligned}$$

We show that the complex $(\Lambda^*\mathbf{n}\otimes S_\infty\mathcal{L}^0[\Gamma\backslash G],d)$ is acyclic. In the proof of Lemma 3.1 we constructed a zero homotopy $H=\int_0^1 H_t$, where $(H_t\omega)(n,a,g)=\Psi_t^*(i_{T_t}\omega(\Psi_t(n),a,g))$.

We claim that $H:\Lambda^p\mathbf{n}\otimes S_\infty\mathcal{L}^0[\Gamma\backslash G]\to\Lambda^{p-1}\mathbf{n}\otimes S_\infty\mathcal{L}^0[\Gamma\backslash G]$. The same discussion as in the proof of Lemma 3.1 leads to the following estimate. Fix $L_1\in\mathcal{U}(\mathbf{n})$, $L_2\in\mathcal{U}(\mathbf{a})$, $L_3\in\mathcal{U}(\mathbf{g})$. Let $r:=\deg(L_1)$. For all ω with

$$\sup_{l\in L_r,na\in NA,g\in G,I\in I_p}\mathrm{e}^{-P\mathrm{dist}_Y(\Gamma\mathcal{O},\Gamma gK)}\mathrm{e}^{-R\mathrm{dist}(\mathcal{O},naK)}|\omega_I(X(l)n,L_2a,L_3g)|=:S(\omega)<\infty$$

we have

$$\sup_{na\in NA,g\in G,J\in I_{p-1}}\mathrm{e}^{-P\mathrm{dist}_Y(\Gamma\mathcal{O},\Gamma gK)}\mathrm{e}^{-(R+Q)\mathrm{dist}(\mathcal{O},naK)}|(H_t\omega)_J(L_1n,L_2a,L_3g)|<CS(\omega)$$

with $C<\infty$ independent of ω. This easily implies the claim. Hence the rows of the double complex are acyclic and their zeroth cohomology is equal to

$$\begin{aligned}V\ :=\ &\{f\in C^\infty(A\times\Gamma\backslash G)\mid\forall L_2\in\mathcal{U}(\mathbf{a}),L_3\in\mathcal{U}(\mathbf{g})\ \ \exists R,Q\in\mathbf{R}\\&\qquad s.t.\ \sup_{a\in A,g\in G}\mathrm{e}^{-R|\log(a)|}\mathrm{e}^{-Q\mathrm{dist}_Y(\Gamma\mathcal{O},\Gamma gK)}|f(L_2a,L_3g)|<\infty\}\ .\end{aligned}$$

The vertical differential given by $H:V\to V$ is surjective. In fact, let $f\in V$ and set

$$F(a,g):=\int_1^a f(b,g)db\ .$$

Then $HF=f$ and $F\in V$. This proves the lemma. $\square$

4 The standard resolution

For the convenience of the reader we repeat here the construction of the standard resolution given in [3].

Let $(\pi,V_{\pi,K})\in\mathcal{HC}(\mathbf{g},K)$ be a Harish-Chandra module. Then $V_{\pi,K}$ decomposes into a direct sum of joint generalized eigenspaces of $\mathcal{Z}(\mathbf{g})$. Since the summands can be treated separately, without loss of generality we can assume that there exist $\mu\in\mathbf{C}$ and $k\in\mathbf{N}$ such that $B:=(\Omega-\mu)^k\in\mathrm{Ann}(V_{\pi,K})$, i.e., $BV_{\pi,K}=0$.

Let W be a finite-dimensional K-stable subspace of the dual $V_{\tilde\pi,K}$ of $V_{\pi,K}$ in the category $\mathcal{HC}(\mathbf{g},K)$, which generates $V_{\tilde\pi,K}$ as a $\mathcal{U}(\mathbf{g})$-module. Let $E_0\to X$ be the homogeneous vector bundle $G\times_K\tilde W$ and $\mathcal{E}_0$ be the space of its smooth sections. Using any globalization V_π of $V_{\pi,K}$ (i.e. a representation of G such that $V_\pi=V_{\pi,K}$) we can define an embedding

$$i:V_{\pi,K}\hookrightarrow\mathcal{E}_0\cong[C^\infty(G)\otimes\tilde W]^K$$

that is characterized by

$$\langle i(v)(g),w\rangle:=\langle w,\pi(g^{-1})v\rangle,\qquad v\in V_{\pi,K},w\in W,g\in G.$$

In fact i maps into $S_\infty\mathcal{E}_0$ and the closure of $i(V_{\pi,K})$ in $S_\infty\mathcal{E}_0$ is contained in $S_\infty\mathcal{E}_0(B)$ and constitutes the distribution vector globalization $V_{\pi,-\infty}$ of $V_{\pi,K}$ (Wallach [20], Ch. 11, Casselman [5]).

We will also consider the space $V_{\pi,for} := V^*_{\tilde{\pi},K}$ of formal power series vectors of $V_{\pi,K}$. There is an exact functor from $\mathcal{HC}(\mathbf{g},K)$ to the category of (not necessarily K-finite) $(\mathbf{g},K)$-modules which sends $V_{\pi,K}$ to $V_{\pi,for}$. Note that $V_{\pi,for} = \prod_{\gamma\in\hat{K}} V_{\pi,K}(\gamma)$.

For homogeneous vector bundles E and F on X we denote by $D(E,F)$ the set of G-invariant differential operators $D : \mathcal{E} \to \mathcal{F}$.

Proposition 4.1 *There exist homogeneous vector bundles $E_1, E_2, \ldots$ on X and G-invariant differential operators $D_i \in D(E_i, E_{i+1})$, $i = 0,1,\ldots$, such that the embedding $i : V_{\pi,-\omega} \hookrightarrow \mathcal{E}_0(B)$ can be extended to a (possibly infinite) exact sequence*

$$0 \to V_{\pi,-\infty} \xrightarrow{i} S_\infty\mathcal{E}_0(B) \xrightarrow{D_0} S_\infty\mathcal{E}_1(B) \xrightarrow{D_1} S_\infty\mathcal{E}_2(B) \xrightarrow{D_2} \ldots \ . \tag{12}$$

This sequence remains to be exact on the level of formal power series :

$$0 \to V_{\pi,for} \xrightarrow{i} \mathcal{E}_0^{for}(B) \xrightarrow{D_0} \mathcal{E}_1^{for}(B) \xrightarrow{D_1} \mathcal{E}_2^{for}(B) \xrightarrow{D_2} \ldots \ . \tag{13}$$

Proof. Let $\mathcal{Z}(E)$ be the image of $\mathcal{Z}(\mathbf{g})$ in $D(E,E)$. For any vector bundle $E \to X$ the $\mathbf{C}[B]$-module $\mathcal{Z}(E)$ is finitely generated ([3], Lemma 2.3).

Lemma 4.2 *For any vector bundle $E \to X$ we have $\mathcal{E}(B)_K \in \mathcal{HC}(\mathbf{g},K)$.*

Proof. Let (γ, V_γ) be the finite dimensional representation of K corresponding to E and $(\tilde{\gamma}, V_{\tilde{\gamma}})$ be its dual. We consider the K-equivariant embedding

$$i : V_{\tilde{\gamma}} \hookrightarrow \widetilde{\mathcal{E}(B)}_K$$

defined by

$$i(\tilde{v})(f) := \langle \tilde{v}, f(e)\rangle \ ,$$

where we identify the fibre of E at $e = [K]$ with V_γ. Let $T := \mathcal{U}(\mathbf{g})(i(V_{\tilde{\gamma}}))$. For any $t \in T$ the dimension of $\mathcal{Z}(\mathbf{g})t$ can be estimated by the dimension of a generating subspace of the $\mathbf{C}[B]$-module $\mathcal{Z}(E)$. Thus T is a locally $Z(\mathbf{g})$-finite and finitely generated $\mathcal{U}(\mathbf{g})$-module. By a theorem of Harish-Chandra ([19], 3.4.7), $T \in \mathcal{HC}(\mathbf{g},K)$. The canonical map $\mathcal{E}(B)_K \to \tilde{T}$ is injective by the analyticity of solutions of the equation $Bf = 0$. In fact, an element in the kernel of this map would have a vanishing Taylor series at e. We obtain that $T \hookrightarrow \widetilde{\mathcal{E}(B)}_K$ is surjective. Thus $T = \widetilde{\mathcal{E}(B)}_K$ and $\mathcal{E}(B)_K \in \mathcal{HC}(\mathbf{g},K)$ since the dual of a Harish-Chandra module is a Harish-Chandra module, too ([19], 4.3.2). □

Lemma 4.3 *Let $V_{\pi,K}$ be a Harish-Chandra submodule of $\mathcal{E}(B)_K$. Then there exist a homogeneous vector bundle F and an operator $D \in D(E,F)$ such that* $\ker D \cap \mathcal{E}(B)_K = V_{\pi,K}$. *We also have* $\ker D \cap S_\infty\mathcal{E}(B) = V_{\pi,-\infty}$.

Proof. According to the proof of Lemma 4.2 there is a surjection

$$\mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{k})} V_{\tilde{\gamma}} \to \mathcal{E}(\widetilde{B})_K \ .$$

Let W be a finite-dimensional K-stable generating subspace of the Harish-Chandra module $V^{\perp}_{\pi,K} \subset \mathcal{E}(\widetilde{B})_K$. Then we choose a K-equivariant map α such that the following diagram

$$\begin{array}{ccc} W & \xrightarrow{\alpha} & \mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{k})} V_{\tilde{\gamma}} \\ \downarrow & & \downarrow \\ V^{\perp}_{\pi,K} & \longrightarrow & \mathcal{E}(\widetilde{B})_K \end{array}$$

commutes. This is possible since $\mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{k})} V_{\tilde{\gamma}}$ is K-semisimple.

We set $F := G \times_K \tilde{W}$. The map α can be considered as an element of

$$[\mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{k})} V_{\tilde{\gamma}} \otimes \tilde{W}]^K \cong [\mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{k})} \mathrm{Hom}(V_\gamma, \tilde{W})]^K \ .$$

The latter space is canonically isomorphic to $D(E,F)$ via the right regular representation R of $\mathcal{U}(\mathbf{g})$ on $C^\infty(G) \otimes V_\gamma$. Thus α defines an element $D \in D(E,F)$. If $\alpha(w) = \sum X_i \otimes v_i$, then

$$\langle w, Df \rangle_F = \sum \langle v_i, R_{X_i} f \rangle_E \in C^\infty(G), \ \ w \in W, v_i \in V_{\tilde{\gamma}}, X_i \in \mathcal{U}(\mathbf{g}) \ .$$

Let $f \in \mathcal{E}(B)$, $X \in \mathcal{U}(\mathbf{g})$ and $w \in W$. Then we have

$$\begin{aligned} \langle w, L_X D f(1) \rangle_F &= \langle w, D L_X f \rangle_F \\ &= \sum \langle v_i, R_{X_i} L_X f(1) \rangle_E \\ &= \langle w, L_X f \rangle_{\mathcal{E}(B)} \\ &= \langle L_{X^{op}} w, f \rangle_{\mathcal{E}(B)} \ , \end{aligned} \qquad (14)$$

where $X \to X^{op}$ is the anti-automorphism of $\mathcal{U}(\mathbf{g})$ induced by the multiplication with -1 on $\mathbf{g}$. By construction $Df = 0$ iff the left hand side of (14) vanishes for all $X \in \mathcal{U}(\mathbf{g})$ and $w \in W$, while $f \in V_{\pi,-\infty}$ iff the right hand side does. The lemma follows. □

In order to construct the bundles E_i and operators D_i of Proposition 4.1 we iterate Lemma 4.3. $D_i(\mathcal{E}_i(B)_K)$ is a Harish-Chandra submodule of $\mathcal{E}_{i+1}(B)_K$. Therefore we find a bundle E_{i+2} and an operator $D_{i+1} \in D(E_{i+1}, E_{i+2})$ such that $\ker D_{i+1} \cap \mathcal{E}_{i+1}(B)_K = D_i(\mathcal{E}_i(B)_K)$. We obtain an exact sequence of Harish-Chandra modules

$$0 \to V_{\pi,K} \xrightarrow{i} \mathcal{E}_0(B)_K \xrightarrow{D_0} \mathcal{E}_1(B)_K \xrightarrow{D_1} \mathcal{E}_2(B)_K \xrightarrow{D_2} \dots \ .$$

Applying the distribution vector globalization functor (which is exact) we end up with (12). Analogously, we want to obtain (13) by taking formal power series vectors. This is possible since for any homogeneous vector bundle E we have

$$(\mathcal{E}(B)_K)_{for} = \mathcal{E}^{for}(B)$$

([3], Lemma 3.5).

The Proposition 4.1 provides a resolution of $V_{\pi,-\infty}$ by spaces $S_\infty \mathcal{E}_i(B)$. We now employ a Koszul complex construction in order to get rid of the eigenspaces. We recall the following fact from [3].

Lemma 4.4 *Let E, F be homogeneous vector bundles on X and $A \in D(E,F)$ such that $A\mathcal{E}(B) = 0$. Then $A = HB$ for some $H \in D(E,F)$.*

Let $V_{\pi,K}, E_i, D_i$ be constructed as in Proposition 4.1.

Proposition 4.5 *There exist $H_i \in D(E_i, E_{i+2})$, $i \geq 0$, making the following into an exact complex:*

$$0 \to V_{\pi,-\infty} \to S_\infty\mathcal{E}_0 \overset{\begin{pmatrix} D_0 \\ B \end{pmatrix}}{\longrightarrow} \begin{matrix} S_\infty\mathcal{E}_1 \\ \oplus \\ S_\infty\mathcal{E}_0 \end{matrix} \overset{\begin{pmatrix} D_1 & H_0 \\ -B & D_0 \end{pmatrix}}{\longrightarrow} \begin{matrix} S_\infty\mathcal{E}_2 \\ \oplus \\ S_\infty\mathcal{E}_1 \end{matrix} \overset{\begin{pmatrix} D_2 & H_1 \\ B & D_1 \end{pmatrix}}{\longrightarrow} \dots \, . \tag{15}$$

We shall call (15) a standard resolution of $V_{\pi,-\infty}$.
Proof. In order to construct the operators H_i we apply Lemma 4.4 for $A = D_{i+1}D_i$. The exactness of (15) is easily reduced to the exactness of (12) and the surjectivity of $B : S_\infty\mathcal{E}_i \to S_\infty\mathcal{E}_i$ proved in Theorem 2.5. □

5 n-cohomology

Let $B = (\Omega_G - \lambda)^l$ for some $\lambda \in \mathbf{C}$, $l \in \mathbf{N}$, where Ω_G is the Casimir operator of G, and $S_\infty\mathcal{E}(B) = \{f \in S_\infty\mathcal{E} \mid Bf = 0\}$.

Lemma 5.1 *We have*

$$H^p(\mathbf{n}, S_\infty\mathcal{E}(B)) = 0, \quad \forall p \geq 1.$$

Proof. By Theorem 2.5 and Lemma 3.1

$$0 \to S_\infty\mathcal{E}(B) \to S_\infty\mathcal{E} \overset{B}{\to} S_\infty\mathcal{E} \to 0$$

is an $\mathbf{n}$-acyclic resolution of $S_\infty\mathcal{E}(B)$. Taking $\mathbf{n}$-invariants we obtain the complex

$$0 \to {}^{\mathbf{n}}S_\infty\mathcal{E}(B) \to S_\infty C^\infty(A) \otimes V_\gamma \overset{{}^{\mathbf{n}}B}{\to} S_\infty C^\infty(A) \otimes V_\gamma \to 0 \, . \tag{16}$$

Here ${}^{\mathbf{n}}B$ is the restriction of B to the subspace of $\mathbf{n}$-invariant vectors. It is a second order translation invariant differential operator on A. The complex (16) is again exact and the Lemma follows. □

Let $(\pi, V_{\pi,K}) \in \mathcal{HC}(\mathbf{g}, K)$. Recall that $H^*(\mathbf{n}, V_{\pi,-\infty})$ carries a natural MA-module structure.

Theorem 5.2 *The inclusion $V_{\pi,-\infty} \hookrightarrow V_{\pi,for}$ induces an isomorphism*

$$H^p(\mathbf{n}, V_{\pi,-\infty}) \overset{\sim}{\longrightarrow} H^p(\mathbf{n}, V_{\pi,for}) \, .$$

Moreover $H^p(\mathbf{n}, V_{\pi,-\infty}) = H^p(\mathbf{n}, V_{\pi,-\omega}) = H^p(\mathbf{n}, V_{\pi,for})$ and all spaces are finite dimensional. The $\mathbf{n}$-cohomology of $V_{\pi,-\infty}$ satisfies Poincaré duality

$$H^p(\mathbf{n}, V_{\pi,-\infty})^* \cong H^{\dim(\mathbf{n})-p}(\mathbf{n}, V_{\tilde{\pi},\infty}) \otimes \Lambda^{\dim \mathbf{n}}\mathbf{n}. \tag{17}$$

We also have

$$H^p(\mathbf{n}, V_{\pi,\infty}) \cong H^p(\mathbf{n}, V_{\pi,K}).$$

Proof. By ([3], Lemma 2.3 and Proposition 4.1), Lemma 3.1 and Proposition 4.1 the cohomology $H^p(\mathbf{n}, V_{\pi,*})$ for $* = -\infty$, for is isomorphic to the cohomology of the subcomplex of $\mathbf{n}$-invariants of (12), (13), respectively. Hence the following lemma implies the first assertion of the theorem.

Lemma 5.3 *For any homogeneous vector bundle $E \to X$ associated to V_γ we have*

$$^{\mathbf{n}}S_\infty\mathcal{E}(B) = {}^{\mathbf{n}}\mathcal{E}^{for}(B) \ .$$

Proof. The $\mathcal{U}(\mathbf{a})$-module

$$^{\mathbf{n}}\mathcal{E}^{for}(B) \cong (\mathcal{E}(\widetilde{B})_K / \mathbf{n}(\mathcal{E}(\widetilde{B})_K))^*$$

is finite dimensional (see [19], Ch.4). Therefore it splits into generalized weight spaces $^{\mathbf{n}}\mathcal{E}^{for}(B)_\mu, \mu \in \mathbf{a}^*_{\mathbf{C}}$. $f \in {}^{\mathbf{n}}\mathcal{E}^{for}(B)_\mu$, considered as a formal power series on $\mathbf{a}$, satisfies the differential equation

$$(H + \mu(H))^k f = 0 \quad \forall H \in \mathbf{a} \tag{18}$$

for a certain $k \in \mathbf{N}$. The solutions of (18) have the form

$$P(H)\mathrm{e}^{-\mu(H)}, \qquad P \in S(\mathbf{a}^*) \otimes V_\gamma \ .$$

They extend to smooth $\mathbf{n}$-invariant sections in $^{\mathbf{n}}S_\infty\mathcal{E}(B)$. □

The proof of the remaining assertions of the theorem is parallel to the proofs of the corresponding facts in [3]. □

6 $S_\infty\mathcal{E}$ is Γ-acyclic

Let $\Gamma \subset G$ be a discrete, torsion-free subgroup of finite covolume. Let $E \to X$ be a G-homogeneous vector bundle and $S_\infty\mathcal{E}$ the space of its sections of moderate growth.

Theorem 6.1 *$S_\infty\mathcal{E}$ is Γ-acyclic, i.e.,*

$$H^p(\Gamma, S_\infty\mathcal{E}) = 0 \quad \forall p \geq 1 \ .$$

Proof. We first consider the space of functions of moderate growth $S_\infty C^\infty(G)$ on G defined by

$$S_\infty C^\infty(G) := \{f \in \mathbf{C}^\infty(G) \mid \forall L \in \mathcal{U}(\mathbf{g})\ \exists R \in \mathbf{R}\ s.t.\ \sup_{g\in G} \mathrm{e}^{-R\mathrm{dist}(gK,\mathcal{O})}|f(Lg)| < \infty\}\ .$$

As a topological vector space $S_\infty C^\infty(G)$ is a limit of Fréchet spaces.

Proposition 6.2 *$S_\infty C^\infty(G)$ is Γ-acyclic.*

Proof. $H^*(\Gamma, S_\infty C^\infty(G))$ is the cohomology of the de Rham complex of $Y = \Gamma\backslash X$ twisted with the flat bundle associated to the Γ-module $S_\infty C^\infty(G)$. In greater detail let $L^* := \Lambda^* T^* X$ and $\mathcal{C}^* := C^\infty(X, L^* \otimes S_\infty C^\infty(G))$. Moreover, let $d : \mathcal{C}^* \to \mathcal{C}^{*+1}$ denote the differential of the de Rham complex. The complex $(\mathcal{C}^*, d)$ is a complex of Γ-modules. If we view $\omega \in \mathcal{C}^p$ as a function on G with values in $\mathcal{L}^p$, then the action of $\gamma \in \Gamma$ on ω is given by $(\gamma\omega)(g) = (L^*_\gamma\omega)(\gamma^{-1}g)$, where L^*_γ is the pull back of forms associated to the diffeomorphism $L_\gamma : X \to X$ given by $L_\gamma(x) := \gamma^{-1}x$. The complex $(\mathcal{C}^*, d)$ is exact (the contraction of X along radial rays induces a zero-homotopy of the de Rham complex) and it consists of Γ-acyclic modules ([3], Lemma 2.4). Hence $H^*(\Gamma, S_\infty C^\infty(G))$ is the cohomology of the complex $({}^\Gamma\mathcal{C}^*, d)$.

Let $\mathcal{S}^* := S_\infty\mathcal{L}^*[S_\infty(G)]$ (see Section 3 for notation). Then $(\mathcal{S}^*, d) \hookrightarrow (\mathcal{C}^*, d)$ is a subcomplex.

Lemma 6.3 *The inclusion $({}^\Gamma\mathcal{S}^*, d) \hookrightarrow ({}^\Gamma\mathcal{C}^*, d)$ induces an isomorphism in cohomology.*

Proof. The manifold Y has finitely many cusps each diffeomorphic to $B_i \times [0, \infty)$, where B_i is some compact nil-manifold. The Borel-Serre compactification of Y is obtained by attaching copies of the cusp bases B_i as a boundary at infinity obtaining a manifold with boundary $\bar{Y}$. There is a natural sheafification $(\underline{{}^\Gamma\mathcal{S}^*}, d) \hookrightarrow (\underline{{}^\Gamma\mathcal{C}^*}, d)$ of the inclusion $({}^\Gamma\mathcal{S}^*, d) \hookrightarrow ({}^\Gamma\mathcal{C}^*, d)$ on $\bar{Y}$. To any finite open covering of $\bar{Y}$ there is an associated partition of unity which is compatible with the sheafs $\underline{{}^\Gamma\mathcal{S}^*}$. Thus $\underline{{}^\Gamma\mathcal{S}^*}$ and $\underline{{}^\Gamma\mathcal{C}^*}$ are acyclic with respect to the global section functor.

The complex of sheaves corresponding to $({}^\Gamma\mathcal{C}^*, d)$ is locally acyclic by the standard Poincaré Lemma. Moreover the inclusion of sheaves $(\underline{{}^\Gamma\mathcal{S}^*}, d) \hookrightarrow (\underline{{}^\Gamma\mathcal{C}^*}, d)$ induces an isomorphism of the zeroth cohomology sheaves. By Lemma 3.2 $(\underline{{}^\Gamma\mathcal{S}^*}, d)$ is locally acyclic. Thus the inclusion $(\underline{{}^\Gamma\mathcal{S}^*}, d) \hookrightarrow (\underline{{}^\Gamma\mathcal{C}^*}, d)$ is a quasi-isomorphism. Since the sheaves $\underline{{}^\Gamma\mathcal{S}^*}, \underline{{}^\Gamma\mathcal{C}^*}$ are acyclic with respect to the global section functor, the induced map of the complexes of global sections $({}^\Gamma\mathcal{S}^*, d) \hookrightarrow ({}^\Gamma\mathcal{C}^*, d)$ is a quasi-isomorphism, too. □

Lemma 6.4 $H^p({}^\Gamma\mathcal{S}^\cdot, d) = 0$, $\forall p \geq 1$.

Proof. The map $T : X \times G \to X \times G$ given by $(x, g) \to (gx, g)$ intertwines the Γ-action on the second factor G with the diagonal Γ-action on the product $X \times G$. We claim that T induces an isomorphism $T^* : S_\infty\mathcal{L}^*[S_\infty(G)] \to S_\infty\mathcal{L}^*[S_\infty(G)]$ intertwining the Γ-module structure given above with the Γ-module structure given by $(\gamma\omega)(g) = \omega(\gamma^{-1}g)$ (again viewing ω as a function from G to $\mathcal{L}^*$). The inverse of T^* is induced by $T^{-1}: X\times G\to X\times G$, $T^{-1}(x,g)=(g^{-1}x,g)$. In fact, $(T^*\omega)(g)=L^*_{g^{-1}}\omega(g)$ and

$$(T^*\gamma\omega)(g)=L^*_{g^{-1}}L^*_\gamma\omega(\gamma^{-1}g))=(T^*\omega)(\gamma^{-1}g)\ .$$

In order to prove the claim we must show that T^*, $(T^*)^{-1}$ are compatible with the weighted spaces. It is at this point that we have to consider the weighted de Rham complex $\mathcal{S}^*$. Since T mixes the G- and the X directions it does not act on $\mathcal{C}^*$.

Let $\Delta:\mathcal{U}(\mathbf{g})\to\mathcal{U}(\mathbf{g})\otimes\mathcal{U}(\mathbf{g})$ be the co-product induced by $X\mapsto X\otimes 1+1\otimes X$, $X\in\mathbf{g}$. Fix $L_1,L\in\mathcal{U}(\mathbf{g})$ and let $(\Delta\otimes\mathrm{id})\Delta(L)=\sum_\alpha A_\alpha\otimes B_\alpha\otimes C_\alpha$. Let $\omega\in\mathcal{S}^p$. Then

$$(T^*\omega)(L_1h,Lg)=\sum_\alpha\omega(B_\alpha gL_1h,C_\alpha g)\circ DL_{g^{-1}A_\alpha^{op}}\ ,$$

where DL_g is the differential of L_g acting on TX.

Let $r=\deg(L_1)+\deg(L)$. Assume that

$$\sup_{g,h\in G,l,l_1\in L_r,}\mathrm{e}^{-R\mathrm{dist}(\mathcal{O},gK)}\mathrm{e}^{-Q\mathrm{dist}(\mathcal{O},hK)}|\omega(X(l)h,X(l_1)g)|<\infty$$

for appropriate $Q,R\in\mathbf{R}$. Here $\{X(l)\}_{l\in L_r}$ is a basis of the differential operators on G of order $\le r$ (similarly to the notation in the proof of Lemma 3.1). Note that $\mathrm{dist}(\mathcal{O},ghK)\le\mathrm{dist}(\mathcal{O},hK)+\mathrm{dist}(\mathcal{O},gK)$. Moreover we have

$$|DL_{g^{-1}A_\alpha^{op}}|\le C\mathrm{e}^{-P\mathrm{dist}(\mathcal{O},gK)}\quad\forall\alpha,\ \forall g\in G$$

and $B_\alpha gL_1h=\sum_{l\in L_r}e_{\alpha,l}(g)X(l)gh$ with

$$|e_{\alpha,l}(g)|\le C\mathrm{e}^{-P\mathrm{dist}(\mathcal{O},gK)}\quad\forall l\in L(r),\quad\forall\alpha,\ \forall g\in G$$

for $P\in\mathbf{R}$ large enough. We conclude that

$$\sup_{g,h\in G}|(T^*\omega)(L_1h,Lg)|\mathrm{e}^{-(R+Q+2P)\mathrm{dist}(\mathcal{O},gK)}\mathrm{e}^{-Q\mathrm{dist}(\mathcal{O},hK)}<\infty\ .$$

Hence $T^*\omega\in\mathcal{S}^p$. In a similar manner one can handle $(T^*)^{-1}$ thus proving the claim.

We see that $H^*({}^\Gamma\mathcal{S}^\cdot,d)$ is isomorphic to the cohomology of $(S_\infty\mathcal{L}^*[\Gamma\backslash G],d)$. Lemma 6.4 now follows from Lemma 3.3. □

We now finish the proof of Theorem 6.1. Note that $S_\infty C^\infty(G)$ carries a right K-module structure which commutes with the left Γ-module structure. This induces a K-action on $({}^\Gamma\mathcal{C}^*,d)$. $H^*(\Gamma,S_\infty\mathcal{E})$ is the cohomology of the complex $([{}^\Gamma\mathcal{C}^*\otimes V_\gamma]^K,d)$.

Let $[z]\in H^p(\Gamma,S_\infty\mathcal{E})$, $p\ge 1$, be represented by the K-invariant cycle $z\in[{}^\Gamma\mathcal{C}^p\otimes V_\gamma]^K$. Then by Lemma 6.2 we have $z=db$ for some possibly non-invariant $p-1$ cochain $b\in{}^\Gamma\mathcal{C}^{p-1}\otimes V_\gamma$. Let $\bar b$ the average of b with respect to K. Then also $d\bar b=z$ and hence $0=[z]\in H^p(\Gamma,S_\infty\mathcal{E})$. This proves Theorem 6.1 . □

7 The cokernel of $B : S_\infty\mathcal{E}_Y \longrightarrow S_\infty\mathcal{E}_Y$

In this section we relate the dimension of the cokernel of $B := \Omega - \mu$ on $S_\infty\mathcal{E}_Y := {}^\Gamma S_\infty\mathcal{E}$ with the dimension of corresponding spaces of cusp forms. Since it is not a priori clear that the range of B is closed note that we consider the algebraic cokernel of B.

Let $E_Y := \Gamma\backslash E$ be the locally homogeneous bundle over Y. A section $f \in C^\infty(Y, E_Y) =: \mathcal{E}_Y$ can be viewed as a function on $f : G \to V_\gamma$ satisfying $f(hgk) = \gamma^{-1}(k)f(g)$ for all $h \in \Gamma$, $k \in K$. The space $S_\infty\mathcal{E}_Y$ has the alternative description

$$S_\infty\mathcal{E}_Y = \{f \in \mathcal{E}_Y \mid \forall L \in \mathcal{U}(\mathbf{g})\ \exists R \in \mathbf{R}\ \ s.t.\ \sup_{g\in G} \mathrm{e}^{R\mathrm{dist}_Y(\Gamma\mathcal{O},\Gamma gK)}|f(Lg)| < \infty\}\ .$$

A section $f \in \mathcal{E}_Y$ which is an eigenfunction of Ω and satisfies

$$\sup_{g\in G} \mathrm{e}^{R\mathrm{dist}_Y(\Gamma\mathcal{O},\Gamma gK)}|f(g)| < \infty \quad \forall R \in \mathbf{R}$$

is called a cusp form. Let V_μ be the space of cusp forms in $\ker(B)$. If $V_\mu \neq \{0\}$, then necessarily $\mu \in\ R$, since Ω is symmetric.

The main result of the present section is the following theorem.

Theorem 7.1

$$\dim\mathrm{coker}(B : S_\infty\mathcal{E}_Y \to S_\infty\mathcal{E}_Y) = \dim V_\mu\ .$$

The proof of the theorem occupies the remainder of the section.

Lemma 7.2

$$\dim\mathrm{coker}(B : S_\infty\mathcal{E}_Y \to S_\infty\mathcal{E}_Y) \geq \dim V_\mu\ .$$

Proof. We show that the projection of V_μ to $\mathrm{coker}(B : S_\infty\mathcal{E}_Y \to S_\infty\mathcal{E}_Y)$ is an inclusion. We can assume that $V_\mu \neq 0$ and hence $\mu \in \mathbf{R}$. Thus B is symmetric.

If $f \in V_\mu$, then it vanishes rapidly and can be integrated against elements of $S_\infty\mathcal{E}_Y$. Thus let $f \in V_\mu$ and assume that $[f] = 0$ in $\mathrm{coker}(B : S_\infty\mathcal{E}_Y \to S_\infty\mathcal{E}_Y)$. Then $f = Bh$ for some $h \in S_\infty\mathcal{E}_Y$. We have

$$\begin{aligned} 0 &= \int_Y \langle (Bf)(y)\ ,\ h(y)\rangle_{E_{Y,y}} dy \\ &= \int_Y \langle f(y)\ ,\ (Bh)(y)\rangle_{E_{Y,y}} dy \\ &= \|f\|^2_{L^2(Y,E_Y)}\ . \end{aligned}$$

It follows $f = 0$. This proves Lemma 7.2. □

We define an increasing sequence of Fréchet spaces $S_R\mathcal{E}_Y$, $R \in \mathbf{R}$, by

$$S_R\mathcal{E}_Y := \{f \in \mathcal{E}_Y \mid p_{Y;-R,L}(f) < \infty\ \ \forall L \in \mathcal{U}(\mathbf{g})\}\ ,$$

where the seminorms are defined by

$$p_{Y;R,L}(f) := \sup_{g\in G} \mathrm{e}^{R\mathrm{dist}(\Gamma\mathcal{O},\Gamma gK)}|f(gL)|\ .$$

Note that (in contrast to the definition of $S_\infty\mathcal{E}$) we employ the right action of $\mathcal{U}(\mathbf{g})$ to define $p_{Y;R,L}(f)$. In fact, to define $S_R\mathcal{E}_Y$ it is sufficient to consider the seminorms $p_{Y;R,\Omega^k}(f)$, $k \in \mathbf{N}_0$. Let

$$S_{-\infty}\mathcal{E}_Y = \bigcap_{R\in\mathbf{R}} S_R\mathcal{E}_Y$$

with the natural topology of the intersection and $S_\infty\mathcal{E}'_Y$ be the topological conjugate dual of $S_{-\infty}\mathcal{E}_Y$. The Hermitian scalar product of E_Y induces an inclusion $S_\infty\mathcal{E}_Y \hookrightarrow S_\infty\mathcal{E}'_Y$.

Lemma 7.3 *The inclusion $S_\infty\mathcal{E}_Y \hookrightarrow S_\infty\mathcal{E}'_Y$ induces an injection*

$$\mathrm{coker}(B : S_\infty\mathcal{E}_Y \to S_\infty\mathcal{E}_Y) \hookrightarrow \mathrm{coker}(B : S_\infty\mathcal{E}'_Y \to S_\infty\mathcal{E}'_Y)\ .$$

Proof. We reduce the proof to Lemma 2.7. Let $\pi : X \to Y$ be the projection. We define a continuous map $\pi_* : S_{-\infty}\mathcal{E} \to S_{-\infty}\mathcal{E}_Y$. For $f \in S_{-\infty}\mathcal{E}$ let $(\pi_* f)(\Gamma g) := \sum_{\gamma\in\Gamma} f(\gamma g)$. We show that the sum converges and π_* is continuous.

There is a $Q \in R$ such that $\sup_{g\in G}\sum_{\gamma\in\Gamma} \mathrm{e}^{Q\mathrm{dist}(\gamma gK,\mathcal{O})} =: D < \infty$. For $k \in \mathbf{N}_0$ we have

$$\begin{aligned} p_{Y;R,\Omega^k}(\pi_* f) &= \sup_{g\in G} \mathrm{e}^{R\mathrm{dist}_Y(\Gamma\mathcal{O},\Gamma gK)} |\sum_{\gamma\in\Gamma} f(\gamma g\Omega^k K)| \\ &\le \sup_{g\in G} \mathrm{e}^{R\mathrm{dist}_Y(\Gamma\mathcal{O},\Gamma gK)} \sum_{\gamma\in\Gamma} |f(\gamma\Omega^k gK)| \\ &\le \sup_{g\in G} \mathrm{e}^{R\mathrm{dist}_Y(\Gamma\mathcal{O},\Gamma gK)} \sum_{\gamma\in\Gamma} \mathrm{e}^{(Q-R)\mathrm{dist}(\gamma gK,\mathcal{O})} p_{(R-Q),\Omega^k}(f) \\ &\le \sup_{g\in G} p_{(R-Q),\Omega^k}(f) \sum_{\gamma\in\Gamma} \mathrm{e}^{Q\mathrm{dist}(\gamma gK,\mathcal{O})} \\ &\le D p_{(R-Q),\Omega^k}(f)\ . \end{aligned}$$

This estimate shows the continuity of π_*. Let $\pi^* : S_\infty\mathcal{E}'_Y \to S_\infty\mathcal{E}'$ be the adjoint of π_*.

To prove the Lemma it suffices to show that if $f \in S_\infty\mathcal{E}'_Y$ with $F = Bf \in S_\infty\mathcal{E}_Y$, then $f \in S_\infty\mathcal{E}_Y$. Let W be the parametrix of B constructed in the proof of Lemma 2.7. Then we have $\pi^* f = WF + S\pi^* f$ (viewing $F \in {}^\Gamma S_\infty\mathcal{E}$). We have already shown that $WF, S\pi^* f \in S_\infty\mathcal{E}$. Since W, S are G-equivariant we obtain $\pi^* f \in {}^\Gamma S_\infty\mathcal{E}$. This finishes the proof of the Lemma. □

Lemma 7.4 *The operator $B^* : S_{-\infty}\mathcal{E}_Y \to S_{-\infty}\mathcal{E}_Y$ has closed range.*

Proof. Let $\{f_i\}$ be a sequence in $S_{-\infty}\mathcal{E}_Y$ such that $B^* f_i =: h_i \to h \in S_{-\infty}\mathcal{E}_Y$. We are to find $f \in S_{-\infty}\mathcal{E}_Y$ with $B^* f = h$. If $V_\mu \neq 0$, then $V_\mu = \ker(B^* : S_{-\infty}\mathcal{E}_Y \to S_{-\infty}\mathcal{E}_Y)$, and we can project f_i to the L^2-orthogonal complement of V_μ. Thus we can assume that $f_i \perp V_\mu$.

We can assume that for some $R \in \mathbf{R}$ the sequence $p_{Y;R,1}(f_i)$ is bounded. If not, we divide f_i by $p_{Y;R,1}(f_i)$ obtaining a sequence $\tilde{f}_i$ with $p_{Y;R,1}(\tilde{f}_i) = 1$ and $B^*\tilde{f}_i \to 0$. We show below that $\tilde{f}_i$ has a subsequence converging to $F \in S_{-\infty}\mathcal{E}_Y$. Now $B^* F = 0$ and $F \perp V_\mu$. Hence $F = 0$ contradicting $p_{Y;R,1}(F) = 1$.

Since for all $k \in \mathbf{N}_0$ the sequence $p_{Y;R,\Omega^k}(h_i)$ is bounded we conclude that $p_{Y;R,\Omega^k}(f_i)$ is bounded, too.

Consider a cusp c_l of Y associated to the minimal parabolic subgroup $P = MAN \subset G$. Then a neighbourhood U of infinity of this cusp can be identified with $\Gamma \cap N\backslash N \times [d_l, \infty)$, $d_l \in A$ large. If $F \in \mathcal{E}_Y$, then we define the constant term $F_P \in C^\infty(U, E_{Y|U})$ by

$$F_P(na) = \frac{1}{\mathrm{vol}(\Gamma \cap N\backslash N)} \int_{\Gamma\cap N\backslash N} F(n'na)dn' \ .$$

Let $\chi \in C^\infty(Y)$ be a cut-off function being one on $\Gamma \cap N\backslash N \times [d_l + 1, \infty)$ and zero outside of U. Then $F_{c_l} = \chi F_P \in \mathcal{E}_Y$. If Y has the cusps c_l, $l = 1, \dots, r$, then we set $F_c := \sum_{l=1}^r F_{c_l}$.

We apply this construction to our sequence f_i obtaining a decomposition $f_i = f_{i,c} + f_{i,r}$ with $f_{i,r} := f_i - f_{i,c}$. Since f_i is bounded in $S_R\mathcal{E}_Y$, by a Lemma of Gelfand [6], Thm. 5, the sequence $f_{i,r}$ is bounded in $S_{-\infty}\mathcal{E}_Y$. Since for $R < R'$ the embedding $S_R\mathcal{E}_Y \hookrightarrow S_{R'}\mathcal{E}_Y$ is compact, any bounded sequence in $S_{-\infty}\mathcal{E}_Y$ has a converging subsequence. Thus by taking a subsequence we can assume that $f_{i,r}$ converges in $S_{-\infty}\mathcal{E}_Y$.

For $\mathrm{dist}_Y(\Gamma\mathcal{O}, y) \ge \max_l d_l + 1$ we have $(B^*f)_c(y) = B^*f_c(y)$. Consider again the cusp c_l and the coordinates $\Gamma \cap N\backslash N \times [d_l, \infty)$. There are commuting $x, y \in \mathrm{End}(V_\gamma)$ such that

$$(B^*F_{c_l})(n, a) = -((\frac{d}{da} + x)(\frac{d}{da} + y)F_{c_l})(n, a) \ ,$$

$a > d_l + 1$. Assume that $R < -\max(\|x\|), \|y\|)$. We set for $F \in S_R\mathcal{E}_Y$

$$(H_{c_l}F)(n, a) := -\chi(a)\mathrm{e}^{-ya} \int_a^\infty \mathrm{e}^{(y-x)a_1} \int_{a_1}^\infty \mathrm{e}^{xb} F_{c_l}(n, b)dbda_1 \ .$$

Again, if Y has cusps c_l, $l = 1, \dots, r$, then we set $H_c := \sum_{l=1}^r H_{c_l}$. Then $H_c : S_R\mathcal{E}_Y \to S_R\mathcal{E}_Y$ is continuous, and $\mathrm{supp}(H_cB^*F - F_c) \subset V$, where $V \subset Y$ is compact and independent of F. The proof of continuity is similar to the corresponding argument in the proof of Lemma 2.3. Notice that $\Omega^k H_c - H_c\Omega^k = W$ is a continuous operator $W : S_R\mathcal{E}_Y \to C_c^\infty(Y, E_Y)$ and $\mathrm{supp} Wf \subset V'$, where $V' \subset Y$ is compact and independent of f.

Thus $F_i := H_cB^*h_i - f_{i,c}$ is a bounded sequence in $S_{-\infty}\mathcal{E}_Y$. Hence F_i has a subsequence converging in $S_{-\infty}\mathcal{E}_Y$. Since $H_cB^*h_i$ converges in $S_{-\infty}\mathcal{E}_Y$ by taking a subsequence we can assume that $f_{i,c}$ converges in $S_{-\infty}\mathcal{E}_Y$, too. Let f be the limit of $f_i = f_{i,c} + f_{i,r}$ for this subsequence. Then $B^*f = h$. This finishes the proof of the lemma. □

We now finish the proof of Theorem 7.1. By Lemma 7.4 we have

$$\dim V_\mu = \dim\ker(B^* : S_{-\infty}\mathcal{E}_Y \to S_{-\infty}\mathcal{E}_Y) = \dim \mathrm{coker}(B : S_\infty\mathcal{E}'_Y \to S_\infty\mathcal{E}'_Y) \ .$$

By Lemma 7.3 we have

$$\dim \mathrm{coker}(B : S_\infty\mathcal{E}_Y \to S_\infty\mathcal{E}_Y) \le \dim V_\mu \ .$$

Combining this with Lemma 7.2 we obtain the theorem. □

8 Γ-cohomology

In this section we discuss properties of the Γ-cohomology of distribution vector globalizations of admissible representations of G.

Let $(\pi, V_{\pi,K}) \in \mathcal{HC}(\mathbf{g}, K)$ and $\Gamma \subset G$ be a discrete torsion-free subgroup of finite covolume. Let $B = (\Omega - \mu)^k$ such that $BV_{\pi,K} = 0$.

Proposition 8.1 *We have*

$$\dim H^p(\Gamma, V_{\pi,-\infty}) < \infty, \quad \forall p \geq 0.$$

Proof. Let

$$0 \to V_{\pi,-\infty} \to S_\infty\mathcal{E}_0 \xrightarrow{\left(\begin{smallmatrix} D_0 \\ B \end{smallmatrix}\right)} \begin{array}{c} S_\infty\mathcal{E}_1 \\ \oplus \\ S_\infty\mathcal{E}_0 \end{array} \xrightarrow{\left(\begin{smallmatrix} D_1 & H_0 \\ -B & D_0 \end{smallmatrix}\right)} \begin{array}{c} S_\infty\mathcal{E}_2 \\ \oplus \\ S_\infty\mathcal{E}_1 \end{array} \xrightarrow{\left(\begin{smallmatrix} D_2 & H_1 \\ B & D_1 \end{smallmatrix}\right)} \dots \tag{19}$$

be a standard resolution (see Proposition 4.5) of $V_{\pi,-\infty}$. It is a Γ-acyclic resolution of $V_{\pi,-\infty}$ by Theorem 6.1. The cohomology of the subcomplex of Γ-invariant vectors is isomorphic to $H^*(\Gamma, V_{\pi,-\infty})$.

For any locally homogeneous vector bundle $E_Y \to Y$ we denote by $\mathcal{E}_Y(B)_{cusp}$ the space of cusp-forms in $\mathcal{E}_Y(B)$ and by $S_\infty\mathcal{E}_Y(B)$ the kernel of B in $S_\infty\mathcal{E}_Y$. We consider the subcomplex of the complex of Γ-invariants of (19)

$$0 \to S_\infty\mathcal{E}_{0,Y}(B) \xrightarrow{\left(\begin{smallmatrix} D_0 \\ 0 \end{smallmatrix}\right)} \begin{array}{c} S_\infty\mathcal{E}_{1,Y}(B) \\ \oplus \\ \mathcal{E}_{0,Y}(B)_{cusp} \end{array} \xrightarrow{\left(\begin{smallmatrix} D_1 & H_0 \\ 0 & D_0 \end{smallmatrix}\right)} \begin{array}{c} S_\infty\mathcal{E}_{2,Y}(B) \\ \oplus \\ \mathcal{E}_{1,Y}(B)_{cusp} \end{array} \xrightarrow{\left(\begin{smallmatrix} D_2 & H_1 \\ 0 & D_1 \end{smallmatrix}\right)} \dots \tag{20}$$

and claim that its cohomology is $H^*(\Gamma, V_{\pi,-\infty})$. In fact, let $(f_i, f_{i-1}) \in S_\infty\mathcal{E}_{i,Y} \oplus S_\infty\mathcal{E}_{i-1,Y}$ be a cochain in the complex of Γ-invariants of (19). By the results of Section 7 there is a unique decomposition $f_{i-1} = f_{i-1}^{im} + f_{i-1}^{cusp}$, where $f_{i-1}^{im} \in BS_\infty\mathcal{E}_{i,Y}$ and $f_{i-1}^{cusp} \in \mathcal{E}_{i-1,Y}(B)_{cusp}$. Notice that $\mathrm{coker} B = \mathrm{coker}(\Omega - \mu)$. Thus modulo a coboundary the cochain (f_i, f_{i-1}) is equivalent to $(\tilde{f}_i, f_{i-1}^{cusp})$. If in addition (f_i, f_{i-1}) and hence $(\tilde{f}_i, f_{i-1}^{cusp})$ is a cocycle, then $B\tilde{f}_i = 0$, since $D_{i-1} : \mathcal{E}_{i-1,Y}(B)_{cusp} \to \mathcal{E}_{i,Y}(B)_{cusp}$ and the range of $B : S_\infty\mathcal{E}_{i,Y} \to S_\infty\mathcal{E}_{i,Y}$ is transverse to $\mathcal{E}_{i,Y}(B)_{cusp}$. We conclude that the cohomology of (20) surjects onto $H^*(\Gamma, V_{\pi,-\infty})$.

Assume now that $f_i = H_{i-2}g_{i-2} + D_{i-1}g_{i-1}$, $f_{i-1} = (-1)^{i-1}Bg_{i-1} + D_{i-2}g_{i-2}$, and $f_{i-1} \in \mathcal{E}_{i-1,Y}(B)_{cusp}$, $g_{i-2} \in \mathcal{E}_{i-2,Y}(B)_{cusp}$. It follows that $Bg_{i-1} = 0$ and thus (g_{i-1}, g_{i-2}) is a $i-1$-cochain of (20). This show that the cohomology of (20) maps injectively to $H^*(\Gamma, V_{\pi,-\infty})$ proving the claim.

The lemma now follows since (20) is a complex of finite-dimensional vector spaces (see e.g. [8], Thm.1). □

If Γ is cocompact, then we can prove a Poincaré duality theorem as in [3], Proposition 5.2. It relates the Γ-cohomology of the distribution vector globalization of an admissible representation of G with the Γ-cohomology of the smooth globalization of its dual.

Proposition 8.2 *Let Γ be cocompact. The Γ-cohomology of $V_{\pi,\pm\infty}$ satisfies Poincaré duality*
$$H^p(\Gamma, V_{\pi,-\infty})^* \cong H^{n-p}(\Gamma, V_{\tilde{\pi},\infty}),$$
where $n = \dim(X)$.

Let $V_{\pi,\pm\omega}$ be the minimal and maximal globalizations of $V_{\pi,K}$, respectively. If Γ is cocompact, then $\mathcal{E}_Y(B)_{cusp} = \mathcal{E}_Y(B) = S_\infty\mathcal{E}_Y(B)$. Combining [3], Proposition 5.1 with Proposition 8.1 we obtain

Corollary 8.3 *Let Γ be cocompact. Then for $(\pi, V_{\pi,K}) \in \mathcal{HC}(\mathbf{g}, K)$ we have*
$$\begin{aligned} H^*(\Gamma, V_{\pi,\omega}) &= H^*(\Gamma, V_{\pi,\infty}) \\ H^*(\Gamma, V_{\pi,-\infty}) &= H^*(\Gamma, V_{\pi,-\omega}) \ , \end{aligned}$$
where the identifications are induced by the natural inclusions of the globalizations.

If Γ has torsion, then there exists a cofinite torsion free normal subgroup $\Gamma' \subset \Gamma$. Employing the spectral sequence for the group cohomology associated to the extension
$$0 \to \Gamma' \to \Gamma \to F \to 0 \ ,$$
where F is some finite group, one easily obtains a generalization of the results of the present section to Γ. The spectral sequence degenerates at the second term since the higher cohomology of a finite group with coefficients in a vector spaces over $\mathbf{C}$ vanishes. It follows $H^*(\Gamma, V) = H^*(\Gamma', V)^F$ for any Γ-module V over $\mathbf{C}$.

9 Fuchsian groups of the first kind

In this section we give a detailed discussion of the Γ-cohomology of a Fuchsian group of the first kind with coefficients in the distribution vector globalization of principal series representations. In this case we know explicit standard resolutions.

Let $\Gamma \subset PSL(2, \mathbf{R}) =: G$ be a discrete torsion-free subgroup of finite covolume. Such a Γ is called a Fuchsian group of the first kind and it acts freely on the hyperbolic plane $X = H^2$. The quotient $Y = \Gamma\backslash X$ is a complete Riemann surface of finite volume.

The group G acts on the circle S^1 which can be identified with the boundary ∂X of X using the Poincaré disc model. Let $T \to S^1$ be the complexified tangent bundle of S^1. It is G-homogeneous and we can form complex powers $T^\lambda \to S^1$, $\lambda \in \mathbf{C}$. The number $\lambda \in \mathbf{C}$ parametrizes a principal series representation (π^λ, H^λ) of G on the Hilbert space $L^2(S^1, T^{\lambda-1/2})$. By $H^\lambda_{-\infty}$ we denote the space of its distribution vectors.

For $\lambda \neq -1/2, -3/2, -5/2, \ldots$ combining a theorem of Helgason ([11], [12] Introduction Thm. 4.3) with the characterization of the distribution vector globalization (Wallach [20] Ch. 11, Casselman [5]) we see that the Poisson transform P_λ is an G-equivariant isomorphism
$$P_\lambda : H^\lambda_{-\infty} \stackrel{\sim}{\to} S_\infty\mathcal{E}(B) \ ,$$

where $B = \Omega - 1/4 + \lambda^2$ and $E = X \times \mathbf{C}$ is the trivial bundle (in this special situation this fact was first obtained by Lewis [14]). Thus a standard resolution of the principal series representation $H^\lambda_{-\infty}$ for $\lambda \neq -1/2, -3/2, -5/2, \ldots$ is simply

$$0 \to H^\lambda_{-\infty} \xrightarrow{P_\lambda} S_\infty\mathcal{E} \xrightarrow{B} S_\infty\mathcal{E} \to 0 \ .$$

The complex (20) reduces to

$$0 \to S_\infty\mathcal{E}_Y(B) \xrightarrow{0} \mathcal{E}_Y(B)_{cusp} \to 0 \ . \tag{21}$$

Proposition 9.1 *For $\lambda \neq -1/2, -3/2, -5/2, \ldots$ we have*

$$\begin{aligned} H^0(\Gamma, H^\lambda_{-\infty}) &= S_\infty\mathcal{E}_Y(B) \\ H^1(\Gamma, H^\lambda_{-\infty}) &= \mathcal{E}_Y(B)_{cusp} \\ H^2(\Gamma, H^\lambda_{-\infty}) &= 0. \end{aligned}$$

Moreover, $\chi(\Gamma, H^\lambda_{-\infty}) = \dim H^0(\Gamma, H^\lambda_{-\infty}) - \dim H^1(\Gamma, H^\lambda_{-\infty}) = r$, where r is the number of cusps of Y. If $H^1(\Gamma, H^\lambda_{-\infty}) \neq 0$, then $\lambda \in \imath\mathbf{R} \cup (-1/2, 1/2)$.

Proof. The first part of the Proposition follows immediately from (21). $\mathcal{E}_Y(B)_{cusp} \subset \ker_{L^2}(B : \mathcal{E}_Y \to \mathcal{E}_Y)$ and $\mathrm{spec}_{L^2}\Delta_Y \subset [0, \infty)$ implies the last assertion. Let $p : S_\infty\mathcal{E}_Y(B) \to \mathbf{C}^{2r}$ be the linear map taking the constant term. For any cusp the constant term has two components (the incoming and the outcoming). It is known that $\dim \mathrm{im}(p) = r$. In fact, the range of p is generated by the constant terms of regular Eisenstein series and their residues. The scattering matrix fixes the relation between the two components of the constant term. Since $\ker(p) = \mathcal{E}_Y(B)_{cusp}$, the assertion about the Euler characteristic follows. □

Now we discuss the case $\lambda = -k/2$, $k = 1, 3, 5 \ldots$ taking the structure of $H^{\pm k/2}_{-\infty}$ as a G-module into account. We first exploit the exact sequence

$$0 \to F_k \to H^{k/2}_{-\infty} \to D^+_k \oplus D^-_k \to 0 \ , \tag{22}$$

where F_k is the finite-dimensional representation of G of dimension k and $D^\pm_k$ are the distribution vectors of holomorphic and anti-holomorphic discrete series representations. Let r denote the number of cusps of Y and g denote the genus. We have $H^i(\Gamma, H^{k/2}_{-\infty}) = 0$, $i \geq 1$, and $\dim H^0(\Gamma, H^{k/2}_{-\infty}) = \dim S_\infty\mathcal{E}_Y(B) = r$ by Proposition 9.1.

The long exact cohomology sequence associated to (22) gives

$$\begin{aligned} 0 \to H^0(\Gamma, F_k) \to H^0(\Gamma, H^{k/2}_{-\infty}) \to H^0(\Gamma, D^+_k \oplus D^-_k) \xrightarrow{\delta} H^1(\Gamma, F_k) \to 0 \\ H^1(\Gamma, D^+_k \oplus D^-_k) = 0 \ . \end{aligned}$$

An investigation of the long exact sequence associated to

$$0 \to D^+_k \oplus D^-_k \to H^{-k/2}_{-\infty} \to F_k \to 0$$

leads to

$$\begin{array}{c} 0 \to H^0(\Gamma, D_k^+ \oplus D_k^-) \to H^0(\Gamma, H_{-\infty}^{-k/2}) \to H^0(\Gamma, F_k) \to 0 \\ H^1(\Gamma, H_{-\infty}^{-k/2}) = H^1(\Gamma, F_k) \ . \end{array}$$

Proposition 9.2 *We have*

$$\begin{aligned} \dim H^0(\Gamma, D_k^+ \oplus D_k^-) &= r + \dim H^1(\Gamma, F_k) - \dim H^0(\Gamma, F_k) \\ \dim H^0(\Gamma, H_{-\infty}^{-k/2}) &= r + \dim H^1(\Gamma, F_k) \\ \dim H^1(\Gamma, H_{-\infty}^{-k/2}) &= \dim H^1(\Gamma, F_k) \\ \chi(\Gamma, H_{-\infty}^{-k/2}) &= r \end{aligned}$$

Since $r > 1$ or $g > 0$ we have for $k > 1$

$$\begin{aligned} H^0(\Gamma, F_1) &= 1 \\ H^1(\Gamma, F_1) &= 2g + r - 1 \\ H^0(\Gamma, F_k) &= 0 \\ H^1(\Gamma, F_k) &= k(2g + r - 2) \ . \end{aligned}$$

It follows that

$$\begin{aligned} \dim H^0(\Gamma, D_1^+ \oplus D_1^-) &= 2g + 2r - 2 \\ \dim H^0(\Gamma, D_k^+ \oplus D_k^-) &= k(2g - 2) + (k+1)r \\ \dim H^0(\Gamma, H_{-\infty}^{-1/2}) &= 2g + 2r - 1 \\ \dim H^1(\Gamma, H_{-\infty}^{-1/2}) &= 2g + r - 1 \\ \dim H^0(\Gamma, H_{-\infty}^{-k/2}) &= k(2g - 2) + (k+1)r \\ \dim H^1(\Gamma, H_{-\infty}^{-k/2}) &= k(2g + r - 2) \ . \end{aligned}$$

The aim of the following discussion is to understand Proposition 9.2 in the framework of standard resolutions.

Let $\mathtt{K}$ be the canonical bundle of X (viewing X as a complex manifold) and $\mathtt{K}^i$ be its i'th power.

Lemma 9.3 *A standard resolution of $H_{-\infty}^{-k/2}$ is given by*

$$0 \longrightarrow H_{-\infty}^{-k/2} \xrightarrow{P} S_\infty\mathcal{K}^{(k+1)/2} \oplus S_\infty\mathcal{K}^{-(k+1)/2}$$

$$\xrightarrow{\begin{pmatrix} \bar\partial^{(k+1)/2} \oplus \partial^{(k+1)/2} \\ \Omega + \frac{k^2-1}{4} \end{pmatrix}} \begin{array}{c} S_\infty\mathcal{K}^0 \\ \oplus \\ S_\infty\mathcal{K}^{(k+1)/2} \oplus S_\infty\mathcal{K}^{-(k+1)/2} \end{array} \longrightarrow$$

$$\xrightarrow{(-\Omega - \frac{k^2-1}{4}, \bar\partial^{(k+1)/2} \oplus \partial^{(k+1)/2})} S_\infty\mathcal{K}^0 \longrightarrow 0 \ .$$

Here $\bar\partial : \mathcal{K}^{(l+1)/2} \to \mathcal{K}^{(l-1)/2}$, $l > 0$ odd, is the contraction of the anti-holomorphic part of the canonical connection with the Kähler form and similarly $\partial : \mathcal{K}^{-(l+1)/2} \to \mathcal{K}^{-(l-1)/2}$ is the contraction of the holomorphic part of the canonical connection with the Kähler form. In abuse of notation we write $\bar\partial^{(k+1)/2}, \partial^{(k+1)/2}$ for the composition of the corresponding number of $\bar\partial$'s, ∂'s, respectively.

Proof. Define $B := \Omega + \frac{k^2-1}{4}$. As G-modules the eigenspaces $S_\infty\mathcal{K}^{\pm(k+1)/2}(B)$ have three-step composition series' with composition factors $D_k^\pm, F_k, D_k^\mp$ (see [15]). The operators $\bar\partial^{(k+1)/2}$, $\partial^{(k+1)/2}$, respectively, annihilate exactly the first composition factor. Above we have seen that $S_\infty\mathcal{K}^0(B) = H_\infty^{k/2}$ with composition factors $D_k^+ \oplus D_k^-, F_k$. This discussion shows that the complex above resolves $H_\infty^{-k/2}$ on a $\mathbf{K}$-theoretic level. In order to show that the complex is in fact a resolution of the specific extension $H_\infty^{-k/2}$ of the composition factors note that there is an injective Poisson transform P. □

Now we take the Γ-invariant vectors in the standard resolution. The complex (20) reduces to

$$0 \longrightarrow S_\infty\mathcal{K}_Y^{(k+1)/2}(B) \oplus S_\infty\mathcal{K}_Y^{-(k+1)/2}(B) \xrightarrow{\begin{pmatrix} \bar\partial^{(k+1)/2} \oplus \partial^{(k+1)/2} \\ 0 \end{pmatrix}} \begin{matrix} S_\infty\mathcal{K}_Y^0(B) \\ \oplus \\ \mathcal{K}_Y^{(k+1)/2}(B)_{cusp} \oplus \mathcal{K}_Y^{-(k+1)/2}(B)_{cusp} \end{matrix} \longrightarrow$$
$$\xrightarrow{(0,\bar\partial^{(k+1)/2} \oplus \partial^{(k+1)/2})} \mathcal{K}_Y^0(B)_{cusp} \longrightarrow 0 \ .$$

In order to make further reductions we assume that Y is not compact (the case of compact Y is an easy exercise and left to the reader). Since for $k > 1$ the operator B on $L^2(Y)$ is strictly positive and

$$\mathcal{K}_Y^0(B)_{cusp} \subset \ker_{L^2}(B : \mathcal{K}_Y^0 \to \mathcal{K}_Y^0)$$

for $k > 1$ we have $\mathcal{K}_Y^0(B)_{cusp} = 0$. This is also true for $k = 1$ since then $\ker_{L^2}(B : \mathcal{K}_Y^0 \to \mathcal{K}_Y^0)$ is generated by the constant function which is not a cusp form.

We claim that for $k > 1$

$$0 = \bar\partial^{(k+1)/2} : S_\infty\mathcal{K}_Y^{(k+1)/2}(B) \to S_\infty\mathcal{K}_Y^0(B) \tag{23}$$
$$0 = \partial^{(k+1)/2} : S_\infty\mathcal{K}_Y^{-(k+1)/2}(B) \to S_\infty\mathcal{K}_Y^0(B) \ . \tag{24}$$

To prove (23) we argue that already

$$0 = \bar\partial : S_\infty\mathcal{K}_Y^{(k+1)/2}(B) \to S_\infty\mathcal{K}_Y^{(k-1)/2}(B) \ .$$

In upper half-plane coordinates (x, y), $y > 0$, $z = x + \imath y$, we trivialize $\mathtt{K}^{(k+1)/2}$ using the section $(\frac{dz}{y})^{(k+1)/2}$. Then the operator B has the form $y^2\Delta + \imath(k+1)\frac{\partial}{\partial x} + \frac{k^2-1}{4}$. If $\phi \in S_\infty\mathcal{K}^{(k+1)/2}(B)$, then its constant term is a linear combination of $y^{-k}dz^{(k+1)/2}$ and $dz^{(k+1)/2}$. It follows that $\bar\partial\phi \in L^2$. But $\ker_{L^2}(B) = 0$ on $\mathtt{K}^{(k-1)/2}$, $k > 1$. Equation (24) follows by complex conjugation.

The spaces

$$\begin{aligned}\mathcal{A}_{k+1} &:= \ker(\bar\partial : S_\infty\mathcal{K}_Y^{(k+1)/2}(B) \to S_\infty\mathcal{K}_Y^{(k-1)/2}(B))\\ \bar{\mathcal{A}}_{k+1} &:= \ker(\partial : S_\infty\mathcal{K}_Y^{-(k+1)/2}(B) \to S_\infty\mathcal{K}_Y^{-(k-1)/2}(B))\end{aligned}$$

are called the holomorphic and anti-holomorphic automorphic forms of weight $k+1$. The subspaces

$$\begin{aligned}\mathcal{S}_{k+1} &:= \mathcal{K}_Y^{(k+1)/2}(B)_{cusp} \subset \mathcal{A}_{k+1}\\ \bar{\mathcal{S}}_{k+1} &:= \mathcal{K}_Y^{-(k+1)/2}(B)_{cusp} \subset \bar{\mathcal{A}}_{k+1}\end{aligned}$$

are the holomorphic and anti-holomorphic cusp-forms of weight $k+1$. Since for $k>1$ the space $S_\infty\mathcal{K}_Y^0(B)$ is generated by Eisenstein series its dimension is equal to the number of cusps r. For $k=3,5,\dots$ we have

$$\begin{aligned}\dim H^0(\Gamma, H_{-\infty}^{-k/2}) &= \dim(\mathcal{A}_{k+1}\oplus\bar{\mathcal{A}}_{k+1})\\ \dim H^1(\Gamma, H_{-\infty}^{-k/2}) &= \dim(\mathcal{S}_{k+1}\oplus\bar{\mathcal{S}}_{k+1}) + r\ .\end{aligned}$$

Employing $\dim\mathcal{A}_{k+1} = \dim\bar{\mathcal{A}}_{k+1} = k(g-1)+r(k+1)/2$ and $\dim\mathcal{S}_{k+1} = \dim\bar{\mathcal{S}}_{k+1} = k(g-1)+r(k-1)/2$ (see [18], Thm. 2.23) we recover the result of Proposition 9.2.

We now consider the case $k=1$. In this case both

$$\bar\partial : S_\infty\mathcal{K}_Y^1(B) \to S_\infty\mathcal{K}_Y^0(B)\ , \tag{25}$$

$$\partial : S_\infty\mathcal{K}_Y^{-1}(B) \to S_\infty\mathcal{K}_Y^0(B) \tag{26}$$

have a one-dimensional range spanned by the constant function. Note the r-dimensional space $S_\infty\mathcal{K}_Y^0(B)$ is generated by the regular Eisenstein series and the constant function. We obtain for $k=1$

$$\begin{aligned}\dim H^0(\Gamma, H_{-\infty}^{-1/2}) &= \dim(\mathcal{A}_2\oplus\bar{\mathcal{A}}_2)+1\\ \dim H^1(\Gamma, H_{-\infty}^{-1/2}) &= \dim(\mathcal{S}_2\oplus\bar{\mathcal{S}}_2) + r - 1\ .\end{aligned}$$

Using $\dim\mathcal{A}_2 = \dim\bar{\mathcal{A}}_2 = g-1+r$, $\dim\mathcal{S}_1 = \dim\bar{\mathcal{S}}_2 = g$ (see [18], Thm. 2.23) we again recover the results of Proposition 9.2.

We finish this section with a discussion of the relation of the sequence (22) with the Eichler homomorphism $E:\mathcal{S}_{k+1}\to H^1(\Gamma, F_k)$ (see e.g. [18]). We first recall its definition. If we restrict F_k to the maximal compact subgroup $K = S^1\subset G$, then it decomposes as $(F_k)_{|K} = \sum_{m=1}^k \mathbf{C}_{m-(k+1)/2}$, where $\mathbf{C}_l$ is the representation of S^1 on $\mathbf{C}$ given by $z\mapsto z^l$. Note that $\mathtt{K}^l = G\times_K \mathbf{C}_{-l}$. The map $(gK, f)\mapsto [g, g^{-1}f]$, $g\in G$, $f\in F_k$ defines an isomorphism

$$G/K\times F_k \xrightarrow{\sim} G\times_K (F_k)_{|K} = \oplus_{m=1}^k \mathtt{K}^{m-(k+1)/2}\ .$$

We denote the canonical G-equivariant embedding $\mathtt{K}^{\pm(k-1)/2}\hookrightarrow X\times F_k\cong L^2\otimes F_k$ by $j_\pm$, where the second identification is given by the Hodge-$*$-operator and L^l is the bundle of

l-forms on X. Taking the tensor product with $\mathtt{K}^{\pm 1}$ we obtain embeddings $i_\pm : \mathtt{K}^{\pm(k+1)/2} \hookrightarrow L^1 \otimes F_k$. If $f \in \mathcal{S}_{k+1} = \mathcal{K}^{(k+1)/2}(B)_{cusp}$, then $i_+(f)$ is a closed form and represents $E(f)$.

The long exact cohomology sequence associated to (22) induces a boundary map

$$\delta : H^0(\Gamma, D_k^+ \oplus D_k^-) \to H^1(\Gamma, F_k) \ .$$

Lemma 9.4 *The map δ restricted to the cusp forms coincides with E.*

Proof. In order to compute δ we need a Γ-acyclic resolution of (22) as a sequence. One possibility is the following:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \to & S_\infty\mathcal{L}^2 \otimes F_k & \stackrel{=}{\to} & S_\infty\mathcal{L}^2 \otimes F_K & \to & 0 & & \\
 & & d\uparrow & & a\uparrow & & \uparrow & & \\
0 & \to & S_\infty\mathcal{L}^1 \otimes F_k & \to & S_\infty\mathcal{L}^1 \otimes F_K \oplus S_\infty\mathcal{K}^{(k-1)/2} \oplus S_\infty\mathcal{K}^{-(k-1)/2} & \to & S_\infty\mathcal{K}^{(k-1)/2} \oplus S_\infty\mathcal{K}^{-(k-1)/2} & \to & 0 \\
 & & d\uparrow & & b\uparrow & & c\uparrow & & \\
0 & \to & S_\infty\mathcal{L}^0 \otimes F_k & \to & S_\infty\mathcal{L}^0 \otimes F_K \oplus S_\infty\mathcal{K}^{(k+1)/2} \oplus S_\infty\mathcal{K}^{-(k+1)/2} & \to & S_\infty\mathcal{K}^{(k+1)/2} \oplus S_\infty\mathcal{K}^{-(k+1)/2} & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & F_k & \to & H^{k/2} & \to & D_k^+ \oplus D_k^- & \to & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & .
\end{array} \tag{27}$$

Here the maps are given by

$$\begin{aligned}
a &:= d - j_+ - j_- \\
b &:= \begin{pmatrix} d & i_+ & i_- \\ 0 & \bar\partial & 0 \\ 0 & 0 & \partial \end{pmatrix} \\
c &:= \bar\partial \oplus \partial \ .
\end{aligned}$$

The horizontal maps are the obvious embeddings and projections, respectively. We leave the verification of the commutativity of the diagram and the exactness of the middle column to the reader.

The boundary map δ is now obtained by the usual diagram chasing. Let $[\alpha] \in H^0(\Gamma, D_k^+)$ be represented by a holomorphic Γ- invariant form $\alpha \in {}^\Gamma S_\infty\mathcal{K}^{(k+1)/2}$. Then $\delta[\alpha] \in H^1(\Gamma, F_k)$ is represented by the closed form $i_+(\alpha) \in S_\infty\mathcal{L}^1 \otimes F_k$. Comparing this with the definition of the Eichler map E given above we finish the proof of the lemma. $\square$

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