Quantum Chains of Hopf Algebras with Quantum Double Cosymmetry

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Abstract

Given a finite dimensional C^*-Hopf algebra H and its dual $\hat{H}$ we construct the infinite crossed product $\mathcal{A} = \ldots \rtimes H \rtimes \hat{H} \rtimes H \rtimes \ldots$ and study its superselection sectors. $\mathcal{A}$ is the observable algebra of a generalized quantum spin chain with H-order and $\hat{H}$-disorder symmetries, where by a duality transformation the role of order and disorder may also appear interchanged. If $H = \mathbb{C}G$ is a group algebra then $\mathcal{A}$ becomes an ordinary G-spin model. We classify all DHR-sectors of $\mathcal{A}$ — relative to some Haag dual vacuum representation — and prove that their symmetry is described by the Drinfeld double $\mathcal{D}(H)$. To achieve this we construct *localized coactions* $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{D}(H)$ and use a certain compressibility property to prove that they are *universal amplimorphisms* on $\mathcal{A}$. In this way the double $\mathcal{D}(H)$ can be recovered from the observable algebra $\mathcal{A}$ as a *universal cosymmetry.*

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1 Introduction and Summary of Results

Quantum chains considered as models of $1+1$-dimensional quantum field theory exhibit many interesting features that are either impossible or unknown in higher ($2+1$ or $3+1$) dimensions. These features include integrability on the one hand and the emergence of braid group statistics and quantum symmetry on the other hand. In this paper we study the second class of phenomena by looking at Hopf spin models as a general class of quantum chains where the quantum symmetry and braid statistics of superselection sectors turns out to be described by Drinfeld's "quantum double" $\mathcal{D}(H)$ of the underlying Hopf algebra H.

Quantum chains on which a quantum group acts are well known for some time; for example the XXZ-chain with the action of $sl(2)_q$ [P,PS] or the lattice Kac–Moody algebras of [AFS]. For a recent paper on the general action of quantum groups on ultralocal quantum chains see [FNW]. However the discovery that — at least for non-integer statistical dimensions —quantum symmetries are described by truncated quasi-Hopf algebras [MS,S] presents new difficulties to this approach. In fact, in such a scenario the field algebras are either non-associative or do not obey commutation relations with c-number coefficients, both properties being tacidly assumed in any "decent" quantum chain.

Here we stress the point of view that an unbiased approach to reveal the quantum symmetry of a model must be based only on the knowledge of the quantum group invariant operators (the "observables") that obey local commutation relations. This is the approach of algebraic quantum field theory (AQFT) [H]. The importance of the algebraic method, in particular the DHR theory of superselection sectors [DHR], in low dimensional QFT has been realized by many authors (see [FRS,BMT,FröGab,F,R] and many others).

The implementation of the DHR theory to quantum chains has been carried out at first for the case of G-spin models in [SzV]. These models have an order-disorder type of quantum symmetry given by the double $\mathcal{D}(G)$ of a finite group G which generalizes the $Z(2)\times Z(2)$ symmetry of the lattice Ising model. Since the disorder part of the double (i.e. the function algebra $\mathcal{C}(G)$) is always Abelian, G-spin models cannot be selfdual in the Kramers-Wannier sense, unless the group is Abelian. Non-Abelian Kramers-Wannier duality can therefore be expected only in a larger class of models.

Here we shall investigate the following generalization of G-spin models. On each lattice site there is a copy of a finite dimensional C^*-Hopf algebra H and on each link there is a copy of its dual $\hat{H}$. Non-trivial commutation relations are postulated only between neighbor links and sites where H and $\hat{H}$ act on each other in the "natural way", so as the link-site and the site-link algebras to form the crossed products $\mathcal{W}(\hat{H})\equiv\hat{H}\rtimes H$ and $\mathcal{W}(H)\equiv H\rtimes\hat{H}$ ("Weyl algebras" in the terminology of [N]). The two-sided infinite crossed product $\ldots\rtimes H\rtimes\hat{H}\rtimes H\rtimes\hat{H}\rtimes\ldots$ defines the observable algebra $\mathcal{A}$ of the Hopf spin model. Its superselection sectors (more precisely those that correspond to charges localized within a finite interval I, the so called DHR sectors) can be created by localized amplimorphisms $\mu\colon\mathcal{A}\to\mathcal{A}\otimes\mathrm{End}\,V$ with V denoting some finite dimensional Hilbert space. The category of localized amplimorphisms $\mathbf{Amp}\,\mathcal{A}$ plays the same role in locally finite dimensional theories as the category $\mathbf{End}\,\mathcal{A}$ of localized endomorphisms in continuum theories. The symmetry of the superselection sectors can be revealed by finding the "quantum group" $\mathcal{G}$, the representation category of which is equivalent to $\mathbf{Amp}\,\mathcal{A}$. In our model we find that $\mathcal{G}$ is the Drinfeld double (also called the quantum double) $\mathcal{D}(H)$ of H.

Finding all endomorphisms or all amplimorphisms of a given observable algebra $\mathcal{A}$ can be a very difficult problem in general. In the Hopf spin model $\mathcal{A}$ possesses a property we call *complete compressibility*, which allows us to do so. Namely if μ is an amplimorphism creating some charge on an arbirary large but finite interval then there exists an amplimorphism ν creating the same charge (i.e. ν is equivalent to μ, written $\nu\sim\mu$) but within an interval I of length 2 (i.e. I consists of a neighbouring site–link pair). Therefore the problem of finding *all* DHR-sectors of the Hopf spin model is reduced to a finite dimensional problem, namely to find all amplimorphisms localized within an interval of length 2. In this way we have proven that all DHR-sectors of $\mathcal{A}$ can be classified by representations of the Drinfeld double.

An important role in this reconstruction is played by the so-called *universal* amplimorphisms in $\mathbf{Amp}\,\mathcal{A}$. These are amplimorphisms $\rho\colon\mathcal{A}\to\mathcal{A}\otimes\mathcal{G}$ where $\mathcal{G}$ is an appropriate (in our approach finite dimensional) "quantum symmetry" C^*-algebra such that for any other amplimorphism μ in $\mathbf{Amp}\,\mathcal{A}$ there exists a representation β_μ of $\mathcal{G}$ such that $\mu\sim(\mathrm{id}_{\mathcal{A}}\otimes\beta_\mu)\circ\rho$. Moreover, the correspondence $\mu\leftrightarrow\beta_\mu$ has to be one-to-one on equivalence classes. We prove that complete compressibility implies that universal amplimorphisms ρ can be chosen to provide *coactions* of $\mathcal{G}$ on $\mathcal{A}$, i.e. there exists a coassociative unital coproduct $\Delta:\mathcal{G}\to\mathcal{G}\otimes\mathcal{G}$ and a counit $\varepsilon:\mathcal{G}\to\mathbb{C}$ such that

$$(\rho\otimes\mathrm{id}_{\mathcal{G}})\circ\rho \;=\; (\mathrm{id}_{\mathcal{A}}\otimes\Delta)\circ\rho \tag{1.1a}$$

$$(\mathrm{id}_{\mathcal{A}}\otimes\varepsilon)\circ\rho \;=\; \mathrm{id}_{\mathcal{A}} \tag{1.1b}$$

Moreover, Δ and ε are uniquely determined by ρ. Thus $\mathcal{G}$ becomes a C^*-Hopf algebra which we call a *universal cosymmetry* of $\mathcal{A}$. $\mathcal{G}$ will in fact be quasitriangular with R-matrix determined by the statistics operator of ρ

$$\epsilon(\rho,\rho) \;=\; \mathbb{1}\otimes P^{12}R \quad\in\mathcal{A}\otimes\mathcal{G}\otimes\mathcal{G} \tag{1.2}$$

where P^{12} is the usual transposition. The antipode S of $\mathcal{G}$ can be recovered by studying conjugate objects $\bar{\rho}$ and intertwiners $\rho\times\bar{\rho}\to\mathrm{id}_{\mathcal{A}}$. In this type of models the statistical dimensions

d_r of the irreducible components ρ_r of ρ are integers: they coincide with the dimensions of the corresponding irreducible representation D_r of $\mathcal{G}$. The statistics phases can be obtained from the universal balancing element $s = S(R_2)R_1 \in \mathrm{Center}\,\mathcal{G}$ evaluated in the representations D_r. For the Hopf spin model this scenario can be verified and calculated explicitely with $\mathcal{G} = \mathcal{D}(H)$.

We emphasize that having established the equivalence $\mathbf{Amp}\,\mathcal{A} \sim \mathbf{Rep}\,\mathcal{G}$ does not mean that $\mathcal{G}$ could be reconstructed from the observable algebra as a *unique* Hopf algebra. Only as a C^*-algebra (together with a distinguished 1-dimensional representation ε) it is uniquely determined. However the quasitriangular Hopf algebra structure on $\mathcal{G}$ can be recovered only up to a twisting by a 2-cocycle: If $u \in \mathcal{G} \otimes \mathcal{G}$ is a 2-cocycle, i.e. a unitary satisfying

$$(u \otimes \mathbf{1}) \cdot (\Delta \otimes \mathrm{id}\,)(u) \quad = \quad (\mathbf{1} \otimes u) \cdot (\mathrm{id}\, \otimes \Delta)(u)\,, \tag{1.3a}$$

$$(\varepsilon \otimes \mathrm{id}\,)(u) \quad = \quad (\mathrm{id}\, \otimes \varepsilon)(u) \;=\; \mathbf{1} \tag{1.3b}$$

then the twisted quasitriangular Hopf algebra with data

$$\begin{aligned} \Delta' &= \mathrm{Ad}\,u \circ \Delta \\ \varepsilon' &= \varepsilon \\ S' &= \mathrm{Ad}\,q \circ S \qquad q := u_1 S(u_2) \\ R' &= u^{op} R u^* \end{aligned}$$

is as good for a symmetry as the original one. In fact, we prove in Section 3.5 that (up to transformations by $\sigma \in \mathrm{Aut}\,(\mathcal{G}, \varepsilon)$) any universal coaction (ρ', Δ') is equivalent to a fixed one (ρ, Δ) by an isometric intertwiner $U \in \mathcal{A} \otimes \mathcal{G}$ satisfying a *twisted cocycle condition*

$$U\rho(A) \quad = \quad \rho'(A)U, \quad A \in \mathcal{A}, \tag{1.4a}$$

$$(U \otimes \mathbf{1}) \cdot (\rho \otimes \mathrm{id}_{\mathcal{G}})(U) \quad = \quad (1\!\!1 \otimes u) \cdot (\mathrm{id}_{\mathcal{A}} \otimes \Delta)(U)\,, \tag{1.4b}$$

$$(\mathrm{id}_{\mathcal{A}} \otimes \varepsilon)(U) \quad = \quad 1\!\!1 \tag{1.4c}$$

implying the identities (1.3) for u. In the Hopf spin model we also have the reverse statement, i.e. for all 2-cocycles u there is a unitary $U \in \mathcal{A} \otimes \mathcal{G}$ and a universal coaction ρ' satisfying (1.4) and therefore (1.1) with Δ' instead of Δ. We point out that (1.4) is a generalization of the usual notion of cocycle equivalence for coactions where one requires $u = \mathbf{1} \otimes \mathbf{1}$ [Ta,NaTa,BaSk,E].

This paper is an extended version of the first part of [NSz1]. In a forthcoming paper we will show [NSz2] that any universal coaction ρ on $\mathcal{A}$ gives rise to a family of complete irreducible field algebra extensions $\mathcal{F} \supset \mathcal{A}$ and that all field algebra extensions of $\mathcal{A}$ arise in this way. Moreover, equivalence classes of complete irreducible field algebra extensions are in one-to-one correspondence with cohomology classes of 2-cocycles $u \in \mathcal{G} \otimes \mathcal{G}$. The Hopf algebra $\mathcal{G}$ will act as a global gauge symmetry on all $\mathcal{F}$'s such that $\mathcal{A} \subset \mathcal{F}$ is precisely the $\mathcal{G}$-invariant subalgebra. Inequivalent field algebras will be shown to be related by Klein transformations involving symmetry operators $Q(X),\ X \in \mathcal{G}$.

The above type of reconstruction of the quasitriangular Hopf algebra $\mathcal{G}$ is a special case of the generalized Tannaka-Krein theorem [U,Maj2]. Namely, any faithful functor $F{:}\,\mathcal{C} \to Vec$ of strict monoidal braided rigid C^*-categories to the category of finite dimensional vector spaces factorizes as $F = f \circ \Phi$ to the forgetful functor f and to an equivalence Φ of $\mathcal{C}$ with the representation category $\mathbf{Rep}\,\mathcal{G}$ of a quasitriangular C^*-Hopf algebra $\mathcal{G}$. In our case $\mathcal{C}$ is the category $\mathbf{Amp}\,\mathcal{A}$ of amplimorphisms of the observable algebra $\mathcal{A}$. The functor F to the vector spaces is given naturally by associating to the amplimorphism $\mu: \mathcal{A} \to \mathcal{A} \otimes \mathrm{End}\, V$ the vector space V. Although the vector spaces V cannot be seen by only looking at the abstract category $\mathbf{Amp}\,\mathcal{A}$, they are "inherently" determined by the amplimorphisms and therefore by the observable algebra itself. In this respect using amplimorphisms one goes somewhat beyond the Tannaka-Krein theorem and approaches a Doplicher-Roberts [DR] type of reconstruction.

We now describe the plan of this paper.

In Section 2.1 we define our model using abstract relations as well as concrete realizations on Hilbert spaces associated to finite lattice intervals. We also discuss duality transformations and the appearence of the Drinfeld double as an order-disorder symmetry. In Section 2.2 we present the notion of a *quantum Gibbs system* on $\mathcal{A}$ and use this to prove (algebraic) Haag duality of our model.

In Section 3 we start with reviewing the category of amplimorphisms $\mathbf{Amp}\,\mathcal{A}$ in Section 3.1 and introduce *localized cosymmetries* $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$ as special kinds of amplimorphisms in Section 3.2. In Section 3.3 we specialize to *effective cosymmetries* and show that $\mathbf{Amp}\,\mathcal{A} \sim \mathbf{Rep}\,\mathcal{G}$ provided $\mathcal{G}$ is also *universal*. In Section 3.4 we introduce and investigate the notion of *complete compressibility* to guarantee the existence of universal cosymmetries. In Section 3.5 we prove that universal cosymmetries are unique up to (twisted) cocycle equivalences. In Section 3.6 we discuss two notions of translation covariance for localized cosymmetries and relate these to the existence of a *coherently translation covariant* structure in $\mathbf{Amp}\,\mathcal{A}$ as introduced for the case of endomorphisms in [DR1].

In Section 4 we apply the general theory to our Hopf spin model. In Section 4.1 we construct localized and strictly translation covariant effective coactions $\rho_I : \mathcal{A} \to \mathcal{A} \otimes \mathcal{D}(H)$ of the Drinfeld double for any interval I of length two and in Section 4.2 we prove that all these coactions are actually universal in $\mathbf{Amp}\,\mathcal{A}$.

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2 The Structure of the Observable Algebra

In this section we describe a canonical method by means of which one associates an observable algebra $\mathcal{A}$ on the 1-dimensional lattice to any finite dimensional C^*-Hopf algebra H. Although a good deal of our construction works for infinite dimensional Hopf algebras as well, we restrict the discussion here to the finite dimensional case. If $H = \mathbb{C}G$ for some finite group G then our construction reproduces the observable algebra of the G-spin model of [SzV].

In Section 2.1 we provide faithful $*$-representations of the local observable algebras $\mathcal{A}(I)$ associated to finite intervals I by placing a Hilbert space $\mathcal{H}_{even} \sim \hat{H}$ on each lattice site. In this way the algebras $\mathcal{A}(I)$ appear as the invariant operators under a global H-symmetry on $\mathcal{H}_{even} \otimes \ldots \otimes \mathcal{H}_{even}$. Similarly, we may represent the local algebras by putting Hilbert spaces $\mathcal{H}_{odd} \sim H$ on each lattice link, such that $\mathcal{A}(I)$ is given by the invariant operators under a global $\hat{H}$-symmetry on $\mathcal{H}_{odd} \otimes \ldots \otimes \mathcal{H}_{odd}$.

This is a generalization of duality transformations to Hopf spin chains. We point out that similarly as in [SzV] both symmetries combine to give the Drinfeld double $\mathcal{D}(H)$ as — what will later be shown to be — the *universal (co-)symmetry* of our model.

In Section 2.2 we view the Hopf spin chain in the more general setting of algebraic quantum field theory (AQFT) as a local net. We then introduce the notion of a *Quantum Gibbs system* as a family of conditional expectations $\eta_I : \mathcal{A} \to \mathcal{A}(I)' \cap \mathcal{A}$ with certain consistency relations, which allow to prove that our model satisfies a lattice version of (algebraic) Haag duality.

2.1 Local Observables and Order-Disorder Symmetries

Consider $\mathbb{Z}$, the set of integers, as the set of cells of the 1-dimensional lattice: even integers represent lattice sites, the odd ones represent links. Let $H = (H, \Delta, \varepsilon, S, *)$ be a finite dimensional C^*-Hopf algebra (see Appendix A). We denote by $\hat{H}$ the dual of H which is then also a C^*-Hopf algebra. We denote the structural maps of $\hat{H}$ by the same symbols Δ, ε, S. Elements of H will be typically denoted as $a, b, \ldots$, while those of $\hat{H}$ by $\varphi, \psi, \ldots$. The canonical pairing between H and $\hat{H}$ is denoted by $a \in H, \varphi \in \hat{H} \mapsto \langle a, \varphi \rangle \equiv \langle \varphi, a \rangle \in \mathbb{C}$. We also identify $\hat{\hat{H}} = H$ and emphasize that H and $\hat{H}$ will always appear on an equal footing. There are natural left and right actions of H on $\hat{H}$ (and vice versa) denoted by Sweedler's arrows:

$$
\begin{aligned}
a \rightharpoonup \varphi &= \varphi_{(1)} \langle a, \varphi_{(2)} \rangle && (2.1a) \\
\varphi \leftharpoonup a &= \langle \varphi_{(1)}, a \rangle \varphi_{(2)} && (2.1b)
\end{aligned}
$$

Here we have used the short cut notations $\Delta(a) = a_{(1)} \otimes a_{(2)}$ and $\Delta(\varphi) = \varphi_{(1)} \otimes \varphi_{(2)}$ implying a summation convention in $H \otimes H$ and $\hat{H} \otimes \hat{H}$, respectively. For a summary of definitions on Hopf algebras and more details on our notation see Appendix A.

We associate to each even integer $2i$ a copy $\mathcal{A}_{2i}$ of the C^*-algebra H and to each odd integer $2i+1$ a copy $\mathcal{A}_{2i+1}$ of $\hat{H}$. We denote the elements of $\mathcal{A}_{2i}$ by $A_{2i}(a),\ a \in H$, and the elements of $\mathcal{A}_{2i+1}$ by $A_{2i+1}(\psi),\ \psi \in \hat{H}$. The quasilocal algebra $\mathcal{A}_{loc}$ is defined to be the unital *-algebra with generators $\mathcal{A}_{2i}(a)$ and $A_{2i+1}(\psi),\ a \in H,\ \psi \in \hat{H},\ i \in \mathbb{Z}$ and commutation relations

$$
\begin{aligned}
AB &= BA, \quad A \in \mathcal{A}_i,\ B \in \mathcal{A}_j,\ |i-j| \geq 2 && (2.2a) \\
A_{2i+1}(\varphi) A_{2i}(a) &= A_{2i}(a_{(1)}) \langle a_{(2)}, \varphi_{(1)} \rangle A_{2i+1}(\varphi_{(2)}) && (2.2b) \\
A_{2i}(a) A_{2i-1}(\varphi) &= A_{2i-1}(\varphi_{(1)}) \langle \varphi_{(2)}, a_{(1)} \rangle A_{2i}(a_{(2)}) && (2.2c)
\end{aligned}
$$

Equation (2.2b) can be inverted to give

$$
\begin{aligned}
A_{2i}(a) A_{2i+1}(\varphi) &= A_{2i}(a_{(3)}) A_{2i+1}(\varphi) A_{2i}(S(a_{(2)}) a_{(1)}) \\
&= A_{2i}(a_{(4)}) A_{2i}(S(a_{(3)})) \langle S(a_{(2)}), \varphi_{(1)} \rangle A_{2i+1}(\varphi_{(2)}) A_{2i}(a_{(1)}) \qquad (2.3) \\
&= \langle S(a_{(2)}), \varphi_{(1)} \rangle A_{2i+1}(\varphi_{(2)}) A_{2i}(a_{(1)})
\end{aligned}
$$

and similarly for (2.2c). Using equ. (A.3) this formula can also be used to check that the relations (2.2b,c) respect the *-involution on $\mathcal{A}_{loc}$. We denote $\mathcal{A}_{n,m} \subset \mathcal{A}_{loc}$ the unital *-subalgebra generated by $\mathcal{A}_i,\ n \leq i \leq m$. For $m < n$ we also put $\mathcal{A}_{n,m} = \mathbb{C}\mathbf{1}$.

The above relations define what can be called a two-sided iterated crossed product, i.e.

$$
\mathcal{A}_{n-1,m+1} = \mathcal{A}_{n-1} \ltimes \mathcal{A}_{n,m} \rtimes \mathcal{A}_{m+1}
$$

where $\mathcal{A}_{m+1}$ acts on $\mathcal{A}_{n,m}$ from the left via

$$
A_{m+1}(a) \triangleright \mathcal{A}_{n,m} = A_{m+1}(a_{(1)}) \mathcal{A}_{n,m} A_{m+1}(S(a_{(2)})) \tag{2.4}
$$

and $\mathcal{A}_{n-1}$ acts on $\mathcal{A}_{n,m}$ from the right via

$$
\mathcal{A}_{n,m} \triangleleft A_{n-1}(a) = A_{n-1}(S(a_{(1)})) \mathcal{A}_{n,m} A_{n-1}(a_{(2)}) \tag{2.5}
$$

and where for all $n \leq m$ these two actions commute.

We now provide a *-representation of $\mathcal{A}_{n,m}$ on finite dimensional Hilbert spaces $\mathcal{H}_{n,m}$ proving that the algebras $\mathcal{A}_{n,m}$ are in fact finite dimensional C^*-algebras and that they arise as the invariant subalgebras in $\mathcal{H}_{n,m}$ under a global H-symmetry. Let $h \in H$ be the unique normalized Haar measure on $\hat{H}$, i.e. $h^2 = h^* = h$ and $h \rightarrow \varphi = \varphi \leftarrow h = \langle h, \varphi \rangle \mathbf{1}$ for all $\varphi \in \hat{H}$. We introduce the Hilbertspace $\mathcal{H} = L^2(\hat{H}, h)$ to be the $\mathbb{C}$- vector space $\hat{H}$ with scalar product

$$\langle \varphi | \psi \rangle := \langle h, \varphi^* \psi \rangle \tag{2.6}$$

Elements of $\mathcal{H}$ are denoted as $|\psi\rangle, \psi \in \hat{H}$. Following the notation of [N] we introduce the following operators in End $\mathcal{H}$

$$\begin{aligned} Q^+(\varphi)|\psi\rangle &:= |\varphi\psi\rangle \\ Q^-(\varphi)|\psi\rangle &:= |\psi\varphi\rangle \\ P^+(a)|\psi\rangle &:= |a \rightarrow \psi\rangle \\ P^-(a)|\psi\rangle &:= |\psi \leftarrow a\rangle \end{aligned} \tag{2.7}$$

where $a \in H$ and $\varphi, \psi \in \hat{H}$. Using the facts that on finite dimensional C^*-Hopf algebras h is tracial, $S(h) = h$ and $S^2 = id$ [W] one easily checks that

$$\begin{aligned} Q^\pm(\varphi)^* &= Q^\pm(\varphi^*) \\ P^\pm(a)^* &= P^\pm(a^*) \end{aligned} \tag{2.8}$$

Moreover $Q^\pm(\hat{H})' = Q^\mp(\hat{H})$ and $P^\pm(H)' = P^\mp(H)$, where the prime denotes the commutant in End $\mathcal{H}$. We also recall the well known fact (see [N] for a review) that $Q^\sigma(\hat{H}) \vee P^{\sigma'}(H) = \mathrm{End}\,\mathcal{H}$ for any choice of $\sigma, \sigma' \in \{+, -\}$.

We now place a copy $\mathcal{H}_n \simeq \mathcal{H}$ at each even lattice site, $n \in 2\mathbb{Z}$, and for $n \le m$ and $n, m \in 2\mathbb{Z}$ we put

$$\mathcal{H}_{n,m} := \mathcal{H}_n \otimes \mathcal{H}_{n+2} \otimes ... \otimes \mathcal{H}_m \tag{2.9}$$

We also use the obvious notations $Q_\nu^\pm(a)$ and $P_\nu^\pm(\varphi)$ to denote the operators acting on the tensor factor $\mathcal{H}_\nu$, $\nu \in 2\mathbb{Z}$. Let now $R_{n,m}$ be the global right action of H on $\mathcal{H}_{n,m}$ given by

$$R_{n,m}(a) = \prod_{i=0}^{\frac{m-n}{2}} P^-_{n+2i}(a_{(1+i)}) \quad , \quad a \in H. \tag{2.10}$$

and put $L_{n,m} := R_{n,m} \circ S$. We then have

Proposition 2.1: *Let $n, m \in 2\mathbb{Z}$, $n \le m$, and let $\pi_{n,m} : \mathcal{A}_{n,m} \rightarrow End\,\mathcal{H}_{n,m}$ be given by*

$$\begin{aligned} \pi_{n,m}(A_{2i}(a)) &= P^+_{2i}(a) \\ \pi_{n,m}(A_{2i+1}(\varphi)) &= Q^-_{2i}(S(\varphi_{(1)}))Q^+_{2i+2}(\varphi_{(2)}) \end{aligned} \tag{2.11}$$

*Then $\pi_{n,m}$ defines a faithful *-representation of $\mathcal{A}_{n,m}$ on $\mathcal{H}_{n,m}$ and $\pi_{n,m}(\mathcal{A}_{n,m}) = L_{n,m}(H)'$.*

Proof: We proceed by induction over $\nu = \frac{m-n}{2}$. For $\nu = 0$ the claim follows from $\pi_{n,n}(\mathcal{A}_{n,n}) = P_n^+(H) = P_n^-(H)'$. For $\nu \ge 1$ we use the Takesaki duality theorem for double cross products [Ta,NaTa] saying that $\mathcal{A}_{n,m+2} \simeq \mathcal{A}_{n,m} \otimes \mathrm{End}\,\mathcal{H} \simeq \mathcal{A}_{n,m} \otimes \mathcal{A}_{m+1,m+2}$ where the isomorphism is given by (see equ. (A.10) of Appendix A)

$$\begin{array}{rcl} \mathcal{T}:\mathcal{A}_{n,m+2} & \to & \mathcal{A}_{n,m}\otimes \mathrm{End}\,\mathcal{H} \\ \mathcal{T}(A) & = & A\otimes \mathbf{1} \\ \mathcal{T}(A_m(a)) & = & A_m(a_{(1)})\otimes P^-(S(a_{(2)})) \\ \mathcal{T}(A_{m+1}(\psi)) & = & \mathbf{1}\otimes Q^+(\psi) \\ \mathcal{T}(A_{m+2}(a)) & = & \mathbf{1}\otimes P^+(a) \end{array} \tag{2.12}$$

where $A \in \mathcal{A}_{n,m-1}$, $a \in H$ and $\psi \in \hat{H}$. Hence, by induction hypothesis $\hat{\pi}_{n,m+2} := (\pi_{n,m} \otimes id) \circ \mathcal{T}$ defines a faithful *-representation of $\mathcal{A}_{n,m+2}$ and $\hat{\pi}_{n,m+2}(\mathcal{A}_{n,m+2}) = (R_{n,m}(H) \otimes \mathbf{1})'$. We now identify $\mathcal{H} \equiv \mathcal{H}_{m+2}$ and construct a unitary $\hat{U} \in End(\mathcal{H}_{n,m+2})$ such that $\pi_{n,m+2} = \mathrm{Ad}\,\hat{U} \circ \hat{\pi}_{n,m+2}$ and $R_{n,m+2}(H) = \hat{U}(R_{n,m}(H) \otimes \mathbf{1})\hat{U}^*$ which proves our claim. To this end we put

$$\begin{array}{c} U:\mathcal{H}_m\otimes\mathcal{H}_{m+2}\to\mathcal{H}_m\otimes\mathcal{H}_{m+2} \\ U|\varphi\otimes\psi\rangle := |\varphi S(\psi_{(1)})\otimes\psi_{(2)}\rangle \end{array} \tag{2.13}$$

and define $\hat{U} = \mathbf{1}_n \otimes ... \otimes \mathbf{1}_{m-2} \otimes U$. We leave it to the reader to check that U is unitary and satisfies [1]

$$U^{-1}|\varphi\otimes\psi\rangle = |\varphi\psi_{(1)}\otimes\psi_{(2)}\rangle$$

Now $\hat{U}$ obviously commutes with $Q^+_m(\hat{H})$ and therefore with $\pi_{n,m}(\mathcal{A}_{n,m-1}) \otimes \mathbf{1}_{m+2}$, proving

$$Ad\,\hat{U}\circ\hat{\pi}_{n,m+2}|\mathcal{A}_{n,m-1} = \pi_{n,m+2}|\mathcal{A}_{n,m-1}$$

Similarly, $\hat{U}$ also commutes with $P^+_{m+2}(H)$, proving

$$Ad\,\hat{U}\circ\hat{\pi}_{n,m+2}|\mathcal{A}_{m+2} = \pi_{n,m+2}|\mathcal{A}_{m+2}$$

Next, we compute

$$\begin{array}{rcl} UQ^+_{m+2}(\chi)|\varphi\otimes\psi\rangle & = & |\varphi S(\psi_{(1)})S(\chi_{(1)})\otimes\chi_{(2)}\psi_{(2)}\rangle \\ & = & Q^-_m(S(\chi_{(1)}))Q^+_{m+2}(\chi_{(2)})U|\varphi\otimes\psi\rangle \end{array}$$

and

$$\begin{array}{rcl} \lefteqn{UP^+_m(a_{(1)})P^-_{m+2}(S(a_{(2)}))\,|\,\varphi\otimes\psi\rangle} & & \\ & = & \langle a_1,\varphi_{(2)}\rangle\langle S(a_{(2)}),\psi_{(1)}\rangle|\varphi_{(1)}S(\psi_{(2)})\otimes\psi_{(3)}\rangle \\ & = & \langle a,\varphi_{(2)}S(\psi_{(1)})\rangle|\varphi_{(1)}S(\psi_{(2)}\otimes\psi_{(3)}\rangle \\ & = & P^+_m(a)U|\varphi\otimes\psi\rangle \end{array}$$

proving that

$$Ad\,\hat{U}\circ\hat{\pi}_{n,m+2}|\mathcal{A}_{m,m+1} = \pi_{n,m+2}|\mathcal{A}_{m,m+1}$$

and therefore $\pi_{n,m+2} = Ad\,\hat{U} \circ \hat{\pi}_{n,m+2}$. Finally

$$\begin{array}{rcl} UP^-_m(a)U^*|\varphi\otimes\psi\rangle & = & \langle a,\varphi_{(1)}\psi_{(1)}\rangle U|\varphi_{(2)}\psi_{(2)}\otimes\psi_{(3)}\rangle \\ & = & \langle a_{(1)},\varphi_{(1)}\rangle\langle a_{(2)},\psi_{(1)}\rangle|\varphi_{(2)}\otimes\psi_{(2)}\rangle \\ & = & P^-_m(a_{(1)})P^-_{m+2}(a_{(2)})|\varphi\otimes\psi\rangle \end{array}$$

[1] Up to a change of left-right conventions U is a version of the pentagon operator (also called Takesaki operator or multiplicative unitary), see, e.g. [BS].

which proves $R_{n,m+2} = \mathrm{Ad}\, \hat{U} \circ (R_{n,m} \otimes \mathbf{1}_{m+2})$.   *Q.e.d.*

We remark at this point that iterated application of the Takesaki duality theorem immediately implies $\mathcal{A}_{i,j} \simeq (End\ \mathcal{H})^{\otimes\nu}$ whenever $j = i + 2\nu + 1$ and therefore the important *split property* of $\mathcal{A}$ (see subsection 2.2). We also remark that we could equally well interchange the role of H and $\hat{H}$ to define faithful *-representations $\pi_{n,m}$ of $\mathcal{A}_{n,m}$ for $n, m \in 2\mathbb{Z} + 1$, where now $\mathcal{H}_{2i+1} = L^2(H, \omega)$, $\omega \in \hat{H}$ being the Haar measure on H. In this way $\pi_{n,m}(\mathcal{A}_{n,m})$ for $n, m \in 2\mathbb{Z} + 1$ would appear as the invariant algebra under a global $\hat{H}$-symmetry.

Hence, depending on how we represent them, our local observable algebras seem to be the invariant algebras under either a global H-symmetry or a global $\hat{H}$-symmetry. It is the purpose of this work to show that in the thermodynamic limit both symmetries can be reconstructed from the category of "physical representations" of $\mathcal{A}$ (i.e. fulfilling an analogue of the Doplicher-Haag-Roberts selection criterion relative to some Haag dual vacuum representation). In a sense to be explained below H and $\hat{H}$ then reappear as *cosymmetries* of $\mathcal{A}$. Generalizing and improving the methods and results of [SzV] we will in fact prove that H and $\hat{H}$ combine to yield the *Drinfeld double* $\mathcal{D}(H)$ (see Appendix B for a review of definitions) as the *universal cosymmetry* of $\mathcal{A}$.

This should be understood as a generalization of the "order-disorder" symmetries in G-spin quantum chains, which are well known to appear for finite abelian groups G and which have been generalized to finite nonabelian groups G by [SzV]. The relation with our present formalism is obtained by letting $H = \mathbb{C}G$ be the group algebra. We then get $\hat{H} = Fun(G)$, the abelian algebra of $\mathbb{C}$-valued functions on G, and $\mathcal{H} = L^2(G, h)$, where $h = |G|^{-1}\Sigma_g\, g \in \mathbb{C}G$ is the Haar measure on $\hat{H}$. Hence $\mathcal{H}_{n,m} \cong L^2(G^{\frac{m-n}{2}})$, $m, n \in 2\mathbb{Z}$, and $\pi_{n,m}$ acts on $\psi \in \mathcal{H}_{n,m}$ by

$$\begin{aligned}(\pi_{n,m}(A_{2i}(a))\psi)(g_n, ..., g_{2i}, ..., g_m) &= \psi(g_n, ..., g_{2i}a, ..., g_m) \\ (\pi_{n,m}(A_{2i+1}(\varphi))\psi)(g_n, ...g_m) &= \varphi(g_{2i}^{-1}g_{2i+2})\psi(g_n, ..., g_m)\end{aligned}$$

These operators are immediately realized to be invariant under the global G-spin rotation

$$(L_{n,m}(a)\psi)(g_n, ..., g_m) = \psi(a^{-1}g_n, ..., a^{-1}g_m), \quad a \in G.$$

which would then be called the "order symmetry".

In this representation a "disorder-symmetry" can be defined as an action $\hat{L}_{n,m}$ of $\hat{H} = Fun(G)$

$$(\hat{L}_{n,m}(\varphi)\psi)(g_n, ..., g_m) := \varphi(g_n g_m^{-1})\psi(g_n, ..., g_m)$$

and it has been shown in [SzV] that $L_{n,m}$ and $\hat{L}_{n,m}$ together generate a representation of the Drinfeld double $\mathcal{D}(G)$. Note that in the limit $(n, m) \to (-\infty, \infty)$ all local observables are also invariant under (i.e. commute with) $\hat{L}_{n,m}(\hat{H})$. The generalization of $\hat{L}_{n,m}$ to arbitrary finite dimensional C^*-Hopf algebras is given by

Lemma 2.2.: *Let $n, m \in 2\mathbb{Z}$, $m \geq n + 2$, and let $\hat{L}_{n,m} : \hat{H} \to End(\mathcal{H}_{n,m})$ be the *-representation given by*

$$\hat{L}_{n,m}(\varphi) = Q_n^+(\varphi_{(1)})Q_m^-(S(\varphi_{(2)})) \tag{2.14}$$

*Then $L_{n,m}(H)$ and $\hat{L}_{n,m}(\hat{H})$ generate a faithful *-representation of the Drinfeld double $\mathcal{D}(H)$ on $\mathcal{H}_{n,m}$.*

Proof: Since $L_{n,m}$ and $\hat{L}_{n,m}$ define faithful *-representations of H and $\hat{H}$, respectively, we are left to show (see eqn. (B.1c)):

$$L_{n,m}(a_{(1)})\langle a_{(2)}, \varphi_{(1)}\rangle \hat{L}_{n,m}(\varphi_{(2)}) = \hat{L}_{n,m}(\varphi_{(1)})\langle \varphi_{(2)}, a_{(1)}\rangle L_{n,m}(a_{(2)})$$

for all $a \in H$ and $\varphi \in \hat{H}$. For $m = n+2$ this is a straight forward calculation using the "Weyl algebra relations" [N]

$$\begin{aligned} P^-(a)Q^+(\varphi) &= Q^+(\varphi_{(2)})P^-(a_{(2)})\langle a_{(1)}, \varphi_{(1)}\rangle \\ P^-(a)Q^-(\varphi) &= Q^-(\varphi_{(2)})P^-(a_{(1)})\langle a_{(2)}, \varphi_{(1)}\rangle \end{aligned}$$

and the identities $\Delta \circ S = (S \otimes S) \circ \Delta_{op}$ and $S^2 = id$. For $m \geq n+4$ we proceed by induction and define the unitary

$$V : \mathcal{H}_{m-2} \otimes \mathcal{H}_m \to \mathcal{H}_{m-2} \otimes \mathcal{H}_m$$

$$V|\varphi \otimes \psi\rangle := |S(\psi_{(1)}) \otimes \psi_{(2)}\varphi\rangle$$

Then $VQ^-_{m-2}(\varphi) = Q^-_m(\varphi)V$ and $VP^-_{m-2}(a) = P^-_{m-2}(a_{(1)})P^-_{m-2}(a_{(2)})V$ for all $\psi \in \hat{H}$ and $a \in H$. Hence

$$\begin{aligned} \mathrm{Ad}\,\hat{V} \circ (L_{n,m-2} \otimes \mathbf{1}_m) &= L_{n,m} \\ \mathrm{Ad}\,\hat{V} \circ (\hat{L}_{n,m-2} \otimes \mathbf{1}_m) &= \hat{L}_{n,m} \end{aligned}$$

where $\hat{V} = \mathbf{1}_n \otimes \cdots \otimes \mathbf{1}_{m-4} \otimes V$, which proves the claim by induction. *Q.e.d.*

We remark that interchanging even and odd lattice sites in Lemma 2.2 we similarly obtain a representation of $\mathcal{D}(\hat{H})$. Now recall that for abelian groups G there is a well known duality transformation which consists of interchanging the role of $H = \mathbb{C}G$ and $\hat{H} = \mathbb{C}\hat{G}$ by simultaneously also interchanging the role of even an odd lattice sites and of order and disorder symmetries, respectively. For nonabelian groups G the dual algebra $\hat{H}$ is no longer a group algebra and at first sight the good use or even the notion of a duality transformation seems to be lost. It is the advantage of our more general Hopf algebraic framework to restore this apparent asymmetry and treat both, H and $\hat{H}$, on a completely equal footing. In particular we also point out that as algebras the Drinfeld doubles $\mathcal{D}(H)$ and $\mathcal{D}(\hat{\mathcal{H}})$ coincide (it is only the coproduct which changes into its opposite, see Appendix B). Hence, from an algebraic point of view there is no intrinsic difference between "order" and "disorder" (co-)symmetries. Distinguishing one from the other only makes sense with respect to a particular choice of the representations given in Lemma 2.2 on the Hilbert spaces associated with even or odd lattice sites, respectively.

2.2 $\mathcal{A}$ as a Haag Dual Net

The local commutation relations (2.3) of the observables suggests that our Hopf spin model can be viewed in the more general setting of algebraic quantum field theory (AQFT) as a local net. More precisely we will use an implementation of AQFT appropriate to study lattice models in which the local algebras are finite dimensional. Although we borrow the language and philosophy of AQFT, the concrete mathematical notions we need on the lattice are quite different from the analogue notions one uses in QFT on Minkowski space.

Let $\mathcal{I}$ denote the set of closed finite subintervals of $I\!\!R$ with endpoints in $Z\!\!\!Z + \frac{1}{2}$. A net of finite dimensional C^*-algebras, or shortly a *net* is a correspondence $I \mapsto \mathcal{A}(I)$ associating to each interval $I \in \mathcal{I}$ a finite dimensional C^*-algebra $\mathcal{A}(I)$ together with unital inclusions $\iota_{J,I}: \mathcal{A}(I) \to \mathcal{A}(J)$, whenever $I \subset J$, such that for all $I \subset J \subset K$ one has $\iota_{K,J} \circ \iota_{J,I} = \iota_{K,I}$. For $I = \emptyset$ we put $\mathcal{A}(\emptyset) = \mathbb{C}\mathbf{1}$.

The inclusions $\iota_{J,I}$ will be suppressed and for $I \subset J$ we will simply write $\mathcal{A}(I) \subset \mathcal{A}(J)$. If Λ is any (possibly infinite) subset of $I\!\!R$ we write $\mathcal{A}(\Lambda)$ for the C^*-inductive limit of $\mathcal{A}(I)$-s with $I \subset \Lambda$:

$$\mathcal{A}(\Lambda) := \vee_{I \subset \Lambda} \mathcal{A}(I).$$

Especially let $\mathcal{A} = \mathcal{A}(I\!\!R)$. As a dense subalgebra of $\mathcal{A}$ we denote

$$\mathcal{A}_{loc} = \cup_{I \in \mathcal{I}} \mathcal{A}(I).$$

The choice of the lattice $Z\!\!\!Z + \frac{1}{2}$ (in place of $Z\!\!\!Z$, say) is merely a matter of notational convenience. In the case of our Hopf spin model we put

$$\mathcal{A}(I) = \vee_{i \in I \cap Z\!\!\!Z}\, \mathcal{A}_i$$

and $\mathcal{A}(I) = \mathbb{C}\mathbf{1}$ if $I \cap Z\!\!\!Z = \emptyset$.

Next, for $\Lambda \subset I\!\!R$ let $\Lambda' = \{x \in I\!\!R | dist(x, \Lambda) \geq 1\}$ which is the analogue of the "spacelike complement" of Λ (for $\Lambda = \emptyset$ put $\Lambda' = I\!\!R$). The net $\{\mathcal{A}(I)\}$ is called *local* if $I \subset J'$ implies $\mathcal{A}(I) \subset \mathcal{A}(J)'$, $\forall I, J \in \mathcal{I}$, where for $\mathcal{B} \subset \mathcal{A}$ we denote $\mathcal{B}' \equiv \mathcal{B}' \cap \mathcal{A}$ the commutant of $\mathcal{B}$ in $\mathcal{A}$. For $\Lambda \subset I\!\!R$ we also denote

$$\begin{aligned} \Lambda^c &:= I\!\!R \setminus \Lambda \\ \bar{\Lambda} &:= \Lambda'^c \\ \operatorname{Int} \Lambda &:= \Lambda^{c'} \\ \partial \Lambda &= \bar{\Lambda} \setminus \operatorname{Int} \Lambda = \bar{\Lambda} \cap \Lambda^c \end{aligned} \tag{2.15}$$

The net $\{\mathcal{A}(I)\}$ is called *split* if for all $I \in \mathcal{I}$ there exists a $J \in \mathcal{I}$ such that $J \supset I$ and $\mathcal{A}(J)$ is simple. The net is called additive, if $\mathcal{A}(I) \vee \mathcal{A}(J) = \mathcal{A}(I \cup J)$ for all $I, J \subset I$, where $M \vee N$ denotes the C^*-subalgebra of $\mathcal{A}$ generated by the subalgebras $M, N \subset \mathcal{A}$. The net is said to satisfy the intersection property if $\mathcal{A}(I) \cap \mathcal{A}(J) = \mathcal{A}(I \cap J)$ for all $I, J \in \mathcal{I}$.

The local observable algebras $\{\mathcal{A}(I)\}$ of the Hopf spin model defined in subsection 2.1 provide an example of a local additive split net with intersection property. What is not so obvious is that this net satisfies *algebraic Haag duality.*

Definition 2.3: The net $\{\mathcal{A}(I)\}$ is said to satisfy (algebraic) Haag duality if

$$\mathcal{A}(I')' = \mathcal{A}(I) \quad \forall I \in \mathcal{I}$$

To prove Haag duality for our model it is useful to introduce a non-commutative analogue of a family of local Gibbs measures in classical statistical lattice models.

Definition 2.4: A *quantum Gibbs system* on the net $\{\mathcal{A}(I)\}$ is a family of conditional expectations $\eta_I : \mathcal{A} \to \mathcal{A}(I)'$ such that for all $I, J \in \mathcal{I}$ the following conditions hold

i)	$\eta_I \circ \eta_J = \eta_I,$	if $J \subset I$
ii)	$\eta_I(\mathcal{A}(J)) \subset \mathcal{A}(I' \cap J),$	if $I \not\subset J$

We will now show that the existence of a quantum Gibbs system on $\{\mathcal{A}(I)\}$ is already sufficient to prove Haag duality. Since we think that our methods might also be useful in higher dimensional models, we will keep our arguments quite general. First we introduce a *wedge* W as the union

$$W = \cup_n I_n$$

where $I_n \subset I_{n+1}$ is an unbounded increasing sequence in $\mathcal{I}$ with the so-called *wedge property* saying that for all $J \in \mathcal{I}$ the sequence $I_n' \cap J$ eventually becomes constant. Putting $W' = \cap_n I_n'$ we now have the following

Proposition 2.5: *Assume that the net $\{\mathcal{A}(I)\}$ admits a quantum Gibbs system $\eta_I : \mathcal{A} \to \mathcal{A}(I)'$. Then $\mathcal{A}$ satisfies*

i) Wedge duality, i.e. $\mathcal{A}(W)' = \mathcal{A}(W')$ for all wedges W.

ii) The intersection property for wedge complements, i.e. $\mathcal{A}(W' \cap \Lambda) = \mathcal{A}(W') \cap \mathcal{A}(\Lambda)$ for all wedges W and intervals or wedges Λ.

iii) Haag duality for intervals, i.e. $\mathcal{A}(I')' = \mathcal{A}(I) \quad \forall I \in \mathcal{I}$.

Proof: i) By locality we have $\mathcal{A}(W') \subset \mathcal{A}(W)'$. Now let $I_n \subset I_{n+1} \in \mathcal{I}$ and $W = \cup_n I_n$. We define

$$\eta_W := \lim_n \ \eta_{I_n}$$

We show that the limit exists on $\mathcal{A}$ and defines a conditional expectation $\eta_W : \mathcal{A} \to \mathcal{A}(W)'$. First the limit exists pointwise on $\mathcal{A}(J)$ for each $J \in \mathcal{I}$, since there exists $n_0 > 0$ such that $I_{n_0} \not\subset J$ and

$$W' \cap J = I_n' \cap J = I_{n_0}' \cap J$$

for all $n \geq n_0$. Hence, by Definition 2.4i), we get for all $n \geq n_0$ and $A \in \mathcal{A}(J)$

$$\eta_{I_n}(A) = \eta_{In} \circ \eta_{I_{n_0}}(A) = \eta_{I_{n_0}}(A)$$

since $\eta_{I_{n_0}}(A) \in \mathcal{A}(I_{n_0}' \cap J) = \mathcal{A}(I_n' \cap J) \subset \mathcal{A}(I_n)'$. Thus $\eta_{I_n}(A)$ eventually becomes constant for all $A \in \mathcal{A}(J)$ and all $J \in \mathcal{I}$ and we get

$$\eta_W(\mathcal{A}(J)) \subset \mathcal{A}(W' \cap J) \quad \forall J \in \mathcal{I}$$

Hence η_W exists on $\mathcal{A}_{loc}$ and is positive and bounded by 1 since all η_{I_n} have this property. Thus η_W may be extended to all of $\mathcal{A}$. A simple 3ε-argument shows that the extension still satisfies

$$\eta_W(A) = \lim_n \eta_{I_n}(A) \quad \forall A \in \mathcal{A}.$$

In particular we conclude

$$\eta_W(\mathcal{A}) \subset \mathcal{A}(W')$$

On the other hand since $I_n \subset W$ we get $\mathcal{A}(W)' \subset \mathcal{A}(I_n)'$ and hence $\eta_W(A) = A$ for all $A \in \mathcal{A}(W)'$. This proves $\mathcal{A}(W)' \subset \mathcal{A}(W')$ and therefore $\mathcal{A}(W)' = \mathcal{A}(W') = \eta_W(\mathcal{A})$.

ii) By the above arguments we have

$$\eta_W(\mathcal{A}(\Lambda)) \subset \mathcal{A}(W' \cap \Lambda) \quad \text{for all } \Lambda \in \mathcal{I}$$

and since η_W is a conditional expectation onto $\mathcal{A}(W') = \mathcal{A}(W)'$ we get $\eta_W(A) = A$ for all $A \in \mathcal{A}(W') \cap \mathcal{A}(\Lambda)$ implying $\mathcal{A}(W') \cap \mathcal{A}(\Lambda) \subset \mathcal{A}(W' \cap \Lambda)$. The inverse inclusion again follows from locality. Continuity of η_W allows to push this argument from intervals Λ to wedges Λ.

iii) Let $I \in \mathcal{I}$ and let W_1 and W_2 be two wedges such that $I' = W_1 \cup W_2'$. Then $\mathcal{A}(W_1) \vee \mathcal{A}(W_2') \subset \mathcal{A}(I')$ and hence $\mathcal{A}(I')' \subset \mathcal{A}(W_1') \cap \mathcal{A}(W_2) = \mathcal{A}(W_1' \cap W_2) = \mathcal{A}(I)$ where we have used wedge duality and the intersection property for wedge complements. *Q.e.d.*

We remark that in Proposition 2.5i) we may put $W = I\!R$ to conclude that $\mathcal{A}$ has trivial center,

$$\mathcal{A}' = \mathcal{A}(W') = \mathcal{A}(\emptyset) = \mathbb{C}1\!\!1 \ .$$

We now provide a quantum Gibbs system on our Hopf spin model by defining for any $I \in \mathcal{I}$ and $A \in \mathcal{A}$

$$\eta_I(A) := \sum_r \frac{1}{n_r} \sum_{a,b=1}^{n_r} e_r^{ab} A e_r^{ba} \tag{2.16}$$

where r runs through the simple components $M_r \simeq Mat(n_r)$ of $\mathcal{A}(I)$ and e_r^{ab} is a system of matrix units in M_r. One immediately checks that $\eta_I : \mathcal{A} \to \mathcal{A}(I)'$ defines a conditional expectation. Moreover $\eta_I(\mathcal{A}(J)) \subset \mathcal{A}(I)' \cap \mathcal{A}(J \cup I)$. We now prove

Lemma 2.6: *The family $(\eta_I)_{I\in\mathcal{I}}$ provides a quantum Gibbs system on the Hopf spin model.*

Proof: By continuity it is enough to prove property i) of Definition 2.2 on $\mathcal{A}_{loc}$. Hence let $J \subset I$ be two intervals and let $A \in \mathcal{A}(\Lambda), \Lambda \in \mathcal{I}$, where without loss $I \cup J \subset \Lambda$. Pick a faithful trace tr_Λ on $\mathcal{A}(\Lambda)$ and define the Hilbert-Schmidt scalar product $\langle A|B\rangle := tr_\Lambda(A^*B),\ A, B \in \mathcal{A}(\Lambda)$. We clearly have $tr_\Lambda(B\eta_I(A)) = tr_\Lambda(BA)$ for all $I \subset \Lambda, B \in \mathcal{A}(I)' \cap \mathcal{A}(\Lambda)$ and $A \in \mathcal{A}(\Lambda)$. Hence, for $I \subset \Lambda$ the restriction $\eta_I|\mathcal{A}(\Lambda)$ is an orthogonal projection onto $\mathcal{A}(\Lambda) \cap \mathcal{A}(I)'$ with respect to $\langle\cdot|\cdot\rangle$. Since $J \subset I$ implies $\mathcal{A}(I)' \subset \mathcal{A}(J)'$ we conclude

$$\eta_I|\mathcal{A}(\Lambda) = \eta_I \circ \eta_J|\mathcal{A}(\Lambda)$$

To prove property ii) let $I \not\subset J$ (implying $I \neq \emptyset$). For $\mathcal{A}(J) = \mathbb{C} \cdot 1\!\!1$ or $\mathcal{A}(I) = \mathbb{C} \cdot 1\!\!1$ the statement is trivial, hence assume $|I| \geq 1$ and $\mathcal{A}(J) = \mathcal{A}_{i,j}$ for some $i \leq j \in \mathbb{Z}$. Using property i) the claim ii) is now equivalent to

$$\begin{aligned} \eta_{i-1}(\mathcal{A}_{i,j}) &= \mathcal{A}_{i+1,j} \\ \eta_{j+1}(\mathcal{A}_{i,j}) &= \mathcal{A}_{i,j-1} \end{aligned} \tag{2.17}$$

where for $I = [i - \frac{1}{2}, i + \frac{1}{2}]$ we write $\eta_I \equiv \eta_i$. Using additivity we have $\mathcal{A}_{i,j} = \mathcal{A}_i \vee \mathcal{A}_{i+1,j} = \mathcal{A}_{i,j-1} \vee \mathcal{A}_j$ and hence (2.17) is equivalent to

$$\eta_i(\mathcal{A}_{i\pm1}) = \mathbb{C} \cdot \mathbf{1}, \qquad \forall i \in \mathbb{Z} \tag{2.18}$$

Let us prove (2.18) for i =even. (For odd i-s the proof is quite analogous.) Choose C^*-matrix units e_r^{ab} of the algebra H. For $r = \varepsilon$, the trivial representation (counit) of H, we have $ae_\varepsilon = e_\varepsilon a = \varepsilon(a)e_\varepsilon$, hence $e_\varepsilon \equiv h$ is just the integral in H (see Appendix A). We now use the following

Lemma 2.7: *Let $\mathbf{B} := (id \otimes S)(\Delta(h)) \in H \otimes H$. Then for finite dimensional C^*-Hopf algebras H we have*

$$\mathbf{B} = (S \otimes \mathrm{id}\,)(\Delta(h)) = \sum_r \frac{1}{n_r} \sum_{a,b} e_r^{ab} \otimes e_r^{ba} \tag{2.19}$$

Proof: By the Appendix A2 of [W] the Haar measure $\omega \in \hat{H}$ is given on H by

$$\omega(e_r^{ab}) = \delta^{ab} \tag{2.20}$$

where the normalization is fixed to $\omega(h) = 1$. Also, $\omega \circ S = \omega$. Let $F_\omega : H \to \hat{H}$ denote the Fourier transformation

$$\langle F_\omega(a), b\rangle := \omega(ab) \equiv \omega(ba) \tag{2.21}$$

Then $F_\omega = \hat{S} \circ F_\omega \circ S$. The inverse Fourier transformation is given by

$$F_\omega^{-1}(\psi) = (\psi \otimes \mathrm{id}\,)(\mathbf{B}) \tag{2.22}$$

(see [N] for a review on Fourier transformations) implying $(S \otimes S)(\mathbf{B}) = \mathbf{B}$. Let $D_r^{ab} \in \hat{H}$ be the basis dual to $\{e_r^{ab}\}$. Then by (2.20)

$$D_r^{ab} = F_\omega(\frac{1}{n_r} e_r^{ab}) \tag{2.23}$$

and Lemma 2.7 follows from (2.22/23) and the identity $S^2 = \mathrm{id}$ [W]. *Q.e.d.*

From equ. (2.19) one recognizes that η_i evaluated on $\mathcal{A}_{i\pm 1}$ is nothing but the adjoint action of the integral h on the dual Hopf algebra $\hat{H}$. Consider the case of $\mathcal{A}_{i-1}$:

$$\begin{aligned} \eta_i(A_{i-1}(\varphi)) &= \sum_r \frac{1}{n_r} \sum_{a,b} A_i(e_r^{ab}) A_{i-1}(\varphi) A_i(e_r^{ba}) \\ &= A_i(h_{(1)}) A_{i-1}(\varphi) A_i(S(h_{(2)})) \\ &= A_{i-1}(h \rightharpoonup \varphi) = \mathbf{1}\langle \varphi | h\rangle \end{aligned}$$

The case of $\mathcal{A}_{i+1}$ can be handled similarly. *Q.e.d.*

Summarizing: The local net $\{\mathcal{A}(I)\}$ of the Hopf spin model is an additive split net satisfying Haag duality and wedge duality. Furthermore the global observable algebra $\mathcal{A}$ is simple, because the split property implies that $\mathcal{A}$ is an UHF algebra and every UHF algebra is simple [Mu].

We finally remark without proof that the inclusion tower $\mathcal{A}_{i,j} \subset \mathcal{A}_{i,j+1}$, $j \geq i$ (or $\mathcal{A}_{i-1,j} \supset \mathcal{A}_{i,j}$, $i \leq j$) together with the family of conditional expectation $\eta_{j+1} : \mathcal{A}_{i,j} \to \mathcal{A}_{i,j-1}$ ($\eta_{i-1} : \mathcal{A}_{i,j} \to \mathcal{A}_{i+1,j}$) precisely arises by the basic Jones construction [J] from the conditonal expectations $\eta_{i\pm 1} : \mathcal{A}_i \to \mathbb{C} \cdot \mathbf{1}$. In particular, putting $e_{2i} = A_{2i}(h)$ and $e_{2i+1} = A_{2i+1}(\omega)$, where $h = h^* = h^2 \in H$ and $\omega = \omega^* = \omega^2 \in \hat{H}$ are the normalized integrals, we find the Temperley-Lieb-Jones algebra

$$\begin{aligned} &e_i^2 = e_i^* = e_i \\ &e_i e_j = e_j e_i, \quad |i - j| \geq 2 \\ &e_i e_{i\pm 1} e_i = (\dim H)^{-1}\, e_i \end{aligned} \tag{2.24}$$

3 Amplimorphisms and Cosymmetries

In this Section we pick up the methods of [SzV] to reformulate the DHR-theory of superselection sectors for locally finite dimensional quantum chains using the category of amplimorphisms $\mathbf{Amp}\,\mathcal{A}$.

In Section 3.1 we shortly review the notions and results of [SzV] and introduce the important concept of *compressibility* saying that up to equivalence all amplimorphisms can be localized in a common finite interval I. In Section 3.2 we consider the special class of amplimorphisms given by localized coactions of some Hopf algebra $\mathcal{G}$ on $\mathcal{A}$. We call such coactions *cosymmetries*.

Sections 3.3 and 3.4 investigate some general conditions under which universal cosymmetries exist on a given net $\mathcal{A}$. Here an amplimorphism ρ is called *universal*, if it is a sum of pairwise inequivalent and irreducible amplimorphisms, one from each equivalence class in $\mathbf{Amp}\,\mathcal{A}$. In Section 3.3 we look at properties of *effective cosymmetries* and use these to show that a universal amplimorphism becomes a cosymmetry (with respect to suitable coproduct on $\mathcal{G}$) if and only if the intertwiner space $(\rho\times\rho|\rho)$ is "scalar", i.e. contained in $1\!\!1_{\mathcal{A}}\otimes\mathrm{Hom}\,(V_\rho\to V_\rho\otimes V_\rho)$. With this result we can prove in Section 3.4 that universal cosymmetries always exist in models which are *completely compressible*. We show that Haag dual split nets (like the Hopf spin chain) are completely compressible iff they are compressible. Compressibility of the Hopf spin chain will then be stated in Theorem 3.12. It will be proven in Section 4.2, where we show that all amplimorphisms of this model are in fact compressible into any interval of length two.

In Section 3.5 we investigate the question of *uniqueness* of universal cosymmetries. We prove that (up to automorphisms of $\mathcal{G}$) universal coactions are always *cocycle equivalent* where we use a more general definition of this terminology as compared to the mathematics literature (e.g. [Ta,NaTa]). In particular this means that the coproduct of a universal cosymmetry $\mathcal{G}$ on $\mathcal{A}$ is only determined up to cocycle equivalence.

In Section 3.6 we discuss two notions of translation covariance for universal coactions and relate these to the existence of a *coherently translation covariant* structure in $\mathbf{Amp}\,\mathcal{A}$.

3.1 The categories $\mathbf{Amp}\,\mathcal{A}$ and $\mathbf{Rep}\,\mathcal{A}$

In this subsection $\{\mathcal{A}(I)\}$ denotes a split net of finite dimensional C^*-algebras which satisfies algebraic Haag duality. Furthermore we assume that the net is *translation covariant*. That is the net is equipped with a *-automorphism $\alpha\in\mathrm{Aut}\,\mathcal{A}$ such that

$$\alpha(\mathcal{A}(I))=\mathcal{A}(I+2)\qquad I\in\mathcal{I}\,. \tag{3.1}$$

At first we recall some notions introduced in [SzV]. An *amplimorphism* of $\mathcal{A}$ is an injective C^*-algebra map

$$\mu:\mathcal{A}\to\mathcal{A}\otimes\mathrm{End}V \tag{3.2}$$

where V is some finite dimensional Hilbert space. If $\mu(\mathbf{1})=\mathbf{1}\otimes 1_V$ then μ is called *unital*. Here we will restrict ourselves to unital amplimorphisms since the localized amplimorphisms in a split net are all equivalent to unital ones (see Thm. 4.13 in [SzV]). An amplimorphism μ is called *localized* within $I\in\mathcal{I}$ if

$$\mu(A)=A\otimes 1_V\qquad A\in\mathcal{A}(I^c)$$

where $I^c:=I\!R\setminus I$. For simplicity, from now on by an amplimorphism we will always mean a localized unital amplimorphism.

The space of *intertwiners* from $\nu : \mathcal{A} \to \mathcal{A} \otimes \mathrm{End}\, W$ to $\mu : \mathcal{A} \to \mathcal{A} \otimes \mathrm{End}\, V$ is

$$(\mu|\nu) := \{ T \in \mathcal{A} \otimes \mathrm{Hom}(W, V) \mid \mu(A)T = T\nu(A),\ A \in \mathcal{A} \} \tag{3.3}$$

Two amplimorphisms μ and ν are called *equivalent*, $\mu \sim \nu$, if there exists an isomorphism $U \in (\mu|\nu)$, that is an intertwiner U satisfying $U^*U = \mathbf{1} \otimes 1_W$ and $UU^* = \mathbf{1} \otimes 1_V$. Let μ be localized within I. Then μ is called *transportable* if for all integer a there exists a ν localized within $I + 2a$ and such that $\nu \sim \mu$. μ is called *translation covariant* if $(\alpha^a \otimes \mathrm{id}_V) \circ \mu \circ \alpha^{-a} \sim \mu$ for all $a \in \mathbb{Z}$. Clearly, α-covariance implies transportability.

Let $\mathbf{Amp}\,\mathcal{A}$ denote the category with objects given by the localized unital amplimorphisms μ and with arrows from ν to μ given by the intertwiners $T \in (\mu|\nu)$. This category has the following *monoidal product* :

$$\begin{array}{rcl} (\mu,\ \nu)\ \mapsto\ \mu \times \nu & := & (\mu \otimes \mathrm{id}_{\mathrm{End}\, W}) \circ \nu \,:\, \mathcal{A} \to \mathcal{A} \otimes \mathrm{End}\, V \otimes \mathrm{End}\, W \\ T_1 \in (\mu_1|\nu_1), T_2 \in (\mu_2|\nu_2) & \mapsto & T_1 \times T_2 := (T_1 \otimes 1_{V_2})(\nu_1 \otimes \mathrm{id}_{\mathrm{Hom}\,(W_2,V_2)})(T_2) \\ & \in & (\mu_1 \times \mu_2|\nu_1 \times \nu_2) \end{array} \tag{3.4}$$

with the monoidal unit being the trivial amplimorphism $\mathrm{id}_{\mathcal{A}}$. The monoidal product $\times$ is a bifunctor therefore we have $(T_1 \times T_2)(S_1 \times S_2) = T_1S_1 \times T_2S_2$, for all intertwiners for which the products are defined, and $1_\mu \times 1_\nu = 1_{\mu\times\nu}$ where $1_\mu := \mathbf{1} \otimes \mathrm{id}_V$ is the unit arrow at the object $\mu : \mathcal{A} \to \mathcal{A} \otimes \mathrm{End}\, V$.

$\mathbf{Amp}\,\mathcal{A}$ contains *direct sums* $\mu \oplus \nu$ of any two objects: $\mu \oplus \nu(A) := \mu(A) \oplus \nu(A)$ defines a direct sum for any orthogonal direct sum $V \oplus W$.

$\mathbf{Amp}\,\mathcal{A}$ has *subobjects*: If $P \in (\mu|\mu)$ is a Hermitean projection then there exists an object ν and an injection $S \in (\mu|\nu)$ such that $SS^* = P$ and $S^*S = 1_\nu$. The existence of subobjects is a trivial statement in the category of all, possibly non-unital, amplimorphisms because ν can be chosen to be $\nu(A) = P\mu(A)$ in that case. In the category $\mathbf{Amp}\,\mathcal{A}$ this is a non-trivial theorem which can be proven [SzV] provided the net is split. An amplimorphism μ is called *irreducible* if the only (non-zero) subobject of μ is μ. Equivalently, μ is irreducible if $(\mu|\mu) = \mathbb{C}1_\mu$. Since the selfintertwiner space $(\mu|\mu)$ of any localized amplimorphism is finite dimensional (use Haag duality to show that any $T \in (\mu|\mu)$ belongs to $\mathcal{A}(\mathrm{Int}\, I) \otimes \mathrm{End}\, V$ where I is the interval where μ is localized, see also Lemma 3.8 below), the category $\mathbf{Amp}\,\mathcal{A}$ is *fully reducible*. That is any object is a finite direct sum of irreducible objects. The category $\mathbf{Amp}\,\mathcal{A}$ is called *rigid* if for any object μ there exists an object $\overline{\mu}$ and intertwiners $C_\mu \in (\overline{\mu} \times \mu \,|\, \mathrm{id}_{\mathcal{A}})$, $\overline{C}_\mu \in (\mu \times \overline{\mu} \,|\, \mathrm{id}_{\mathcal{A}})$ satisfying

$$\begin{array}{rcl} (\overline{C}^*_\mu \times \mathbf{1}_\mu)(\mathbf{1}_\mu \times C_\mu) & = & \mathbf{1}_\mu \\ (\mathbf{1}_{\overline{\mu}} \times \overline{C}^*_\mu)(C_\mu \times \mathbf{1}_{\overline{\mu}}) & = & \mathbf{1}_{\overline{\mu}} \end{array} \tag{3.5}$$

Two full subcategories $\mathbf{Amp}_1\mathcal{A}$ and $\mathbf{Amp}_2\mathcal{A}$ of $\mathbf{Amp}\,\mathcal{A}$ are called *equivalent*, $\mathbf{Amp}_1\mathcal{A} \sim \mathbf{Amp}_2\mathcal{A}$, if any object in $\mathbf{Amp}_1\mathcal{A}$ is equivalent to an object in $\mathbf{Amp}_2\mathcal{A}$ and vice versa. For $I \in \mathcal{I}$ we denote $\mathbf{Amp}\,(\mathcal{A}, I) \subset \mathbf{Amp}\,\mathcal{A}$ the full subcategory of amplimorphisms localized in I. We say that $\mathbf{Amp}\,\mathcal{A}$ is *compressible* (into I) if there exists $I \in \mathcal{I}$ such that $\mathbf{Amp}\,\mathcal{A} \sim \mathbf{Amp}\,(\mathcal{A}, I)$. Clearly, if $\mathbf{Amp}\,\mathcal{A}$ is compressible into I then it is compressible into $I + 2a$, $\forall a \in \mathbb{Z}$. This follows, since the translation automorphism $\alpha \in \mathrm{Aut}\,\mathcal{A}$ induces an autofunctor $\underline{\alpha}$ on $\mathbf{Amp}\,\mathcal{A}$ given on objects by $\rho \mapsto \rho^\alpha := (\alpha \otimes \mathrm{id}) \circ \rho \circ \alpha^{-1}$ and on intertwiners by $T \mapsto (\alpha \otimes \mathrm{id})(T)$. Hence $\underline{\alpha}(\mathbf{Amp}\,(\mathcal{A}, I)) = \mathbf{Amp}\,(\mathcal{A}, I + 2)$. Moreover, we have

Lemma 3.1: *Let* $\mathbf{Amp}\,\mathcal{A}$ *be compressible into* $I \in \mathcal{I}$ *and let* $J \supset I + 2a$ *for some* $a \in \mathbb{Z}$. *Then all amplimorphisms in* $\mathbf{Amp}\,(\mathcal{A}, J)$ *are transportable.*

Proof: Let $\{\rho_r : \mathcal{A} \to \mathcal{A} \otimes \mathrm{End}\, V_r\}$ be a complete list of pairwise inequivalent irreducible amplimorhisms in $\mathbf{Amp}\,(\mathcal{A}, I)$ and put $\rho = \oplus_r \rho_r$. Then $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$, $\mathcal{G} := \oplus_r \mathrm{End}\, V_r$, is *universal* in $\mathbf{Amp}\,\mathcal{A}$, i.e. every $\mu \in \mathbf{Amp}\,\mathcal{A}$ is equivalent to $(\mathrm{id}_{\mathcal{A}} \otimes \beta) \circ \rho$ for some $\beta \in \mathbf{Rep}\,\mathcal{G}$. Moreover, $\rho^\alpha \in \mathbf{Amp}\,(\mathcal{A}, I+2)$ is also universal and therefore $\rho^\alpha = \mathrm{Ad}\, W \circ (\mathrm{id} \otimes \sigma) \circ \rho$ for some unitary $W \in \mathcal{A} \otimes \mathcal{G}$ and some $\sigma \in \mathrm{Aut}\, \mathcal{G}$. Let now $J \supset I$ and $\mu = \mathrm{Ad}\, U \circ (\mathrm{id}_{\mathcal{A}} \otimes \beta) \circ \rho \in \mathbf{Amp}\,(\mathcal{A}, J)$. Then, by Haag duality, $U \in \mathcal{A}(\mathrm{Int}\, J) \otimes \mathrm{End}\, V_\mu$, since U must commute with $\mathcal{A}(J^c) \otimes \mathbf{1}$. With $\sigma \in \mathrm{Aut}\, \mathcal{G}$ defined as above put $\tilde{\mu} := \mathrm{Ad}\, U \circ (\mathrm{id}_{\mathcal{A}} \otimes \tilde{\beta}) \circ \rho \in \mathbf{Amp}\,(\mathcal{A}, J)$, where $\tilde{\beta} := \beta \circ \sigma^{-1}$. Then $\tilde{\mu}^\alpha \equiv (\alpha \otimes \mathrm{id}\,) \circ \tilde{\mu} \circ \alpha^{-1} \in \mathbf{Amp}\,(\mathcal{A}, J+2)$ satifies

$$\tilde{\mu}^\alpha = \mathrm{Ad}\, \tilde{U} \circ (\mathrm{id}_{\mathcal{A}} \otimes \beta) \circ \rho = \mathrm{Ad}\, (\tilde{U} U^*) \circ \mu,$$

where $\tilde{U} = (\alpha \otimes \mathrm{id}\,)(U)(\mathrm{id}_{\mathcal{A}} \otimes \tilde{\beta})(W) \in \mathcal{A} \otimes \mathrm{End}\, V_\mu$ is unitary. Thus μ is transportable into $J+2$ and analogously into $J-2$ and therefore into $J + 2a$, $a \in 2\mathbb{Z}$. *Q.e.d.*

We remark that even if μ was localized in $J_0 \subset I$, its transported version may in general only be expected to be smeared over all of $I + 2a$.

Next, we recall that the full subcategory $\mathbf{Amp}^{tr}\mathcal{A}$ of transportable amplimorphisms is a *braided category.* The braiding structure is provided by the *statistics operators*

$$\epsilon(\mu, \nu) \ \in \ (\nu \times \mu | \mu \times \nu) \tag{3.6}$$

defined by

$$\epsilon(\mu, \nu) := (U^* \otimes \mathbf{1})(1\!\!1 \otimes P)(\mu \otimes \mathrm{id}\,)(U) \tag{3.7}$$

where $P : \mathrm{End}\, V_\mu \otimes \mathrm{End}\, V_\nu \to \mathrm{End}\, V_\nu \otimes \mathrm{End}\, V_\mu$ denotes the permutation and where U is any isomorphism from ν to some $\tilde{\nu}$ such that the localization region of $\tilde{\nu}$ lies to the left from that of μ. The statistics operator satisfies

$$\text{naturality:} \quad \epsilon(\mu_1, \mu_2)\ (T_1 \times T_2) \ = \ (T_2 \times T_1)\ \epsilon(\nu_1, \nu_2) \tag{3.8a}$$

$$\text{pentagons:} \quad \begin{cases} \epsilon(\lambda \times \mu, \nu) & = \ (\epsilon(\lambda, \nu) \times 1_\mu)(1_\lambda \times \epsilon(\mu, \nu)) \\ \epsilon(\lambda, \mu \times \nu) & = \ (1_\mu \times \epsilon(\lambda, \nu))(\epsilon(\lambda, \mu) \times 1_\nu) \end{cases} \tag{3.8b}$$

The relevance of the category $\mathbf{Amp}\,\mathcal{A}$ to the representation theory of the observable algebra $\mathcal{A}$ can be summarized in the following theorem taken over from [SzV].

Theorem 3.1. *Let π_0 be a faithful irreducible representation of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_0$ that satisfies Haag duality (here the second prime denotes the commutant in $\mathcal{L}(\mathcal{H}_0)$):*

$$\pi_0(\mathcal{A}(I'))' = \pi_0(\mathcal{A}(I)) \qquad I \in \mathcal{I}\,. \tag{3.9}$$

and let $\mathbf{Rep}\,\mathcal{A}$ be the category of representations π of $\mathcal{A}$ that satisfy the following selection criterion (analogue of the DHR-criterion):

$$\exists I \in \mathcal{I},\ n \in \mathbb{N} : \qquad \pi|_{\mathcal{A}(I')} \simeq n \cdot \pi_0|_{\mathcal{A}(I')} \tag{3.10}$$

where $\simeq$ denotes unitary equivalence. Then $\mathbf{Rep}\,\mathcal{A}$ is isomorphic to $\mathbf{Amp}\,\mathcal{A}$. If we add the condition that π_0 is α-covariant and denote by $\mathbf{Rep}^\alpha\mathcal{A}$ the full subcategory in $\mathbf{Rep}\,\mathcal{A}$ of α-covariant representations then $\mathbf{Rep}^\alpha\mathcal{A}$ is isomorphic to the category $\mathbf{Amp}^\alpha\mathcal{A}$ of α-covariant amplimorphisms.

In general $\mathbf{Amp}^\alpha\mathcal{A} \subset \mathbf{Amp}^{tr}\mathcal{A} \subset \mathbf{Amp}\,\mathcal{A}$. In the Hopf spin model we shall see in Section 4 that $\mathbf{Amp}^\alpha\mathcal{A} = \mathbf{Amp}\,\mathcal{A}$ and that $\mathbf{Amp}\,\mathcal{A}$ is equivalent to $\mathbf{Rep}\,\mathcal{D}(H)$.

3.2 Localized Cosymmetries

For simplicity we assume from now on that $\mathbf{Amp}\,\mathcal{A}$ contains only finitely many equivalence classes of irreducible objects. For the Hopf spin model this will follow from compressibility, see Theorem 3.12 in Section 3.4. Let $\{\mu_r\}$ be a list of irreducible amplimorphisms in $\mathbf{Amp}\,\mathcal{A}$ containing exactly one from each equivalence class . Then an object ρ is called *universal* if it is equivalent to $\oplus_r \mu_r$. Define the C^*-algebra $\mathcal{G}$ by

$$\mathcal{G} := \oplus_r \ \mathrm{End}\, V_r$$

then every universal object is a unital C^*-algebra morphism $\rho: \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$. We denote by e_r the minimal central projections in $\mathcal{G}$. There is a distinguished 1-dimensional block $r = \varepsilon$, i.e. $\mathrm{End}\, V_\varepsilon \cong \mathbb{C}$ associated with the identity morphism $\mathrm{id}_\mathcal{A} \equiv \rho_\varepsilon$ as a subobject of ρ. We also denote $\varepsilon: \mathcal{G} \to \mathbb{C}$ the associated 1-dimensional representation of $\mathcal{G}$. Note that by construction $\mathcal{G}$ is uniquley determined up to isomorphisms leaving e_ε invariant. We also remark that if ε is the counit with respect to some coproduct $\Delta: \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ then e_ε is the two-sided *integral* in $\mathcal{G}$, since $x e_\varepsilon = e_\varepsilon x = \varepsilon(x) e_\varepsilon$ for all $x \in \mathcal{G}$.

Universality of ρ implies that any amplimorphism μ is equivalent to $(id \otimes \beta_\mu) \circ \rho$ for some representation β_μ of $\mathcal{G}$. In particular, there must exist a $*$-algebra morphism $\Delta_\rho : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ such that $\rho \times \rho$ is equivalent to $(\mathrm{id} \otimes \Delta_\rho) \circ \rho$ [2]. As a characteristic feature of a Hopf algebra symmetry we now investigate the question whether there exists an appropriate choice of ρ such that $\rho \times \rho = (\mathrm{id}_\mathcal{A} \otimes \Delta) \circ \rho$ for some *coassociative coproduct* $\Delta: \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$. If ρ can be chosen in such a way then we arrive to the very useful notion of a comodule algebra action.

Definition 3.2: Let $\mathcal{G}$ be a C^*-bialgebra with coproduct Δ and counit ε. A *localized comodule algebra action* of $\mathcal{G}$ on $\mathcal{A}$ is a localized amplimorphism $\rho: \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$ that is also a coaction on $\mathcal{A}$ with respect to the coalgebra $(\mathcal{G}, \Delta, \varepsilon)$. In other words: ρ is a linear map satisfying the axioms:

$$\begin{aligned}
\rho(A)\rho(B) &= \rho(AB) && \text{(3.11a)}\\
\rho(\mathbb{1}) &= \mathbb{1} \otimes \mathbf{1} && \text{(3.11b)}\\
\rho(A^*) &= \rho(A)^* && \text{(3.11c)}\\
\rho \times \rho \equiv (\rho \otimes \mathrm{id}\,) \circ \rho &= (\mathrm{id}\, \otimes \Delta) \circ \rho && \text{(3.11d)}\\
(\mathrm{id}_\mathcal{A} \otimes \varepsilon) \circ \rho &= \mathrm{id}_\mathcal{A} && \text{(3.11e)}\\
\exists I \in \mathcal{I}\ :\ \rho(A) &= A \otimes \mathbf{1} \quad A \in \mathcal{A}(I^c) && \text{(3.11f)}
\end{aligned}$$

The coaction ρ is said to be *universal* if it is — as an amplimorphism — a universal object of $\mathbf{Amp}\,\mathcal{A}$.

For brevity by a coaction we will from now on mean a localized comodule algebra action in the sense of Definition 3.2. If $\mathcal{A}$ admits a coaction of $(\mathcal{G}, \varepsilon, \Delta)$ then we also call $\mathcal{G}$ a *localized cosymmetry* of $\mathcal{A}$. Examples of universal localized cosymmetries for the Hopf spin chain will be given in Section 4.

Next, we recall that every coaction $\rho: \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$ uniquely determines an action of the dual $\hat{\mathcal{G}}$ on $\mathcal{A}$, also denoted by ρ, as follows (for simplicity assume $\mathcal{G}$ to be finite dimensional):

[2] This argument fails in locally infinite theories where one may have $\mathcal{A}(I) \cong \mathcal{A}(I) \otimes \mathrm{Mat}\,(n)$, $\forall n \in \mathbb{N}$, in which case the dimensions $\dim V_\mu$ are not an invariant of the equivalence classes $[\mu]$.

$$\begin{aligned}\rho_\xi &: \mathcal{A}\to\mathcal{A} \qquad \xi\in\hat{\mathcal{G}} \\ \rho_\xi(A) &:= (\mathrm{id}_\mathcal{A}\otimes\xi)(\rho(A))\end{aligned} \tag{3.12}$$

The following axioms for a localized action of the bialgebra $\hat{\mathcal{G}}$ on the C^*-algebra $\mathcal{A}$ are easily verified

$$\rho_\xi(AB) = \rho_{\xi_{(1)}}(A)\rho_{\xi_{(2)}}(B) \tag{3.13a}$$

$$\rho_\xi(\mathbb{1}) = \hat{\varepsilon}(\xi)\mathbb{1} \tag{3.13b}$$

$$\rho_\xi(A)^* = \rho_{\xi_*}(A^*) \tag{3.13c}$$

$$\rho_\xi\circ\rho_\eta = \rho_{\xi\eta} \tag{3.13d}$$

$$\rho_\varepsilon = \mathrm{id}_\mathcal{A} \tag{3.13e}$$

$$\exists I\in\mathcal{I}\ :\ \rho_\xi(A) = \hat{\varepsilon}(\xi)A\,, \quad \forall A\in\mathcal{A}(I^c) \tag{3.13f}$$

Here $\hat{\varepsilon}\equiv\mathbf{1}\in\mathcal{G}$ denotes the counit on $\hat{\mathcal{G}}$. Converseley, if ρ_ξ satisfies (3.13) then

$$A\mapsto\rho(A)=\sum_s \rho_{\eta_s}(A)\otimes Y^s \ \in\ \mathcal{A}\otimes\mathcal{G}$$

defines a coaction, where $\{\eta_s\}$ and $\{Y^s\}$ denote a pair of dual bases of $\hat{\mathcal{G}}$ and $\mathcal{G}$, respectively. In (3.13c) we used the notation $\xi\mapsto\xi_*$ for the antilinear involutive algebra automorphism defined by $\langle\xi_*|a\rangle=\overline{\langle\xi|a^*\rangle}$. If $\mathcal{G}$ (and therefore also $\hat{\mathcal{G}}$) has an antipode S, then $\xi^*:=S(\xi_*)\equiv S^{-1}(\xi)_*$ defines a $*$-structure on $\hat{\mathcal{G}}$.

One can also check that for $\langle\xi|a\rangle:=D_r^{kl}(a)$, the representation matrix of the unitary irrep r of $\mathcal{G}$, the matrix $\rho_\xi(A)$ determines an ordinary matrix amplimorphism $\rho_r\colon\mathcal{A}\to\mathcal{A}\otimes M_{n_r}$. Whether such a ρ_r is irreducible is not guaranteed in general, so we will call it a *component* of ρ.

3.3 Effective Cosymmetries

To investigate the conditions under which the components of a given coaction are pairwise inequivalent and irreducible we introduce the following

Definition 3.3 Let $\rho:\mathcal{A}\to\mathcal{A}\otimes\mathrm{End}\,V_\rho$ be an amplimorphism and let $\mathcal{A}$ have trivial center. A unital *-subalgebra $\mathcal{G}\subset\mathrm{End}\,V_\rho$ is called *effective* for ρ, if $\rho(\mathcal{A})\subset\mathcal{A}\otimes\mathcal{G}$ and $(\rho_r|\rho_s)=\delta_{rs}\mathbb{C}(\mathbf{1}_\mathcal{A}\otimes\mathbf{1}_{V_r})$, where r,s run through a complete set of pairwise inequivalent representations of $\mathcal{G}$ and where $\rho_r=(id\otimes r)\circ\rho$. A coaction $\rho:\mathcal{A}\to\mathcal{A}\otimes\mathcal{G}$ is called effective, if $\mathcal{G}$ is effective for ρ (with respect to some unital inclusion $\mathcal{G}\subset\mathrm{End}\,V_\rho$).

To see whether an effective $\mathcal{G}\subset\mathrm{End}\,V_\rho$ exists for a given amplimorphism ρ, we now introduce $\mathbf{Amp}_\rho\mathcal{A}$ as the full subcategroy of $\mathbf{Amp}\,\mathcal{A}$ generated by objects which are equivalent to direct sums of the irreducibles ρ_r ocurring in ρ as a subobject. We also put $\mathbf{Amp}_\rho^\circ\mathcal{A}\subset\mathbf{Amp}_\rho\mathcal{A}$ as the full subcategory consisting of objects μ, such that all intertwiners in $(\mu|\rho)$ are "scalar", i.e.

$$(\mu|\rho)\subset\mathbf{1}_\mathcal{A}\otimes\mathrm{Hom}\,(V_\rho,V_\mu)$$

Note that the amplimorphism ρ itself belongs to $\mathbf{Amp}_\rho^\circ\mathcal{A}$ iff $(\rho|\rho)\equiv\rho(\mathcal{A})'=\mathbf{1}_\mathcal{A}\otimes\mathcal{C}_\rho$ for some unital $*$-subalgebra $\mathcal{C}_\rho\subset\mathrm{End}\,V_\rho$, which also implies $\mathcal{A}\otimes\mathcal{C}_\rho'\cap\mathrm{End}\,V_\rho\subset\rho(\mathcal{A})$. We now have

Proposition 3.4: *Let $\mathcal{A}$ have trivial center and let $\rho : \mathcal{A} \to \mathcal{A} \otimes End\, V_\rho$ be an amplimorphism. For a unital $*$-subalgebra $\mathcal{G} \subset End\, V_\rho$ the following conditions are equivalent:*

i) $\mathcal{G}$ is effective for ρ

ii) $(\rho|\rho) = \mathbf{1}_\mathcal{A} \otimes \mathcal{C}_\rho$ and $\mathcal{G} = \mathcal{C}'_\rho \cap End\, V_\rho$

iii) $\rho(\mathcal{A}) \subset \mathcal{A} \otimes \mathcal{G}$ and $\mathbf{Rep}\,(\mathcal{G}) \cong \mathbf{Amp}\,^\circ_\rho(\mathcal{A})$, where the isomorphism is given on objects by $\beta \to (id \otimes \beta) \circ \rho$ and on intertwiners by $t \to \mathbf{1}_\mathcal{A} \otimes t$.

Proof: Denote V_r the representation spaces of a complete set of pairwise inequivalent irreducible representations r of $\mathcal{G}$. Decomposing V_ρ into irreducible subspaces under the action of $\mathcal{G}$ we get a family of isometries

$$u_r : V_r \otimes \mathbb{C}^{N_\rho^r} \to V_\rho$$

where $N_\rho^r \in I\!N$ are nonvanishing multiplicities and where $u_r^* u_s = \delta_{rs}$, $\sum_r u_r u_r^* = \mathbf{1}_{V_\rho}$ and

$$g u_r = u_r (r(g) \otimes \mathbf{1}_{N_\rho^r}) \quad , \quad \forall g \in \mathcal{G}.$$

Putting $u = \oplus_r u_r : \oplus_r (V_r \otimes \mathbb{C}^{N_\rho^r}) \to V_\rho$ we conclude that u is an isomorphism obeying

$$\begin{aligned} u^* \mathcal{G} u &= \oplus_r (\mathrm{End}\, V_r \otimes \mathbf{1}_{N_\rho^r}) \\ u^* (\mathcal{G}' \cap End\, V_\rho) u &= \oplus_r (\mathbf{1}_{V_r} \otimes Mat(N_\rho^r)) \end{aligned}$$

and

$$(\mathbf{1}_\mathcal{A} \otimes u^*) \rho(A) (\mathbf{1}_\mathcal{A} \otimes u) = \oplus_r (\rho_r(A) \otimes \mathbf{1}_{N_\rho^r}) \quad , \quad \forall A \in \mathcal{A}$$

We now prove the equivalence i)$\Leftrightarrow$ ii).

i) $\Rightarrow$ ii): Let $(\rho_r|\rho_s) = \delta_{rs} \mathbb{C}(\mathbf{1}_\mathcal{A} \otimes \mathbf{1}_{V_r})$. Then

$$(\mathbf{1}_\mathcal{A} \otimes u^*)(\rho|\rho)(\mathbf{1}_\mathcal{A} \otimes u) = \oplus_r (\mathbf{1}_A \otimes \mathbf{1}_{V_r} \otimes Mat(N_\rho^r))$$

which proves $(\rho|\rho) = \mathbf{1}_\mathcal{A} \otimes \mathcal{C}_\rho$ where $\mathcal{C}_\rho = \mathcal{G}' \cap End\, V_\rho$ and therefore $\mathcal{G} = \mathcal{C}'_\rho \cap End\, V_\rho$.

ii)$\Rightarrow$ i): If $\rho(\mathcal{A})' \equiv (\rho|\rho) = \mathbf{1}_\mathcal{A} \otimes \mathcal{C}_\rho$ then $\rho(\mathcal{A}) \subset \rho(\mathcal{A})'' = \mathcal{A} \otimes (\mathcal{C}'_\rho \cap End\, V_\rho) = \mathcal{A} \otimes \mathcal{G}$. Let now $M \in \mathrm{Hom}\,(\mathbb{C}^{N_\rho^s}, \mathbb{C}^{N_\rho^r})$ and $T \in (\rho_r|\rho_s)$ and put

$$T_M := (\mathbf{1}_\mathcal{A} \otimes u_r)(T \otimes M)(\mathbf{1}_\mathcal{A} \otimes u_s^*)$$

Then $T_M \in (\rho|\rho)$ and therefore $T_M = \mathbf{1}_\mathcal{A} \otimes t_M$ for some $t_M \in \mathcal{C}_\rho$. Now $\mathcal{C}_\rho = \mathcal{G}' \cap End\, V_\rho$ implies $u_r^* \mathcal{C}_\rho u_s = \delta_{rs} (\mathbf{1}_{V_r} \otimes Mat(N_\rho^r))$ and therefore

$$T \otimes M = \mathbf{1}_\mathcal{A} \otimes u_r^* t_M u_s \in \delta_{rs} (\mathbf{1}_\mathcal{A} \otimes \mathbf{1}_{V_r} \otimes Mat(N_\rho^r))$$

which finally yields $T \in \delta_{rs} \mathbb{C}(\mathbf{1}_\mathcal{A} \otimes \mathbf{1}_{V_r})$.

Next we prove the equivalence i)+ii) $\Leftrightarrow$ iii) by first noting that the implication iii) $\Rightarrow$ i) is trivial. We are left with

i)+ii) $\Rightarrow$ iii): We first show that $\mu \in \mathbf{Amp}\,^0_\rho \mathcal{A}$ implies $(\mu|\rho_r) \subset \mathbf{1}_\mathcal{A} \otimes \mathrm{Hom}\,(V_r, V_\mu)\ \forall r$. To this end let $e \in \mathbb{C}^{N_\rho^r}$ be a unit vector and define $1\!\!1_\mathcal{A} \otimes u_{r,e} \in (\rho|\rho_r)$ by

$$u_{r,e} : V_r \to V_\rho, \quad v \mapsto u_r(v \otimes e)$$

For any $T \in (\mu|\rho_r)$ we then put

$$T_e := T(\mathbf{1}_\mathcal{A} \otimes u_{r,e}^*)$$

Then $T_e \in (\mu|\rho)$ and therefore, by assumption ii), $T_e = \mathbf{1}_{\mathcal{A}} \otimes t_e$ for some $t_e \in \mathrm{Hom}\,(V_\rho, V_\mu)$. Using $u_{r,e}^* u_{r,e} = \mathbf{1}_{V_r}$ we conclude $T = \mathbf{1}_{\mathcal{A}} \otimes t_e u_{r,e}$ and hence $(\mu|\rho_r)$ is scalar. Now μ being equivalent to a direct sum of ρ_r's we must have a family of isometries

$$w_r : V_r \otimes \mathbb{C}^{N_\mu^r} \to V_\mu$$

where $N_\mu^r \in \mathbb{N}_o$ are possibly vanishing multiplicities and where $w_r^* w_s = \delta_{rs}$ (if $N_\mu^s \neq 0$), $\Sigma_r w_r w_r^* = \mathbf{1}_{V_\mu}$ and

$$\mu(A)(\mathbf{1}_{\mathcal{A}} \otimes w_r) = (\mathbf{1}_{\mathcal{A}} \otimes w_r)(\rho_r(A) \otimes \mathbf{1}_{N_\mu^r}), \quad A \in \mathcal{A}.$$

Hence we get $\mu = (id \otimes \beta_\mu) \circ \rho$, where $\beta_\mu \in \mathbf{Rep}\,\mathcal{G}$ is given by

$$\beta_\mu(g) = \Sigma_r w_r (r(g) \otimes \mathbf{1}_{N_\mu^r}) w_r^*$$

Next, to show that $\beta \in \mathbf{Rep}\,\mathcal{G}$ is uniquely determined by $\mu = (id \otimes \beta) \circ \rho \in \mathbf{Amp}_\rho^0(\mathcal{A})$ we define

$$\mathcal{G}_\rho := \{(\omega \otimes id_{\mathcal{G}})(\rho(\mathcal{A}))|\ \omega \in \hat{\mathcal{A}}\} \subset \mathcal{G}$$

where $\hat{\mathcal{A}}$ is the dual of $\mathcal{A}$. Clearly the restriction $\beta|\mathcal{G}_\rho$ is uniquely determined by μ. Moreover

$$\mathbf{1}_{\mathcal{A}} \otimes (\mathcal{G}_\rho' \cap End\ V_\rho) = (\mathbf{1}_{\mathcal{A}} \otimes End\ V_\rho) \cap \rho(\mathcal{A})'.$$

Since, by assumption ii), $\rho(\mathcal{A})' \equiv (\rho|\rho) = \mathbf{1}_{\mathcal{A}} \otimes (\mathcal{G}' \otimes End\ V_\rho)$ we conclude

$$\mathcal{G}_\rho' \cap End\ V_\rho = \mathcal{G}' \cap End\ V_\rho$$

and therefore the algebraic closure of $\mathcal{G}_\rho$ coincides with $\mathcal{G}$. Hence, being an algebra homomorphism β is uniquely determined by its restriction $\beta|\mathcal{G}_\rho$ and therefore by μ.

Finally we show that $\mathbf{1}_{\mathcal{A}} \otimes (\beta|\gamma) = ((id \otimes \beta) \circ \rho|(id \otimes \gamma) \circ \rho)$ for all $\beta, \gamma \in \mathbf{Rep}\ \mathcal{G}$, which in particular implies $(id \otimes \beta) \circ \rho \in \mathbf{Amp}_\rho^0 \mathcal{A}$ for all $\beta \in \mathbf{Rep}\ \mathcal{G}$ (put $\gamma = id$). By decomposing β and γ we get unitary isomorphisms

$$\begin{aligned} w_\beta &: \oplus_r (V_r \otimes \mathcal{C}^{N_\beta^r}) \to V_\beta \\ w_\gamma &: \oplus_r (V_r \otimes C^{N_\gamma^r}) \to V_\gamma \end{aligned}$$

obeying for $x = \beta, \gamma$

$$x(g) w_x = w_x \oplus_r (r(g) \otimes \mathbf{1}_{N_x^r}) \quad \forall g \in \mathcal{G}.$$

Hence

$$\begin{aligned} (\mathbf{1}_{\mathcal{A}} \otimes w_\beta^*) \cdot ((id \otimes \beta) \circ \rho\,|\,(id \otimes \gamma) \circ \rho) \cdot (\mathbf{1}_{\mathcal{A}} \otimes w_\gamma) & \\ = \ & (\oplus_r N_\beta^r \rho_r |\oplus_s N_\gamma^s \rho_s) \\ = \ & \oplus_r (\mathbf{1}_{\mathcal{A}} \otimes \mathbf{1}_{V_r} \otimes \mathrm{Hom}\,(\mathbb{C}^{N_\gamma^r}, \mathbb{C}^{N_\beta^r})) \end{aligned}$$

by assumption i), which proves $((id \otimes \beta) \circ \rho|(id \otimes \gamma) \circ \rho) = \mathbf{1}_{\mathcal{A}} \otimes (\beta|\gamma)$. *Q.e.d.*

We are now in the position to give a rather complete characterization of effective cosymmetries.

Theorem 3.5: *Let $\rho : \mathcal{A} \to \mathcal{A} \otimes End\ V_\rho$ be an amplimorphism and assume $\mathcal{G} \subset End\ V_\rho$ to be effective for ρ. Let furthermore $\varepsilon : \mathcal{G} \to \mathbb{C}$ be a distinguished one-dimensional representation such that $\rho_\varepsilon := (id \otimes \varepsilon) \circ \rho = id_{\mathcal{A}}$. Then the following conditions A)-C) are equivalent*

A) $\mathbf{Amp}_\rho^\circ(\mathcal{A})$ *closes under the monoidal product*

B) $\rho \times \rho \in \mathbf{Amp}_\rho^\circ(\mathcal{A})$

C) There exists a coassociative coproduct Δ *on* $(\mathcal{G}, \varepsilon)$ *such that* (ρ, Δ) *provides an effective coaction of* $(\mathcal{G}, \varepsilon)$ *on* $\mathcal{A}$.

Moreover, under these conditions we have

i) Δ *is uniquely determined by* ρ.

ii) $\mathbf{Amp}_\rho(\mathcal{A})$ *is rigid iff* $\mathcal{G}$ *admits an antipode.*

iii) $\mathbf{Amp}_\rho(\mathcal{A})$ *is braided, iff there exists a quasitriangular element* $R \in \mathcal{G} \otimes \mathcal{G}$.

iv) $\mathbf{Amp}_\rho(\mathcal{A}) \sim \mathbf{Rep}\,(\mathcal{G})$ *as strict monoidal, (rigid, braided) categories.*

Proof: The implication $A) \Rightarrow B)$ is obvious, since $\rho \in \mathbf{Amp}_\rho^\circ(\mathcal{A})$ by Proposition 3.4ii). To prove $B) \Rightarrow C)$ let $\Delta : \mathcal{G} \to \mathrm{End}\,(V_\rho \otimes V_\rho)$ such that $\rho \times \rho = (id \otimes \Delta) \circ \rho$. Then Δ uniquely exists by Proposition 3.4iii). Moreover $\mathbf{1}_\mathcal{A} \otimes \mathcal{G}' \otimes \mathcal{G}' \subset (\rho \times \rho | \rho \times \rho)$ which again by Proposition 3.4iii) implies $\mathcal{G}' \otimes \mathcal{G}' \subset \Delta(\mathcal{G})'$ and therefore $\Delta(\mathcal{G}) \subset \mathcal{G} \otimes \mathcal{G}$. The identity $\rho_\varepsilon = id_\mathcal{A}$ implies the counit property $(id_\mathcal{G} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes id_\mathcal{G}) \circ \Delta = id_\mathcal{G}$ and the identity $\rho \times (\rho \times \rho) = (\rho \times \rho) \times \rho$ implies the coassociativity $(id_\mathcal{G} \otimes \Delta) \circ \Delta = (\Delta \otimes id_\mathcal{G}) \circ \Delta$. Here we have again used that any $\beta \in \mathbf{Rep}\,\mathcal{G}$ is uniquely determined by $(id_\mathcal{A} \otimes \beta) \circ \rho$. To prove $C) \Rightarrow A)$ we note $\mathbf{Amp}_\rho^\circ(\mathcal{A}) \cong \mathbf{Rep}\,\mathcal{G}$ by Proposition 3.4iii) and recall that $\mathbf{Rep}\,\mathcal{G}$ becomes monoidal for any bialgebra $(\mathcal{G}, \Delta, \varepsilon)$.

Next, part i) has already been pointed out above and part iv) follows since any object in $\mathbf{Amp}_\rho(\mathcal{A})$ is equivalent to an object in $\mathbf{Amp}_\rho^\circ(\mathcal{A})$ and therefore $\mathbf{Amp}_\rho(\mathcal{A}) \sim \mathbf{Amp}_\rho^\circ(\mathcal{A}) \cong \mathbf{Rep}\,\mathcal{G}$ by Proposition 3.4iii). By the same argument, it is enough to prove parts ii)+iii) with $\mathbf{Amp}_\rho(\mathcal{A})$ replaced by $\mathbf{Rep}\,\mathcal{G}$. However, for $\mathbf{Rep}\,\mathcal{G}$ these statements become standard (see e.g. [Maj2,U]) and we only give a short sketch of proofs here. So if $\beta \in \mathbf{Rep}\,\mathcal{G}$ and $S : \mathcal{G} \to \mathcal{G}$ is the antipode then one defines the conjugate representation $\bar\beta := \beta^T \circ S$, where β^T is the transpose of β acting on the dual vector space $\hat V_\beta$. Since on finite dimensional C^*-Hopf algebras $\mathcal{G}$ the antipode is involutive, $S^2 = \mathrm{id}_\mathcal{G}$ [W], the left and right evaluation maps which make $\mathbf{Rep}\,\mathcal{G}$ rigid are given by the natural pairings $\hat V_\beta \otimes V_\beta \to \mathbb{C}$ and $V_\beta \otimes \hat V_\beta \to \mathbb{C}$,respectively. Conversely, let $\mathbf{Rep}\,\mathcal{G}$ be rigid and identify $\mathcal{G} = \oplus_r \mathrm{End}\, V_r$, where r labels the simple ideals — and therefore the (equivalence classes of) irreducible representations — of $\mathcal{G}$. For $X \in \mathrm{End}\, V_r \subset \mathcal{G}$ let $S(X) \in \mathrm{End}\, V_{\bar r}$ be given by

$$S(X) = (\mathbf{1}_{\bar r} \otimes \overline{C}_r^*)(\mathbf{1}_{\bar r} \otimes X \otimes \mathbf{1}_{\bar r})(C_r \otimes \mathbf{1}_{\bar r})$$

We now use that for $X \in \mathrm{End}\, V_r \subset \mathcal{G}$ the coproduct may be written as $\Delta(X) = \sum_{p,q} \Delta_{p,q}(X)$ where $\Delta_{p,q}(X) \in \mathrm{End}\, V_p \otimes \mathrm{End}\, V_q$ is given by

$$\Delta_{p,q}(X) = \sum_{i=1}^{N_{pq}^r} t_{pq,i}^r \; X \; t_{pq,i}^{r\,*}$$

where $t_{pq,i}^r \in (p \times q | r)$, $i = 1, .., N_{pq}^r$, is an orthonormal basis of intertwiners in $\mathbf{Rep}\,\mathcal{G}$. Choosing a basis in V_p and using the rigidity properties (3.5) it is now not difficult to verify the defining properties of the antipode

$$S(X_{(1)})X_{(2)} = X_{(1)}S(X_{(2)}) = \varepsilon(X)\mathbf{1}$$

To prove iii) let $R \in \mathcal{G} \otimes \mathcal{G}$ be quasitriangular and let $\alpha, \beta \in \mathbf{Rep}\,\mathcal{G}$. Then

$$\epsilon(\alpha, \beta) := \sigma_{\alpha,\beta} \circ (\alpha \otimes \beta)(R)$$

defines a braiding on $\mathbf{Rep}\,\mathcal{G}$, where $\sigma_{\alpha,\beta} : V_\alpha \otimes V_\beta \to V_\beta \otimes V_\alpha$ denotes the permutation. Conversely, let $\epsilon(\alpha, \beta) \in (\beta \times \alpha | \alpha \times \beta)$ be a braiding and denote

$$R_{r,r'} := \sigma_{r',r} \circ \epsilon(r, r') \in \mathrm{End}\, V_r \otimes \mathrm{End}\, V_{r'}$$

Putting $R := \oplus_{r,r'} R_{r,r'}$ and using the above formula for the coproduct it is again straightforward to check that R is quasitriangular, i.e.

$$\begin{aligned} (\Delta \otimes \mathrm{id}\,)(R) &= R_{13} R_{23} \\ (\mathrm{id}\, \otimes \Delta)(R) &= R_{13} R_{12}, \end{aligned}$$

This concludes the proof of Theorem 3.5. *Q.e.d.*

Corollary 3.6: *Necessary for a localized effective coaction (ρ, Δ) of $(\mathcal{G}, \varepsilon)$ on a net $\{\mathcal{A}(I)\}$ to be transportable is that $\mathcal{G}$ be quasitriangular.*

Proof: If ρ is transportable then any irreducible component ρ_r is transportable and hence $\mathbf{Amp}\,_\rho\mathcal{A}$ is braided, see equs. (3.6-8) and [SzV]. *Q.e.d.*

3.4 Universal Cosymmetries and Complete Compressibility

Theorem 3.5 implies that $\mathbf{Amp}\,\mathcal{A} \sim \mathbf{Rep}\,\mathcal{G}$ for a suitable C^*-bialgebra $(\mathcal{G}, \varepsilon, \Delta)$, provided we can find a universal object $\rho = \oplus_r \rho_r$ in $\mathbf{Amp}\,\mathcal{A}$, such that $\rho \times \rho \in \mathbf{Amp}\,^0_\rho\mathcal{A}$. In this case we call ρ a *universal coaction* on $\mathcal{A}$ and $\mathcal{G}$ a *universal cosymmetry* of $\mathcal{A}$. In other words, a localized coaction $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$ is universal, if and only if it is effective and for any $\mu \in \mathbf{Amp}\,\mathcal{A}$ there exists a representation $\beta_\mu \in \mathbf{Rep}\,\mathcal{G}$ such that μ is equivalent to $(id \otimes \beta_\mu) \circ \rho$.

We note that a priorily universal coactions need not exist on $\mathcal{A}$. However, if they do, then as an algebra $\mathcal{G}$ is determined up to isomorphisms, i.e.

$$G \simeq \oplus_r End\ V_r$$

where $\rho_r : \mathcal{A} \to \mathcal{A} \otimes \mathrm{End}\, V_r$ are the irreducible components of ρ. Moreover, as will be shown in Section 3.5, universal coactions ρ - and hence the coproduct Δ on $\mathcal{G}$ - are determined up to cocycle equivalence provided they exist.

In this subsection we investigate the question of *existence* of universal coactions ρ by analysing the condition $\rho \times \rho \in \mathbf{Amp}\,^\circ_\rho\mathcal{A}$. To this end we introduce the ρ-stable subalgebra $\mathcal{A}_\rho \subset \mathcal{A}$

$$\mathcal{A}_\rho := \{A \in \mathcal{A} |\ \rho(A) = A \otimes \mathbf{1}\} \tag{3.14}$$

If $\mathcal{B} \subset \mathcal{A}$ is a unital $*$-subalgebra, then we say that ρ is localized away from $\mathcal{B}$, if $\mathcal{B} \subset \mathcal{A}_\rho$, and we denote the full subcategory

$$\mathbf{Amp}\,(\mathcal{A}|\mathcal{B}) = \{\rho \in \mathbf{Amp}\,\mathcal{A} |\ \mathcal{B} \subset \mathcal{A}_\rho\}$$

We note that intertwiners between amplimorphisms in $\mathbf{Amp}\,(\mathcal{A}|\mathcal{B})$ are always in $(\mathcal{B}'\cap\mathcal{A})\otimes\mathrm{End}\,V_\rho$. This follows from the more general and obvious fact that for any two amplimorphisms $\rho_i:\mathcal{A}\to\mathcal{A}\otimes\mathrm{End}\,V_i,\ i=1,2$, we have

$$(\rho_1|\rho_2)\subset((\mathcal{A}_{\rho_1}\cap\mathcal{A}_{\rho_2})'\cap\mathcal{A})\otimes\mathrm{Hom}\,(V_2,V_1)$$

We also note that $\mathbf{Amp}\,(\mathcal{A}|B)$ clearly closes under the monoidal product. Hence we get the immediate

Corollary 3.7: *Assume $\mathcal{B}\subset\mathcal{A}$ and $\mathcal{B}'\cap\mathcal{A}=\mathbb{C}\cdot\mathbf{1}_\mathcal{A}$ and let $\rho\in\mathbf{Amp}\,(\mathcal{A}|\mathcal{B})$ be universal in $\mathbf{Amp}\,(\mathcal{A}|\mathcal{B})$. Then $(\rho|\rho)=\mathbf{1}_\mathcal{A}\otimes\mathcal{C}_\rho$ and $\rho\times\rho\in\mathbf{Amp}_\rho^0\mathcal{A}$ and therefore $\rho:\mathcal{A}\to\mathcal{A}\otimes\mathcal{G}$ provides an effective coaction, where $\mathcal{G}=\mathcal{C}_\rho'\cap\mathrm{End}\,V_\rho$.*

It is suggestive to call the resulting bialgebra $\mathcal{G}=:Gal(\mathcal{A}|\mathcal{B})$ the universal cosymmetry or "Galois coalgebra" (since the dual bialgebra $\hat{\mathcal{G}}$ would be the analogue of a Galois group) associated with the irreducible inclusion $\mathcal{B}\subset\mathcal{A}$. If under the conditions of Corollary 3.7 $\mathcal{B}=\mathcal{A}_\rho$, then one might also call $\mathcal{B}\subset\mathcal{A}$ a Galois extension (recall $\mathcal{B}\subset\mathcal{A}_\rho$ by definition).

Motivated by these considerations we call $\mathbf{Amp}\,\mathcal{A}$ *compressible* relative to $\mathcal{B}$, if any object in $\mathbf{Amp}\,\mathcal{A}$ is equivalent to an object in $\mathbf{Amp}\,(\mathcal{A}|\mathcal{B})$.

Coming back to our net of local algebras $\mathcal{A}(I)$ this fits with our previous terminology, i.e. $\mathbf{Amp}\,\mathcal{A}$ is compressible (i.e. compressible into $\mathcal{A}(I)$ for some $I\in\mathcal{I}$), iff it is compressible relative to $\mathcal{A}(I^c)$ for some $I\in\mathcal{I}$. Also, ρ is localized in Λ (or equivalently on $\mathcal{A}(\Lambda)$), iff it is localized away from $\mathcal{B}=\mathcal{A}(\Lambda^c)$. We say that ρ is compressible into Λ, if it is equivalent to an amplimorphism localized in Λ. We also recall our previous notation

$$\mathbf{Amp}\,(\mathcal{A},\Lambda)\equiv\mathbf{Amp}\,(\mathcal{A}|\mathcal{A}(\Lambda^c))$$

Our strategy for constructing localized universal coactions in $\mathbf{Amp}\,\mathcal{A}$ will now be to find a suitable bounded region $\Lambda=\cup_n I_n, I_n\in\mathcal{I}$, such that $\mathbf{Amp}\,\mathcal{A}$ is compressible into Λ and $\mathcal{A}(\Lambda^c)'\cap\mathcal{A}=\mathbb{C}\cdot\mathbf{1}$. In this case we call $\mathbf{Amp}\,\mathcal{A}$ *completely compressible*. By Corollary 3.7 we are then only left with constructing a universal object in $\mathbf{Amp}\,(\mathcal{A},\Lambda)$. First we note

Lemma 3.8: *For $i=1,2$ let $\rho_i\in\mathbf{Amp}\,(\mathcal{A},I), I\in\mathcal{I}$, and let the net $\{\mathcal{A}(I)\}$ satisfy Haag duality. Then $\rho_i(\mathcal{A}(I))\subset\mathcal{A}(I)\otimes\mathrm{End}\,V_{\rho_i}$ and $(\rho_1|\rho_2)\subset\mathcal{A}(\mathrm{Int}\,I)\otimes\mathrm{Hom}\,(V_{\rho_2},V_{\rho_1})$.*

Proof: We use the general identiy $\rho(\mathcal{A}(I))\subset\rho(\mathcal{A}(I)')'$ and the locality property $\mathcal{A}(I)'\supset\mathcal{A}(I')$ to conclude

$$\begin{aligned}\rho(\mathcal{A}(I)) &\subset \rho(\mathcal{A}(I'))' \\ &= \mathcal{A}(I')'\otimes\mathrm{End}\,V_\rho \\ &= \mathcal{A}(I)\otimes\mathrm{End}\,V_\rho,\end{aligned}$$

where we have used $\mathcal{A}(I')\subset\mathcal{A}(I^c)\subset\mathcal{A}_\rho$ in the second line and Haag duality in the third line. Since $I^c=(\mathrm{Int}\ I)'$ we have $\mathcal{A}((\mathrm{Int}\,I)')\subset\mathcal{A}_\rho$ for all $\rho\in\mathbf{Amp}\,(\mathcal{A},I)$ and therefore $\mathcal{A}_{\rho_i}'\subset\mathcal{A}(\mathrm{Int}\,I)$ by Haag duality, from which $(\rho_1|\rho_2)\subset\mathcal{A}(\mathrm{Int}\,I)\otimes\mathrm{Hom}\,(V_{\rho_2},V_{\rho_1})$ follows.

Q.e.d.

We remark that for additive Haag dual nets Lemma 3.8 implies that $\mathbf{Amp}\,(\mathcal{A},I)$ is uniquely determined by $\mathbf{Amp}\,(\mathcal{A}(I),I)$, with arrows given by the set of intertwiners localized in $\mathrm{Int}\,I$.

Next, if the Haag dual net $\{\mathcal{A}(I)\}$ is also split, then for any localized amplimorhpism ρ there exists $I \in \mathcal{I}$ such that $\mathcal{A}(I)$ is simple and ρ is localized in $\mathcal{A}(I)$. By Lemma 3.8, ρ restricts to an amplimorphism on $\mathcal{A}(I)$ and by simplicity of $\mathcal{A}(I)$ this restriction must be inner, i.e.

$$\rho(A) = U(A \otimes \mathbf{1})U^{-1}$$

for some unitary $U \in \mathcal{A}(I) \otimes \operatorname{End} V_\rho$ and all $A \in \mathcal{A}(I)$. Hence $\rho' := \operatorname{Ad} U^{-1} \circ \rho$ is localized in ∂I and we have

Corollary 3.9: *Let $\{\mathcal{A}(I)\}$ be a split net satisfying Haag duality. Then for any localized amplimorphism ρ there exists $I \in \mathcal{I}$ such that $\mathcal{A}(I)$ is simple and ρ is compressible into ∂I. In particular $\mathbf{Amp}\,\mathcal{A}$ is completely compressible if and only if it is compressible.*

Proof: The second statement follows by noting that if $\mathcal{A}(I)$ is simple then $\mathcal{A}((\partial I)^c)' \cap \mathcal{A} = \mathbb{C}\mathbf{1}$, which follows more generally from

Lemma 3.10: *Assume Haag duality and let $I \in \mathcal{I}$. Then*

$$\mathcal{A}((\partial I)^c)' = \mathcal{A}(I)' \cap \mathcal{A}(I)$$

Proof: We have $(\partial I)^c = I \cup I'$. Hence $\mathcal{A}((\partial I)^c)' = \mathcal{A}(I)' \cap \mathcal{A}(I')' = \mathcal{A}(I)' \cap \mathcal{A}(I)$. *Q.e.d.*

Compressibility of $\mathbf{Amp}\,\mathcal{A}$ for example holds, if $\mathbf{Amp}\,\mathcal{A}$ contains only finitely many equivalence classes of irreducible objects. Since in general we do not know this let us now look at the obvious inclusions $\mathbf{Amp}\,(\mathcal{A}, I) \subset \mathbf{Amp}\,(\mathcal{A}, J)$ for all $I \subset J$. If $\mathcal{A}(I)$ is simple then by Corollary 3.9 $\mathbf{Amp}\,(\mathcal{A}, I) \sim \mathbf{Amp}\,(\mathcal{A}, \partial I)$. Hence we get

Corollary 3.11: *Under the conditions of Corollary 3.9 let $I_n \subset I_{n+1} \in \mathcal{I}$ be a sequence such that $\mathcal{A}(I_n)$ is simple for all n and $\cup_n I_n = \mathbb{R}$. If the sequence $\mathbf{Amp}\,(\mathcal{A}, \partial I_n)$ becomes constant (up to equivalence) for $n \geq n_0$ then $\mathbf{Amp}\,\mathcal{A}$ is completely compressible, i.e. compressible into ∂I_{n_0}.*

We now recall that in the case of our Hopf Spin model the local algebras $\mathcal{A}(I)$ are simple for all intervals I of even length, $|I| = 2n,\ n \in \mathbb{N}_o$. In particular this holds for "one-point-intervals" $I = \{i + \frac{1}{2}\}$, where $|I| = 0,\ \mathcal{A}(I) = \mathbb{C}\mathbf{1}$ and $\mathcal{A}(\partial I) = \mathcal{A}(\bar{I}) = \mathcal{A}_{i,i+1}$ (since $\operatorname{Int} I = \emptyset$). The following Theorem implies that in this model the conditions of Corollary 3.11 hold in fact for *any* choice of one-point-intervals $I_{n_0} \subset I_n$.

Theorem 3.12: *If $\mathcal{A}$ is the observable algebra of the Hopf spin model then $\mathbf{Amp}\,\mathcal{A}$ is compressible into any interval of length two.*

Theorem 3.12 will be proven in Section 4.2. In Section 4.1 we will completely analyse $\mathbf{Amp}\,(\mathcal{A}, I)$ for all $|I| = 2$ (i.e. $\mathcal{A}(I) = \mathcal{A}_{i,i+1},\ i \in \mathbb{Z}$), showing that its universal cosymmetry is given by the Drinfeld double $\mathcal{G} = \mathcal{D}(H)$. We also construct a universal intertwiner from $\mathbf{Amp}\,(\mathcal{A}, I)$ to $\mathbf{Amp}\,(\mathcal{A}, I - 1)$ and thereby prove that $\mathbf{Amp}\,(\mathcal{A}, I)$ (and therefore $\mathbf{Amp}\,\mathcal{A}$) is not only transportable, but even *coherently translation covariant* (see Def. 3.17 below and [DR1, Sec.8]).

3.5 Cocycle Equivalences

Given two amplimorphisms $\rho, \rho' \in \mathbf{Amp}\,(\mathcal{A}, \Lambda)$ which are both universal in $\mathbf{Amp}\,(\mathcal{A}, \Lambda)$ we may without loss consider both of them as maps $\mathcal{A} \to \mathcal{A} \otimes \mathcal{G}$, with a fixed $*$-algebra $\mathcal{G} = \oplus_r End\ V_r$ and a fixed 1-dimensional representation $\varepsilon : \mathcal{G} \to End\ V_\varepsilon = \mathbb{C}$ such that $\rho_\varepsilon = id_{\mathcal{A}}$. However, even if ρ and ρ' are both effective coactions, they may lead to different coproducts, Δ and Δ', on $(\mathcal{G}, \varepsilon)$. Coactions with $(\mathcal{G}, \varepsilon)$ fixed, but with varying coproduct Δ, will be denoted as a pair (ρ, Δ). In order to compare such coactions we first identify coactions (ρ, Δ) and (ρ', Δ') whenever $\rho' = (id \otimes \sigma) \circ \rho$ and $\Delta' = (\sigma \otimes \sigma) \circ \Delta \circ \sigma^{-1}$ for some *-algebra automorphism $\sigma : \mathcal{G} \to \mathcal{G}$ satisfying $\varepsilon \circ \sigma = \varepsilon$. In other words, given an effective coaction (ρ, Δ) of $(\mathcal{G}, \varepsilon)$ on $\mathcal{A}$, then up to a transformation by $\sigma \in \mathrm{Aut}\,(\mathcal{G}, \varepsilon)$ any universal amplimorphism in $\mathbf{Amp}_\rho(\mathcal{A})$ will be considered to be of the form

$$\rho' = Ad\ U \circ \rho$$

where $U \in \mathcal{A} \otimes \mathcal{G}$ is a unitary satisfying $(id \otimes \varepsilon)(U) = \mathbf{1}_{\mathcal{A}}$. Decomposing $\rho = \oplus_r \rho_r$ and $\rho' = \oplus_r \rho'_r$ this implies $\rho_r \simeq \rho'_r$ for all r, i.e. we have fixed an ordering convention among the irreducibles r of coinciding dimensions $d_r = dim\ V_r$.

We now introduce the notion of cocycle equivalence for coactions (ρ, Δ). First, we recall that two coproducts, Δ and Δ', on $(\mathcal{G}, \varepsilon)$ are called *cocycle equivalent*, if $\Delta' = Ad\, u \circ \Delta$, where $u \in \mathcal{G} \otimes \mathcal{G}$ is a unitary *left* Δ*-cocycle*, i.e. $u^* = u^{-1}$ and

$$\begin{aligned} (\mathbf{1} \otimes u)(id \otimes \Delta)(u) &= (u \otimes \mathbf{1})(\Delta \otimes id)(u) && (3.15a) \\ (id \otimes \varepsilon)(u) &= (\varepsilon \otimes id)(u) = \mathbf{1} && (3.15b) \end{aligned}$$

The most familiar case is the one where $\Delta' = \Delta_{op}$, the opposite coproduct, and where $u = R$ is quasitriangular. We call u a *right* Δ*-cocycle*, if u^{-1} is a left Δ-cocycle. Note that if u is a left Δ-cocycle then $\Delta' := \mathrm{Ad}\, u \circ \Delta$ is a coassociative coproduct on $(\mathcal{G}, \varepsilon)$. If in this case S is an antipode for Δ then $S' = \mathrm{Ad}\, q \circ S$ is an antipode fore Δ', where $q := \sum_i a_i S(b_i)$ if $u = \sum_i a_i \otimes b_i$. Moreover, v is a left Δ'-cocycle iff vu is a left Δ-cocycle. In particular, u^{-1} is a left Δ'-cocycle. Two left Δ-cocycles u, v are called *cohomologous* , if

$$u = (x^{-1} \otimes x^{-1})\, v\, \Delta(x) \tag{3.16}$$

for some unitary $x \in \mathcal{G}$ obeying $\varepsilon(x) = 1$. A left Δ-cocycle cohomologous to $\mathbf{1} \otimes \mathbf{1}$ is called a left Δ*-coboundary.* We now give the following

Definition 3.13: Let (ρ, Δ) and (ρ', Δ') be two coactions of $(\mathcal{G}, \varepsilon)$ on $\mathcal{A}$. Then a pair (U, u) of unitaries $U \in \mathcal{A} \otimes \mathcal{G}$ and $u \in \mathcal{G} \otimes \mathcal{G}$ is called a *cocycle equivalence* from (ρ, Δ) to (ρ', Δ') if

$$\begin{aligned} U\rho(A) &= \rho'(A)U \qquad A \in \mathcal{A} && (3.17a) \\ u\Delta(X) &= \Delta'(X)u \qquad X \in \mathcal{G} && (3.17b) \\ U \times_\rho U &= (\mathbb{1} \otimes u) \cdot (\mathrm{id}_{\mathcal{A}} \otimes \Delta)(U) && (3.17c) \\ (\mathrm{id}_{\mathcal{A}} \otimes \varepsilon)(U) &= \mathbb{1}_{\mathcal{A}} && (3.17d) \end{aligned}$$

where we have used the notation

$$U \times_\rho U = (U \otimes \mathbf{1})(\rho \otimes \mathrm{id}_{\mathcal{G}})(U) \in \mathcal{A} \otimes \mathcal{G} \otimes \mathcal{G} \tag{3.18}$$

The pair (U, u) is called a *coboundary equivalence* if in addition to (a–d) u is a left Δ- coboundary. If $u = \mathbf{1} \otimes \mathbf{1}$, then (ρ, Δ) and (ρ', Δ') are called *strictly equivalent.*

Note that equs. (3.17 c,d) imply the left Δ-cocycle conditions (3.15) for u. We leave it to the reader to check that the above definitions indeed provide equivalence relations which are preserved under transformations by $\sigma \in \mathrm{Aut}\,(\mathcal{G},\varepsilon)$. We also remark, that to our knowledge in the literature the terminology "cocycle equivalence for coactions" is restricted to the case $u = \mathbf{1}\otimes\mathbf{1}$ and hence $\Delta' = \Delta$ [Ta,NaTa]. (If in this case $U = (V^{-1}\otimes\mathbf{1})\rho(V)$ for some unitary $V \in \mathcal{A}$ then U would be called a ρ-coboundary.)

We now have

Proposition 3.14: *Let (ρ,Δ) be an effective coaction of $\mathcal{G} = \oplus_r End\ V_r$ on $\mathcal{A}$. Then up to transformations by $\sigma \in Aut\,(\mathcal{G},\varepsilon)$ all universal coactions (ρ',Δ') in $\mathbf{Amp}_\rho(\mathcal{A})$ ($\mathbf{Amp}^0_\rho(\mathcal{A})$) are cocycle equivalent (coboundary equivalent) to (ρ,Δ).*

Proof: Let $\rho' = \mathrm{Ad}\,U\circ\rho$ where $U \in \mathcal{A}\otimes\mathcal{G}$ is unitary and satisfies $(id\otimes\varepsilon)(U) = \mathbf{1}_\mathcal{A}$. We then have two unitary intertwiners

$$\begin{aligned}(id\otimes\Delta)(U) : \rho\times\rho &\to (id\otimes\Delta)\circ\rho' \\ U\times_\rho U : \rho\times\rho &\to \rho'\times\rho' = (id\otimes\Delta')\circ\rho'\end{aligned}$$

Now $\mathcal{G}$ is also effective for ρ' and therfore any intertwiner from $(id\otimes\Delta')\circ\rho'$ to $(id\otimes\Delta)\circ\rho'$ must be a scalar by Proposition 3.4iii (consider Δ and Δ' as representations of $\mathcal{G}$ on $\oplus_{r,s}(V_r\otimes V_s)$). Hence there exists a unitary $u \in \mathcal{G}\otimes\mathcal{G}$ such that

$$U\times_\rho U = (\mathbf{1}_\mathcal{A}\otimes u)(id\otimes\Delta)(U)$$

Consequently (U,u) provides a cocycle for (ρ,Δ) and $(id\otimes\Delta')\circ\rho' = (id\otimes(\mathrm{Ad}\,u\circ\Delta))\circ\rho'$. By Theorem 3.5i) we conclude $\Delta' = \mathrm{Ad}\,u\circ\Delta$ and therefore (ρ',Δ') is cocycle equivalent to (ρ,Δ). If in addition $\rho' \in \mathbf{Amp}^0_\rho(\mathcal{A})$ then $U = \mathbf{1}_\mathcal{A}\otimes x$ for some unitary $x \in \mathcal{G}$. Hence $u = (x\otimes x)\Delta(x^{-1})$ is a coboundary. *Q.e.d.*

3.6 Translation Covariance

In this section we study transformation properties of universal coactions under the translation automorphisms $\alpha^a : \mathcal{A}\to\mathcal{A},\ a \in \mathbb{Z}$.

First note that if (ρ,Δ) is a localized coaction on $\mathcal{A}$ then (ρ^α,Δ) also is a localized coaction, where $\rho^\alpha := (\alpha\otimes\mathrm{id}\,)\circ\rho\circ\alpha^{-1}$.

Definition 3.15: A coaction (ρ,Δ) is called *translation covariant* if (ρ,Δ) and (ρ^α,Δ) are cocycle equivalent. It is called *strictly translation covariant* if (ρ,Δ) and (ρ^α,Δ) are strictly equivalent.

If (ρ,Δ) is a universal coaction in $\mathbf{Amp}\,\mathcal{A}$, then (ρ^α,Δ) is also universal. By Proposition 3.14, (ρ,Δ) and (ρ^α,Δ) must be cocycle equivalent up to a transformation by $\sigma \in \mathrm{Aut}\,(\mathcal{G},\varepsilon)$. Thus, ρ is translation covariant iff we can choose $\sigma = \mathrm{id}_\mathcal{G}$. The following Lemma shows that this property is actually inherent in $\mathbf{Amp}\,\mathcal{A}$, i.e. independent of the choice of ρ.

Lemma 3.16: *Let (ρ,Δ) be a universal and (strictly) translation covariant coaction on $\mathcal{A}$. Then all universal coactions in $\mathbf{Amp}\,\mathcal{A}$ are (strictly) translation covariant.*

Proof: By the remark after Definition 3.13 (strict) translation covariance is preserved under transformations by $\sigma \in \mathrm{Aut}\,(\mathcal{G},\varepsilon)$. Let now (W,w) be a cocycle equivalence from ρ to ρ^α and let (U, u) be a cocycle equivalence from ρ to ρ'. Then $((\alpha\otimes\mathrm{id}_{\mathcal{G}})(U)WU^{-1}, uwu^{-1})$ is a cocycle equivalence from ρ' to ρ'^{α}. *Q.e.d.*

In [NSz2] we will show (see also [NSz1]) that strict translation covariance of a universal coaction ρ is necessary and sufficient for the existence of a lift of the translation automorphism α on $\mathcal{A}$ to an automorphism $\hat{\alpha}$ on the field algebra $\mathcal{F}_\rho \supset \mathcal{A}$ constructed from ρ, such that $\hat{\alpha}$ commutes with the global $\mathcal{G}$-gauge symmetry acting on $\mathcal{F}_\rho$. In continuum theories with a global gauge symmetry under a compact group there is a related result [DR1, Thm 8.4] stating that such a lift exists if and only if the category of translation covariant localized endomorphisms of $\mathcal{A}$ is *coherently translation covariant.*

We now show that in our formalism these conditions actually concide, i.e. a universal coaction (ρ, Δ) on $\mathcal{A}$ is strictly translation covariant if and only if $\mathbf{Amp}\,\mathcal{A}$ is coherently translation covariant. Here we follow [DR1, Sec.8] (see also [DHR4, Sec.2]) and define

Definition 3.17: We say that $\mathbf{Amp}\,\mathcal{A}$ is *translation covariant* if for any amplimorphism μ on $\mathcal{A}$ there exists an assignment $\mathbb{Z} \ni a \to W_\mu(a) \in \mathcal{A}\otimes \mathrm{End}\, V_\mu$ satisfying properties i)-iv) below. If also v) holds, then $\mathbf{Amp}\,\mathcal{A}$ is called *coherently translation covariant*:

$$
\begin{aligned}
&i) && W_\mu(a) \in (\mu^{\alpha^a} \mid \mu) && (3.19)\\
&ii) && W_\mu(a+b) = (\alpha^a\otimes\mathrm{id}\,)(W_\mu(b))W_\mu(a) && (3.20)\\
&iii) && W_\mu(a)^* = W_\mu(a)^{-1} = (\alpha^a\otimes\mathrm{id}\,)(W_\mu(-a)) && (3.21)\\
&iv) && W_\mu(a)T = (\alpha^a\otimes\mathrm{id}\,)(T)W_\nu(a), \quad \forall T\in(\mu\mid\nu) && (3.22)\\
&v) && W_{\mu\times\nu}(a) = (W_\mu(a)\otimes\mathbf{1}_\nu)(\mu\otimes\mathrm{id}\,_\nu)(W_\nu(a)) && (3.23)
\end{aligned}
$$

To illuminate these axioms let $\pi_0 : \mathcal{A}\to\mathcal{L}(\mathcal{H}_0)$ be a faithful Haag dual "vacuum" representation and let $\mathbb{Z}\ni a\to U_0(a)\in\mathcal{L}(\mathcal{H}_0)$ be a unitary representation implementing the translations α^a, i.e.

$$\mathrm{Ad}\,U_0(a)\circ\pi_0 = \pi_0\circ\alpha^a \ . \tag{3.24}$$

Then given $W_\mu(a)$ satisfying i)-iii) above the "charged" representation $\pi_\mu = (\pi_0\otimes\mathrm{id}\,_\mu)\circ\mu$ is also translation covariant, i.e.

$$\mathrm{Ad}\,U_\mu(a)\circ\pi_\mu = \pi_\mu\circ\alpha^a \ , \tag{3.25}$$

where the representation $\mathbb{Z}\ni a\to U_\mu(a)\in\mathcal{L}(\mathcal{H}_0)\otimes\mathrm{End}\,V_\mu$ is given by

$$U_\mu(a) = (\pi_0\otimes\mathrm{id}\,)(W_\mu(a)^*)(U_0(a)\otimes\mathbf{1}_\mu) \quad . \tag{3.26}$$

Conversely, if $U_\mu(a)$ is a representation of $\mathbb{Z}$ satisfying (3.25) then we may define $W_\mu(a)$ satisfying i)-iii) of Definition 3.17 by

$$(\pi_0\otimes\mathrm{id}\,)(W_\mu(a)) = (U_0(a)\otimes\mathbf{1}_\mu)U_\mu(a)^* \tag{3.27}$$

Note that by faithfulness and Haag duality of π_0 this is well defined, since if μ is localized in $I\in\mathcal{I}$ and if $J\in\mathcal{I}$ contains I and $I-a$ then the r.h.s. of (3.27) commutes with $\pi_0(\mathcal{A}(J'))\otimes\mathbf{1}_\mu$ and therefore is in $\pi_0(\mathcal{A}(J))\otimes\mathrm{End}\,V_\mu$. In this case property iv) of Definition 3.17 is equivalent to

$$(\pi_0\otimes\mathrm{id}\,)(T)U_\mu(a) = U_\nu(a)(\pi_0\otimes\mathrm{id}\,)(T), \quad \forall T\in(\nu|\mu) \tag{3.28}$$

and property v) is equivalent to

$$U_{\mu\times\nu}(a) = (\pi_\mu \otimes \mathrm{id}\,)(W_\nu(a)^*)(U_\mu(a) \otimes \mathbf{1}_\nu) \tag{3.29}$$

Proposition 3.18: *Let ρ be a universal coaction of $(\mathcal{G}, \Delta, \varepsilon)$ on $\mathcal{A}$. Then ρ is (strictly) translation covariant if and only if* $\mathbf{Amp}\,\mathcal{A}$ *is (coherently) translation covariant.*

Proof: Let (W, w) be a cocycle equivalence from (ρ, Δ) to (ρ^α, Δ) and define $\mathbb{Z} \ni a \to W_\rho(a) \in \mathcal{A} \otimes \mathcal{G}$ inductively by putting $W_\rho(0) = 1\!\!1 \otimes \mathbf{1}$ and

$$W_\rho(a+1) = (\alpha \otimes \mathrm{id}\,)(W_\rho(a))W\ . \tag{3.30}$$

Then $(W_\rho(a), w^a)$ is a cocycle equivalence from (ρ, Δ) to $(\rho^{\alpha^a}, \Delta)$, $\forall a \in \mathbb{Z}$. Moreover,

$$\begin{aligned} W_\rho(a+b) &= (\alpha^a \otimes \mathrm{id}\,)(W_\rho(b))W_\rho(a) && (3.31)\\ W_\rho(a)^* &= W_\rho(a)^{-1} = (\alpha^a \otimes \mathrm{id}\,)(W_\rho(-a)) && (3.32) \end{aligned}$$

as one easily verifies. For an amplimorphism $\mu \in \mathbf{Amp}\,\mathcal{A}$ let now $\beta_\mu \in \mathbf{Rep}\,\mathcal{G}$ and let $T_\mu \in \mathcal{A} \otimes \mathrm{End}\,V_\mu$ be a unitary such that

$$\mu = \mathrm{Ad}\,T_\mu \circ (\mathrm{id}\,\otimes \beta_\mu) \circ \rho\ . \tag{3.33}$$

We then define

$$W_\mu(a) := (\alpha^a \otimes \mathrm{id}\,)(T_\mu)(\mathrm{id}\,\otimes \beta_\mu)(W_\rho(a))T_\mu^{-1}\ . \tag{3.34}$$

Since β_μ is determined by μ up to equivalence, the definition (3.34) of $W_\mu(a)$ is actually independent of the particular choice of T_μ and β_μ. Moreover, $W_\mu(a)$ clearly intertwines μ and μ^{α^a} and equs. (3.20/21) follow from equs. (3.31/32). To prove (3.22) let $T \in (\mu|\nu)$. Then

$$T_\mu^{-1} T T_\nu \in ((\mathrm{id}_\mathcal{A} \otimes \beta_\mu) \circ \rho \,|\, (\mathrm{id}_\mathcal{A} \otimes \beta_\nu) \circ \rho) = 1\!\!1_\mathcal{A} \otimes (\beta_\mu|\beta_\nu)$$

by the effectiveness of ρ. Therefore

$$T = T_\mu(1\!\!1 \otimes t)T_\nu^{-1} \tag{3.35}$$

for some $t \in (\beta_\mu|\beta_\nu)$, and (3.22) follows from (3.34/35).

If ρ is even strictly translation covariant then

$$(W_\rho(a) \otimes \mathbf{1})(\rho \otimes \mathrm{id}\,)(W_\rho(a)) = (\mathrm{id}\,\otimes \Delta)(W_\rho(a))\ . \tag{3.36}$$

We show that this implies (3.23) for all objects in $\mathbf{Amp}_\rho^0\mathcal{A}$. By Proposition 3.4iii) the amplimorphisms in $\mathbf{Amp}_\rho^0\mathcal{A}$ are all of the form $\mu = (\mathrm{id}_\mathcal{A} \otimes \beta_\mu) \circ \rho$ for some $\beta_\mu \in \mathbf{Rep}\,\mathcal{G}$ uniquely determined by μ. Hence, by (3.34)

$$W_\mu(a) = (\mathrm{id}_\mathcal{A} \otimes \beta_\mu)(W_\rho(a))\ .$$

Moreover, using the coaction property $\rho \times \rho = (\mathrm{id}_\mathcal{A} \otimes \Delta) \circ \rho$ we get $\mu \times \nu = (\mathrm{id}_\mathcal{A} \otimes \beta_{\mu\times\nu}) \circ \rho$ where $\beta_{\mu\times\nu} = (\beta_\mu \otimes \beta_\nu) \circ \Delta$. Hence

$$\begin{aligned} W_{\mu\times\nu}(a) &= (\mathrm{id}_\mathcal{A} \otimes \beta_{\mu\times\nu})(W_\rho(a))\\ &= (\mathrm{id}_\mathcal{A} \otimes \beta_\mu \otimes \beta_\nu) \circ (\mathrm{id}_\mathcal{A} \otimes \Delta)(W_\rho(a))\\ &= (W_\mu(a) \otimes \mathbf{1}_\nu)(\mu \otimes \mathrm{id}\,_\nu)(W_\nu(a)) && (3.37) \end{aligned}$$

where we have used (3.36). This proves (3.32) in $\mathbf{Amp}_\rho^0\mathcal{A}$. The extension to $\mathbf{Amp}\,\mathcal{A} \sim \mathbf{Amp}_\rho^0\mathcal{A}$ follows straightforwardly from (3.22).

Conversely, let now $\mathbf{Amp}\,\mathcal{A}$ be translation covariant and identify $\mathcal{G}$ with the direct sum of its irreducible representations, $\mathcal{G} = \oplus_r \mathrm{End}\, V_r$. Then $\rho = \oplus_r \rho_r$ is a special amplimorphism and $W_\rho(a) = \oplus_r W_r(a) \in \mathcal{A} \otimes \mathcal{G}$ is an equivalence from ρ to ρ^{α^a}, which must be a cocycle equivalence by Proposition 3.14. Hence ρ is translation covariant. If moreover $\mathbf{Amp}\,\mathcal{A}$ is coherently translation covariant then by (3.18) and (3.23)

$$W_{\rho\times\rho}(a) = W_\rho(a) \times_\rho W_\rho(a) \tag{3.38}$$

On the other hand, similarly as in the proof of Proposition 3.4iii) equ. (3.22) implies

$$W_{(\mathrm{id}_{\mathcal{A}}\otimes\beta)\circ\rho}(a) = (\mathrm{id}_{\mathcal{A}} \otimes \beta)(W_\rho(a))$$

for all $\beta \in \mathbf{Rep}\,\mathcal{G}$. Putting $\beta = \Delta : \mathcal{G} \to \mathcal{G} \otimes \mathcal{G}$ this gives

$$W_{\rho\times\rho}(a) \equiv W_{(\mathrm{id}_{\mathcal{A}}\otimes\Delta)\circ\rho}(a) = (\mathrm{id}_{\mathcal{A}} \otimes \Delta)(W_\rho(a)) \tag{3.39}$$

and by (3.38/39) ρ is strictly translation covariant. *Q.e.d.*

4 The Drinfeld Double as a Universal Cosymmetry

In this section we prove that the Drinfeld double $\mathcal{D}(H)$ is a universal cosymmetry of the Hopf spin chain. To this end we construct in Section 4.1 a family of "two-point" coactions $\rho_I : \mathcal{A}(I) \to \mathcal{A}(I) \otimes \mathcal{D}(H)$ for any interval $I \in \mathcal{I}$ of length two. We then prove that ρ_I extends to a universal coaction in $\mathbf{Amp}\,(\mathcal{A}, I)$. We also explicitely provide the cocycle equivalences from ρ_I to ρ_{I-1} and show that ρ_I and ρ_{I-2} are strictly equivalent and therefore — being translates of each other — also strictly translation covariant. Moreover, the statistics operators $\epsilon(\rho_I, \rho_I)$ are given in terms of the standard quasitriangular R-matrix in $\mathcal{D}(H) \otimes \mathcal{D}(H)$. Finally, for any left 2-cocycle $u \in \mathcal{D}(H) \otimes \mathcal{D}(H)$ we construct a unitary $U \in \mathcal{A} \otimes \mathcal{D}(H)$ and a universal coaction (ρ', Δ') on $\mathcal{A}$ such that (U, u) provides a cocycle equivalence from ρ_I to ρ'. The statistics operator for ρ' is given in terms of the twisted R-matrix $u^{op} R u^*$.

In Section 4.2 we proceed with constructing "edge" amplimorphisms $\rho_{\partial I} : \mathcal{A}(\partial I) \to \mathcal{A} \otimes \mathcal{D}(H)$ for all intervals I of (nonzero) even length, which extend to universal ampimorphisms in $\mathbf{Amp}\,(\mathcal{A}, \partial I)$. We then show that these edge amplimorphisms are all equivalent to the previous two-point amplimorphisms. By Corollary 3.11 this proves complete compressibility of the Hopf spin chain as stated in Theorem 3.12. Thus the double $\mathcal{D}(H)$ is the universal cosymmetry of our model.

4.1 The Two-Point Amplimorphisms

In this subsection we provide a universal and strictly translation covariant coaction $\rho_I \in \mathbf{Amp}\,(\mathcal{A}, I)$ of the Drinfeld double $\mathcal{D}(H)$ on our Hopf spin chain $\mathcal{A}$ for any interval I of length $|I| = 2$. Anticipating the proof of Theorem 3.12 this proves that $\mathcal{D}(H)$ is the universal cosymmetry of $\mathcal{A}$.

A review of the Drinfeld $\mathcal{D}(H)$ double is given in Appendix B. Here we just note that it is generated by H and $\hat{H}_{cop}$ which are both contained as Hopf subalgebras in $\mathcal{D}(H)$, where $\hat{H}_{cop}$ is the Hopf algebra $\hat{H}$ with opposite coproduct. We denote the generators of $\mathcal{D}(H)$ by $\mathcal{D}(a)$, $a \in H$, and $\mathcal{D}(\varphi)$, $\varphi \in \hat{H}$, respectively.

Theorem 4.1: *On the Hopf spin chain define* $\rho_I : \mathcal{A}(I) \to \mathcal{A}(I) \otimes \mathcal{D}(H)$, $|I| = 2$, *by* [3]

$$\begin{aligned}
\rho_{2i,2i+1}(A_{2i}(a)A_{2i+1}(\varphi)) \;&:=\; A_{2i}(a_{(1)})A_{2i+1}(\varphi_{(2)}) \;\otimes\; \mathcal{D}(a_{(2)})\mathcal{D}(\varphi_{(1)}) && (4.1a)\\
\rho_{2i-1,2i}(A_{2i-1}(\varphi)A_{2i}(a)) \;&:=\; A_{2i-1}(\varphi_{(1)})A_{2i}(a_{(2)}) \;\otimes\; \mathcal{D}(\varphi_{(2)})\mathcal{D}(a_{(1)}) && (4.1b)
\end{aligned}$$

Then:

i) $\rho_{i,i+1}$ *provides a coaction of* $\mathcal{D}(H)$ *on* $\mathcal{A}_{i,i+1}$ *with respect to the natural coproducts* $\Delta_{\mathcal{D}}$ *(if* i *is even) or* $\Delta_{\mathcal{D}}^{op}$ *(if* i *is odd) on* $\mathcal{D}(H)$.

ii) $\rho_{i,i+1}$ *extends to a coaction in* $\mathbf{Amp}(\mathcal{A}, I)$ *which is universal in* $\mathbf{Amp}(\mathcal{A}, I)$.

Proof: i) Since interchanging even and odd sites amounts to interchaning H and $\hat{H}$ and since $\mathcal{D}(\hat{H}) = \mathcal{D}(H)_{cop}$ it is enough to prove all statements for i even. It is obvious that the restrictions $\rho_{2i,2i+1}|\mathcal{A}_{2i}$ and $\rho_{2i,2i+1}|\mathcal{A}_{2i+1}$ define *-algebra homomorphisms. Hence, to prove that $\rho_{2i,2i+1} : \mathcal{A}_{2i,2i+1} \to \mathcal{A}_{2i,2i+1} \otimes \mathcal{D}(H)$ is a well defined amplimorphism we are left to check that the commutation relations (2.2) are respected, i.e.

$$\rho_{2i,2i+1}(A_{2i+1}(\varphi))\rho_{2i,2i+1}(A_{2i}(a)) = \rho_{2i,2i+1}\left(A_{2i}(a_{(1)})\langle a_{(2)}, \varphi_{(1)}\rangle A_{2i+1}(\varphi_{(2)})\right)$$

Using eqn. (B.2) this is straightforward and is left to the reader. Using equs. (B.3a,b) the identities $(\mathrm{id}_{\mathcal{A}} \otimes \varepsilon_{\mathcal{D}}) \circ \rho_{2i,2i+1} = \mathrm{id}_{\mathcal{A}}$ and $(\rho_{2i,2i+1} \times \rho_{2i,2i+1}) = (\mathrm{id} \otimes \Delta_{\mathcal{D}}) \circ \rho_{2i,2i+1}$ are nearly trivial and are also left to the reader.

ii) To show that ρ_I extends to an amplimorphism in $\mathbf{Amp}(\mathcal{A}, I)$ (still denoted by ρ_I) we have to check that together with the definition $\rho_I(A) := A \otimes \mathbf{1}_{\mathcal{D}(H)}$, $A \in \mathcal{A}(I^c)$, we get a well defined *-algebra homomorphism $\rho_I : \mathcal{A} \to \mathcal{A} \otimes \mathcal{D}(H)$. Clearly, this holds if and only if $\rho_{i,i+1}|\mathcal{A}_{i,i+1}$ commutes with the left adjoint action of $\mathcal{A}_{i+2}$ and the right adjoint action of $\mathcal{A}_{i-1}$, respectively, on $\mathcal{A}_{i,i+1}$, where these actions are defined on $B \in \mathcal{A}_{i,i+1}$ by

$$\begin{aligned}
A_{2i+2}(a) \triangleright B \;&:=\; A_{2i+1}(a_{(1)})BA_{2i+1}(S(a_{(2)}))\\
B \triangleleft A_{2i-1}(\varphi) \;&:=\; A_{2i-1}(S(\varphi_{(1)}))BA_{2i-1}(\varphi_{(2)})
\end{aligned}$$

Now $\mathcal{A}_{2i+2}$ commutes with $\mathcal{A}_{2i}$ and $\mathcal{A}_{2i-1}$ commutes with $\mathcal{A}_{2i+1}$ and

$$\begin{aligned}
A_{2i+2}(a) \triangleright A_{2i+1}(\varphi) \;&=\; A_{2i+1}(a \to \varphi) && (4.2a)\\
A_{2i}(a) \triangleleft A_{2i-1}(\varphi) \;&=\; A_{2i}(a \leftarrow \varphi) && (4.2b)
\end{aligned}$$

Hence $\rho_{2i,2i+1}$ commutes with these actions, since by coassociativity

$$\begin{aligned}
A_{2i}((a \leftarrow \varphi)_{(1)}) \otimes \mathcal{D}((a \leftarrow \varphi)_{(2)}) \;&=\; A_{2i}(a_{(1)} \leftarrow \varphi) \otimes \mathcal{D}(a_{(2)})\\
A_{2i+1}((a \to \varphi)_{(2)}) \otimes \mathcal{D}((a \to \varphi)_{(1)}) \;&=\; A_{2i+1}(a \to \varphi_{(2)}) \otimes \mathcal{D}(\varphi_{(1)})
\end{aligned}$$

Next we identify $\mathcal{D}(H) = \oplus_r \mathrm{End}\, V_r \subset \mathrm{End}\, V$, where r runs through a complete set of pairwise inequivalent irreducible representations of $\mathcal{D}(H)$ and where $V := \oplus_r V_r$. Since $|I| = 2$ implies $\mathcal{A}(\mathrm{Int}\, I) = \mathbb{C} \cdot \mathbb{1}_{\mathcal{A}}$ we conclude by Lemma 3.8

$$\rho_{2i,2i+1}(\mathcal{A})' \cap (\mathcal{A} \otimes \mathrm{End}\, V) = \mathbb{1}_{\mathcal{A}} \otimes \mathcal{C}$$

[3] Here we identify I with $I \cap \mathbb{Z}$.

for some unital *-subalgebra $\mathcal{C} \subset \mathrm{End}\, V$. Hence, by Proposition 3.4ii, $\mathcal{D}(H)$ is effective for $\rho_{2i,2i+1}$ provided $\mathcal{C} = \mathcal{D}(H)' \cap \mathrm{End}\, V$. To show this we now compute for $a \in H$ and $\varphi \in \hat{H}$

$$\begin{aligned}
&\Big[A_{2i+1}(S(\varphi_{(2)}))A_{2i}(S(a_{(1)})) \otimes \mathbf{1}_{\mathcal{D}(H)}\Big] \cdot \rho_{2i,2i+1}\Big(A_{2i}(a_{(2)})A_{2i+1}(\varphi_{(1)})\Big) \\
&\quad = A_{2i+1}(S(\varphi_{(3)}))A_{2i}(S(a_{(1)})a_{(2)})A_{2i+1}(\varphi_{(2)}) \otimes \mathcal{D}(a_{(3)})\mathcal{D}(\varphi_{(1)}) \\
&\quad = \mathbb{1}_{\mathcal{A}} \otimes \mathcal{D}(a)\mathcal{D}(\varphi).
\end{aligned}$$

Hence, $\mathcal{A} \otimes \mathcal{D}(H) = (\mathcal{A} \otimes \mathbf{1}_{\mathcal{D}(H)}) \vee \rho_{2i,2i+1}(\mathcal{A})$ and therefore

$$\begin{aligned}
\mathbb{1}_{\mathcal{A}} \otimes (\mathcal{D}(H)' \cap \mathrm{End}\, V) &\equiv (\mathcal{A} \otimes \mathcal{D}(H))' \cap (\mathcal{A} \otimes \mathrm{End}\, V) \\
&= (\mathcal{A} \otimes \mathbf{1}_{\mathcal{D}(H)})' \cap \rho_{2i,2i+1}(\mathcal{A})' \cap (\mathcal{A} \otimes \mathrm{End}\, V) \\
&= \mathbf{1}_{\mathcal{A}} \otimes \mathcal{C}
\end{aligned}$$

which proves that $\mathcal{D}(H)$ is effective for $\rho_{2i,2i+1}$. To prove that ρ_I is universal in $\mathbf{Amp}\,(\mathcal{A}, I)$ we now show $\mathbf{Amp}\,(\mathcal{A}, I) \subset \mathbf{Amp}^{\,0}_{\rho_I}(\mathcal{A})$. Hence let $\mu \in \mathbf{Amp}\,(\mathcal{A}, I)$, $I \cap \mathbb{Z} = \{2i, 2i+1\}$. Then $\mu(\mathcal{A}_{2i,2i+1}) \subset \mathcal{A}_{2i,2i+1} \otimes \mathcal{D}(H)$ by Lemma 3.8 and the restriction $\mu|\mathcal{A}_{2i,2i+1}$ commutes with the left adjoint action of $\mathcal{A}_{2i+2}$ and the right adjoint action of $\mathcal{A}_{2i-1}$, respectively, on $\mathcal{A}_{2i,2i+1}$. This allows to construct a representation $\beta_\mu : \mathcal{D}(H) \to \mathrm{End}\, V_\mu$ such that $\mu = (id \otimes \beta_\mu) \circ \rho_{2i,2i+1}$ and therefore, by Proposition 3.4iii), $\mu \in \mathbf{Amp}^{\,0}_{\rho_{2i,2i+1}}(\mathcal{A})$, as follows. First we use the above commutation properties together with eqn (2.17) to conclude

$$\begin{aligned}
\mu(\mathcal{A}_{2i}) &\subset (\mathcal{A}_{2i,2i+1} \cap \mathcal{A}'_{2i+2}) \otimes \mathrm{End}\, V_\mu &= \mathcal{A}_{2i} \otimes \mathrm{End}\, V_\mu \\
\mu(\mathcal{A}_{2i+1}) &\subset (\mathcal{A}_{2i,2i+1} \cap \mathcal{A}'_{2i+1}) \otimes \mathrm{End}\, V_\mu &= \mathcal{A}_{2i+1} \otimes \mathrm{End}\, V_\mu
\end{aligned}$$

Now we define, for $a \in H \subset \mathcal{D}(H)$ and $\varphi \in \hat{H} \subset \mathcal{D}(H)$,

$$\beta_\mu(\mathcal{D}(a)) := (A_{2i}(S(a_{(1)})) \otimes \mathbf{1})\, \mu(A_{2i}(a_{(2)})) \tag{4.3a}$$

$$\beta_\mu(\mathcal{D}(\varphi)) := \mu(A_{2i+1}(\varphi_{(1)}))\, (A_{2i+1}(S(\varphi_{(2)})) \otimes \mathbf{1}) \tag{4.3b}$$

Using that μ commutes with the (left or right) adjont actions of $\mathcal{A}_{2i-1}$ and $\mathcal{A}_{2i+1}$, respectivley, it is straightforward to check that $\beta_\mu(H) \subset \mathcal{A}_{2i} \otimes \mathrm{End}\, V_\mu$ commutes with $\mathcal{A}_{2i-1} \otimes \mathbf{1}$ and $\beta_\mu(\hat{H}) \subset \mathcal{A}_{2i+1} \otimes \mathrm{End}\, V_\mu$ commutes with $\mathcal{A}_{2i+2} \otimes \mathbf{1}$. Hence, by eqn. (2.18), $\beta_\mu|H$ and $\beta_\mu|\hat{H}$ take values in $\mathbf{1}_{\mathcal{A}} \otimes \mathrm{End}\, V_\mu$ and therefore (identifying $\mathcal{A}_{2i} = H$ and $\mathcal{A}_{2i+1} = \hat{H}$)

$$\begin{aligned}
\beta_\mu|H &= (\varepsilon_H \otimes id) \circ \beta_\mu|H = (\varepsilon_H \otimes id) \circ \mu|\mathcal{A}_{2i} \\
\beta_\mu|\hat{H} &= (\varepsilon_{\hat{H}} \otimes id) \circ \beta_\mu|\hat{H} = (\varepsilon_{\hat{H}} \otimes id) \circ \mu|\mathcal{A}_{2i+1}
\end{aligned}$$

where ε_H and $\varepsilon_{\hat{H}}$ denote the counits on H and $\hat{H}$, respectively, and where the second identities follow from the definition (4.3). Thus, identifying $\mathbb{1}_{\mathcal{A}} \otimes \mathrm{End}\, V_\mu = \mathrm{End}\, V_\mu$, the maps $\beta_\mu|H$ and $\beta_\mu|\hat{H}$ define *-representations of H and $\hat{H}$, respectively, on V_μ. Moreover, inverting (4.3) we get

$$\mu(A_{2i}(a)) = A_{2i}(a_{(1)}) \otimes \beta_\mu(\mathcal{D}(a_{(2)})) \tag{4.4a}$$

$$\mu(A_{2i+1}(\varphi)) = A_{2i+1}(\varphi_{(2)}) \otimes \beta_\mu(\mathcal{D}(\varphi_{(1)})) \tag{4.4b}$$

Thus $\mu = (id \otimes \beta_\mu) \circ \rho_I$, provided that β_μ actually extends to a representation of all of $\mathcal{D}(H)$. To see this we have to check that β_μ respects the commutation relations (B.1c). Recalling the identity $A_{2i+1}(\varphi)A_{2i}(a) = A_{2i}(a_{(1)})\langle a_{(2)}, \varphi_{(1)}\rangle A_{2i+1}(\varphi_{(2)})$ and the definition (4.3) we compute

$$
\begin{aligned}
&\mathbf{1}_{\mathcal{A}} \otimes \beta_\mu(\mathcal{D}(a_{(1)}))\,\langle a_{(2)}, \varphi_{(1)}\rangle\,\beta_\mu(\mathcal{D}(\varphi_{(2)}))\\
&= \Big(A_{2i}(S(a_{(1)}))\otimes \mathbf{1}\Big)\;\mu\,\Big(A_{2i+1}(\varphi_{(1)})A_{2i}(a_{(2)})\Big)\;\Big(A_{2i+1}(S(\varphi_{(2)}))\otimes \mathbf{1}\Big)\\
&= A_{2i}(S(a_{(1)}))\,A_{2i+1}(\varphi_{(2)})\,A_{2i}(a_{(2)})\,A_{2i+1}(S(\varphi_{(3)}))\,\otimes\,\beta_\mu(\mathcal{D}(\varphi_{(1)}))\,\beta_\mu(\mathcal{D}(a_{(3)}))\\
&= \mathbf{1}_{\mathcal{A}} \otimes \beta_\mu(\mathcal{D}(\varphi_{(1)}))\,\langle \varphi_{(2)}, a_{(1)}\rangle\,\beta_\mu(\mathcal{D}(a_{(2)}))
\end{aligned}
$$

where in the third line we have used (4.4). Hence, by (B.1c) β_μ extends to a representation of $\mathcal{D}(H)$ and therefore $\mu \in \mathbf{Amp}^{\,0}_{\rho_I}(\mathcal{A})$. This proves that ρ_I is universal in $\mathbf{Amp}\,(\mathcal{A}, I)$. *Q.e.d.*

We now show that the coactions $\rho_{i,i+1}$ are all cocycle equivalent and strictly translation covariant. To this end let $\{b_A\}$ be a basis in H with dual basis $\{\beta^A\}$ in $\hat{H}$ and define the *charge transporters* $T_i \in \mathcal{A}_i \otimes \mathcal{D}(H)$ by

$$
T_i := \begin{cases} A_i(b_A)\otimes D(\beta^A) & i = \text{even} \\ A_i(\beta^A)\otimes D(b_A) & i = \text{odd} \end{cases} \tag{4.5}
$$

Also recall that the canonical quasitriangular R-matrix in $\mathcal{D}(H)\otimes\mathcal{D}(H)$ is given by

$$
R = \mathcal{D}(b_A)\otimes\mathcal{D}(\beta^A)
$$

We then have

Proposition 4.2: *The charge transporters T_i are unitary intertwiners from $\rho_{i,i+1}$ to $\rho_{i-1,i}$, i.e.*

$$
T_i\rho_{i,i+1}(A) \;=\; \rho_{i-1,i}(A)T_i\,, \qquad A\in\mathcal{A} \tag{4.6}
$$

and they satisfy the cocycle condition

$$
\begin{aligned}
T_i \times_{\rho_{i,i+1}} T_i \;&\equiv\; (T_i\otimes\mathbf{1})\cdot(\rho_{i,i+1}\otimes id\,)(T_i) \;=\\
&= \begin{cases} (\mathbf{1}\otimes R)\cdot(id\,\otimes\Delta_{\mathcal{D}})(T_i) & i = even \\ (\mathbf{1}\otimes R^{op})\cdot(id\,\otimes\Delta_{\mathcal{D}}^{op})(T_i) & i = odd \end{cases}
\end{aligned} \tag{4.7}
$$

Proof: This is a lengthy but straightforward calculation, which we leave to the reader. *Q.e.d.*

Iterating the identities (4.6/7) we get an infinite sequence of cocycle equivalences

$$
\ldots(\rho_{2i,2i+1},\Delta_{\mathcal{D}}) \overset{(T_{2i+1},R^{op})}{\longleftarrow} (\rho_{2i+1,2i+2},\Delta_{\mathcal{D}}^{op}) \overset{(T_{2i+2},R)}{\longleftarrow} (\rho_{2i+2,2i+3},\Delta_{\mathcal{D}})\ldots
$$

Composing two such arrows we obtain a coboundary equivalence $(T_{2i+1}T_{2i+2}, R^{op}R)$ because $R^{op}R = (s\otimes s)\Delta_{\mathcal{D}}(s^{-1})$ according to [Dr], where $s\in\mathcal{D}(H)$ is the central unitary $s = S_{\mathcal{D}}(R_2)R_1 = \mathcal{D}(S(\beta^A))\mathcal{D}(b_A)$. Likewise $(T_{2i}T_{2i+1}, RR^{op})$ yields a coboundary equivalence. Therefore introducing

$$
U_{i,i+1} := (\mathbf{1}\otimes s^{-1})T_iT_{i+1} \quad\in\; (\rho_{i-1,i}|\rho_{i+1,i+2}) \tag{4.8}
$$

we obtain unitary charge transporters localized within $\{i, i+1\}$ that satisfiy the *trivial cocycle* conditions

$$
\begin{aligned}
U_{2i-1,2i}\;\times_{\rho_{2i,2i+1}}\;U_{2i-1,2i} \;&=\; (\mathrm{id}_{\mathcal{A}}\otimes\Delta_{\mathcal{D}})(U_{2i-1,2i})\\
U_{2i-2,2i-1}\;\times_{\rho_{2i-1,2i}}\;U_{2i-2,2i-1} \;&=\; (\mathrm{id}_{\mathcal{A}}\otimes\Delta_{\mathcal{D}}^{op})(U_{2i-2,2i-1})
\end{aligned} \tag{4.9}
$$

Hence, summarizing the above results (and anticipating the result of Theorem 3.12) we have shown

Corollary 4.3: *The coactions $\rho_{i,i+1}$ are all strictly translation covariant and universal in* $\mathbf{Amp}\,\mathcal{A}$.

Proof: Universality follows from Theorem 4.1ii) and Theorem 3.12 and strict translation covariance (Definition 3.15) follows from (4.8/9), since $\rho_{i+1,i+2} = (\alpha \otimes \mathrm{id}\,) \circ \rho_{i-1,i} \circ \alpha^{-1}$. *Q.e.d.*

Proposition 4.2 also enables us to compute the statistics operator of ρ_I.

Theorem 4.4: *Let ρ_I be given as in Theorem 4.1 and let $\epsilon(\rho_I, \rho_I)$ be the associated statistics operator (3.7). Then*

$$\epsilon(\rho_I, \rho_I) = 1\!\!1 \otimes PR_I \tag{4.10}$$

where $P : \mathcal{D}(H) \otimes \mathcal{D}(H) \to \mathcal{D}(H) \otimes \mathcal{D}(H)$ denotes the permutation and

$$R_{i,i+1} = \begin{cases} R & , \quad i = \text{even} \\ R^{op} & , \quad i = \text{odd} \end{cases} \tag{4.11}$$

Moreover, if (U, u) is a cocycle equivalence from $(\rho_I, \Delta_{\mathcal{D}}^{(op)})$ to (ρ', Δ') then $\epsilon(\rho', \rho') = 1\!\!1 \otimes PR'$ where $R' = u_{op} R_I u^*$.

Proof: Putting $I \cap \mathbb{Z} = \{i, i+1\}$ and using (3.7) and (4.8) we get

$$\begin{aligned} (1\!\!1 \otimes P)\epsilon(\rho_I, \rho_I) &= (U^*_{i-1,i})^{02}(\rho_{i,i+1} \otimes \mathrm{id}_{\mathcal{G}})(U_{i-1,i}) \\ &= (T_i^*)^{02}(T_i^*)^{01}(T_i \times_{\rho_{i,i+1}} T_i)\,, \end{aligned} \tag{4.12}$$

where the superfix 01/02 refers to the obvious inclusions of $\mathcal{A} \otimes \mathcal{D}(H)$ into $\mathcal{A} \otimes \mathcal{D}(H) \otimes \mathcal{D}(H)$, and where the second line follows since s is central and $(\rho_{i,i+1} \otimes \mathrm{id}_{\mathcal{G}})(T_{i-1}) = T_{i-1}^{02}$. Now (4.10/11) follows from (4.7) and (4.12) by using $\Delta_{\mathcal{D}}^{op} = \mathrm{Ad}\,R \circ \Delta_{\mathcal{D}}$ and the identities

$$(\mathrm{id}_{\mathcal{A}} \otimes \Delta_{\mathcal{D}})(T_i) = \begin{cases} T_i^{02} T_i^{01} & , \quad i = \text{even} \\ T_i^{01} T_i^{02} & , \quad i = \text{odd} \end{cases}$$

which follow straightforwardly from (4.5).

Let now (U, u) be a cocycle equivalence from (ρ, Δ) to (ρ', Δ'). Then by (3.8a) and (3.17c)

$$\begin{aligned} (1\!\!1 \otimes P)\epsilon(\rho', \rho') &= (1\!\!1 \otimes P)(U \times_\rho U)\epsilon(\rho, \rho)(U \times_\rho U)^* \\ &= (1\!\!1 \otimes u^{op})(\mathrm{id}_{\mathcal{A}} \otimes \Delta^{op})(U)(1\!\!1 \otimes R)(\mathrm{id}_{\mathcal{A}} \otimes \Delta)(U^*)(1\!\!1 \otimes u^*) \\ &= 1\!\!1 \otimes (u^{op} R u^*)\,. \end{aligned}$$

Q.e.d.

We conclude this subsection by demonstrating that for any left 2-cocycle $u \in \mathcal{D}(H) \otimes \mathcal{D}(H)$ there exists a coaction (ρ', Δ') which is cocycle equivalent to $(\rho_I, \Delta^{(op)})$. To this end we first note that there exist $*$-algebra inclusions $\Lambda_{i,i+1} : \mathcal{D}(H) \to \mathcal{A}$ given by

$$\begin{aligned} \Lambda_{2i,2i+1}(\mathcal{D}(a)) &:= A_{2i}(a) \\ \Lambda_{2i,2i+1}(\mathcal{D}(\varphi)) &:= A_{2i-1}(\varphi_{(2)}) A_{2i+1}(\varphi_{(1)}) \end{aligned}$$

and analogously for $\Lambda_{2i-1,2i}$. Moreover, the following identities are straightforwardly checked

$$\rho_I \circ \Lambda_I = (\Lambda_I \otimes \mathrm{id}\,) \circ \Delta_{\mathcal{D}}^{(op)}$$

For a given 2-cocycle $u \in \mathcal{D}(H) \otimes \mathcal{D}(H)$ we now put $\Delta' = \mathrm{Ad}\, u \circ \Delta_{\mathcal{D}}^{(op)}$, $U = (\Lambda_I \otimes \mathrm{id}\,)(u)$ and $\rho' = \mathrm{Ad}\, U \circ \rho_I$, from which it is not difficult to see that (U, u) provides a cocycle equivalence from $(\rho_I, \Delta_{\mathcal{D}}^{(op)})$ to (ρ', Δ').

4.2 Edge Amplimorphisms and Complete Compressibility

This subsection is devoted to the construction of universal edge amplimorphisms and thereby to the proof of Theorem 3.12. As a preparation we first need

Proposition 4.5: *Let $j = i + 2n + 1,\ i \in \mathbb{Z},\ n \in \mathbb{N}_0$. Then there exist *-algebra inclusions*

$$\begin{array}{rcl} L_{i,j} & : & \mathcal{A}_{i-1} \to \mathcal{A}_{i,j} \cap \mathcal{A}'_{i+1,j} \\ R_{i,j} & : & \mathcal{A}_{j+1} \to \mathcal{A}_{i,j} \cap \mathcal{A}'_{i,j-1} \end{array}$$

such that for all $A_{i-1}(a) \in \mathcal{A}_{i-1}$ and all $A_{j+1}(\varphi) \in \mathcal{A}_{j+1}$

$$i) \qquad A_{i-1}(a_{(1)}) L_{i,j}(S(a_{(2)})) \in \mathcal{A}_{i-1,j} \cap \mathcal{A}'_{i,j} \tag{4.13}$$

$$ii) \qquad R_{i,j}(S(\varphi_{(1)})) A_{j+1}(\varphi_{(2)}) \in \mathcal{A}_{i,j+1} \cap \mathcal{A}'_{i,j} \tag{4.14}$$

$$iii) \qquad L_{i,j}(a) R_{i,j}(\varphi) = R_{i,j}(\varphi_{(1)}) \langle \varphi_{(2)}\,,\, a_{(1)} \rangle L_{i,j}(a_{(2)}) \tag{4.15}$$

Proof: We first use the left action (2.4) of $\mathcal{A}_{j+1}$ on $\mathcal{A}_{i,j}$ and the right action (2.5) of $\mathcal{A}_{i-1}$ on $\mathcal{A}_{i,j}$ to point out that the assertions (4.13) and (4.14) are equivalent, respectively, to

$$A_{i,j} \triangleleft A_{i-1}(a) = L_{i,j}(S(a_{(1)}) A_{i,j} L_{i,j}(a_{(2)}) \tag{4.16a}$$

$$A_{j+1}(\varphi) \triangleright A_{i,j} = R_{i,j}(\varphi_{(1)}) A_{i,j} R_{i,j}(S(\varphi_{(2)})) \tag{4.16b}$$

for all $A_{i-1}(a) \in \mathcal{A}_{i-1}$, $A_{j+1}(\varphi) \in \mathcal{A}_{j+1}$ and $A_{i,j} \in \mathcal{A}_{i,j}$. Note that equs. (4.16) say that these actions are inner in $\mathcal{A}_{i,j}$, as they must be since $\mathcal{A}_{i,j}$ is simple for $j - i = 2n + 1$.

Given that $L_{i,j}$ commutes with $\mathcal{A}_{i+1,j}$ and $R_{i,j}$ commutes with $\mathcal{A}_{i,j-1}$ eqns. (4.16) may also be rewritten as

$$A_i(\psi) L_{i,j}(a) = L_{i,j}(a_{(1)}) A_i(\psi \leftarrow a_{(2)}) \tag{4.17a}$$

$$R_{i,j}(\varphi) A_j(b) = A_j(\varphi_{(1)} \rightarrow b) R_{i,j}(\varphi_{(2)}) \tag{4.17b}$$

We now use the *-algebra isomorphism (2.12)

$$\mathcal{T}_{i,j} : \mathcal{A}_{i,j} \to \mathcal{A}_{i,j-2} \otimes \mathrm{End}\, \mathcal{H}$$

(assume without loss $\mathcal{A}_i \cong \hat{H}$) and proceed by induction over $n \in \mathbb{N}_0$. For $n = 0$ we have $\mathcal{T}_{i,i+1}(\mathcal{A}_{i,i+1}) = \mathrm{End}\, \mathcal{H}$,

$$\mathcal{T}_{i,i+1}(A_i(\psi)) = Q^+(\psi) \tag{4.18a}$$

$$\mathcal{T}_{i,i+1}(A_{i+1}(b)) = P^+(b) \tag{4.18b}$$

and we put

$$
\begin{aligned}
L_{i,i+1}(a) &:= T^{-1}_{i,i+1}\left(P^-(S^{-1}(a))\right) &\quad (4.19a)\\
R_{i,i+1}(\varphi) &:= T^{-1}_{i,i+1}\left(Q^-(S^{-1}(\varphi))\right) &\quad (4.19b)
\end{aligned}
$$

Then $L_{i,i+1}$ and $R_{i,i+1}$ define *-algebra inclusions and (4.15) follows straightforwardly from the definitions (2.7). Moreover, $L_{i,i+1}(a)$ commutes with $\mathcal{A}_{i+1} = \mathcal{T}^{-1}_{i,i+1}(P^+(H))$ and $R_{i,i+1}(\varphi)$ commutes with $\mathcal{A}_i = \mathcal{T}^{-1}_{i,i+1}(Q^+(\hat{H}))$. Finally, using (4.18/19) and (2.7) we get for $j = i+1$

$$
\begin{aligned}
L_{i,i+1}(S(a_{(1)}))A_i(\psi)L_{i,i+1}(a_{(2)}) &= A_i(\psi \leftarrow a) &= A_i(\psi) \triangleleft A_{i-1}(a)\\
R_{i,i+1}(\varphi_{(1)})A_{i+1}(b)R_{i,i+1}(S(\varphi_{(2)})) &= A_{i+1}(\varphi \rightarrow b) &= A_{i+2}(\varphi) \triangleright A_{i+1}(b)
\end{aligned}
$$

where the second equalities follow from (2.2), see also (4.2). This proves (4.16) and therefore Proposition 4.5i)-iii) for $n = 0$.

Assume now the claim holds for $j = i + 2n + 1$ and put

$$
\begin{aligned}
L_{i,j+2}(a) &:= \mathcal{T}^{-1}_{i,j+2}\left(L_{i,j}(a) \otimes \mathbf{1}\right) &\quad (4.20a)\\
R_{i,j+2}(\varphi) &:= \mathcal{T}^{-1}_{i,j+2}\left(R_{i,j}(\varphi_{(2)}) \otimes Q^-(S^{-1}(\varphi_{(1)}))\right) &\quad (4.20b)
\end{aligned}
$$

Then $L_{i,j+2}$ and $R_{i,j+2}$ again define *-algebra inclusions and (4.15) immediately follows from the induction hypothesis. Also, since $\mathcal{T}_{i,j+2}(\mathcal{A}_{j+1,j+2}) = \mathbf{1}_{\mathcal{A}} \otimes \mathrm{End}\,\mathcal{H}$ we have

$$
L_{i,j+2}(a) \in \mathcal{A}_{i,j+2} \cap \mathcal{A}'_{j+1,j+2}
$$

Moreover, $\mathcal{T}_{i,j+2}(\mathcal{A}_{i+1,j}) \subset \mathcal{A}_{i+1,j} \otimes P^-(H)$ commutes with $L_{i,j}(a) \otimes \mathbf{1}$ by the induction hypothesis, and therefore $L_{i,j+2}(a) \in \mathcal{A}'_{i+1,j}$ implies

$$
L_{i,j+2}(a) \in \mathcal{A}_{i,j+2} \cap \mathcal{A}'_{i+1,j+2}. \quad (4.21)
$$

Next, to show that $R_{i,j}(\varphi)$ commutes with $\mathcal{A}_{i,j+1}$ we first note that $\mathcal{T}_{i,j+2}(\mathcal{A}_{i,j-1}) = \mathcal{A}_{i,j-1} \otimes \mathbf{1}$ and $\mathcal{T}_{i,j+2}(\mathcal{A}_{j+1}) = \mathbf{1}_{\mathcal{A}} \otimes Q^+(\hat{H})$ and therefore

$$
R_{i,j+2}(\varphi) \in \mathcal{A}_{i,j+2} \cap \mathcal{A}'_{i,j-1} \cap \mathcal{A}'_{j+1}
$$

by (4.20b) and the induction hypothesis. To show that $R_{i,j+2}(\varphi)$ also commutes with $\mathcal{A}_j$ we compute

$$
\begin{aligned}
\mathcal{T}_{i,j+2}(R_{i,j+2}(\varphi)A_j(b)) &= R_{i,j}(\varphi_{(2)})A_j(b_{(1)}) \otimes Q^-(S^{-1}(\varphi_{(1)}))P^-(S(b_{(2)}))\\
&= A_j(b_{(1)})R_{i,j}(\varphi_{(3)}) \otimes \langle \varphi_{(2)}, b_{(2)}\rangle Q^-(S^{-1}(\varphi_{(1)}))P^-(S(b_{(3)}))\\
&= A_j(b_{(1)})R_{i,j}(\varphi_{(2)}) \otimes P^-(S(b_{(2)}))Q^-(S^{-1}(\varphi_{(1)}))\\
&= \mathcal{T}_{i,j+2}(A_j(b)R_{i,j+2}(\varphi))
\end{aligned}
$$

where in the second line we have used the induction hypothesis in the form (4.17b) and in the third line the Weyl algebra identity $P^-(b)Q^-(\varphi) = Q^-(\varphi_{(2)})P^-(b_{(1)})\langle\varphi_{(1)}, b_{(2)}\rangle$. Hence $R_{i,j+2}(\varphi)$ also commutes with $\mathcal{A}_j$ and therefore

$$
R_{i,j+2}(\varphi) \in \mathcal{A}_{i,j+2} \cap \mathcal{A}'_{i,j+1} \quad (4.22)
$$

To prove (4.13) for $L_{i,j+2}$ we note that $\mathcal{T}_{i,j+2} = \mathcal{T}_{i-1,j+2}|\mathcal{A}_{i,j+2}$ and $\mathcal{T}_{i-1,j+2}(A_{i-1}(a)) = A_{i-1}(a) \otimes \mathbf{1}$, and therefore

$$\begin{aligned} \mathcal{T}_{i-1,j+2}\left(A_{i-1}(a_{(1)})L_{i,j+2}(S(a_{(2)}))\right) &= A_{i-1}(a_{(1)})L_{i,j}(S(a_{(2)})) \otimes \mathbf{1} \\ &\in (\mathcal{A}_{i-1,j} \cap \mathcal{A}'_{i,j}) \otimes \mathbf{1} \equiv \mathcal{T}_{i-1,j+2}(\mathcal{A}_{i-1,j+2} \cap \mathcal{A}'_{i,j+2}) \end{aligned}$$

by the induction hypothesis. To prove (4.14) for $R_{i,j+2}$ we equivalently prove (4.17b) for $R_{i,j+2}$ by computing

$$\begin{aligned} \mathcal{T}_{i,j+2}\left(R_{i,j+2}(\varphi)A_{j+2}(b)\right) &= R_{i,j}(\varphi_{(2)}) \otimes Q^-(S^{-1}(\varphi_{(1)}))P^+(b) \\ &= R_{i,j}(\varphi_{(3)}) \otimes P^+(\varphi_{(1)} \rightarrow b)Q^-(S^{-1}(\varphi_{(2)})) \\ &= \mathcal{T}_{i,j+2}\left(A_{j+2}(\varphi_{(1)} \rightarrow b)R_{i,j+2}(\varphi_{(2)})\right) \end{aligned}$$

where the Weyl algebra identity used in the second line follows again straightforwardly from (2.7). This concludes the proof of Proposition 4.5. *Q.e.d.*

As a particular consequence of Proposition 4.5 we also need

Corollary 4.6: *For all $A_j(a) \in \mathcal{A}_j$ and $A_{j+1}(\varphi) \in \mathcal{A}_{j+1}$ we have*

$$\begin{aligned} &i) \quad A_{j+1}(S(\varphi_{(1)}))R_{i,j}(\varphi_{(2)}) = R_{i,j}(\varphi_{(2)})A_{j+1}(S(\varphi_{(1)})) \in \mathcal{A}_{i,j+1} \cap \mathcal{A}'_{i,j} && (4.23) \\ &ii) \quad R_{i,j}(\varphi)A_j(a) = A_j(a_{(1)})R_{i,j}(\varphi \leftarrow a_{(2)}) && (4.24) \end{aligned}$$

Proof:

$$\begin{aligned} i) \quad A_{j+1}(S(\varphi_{(1)}))R_{i,j}(\varphi_{(2)}) &= R_{i,j}\left(S(S(\varphi_{(2)})\varphi_{(3)})\right)A_{j+1}(S(\varphi_{(1)}))R_{i,j}(\varphi_{(4)}) \\ &= R_{i,j}(S^2(\varphi_{(2)}))A_{j+1}(S(\varphi_{(1)}))R_{i,j}(S(\varphi_{(3)})\varphi_{(4)}) \\ &= R_{i,j}(\varphi_{(2)})A_{j+1}(S(\varphi_{(1)})) \in \mathcal{A}_{i,j+1} \cap \mathcal{A}'_{i,j} \end{aligned}$$

where in the second line we have used (4.14) and in last line $S^2 = \mathrm{id}$.

$$\begin{aligned} ii) \quad R_{i,j}(\varphi)A_j(a) &= A_{j+1}(\varphi_{(1)}S(\varphi_{(2)}))R_{i,j}(\varphi_{(3)})A_j(a) \\ &= A_{j+1}(\varphi_{(1)})A_j(a)A_{j+1}(S(\varphi_{(2)}))R_{i,j}(\varphi_{(3)}) \\ &= A_j(a_{(1)})A_{j+1}(\varphi_{(1)} \leftarrow a_{(2)})A_{j+1}(S(\varphi_{(2)}))R_{i,j}(\varphi_{(3)}) \\ &= A_j(a_{(1)})R_{i,j}(\varphi \leftarrow a_{(2)}) \end{aligned}$$

where in the second line we have used (4.23) and the the third line (2.2b). *Q.e.d.*

Using Proposition 4.5 and Corollary 4.6 we are now in the position to prove Theorem 3.12 as a particular consequence of the following

Theorem 4.7: *Let $j = i + 2n + 1,\ n \in \mathbb{N}_0,\ i \in \mathbb{Z}$, and let $I = [i - \frac{1}{2}, j + \frac{1}{2}] \in \mathcal{I}$. Define $\rho_{i-1,j+1} : \mathcal{A}(\partial I) \rightarrow \mathcal{A}_{i-1,j+1} \otimes \mathcal{D}(H)$ by*

$$\begin{aligned} \rho_{i-1,j+1}(A_{j+1}(\varphi)) &:= R_{i,j}(\varphi_{(1)}S(\varphi_{(3)}))A_{j+1}(\varphi_{(4)}) \otimes \mathcal{D}(\varphi_{(2)}) && (4.25a) \\ \rho_{i-1,j+1}(A_{i-1}(a)) &:= A_{i-1}(a_{(1)})L_{i,j}(S(a_{(2)})a_{(4)}) \otimes \mathcal{D}(a_{(3)}) && (4.25b) \end{aligned}$$

Then

i) $\rho_{i-1,j+1}$ *extends to a coaction* $\hat\rho_{i-1,j+1} \in \mathbf{Amp}\,(\mathcal{A}, \partial I)$, *which is strictly equivalent to* $\rho_{i-1,i}$.

ii) The coaction $\hat\rho_{i-1,j+1}$ *is universal in* $\mathbf{Amp}\,(\mathcal{A}, \partial I)$.

Proof: Assume without loss $\mathcal{A}_i \simeq \hat H$ and define

$$T_{i,j} := \sum_k L_{i,j}(b_k) \otimes \mathcal{D}(\xi^k) \in \mathcal{A}_{i,j} \otimes \mathcal{D}(H) \tag{4.26a}$$

where $b_k \in H$ is a basis with dual basis $\xi^k \in \hat H$. Then $T_{i,j}$ is unitary,

$$T_{i,j}^* = T_{i,j}^{-1} = \sum_k L_{i,j}(b_k) \otimes \mathcal{D}(S(\xi^k)) \tag{4.26b}$$

and we put

$$\hat\rho_{i-1,j+1} := \mathrm{Ad}\, T_{i,j} \circ \rho_{i-1,i} \tag{4.27}$$

To prove i) we first show

$$\hat\rho_{i-1,j+1} \in \mathbf{Amp}\,(\mathcal{A}, \partial I) \tag{4.28}$$

and

$$\hat\rho_{i-1,j+1} | \mathcal{A}(\partial I) = \rho_{i-1,j+1}. \tag{4.29}$$

To this end we use that $L_{i,j}(a) \in \mathcal{A}_{i,j} \cap \mathcal{A}'_{i+1,j}$ to conclude

$$T_{i,j} \in (\mathcal{A}'_{-\infty,i-2} \cap \mathcal{A}'_{i+1,j} \cap \mathcal{A}'_{j+2,\infty}) \otimes \mathcal{D}(H)$$

Now $\mathcal{A}((\partial I)^c) = \mathcal{A}_{-\infty,i-2} \vee \mathcal{A}_{i,j} \vee \mathcal{A}_{j+2,\infty}$ and since $\rho_{i-1,i}$ is localized on $\mathcal{A}_{i-1,i}$ the claim (4.28) follows provided

$$(A_i(\varphi) \otimes \mathbf{1})\, T_{i,j} = T_{i,j}\, \rho_{i-1,i}(A_i(\varphi)), \quad \forall \varphi \in \hat H. \tag{4.30}$$

To check (4.30) we compute

$$\begin{aligned}
(A_i(\varphi) \otimes \mathbf{1}) T_{i,j} &= \sum_k A_i(\varphi) L_{i,j}(b_k) \otimes \mathcal{D}(\xi^k) \\
&= \sum_{k_1,k_2} L_{i,j}(b_{k_1}) A_i(\varphi \leftarrow b_{k_2}) \otimes \mathcal{D}(\xi^{k_1}\xi^{k_2}) \\
&= \sum_k L_{i,j}(b_k) A_i(\varphi_{(2)}) \otimes \mathcal{D}(\xi^k \varphi_{(1)}) \\
&= T_{i,j} \rho_{i-1,i}(A_i(\varphi))
\end{aligned}$$

where in the second line we have used (4.17a). Thus we have proven (4.28). To prove (4.29) we compute

$$\begin{aligned}
&\rho_{i-1,j+1}(A_{j+1}(\varphi)) T_{i,j} = \\
&\quad = \sum_k R_{i,j}(\varphi_{(1)} S(\varphi_{(3)})) A_{j+1}(\varphi_{(4)}) L_{i,j}(b_k) \otimes \mathcal{D}(\varphi_{(2)}\xi^k) && (4.31a) \\
&\quad = \sum_k R_{i,j}(\varphi_{(1)}) L_{i,j}(b_k) R_{i,j}(S(\varphi_{(3)})) A_{j+1}(\varphi_{(4)}) \otimes \mathcal{D}(\varphi_{(2)}\xi^k) \\
&\quad = \sum_{k_1,k_2} L_{i,j}(b_{k_2}) R_{i,j}(S^{-1}(b_{k_1}) \rightarrow \varphi_{(1)}) R_{i,j}(S(\varphi_{(3)})) A_{j+1}(\varphi_{(4)}) \otimes \mathcal{D}(\varphi_{(2)}\xi^{k_1}\xi^{k_2})
\end{aligned}$$

$$
\begin{aligned}
&= \sum_k L_{i,j}(b_k) R_{i,j}(\varphi_{(1)} S(\varphi_{(4)})) A_{j+1}(\varphi_{(5)}) \otimes \mathcal{D}(\varphi_{(3)} S^{-1}(\varphi_{(2)}) \xi^k) \\
&= \sum_k L_{i,j}(b_k) A_{j+1}(\varphi) \otimes \mathcal{D}(\xi^k) \\
&= T_{i,j}\,(A_{j+1}(\varphi) \otimes \mathbf{1}) && (4.31b) \\
&= T_{i,j}\,\rho_{i-1,i}(A_{j+1}(\varphi)) && (4.31c)
\end{aligned}
$$

where in the second equation we have used (4.14) and in the third equation the inverse of (4.15). Next we compute

$$
\begin{aligned}
T_{i,j}\,\rho_{i-1,i}(A_{i-1}(a)) &= T_{i,j}\,[A_{i-1}(a_{(1)}) \otimes \mathcal{D}(a_{(a)})] \\
&= T_{i,j}\,[A_{i-1}(a_{(1)}) L_{i,j}(S(a_{(2)}) a_{(3)}) \otimes \mathcal{D}(a_{(4)})] \\
&= [A_{i-1}(a_{(1)}) L_{i,j}(S(a_{(2)})) \otimes \mathbf{1}]\, T_{i,j}\,[L_{i,j}(a_{(3)}) \otimes \mathcal{D}(a_{(4)})] \\
&= [A_{i-1}(a_{(1)}) L_{i,j}(S(a_{(2)}) a_{(4)}) \otimes \mathcal{D}(a_{(3)})]\, T_{i,j} \\
&= \rho_{i-1,j+1}(A_{i-1}(a))\, T_{i,j}
\end{aligned}
$$

where in the third line we have used (4.13) and in the fourth line the identity

$$
T_{i,j}\,[L_{i,j}(a_{(1)}) \otimes \mathcal{D}(a_{(2)}] = [L_{i,j}(a_{(2)}) \otimes \mathcal{D}(a_{(1)})]\, T_{i,j} \tag{4.32}
$$

which follows straightforwardly from equ. (B.2) in Appendix B. Thus we have proven (4.29). To complete the proof of part i) we are left to show that $\rho_{i-1,j+1}$ provides a coaction which is strictly equivalent to $\rho_{i-1,i}$. This follows provided

$$
T_{i,j} \times_{\rho_{i-1,i}} T_{i,j} = (id \otimes \Delta_{\mathcal{D}}^{(op)})(T_{i,j}) \tag{4.33}
$$

To prove (4.33) we use that $L_{i,j}(b_k)$ lies in $\mathcal{A}_{i,j}$ and therefore $(\hat{\rho}_{i-1,j+1} \otimes \mathrm{id}\,)(T_{i,j}) = T_{i,j}^{02}$ implying

$$
\begin{aligned}
T_{i,j} \times_{\rho_{i-1,i}} T_{i,j} &= (\hat{\rho}_{i-1,j+1} \otimes \mathrm{id}\,)(T_{i,j})(T_{i,j} \otimes \mathbf{1}) \\
&= T_{i,j}^{02} T_{i,j}^{01} \\
&= (\mathrm{id}\, \otimes \Delta_{\mathcal{D}}^{(op)})(T_{i,j})
\end{aligned}
$$

Thus we have proven part i) of Theorem 4.7.

To prove part ii) first recall that $\rho_{i-1,i}$ is effective and therefore $\hat{\rho}_{i-1,j+1} = \mathrm{Ad}\, T_{i,j} \circ \rho_{i-1,i}$ is effective. Let now $\mu \in \mathbf{Amp}\,(\mathcal{A}, \partial I)$ and define $\hat{\mu} : \mathcal{A}_{j+1} \to \mathcal{A} \otimes \mathrm{End}\, V_\mu$ by

$$
\hat{\mu}(A_{j+1}(\varphi)) := \mu(A_{j+1}(\varphi_{(2)}))[A_{j+1}(S(\varphi_{(3)})) R_{i,j}(\varphi_{(4)} S^{-1}(\varphi_{(1)})) \otimes \mathbf{1}] \tag{4.34a}
$$

Then μ may be expressed in terms of $\hat{\mu}$

$$
\begin{aligned}
\mu(A_{j+1}(\varphi)) &= \mu(A_{j+1}(\varphi_{(3)}))\,[R_{i,j}(S^{-1}(\varphi_{(2)})) A_{j+1}(S(\varphi_{(4)})) R_{i,j}(\varphi_{(5)}) \otimes \mathbf{1}] \\
&\quad \times [R_{i,j}(S(\varphi_{(6)})) A_{j+1}(\varphi_{(7)}) R_{i,j}(\varphi_{(1)}) \otimes \mathbf{1}] \\
&= \hat{\mu}(A_{j+1}(\varphi_{(2)}))\,[R_{i,j}(\varphi_{(1)} S(\varphi_{(3)})) A_{j+1}(\varphi_{(4)}) \otimes \mathbf{1}]
\end{aligned} \tag{4.34b}
$$

where in the second equation we have used (4.14). In Lemma 4.8 below we show that there exists a *-representation $\beta_\mu : \hat{H} \to \mathrm{End}\, V_\mu$ such that

$$
\hat{\mu}(A_{j+1}(\varphi)) = \mathbf{1}_{\mathcal{A}} \otimes \beta_\mu(\varphi) \tag{4.35}
$$

Then (4.34b) implies

$$\mu(A_{j+1}(\varphi)) = R_{i,j}(\varphi_{(1)}S(\varphi_{(3)}))A_{j+1}(\varphi_{(4)}) \otimes \beta_\mu(\varphi_{(2)}) \quad . \tag{4.36}$$

Putting

$$V_{i,j} = \sum_k L_{i,j}(b_k) \otimes \beta_\mu(\xi^k) \tag{4.37}$$

and repreating the calculation from (4.31a) to (4.31b) with $\rho_{i-1,j+1}$ replaced by μ, $T_{i,j}$ replaced by $V_{i,j}$ and $\mathcal{D}(\varphi)$ replaced by $\beta_\mu(\varphi)$ we get

$$\mu(A_{j+1}(\varphi))V_{i,j} = V_{i,j}(A_{j+1}(\varphi) \otimes \mathbf{1}). \tag{4.38}$$

Moreover, similarly as for $T_{i,j}$ we have

$$V_{i,j} \in (\mathcal{A}'_{-\infty,i-2} \cap \mathcal{A}'_{i+1,j} \cap \mathcal{A}'_{j+2,\infty}) \otimes \mathrm{End}\, V_\mu \quad . \tag{4.39}$$

By (4.38) and (4.39) $\mathrm{Ad}\, V^*_{i,j} \circ \mu$ is localized on $\mathcal{A}_{i-1,i}$. In particular

$$V^*_{i,j}\, \mu(A_i(\varphi))V_{i,j} \equiv V^*_{i,j}(A_i(\varphi) \otimes \mathbf{1})V_{i,j} = A(\varphi_{(2)}) \otimes \beta_\mu(\varphi_{(1)}) \tag{4.40}$$

which one proves in the same way as (4.30). Hence, by Theorem 4.1ii) β_μ extends to a represenation $\hat\beta_\mu : \mathcal{D}(H) \to \mathrm{End}\, V_\mu$ such that

$$\mathrm{Ad}\, V^*_{i,j} \circ \mu = (id \otimes \hat\beta_\mu) \circ \rho_{i-1,i}$$

and therefore

$$\mu = (id \otimes \hat\beta_\mu) \circ \rho_{i-1,j+1}. \tag{4.41}$$

This proves that $\rho_{i-1,j+1}$ is universal in $\mathbf{Amp}\,(\mathcal{A}, \partial I)$ and therefore part ii) of Theorem 4.7. *Q.e.d.*

Since by Proposition 4.2 the coactions $\rho_{i-1,i}$, $i \in \mathbb{Z}$, are all (cocycle) equivalent and since by Corollary 3.9 any amplimorphism $\mu \in \mathbf{Amp}\,\mathcal{A}$ is compressible into ∂I for some interval $I \in \mathcal{I}$ of even length, Theorem 4.7 implies that $\mathbf{Amp}\,\mathcal{A}$ is compressible into *any* interval of length two. In particular, $\mathbf{Amp}\,\mathcal{A}$ is completely compressible. This concludes the proof of Theorem 3.12.

We are left to prove the claim (4.35).

Lemma 4.8: *Under the conditions of Theorem 4.7 let $\mu \in \mathbf{Amp}\,(\mathcal{A}, \partial I)$ and let $\hat\mu : \mathcal{A}_{j+1} \to \mathcal{A}_{i,j+1} \otimes \mathrm{End}\, V_\mu$ be given by (4.34a). Then there exists a *-representation $\beta_\mu : \mathcal{A}_{j+1} \to \mathrm{End}\, V_\mu$ such that $\hat\mu = \mathbf{1}_\mathcal{A} \otimes \beta_\mu$.*

Proof: Since $\partial I \subset \overline{I}$ we have by Lemma 3.8

$$\mu(\mathcal{A}(\partial I)) \subset \mathcal{A}_{i-1,j+1} \otimes \mathrm{End}\, V_\mu$$

Using $\mathcal{A}_{j+1} \subset \mathcal{A}(\partial I) \cap \mathcal{A}'_{i-2} \cap \mathcal{A}'_{i,j-1}$ we conclude

$$\begin{aligned} \mu(\mathcal{A}_j) &\subset (\mathcal{A}_{i-1,j+1} \otimes \mathrm{End}\, V_\mu) \cap \mu(\mathcal{A}_{i-2})' \cap \mu(\mathcal{A}_{i,j-1})' \\ &= (\mathcal{A}_{i-1,j+1} \cap \mathcal{A}'_{i-2} \cap \mathcal{A}'_{i,j-1}) \otimes \mathrm{End}\, V_\mu \\ &= (\mathcal{A}_{i,j+1} \cap \mathcal{A}'_{i,j-1}) \otimes \mathrm{End}\, V_\mu \end{aligned}$$

Let now

$$\lambda(\varphi) := \mu(A_{j+1}(\varphi_{(1)}))[A_{j+1}(S(\varphi_{(2)})) \otimes \mathbf{1}] \tag{4.42}$$

Using that $\mu|\mathcal{A}_{j+2} = \mathrm{id} \otimes \mathbf{1}$ we conclude

$$\begin{aligned}
[A_{j+2}(a) \otimes \mathbf{1}]\lambda(\varphi) &= \mu(A_{j+1}(a_{(1)} \to \varphi_{(1)}))[A_{j+1}(a_{(2)} \to S(\varphi_{(2)}))A_{j+2}(a_{(3)}) \otimes \mathbf{1}] \\
&= \mu(A_{j+1}(\varphi_{(1)}))[A_{j+1}(S(\varphi_{(4)}))A_{j+2}(a_{(2)})\langle a_{(1)}, \varphi_{(2)}S(\varphi_{(3)})\rangle \otimes \mathbf{1}] \\
&= \lambda(\varphi)[A_{j+2}(a) \otimes \mathbf{1}]
\end{aligned}$$

and therefore

$$\begin{aligned}
\lambda(\varphi) &\in (\mathcal{A}_{i,j+1} \cap \mathcal{A}'_{j+2} \cap \mathcal{A}'_{i,j-1}) \otimes \mathrm{End}\, V_\mu \\
&= (\mathcal{A}_{i,j} \cap \mathcal{A}'_{i,j-1}) \otimes \mathrm{End}\, V_\mu
\end{aligned}$$

Thus we get

$$\begin{aligned}
\hat{\mu}(\varphi) &\equiv \lambda(\varphi_{(2)})[R_{i,j}(\varphi_{(3)}S^{-1}(\varphi_{(1)})) \otimes \mathbf{1}] \\
&\in (\mathcal{A}_{i,j} \cap \mathcal{A}'_{i,j-1}) \otimes \mathrm{End}\, V_\mu
\end{aligned} \tag{4.43}$$

We claim that $\hat{\mu}(\varphi)$ commutes with $\mathcal{A}_j \otimes \mathbf{1}$ and therefore

$$\begin{aligned}
\hat{\mu}(\varphi) &\in (\mathcal{A}_{i,j} \cap \mathcal{A}'_{i,j}) \otimes \mathrm{End}\, V_\mu \\
&= \mathbf{1}_{\mathcal{A}} \otimes \mathrm{End}\, V_\mu
\end{aligned} \tag{4.44}$$

by the simplicity of $\mathcal{A}_{i,j}$. To this end we use (4.23) and (4.24) and $\mu(A_j(a)) = A_j(a) \otimes \mathbf{1}$ to compute

$$\begin{aligned}
&\hat{\mu}(\varphi)\,[A_j(a) \otimes \mathbf{1}] = \\
&= \mu(A_{j+1}(\varphi_{(2)}))\,[R_{i,j}(S^{-1}(\varphi_{(1)}))A_j(a)A_{j+1}(S(\varphi_{(3)}))R_{i,j}(\varphi_{(4)}) \otimes \mathbf{1}] \\
&= [A_j(a_{(1)}) \otimes \mathbf{1}]\,\mu(A_{j+1}(\varphi_{(2)} \leftarrow a_{(2)}))\,[R_{i,j}(S^{-1}(\varphi_{(1)}) \leftarrow a_{(3)})A_{j+1}(S(\varphi_{(3)}))R_{i,j}(\varphi_{(4)}) \otimes \mathbf{1}] \\
&= [A_j(a_{(1)})\,\langle a_{(2)}, \varphi_{(3)}S^{-1}(\varphi_{(2)})\rangle \otimes \mathbf{1}]\,\mu(A_{j+1}(\varphi_{(4)}))\,[A_{j+1}(S(\varphi_{(5)}))R_{i,j}(\varphi_{(6)}S^{-1}(\varphi_{(1)})) \otimes \mathbf{1}] \\
&= [A_j(a) \otimes \mathbf{1}]\,\hat{\mu}(\varphi).
\end{aligned} \tag{4.45}$$

From (4.43) and (4.45) we get (4.44) and therefore

$$\hat{\mu}(\varphi) = \mathbf{1}_{\mathcal{A}} \otimes \beta_\mu(\varphi)$$

for some linear map $\beta_\mu : \mathcal{A}_{j+1} \to \mathrm{End}\, V_\mu$. We are left to check that β_μ provides a *-representation:

$$\begin{aligned}
\hat{\mu}(\varphi)\hat{\mu}(\psi) &= (A_{j+1}(\varphi_{(2)}))\,\hat{\mu}(\psi)\,[A_{j+1}(S(\varphi_{(3)}))R_{i,j}(\varphi_{(4)}S^{-1}(\varphi_{(1)})) \otimes \mathbf{1}] \\
&= \mu(A_{j+1}(\varphi_{(2)}\psi_{(2)})\,[A_{j+1}(S(\varphi_{(3)}\psi_{(3)}))R_{i,j}(\varphi_{(4)}\psi_{(4)}S^{-1}(\psi_{(1)})S^{-1}(\varphi_{(1)}) \otimes \mathbf{1}] \\
&= \hat{\mu}(\varphi\psi)
\end{aligned}$$

where in the second line we have used (4.23).

$$\begin{aligned}
\hat{\mu}(\psi^*)^* &= [R_{i,j}(S(\psi_{(1)})\psi_{(4)})A_{j+1}(S^{-1}(\psi_{(3)})) \otimes \mathbf{1}]\,\mu(A_{j+1}(\psi_{(2)})) \\
&= R_{i,j}(S(\psi_{(1)})\psi_{(7)})A_{j+1}(S^{-1}(\psi_{(6)}))R_{i,j}(\psi_{(2)}S(\psi_{(4)}))A_{j+1}(\psi_{(5)}) \otimes \beta_\mu(\psi_{(3)}) \\
&= R_{i,j}(S(\psi_{(1)})\psi_{(2)}S(\psi_{(4)})\psi_{(7)})A_{j+1}(S^{-1}(\psi_{(6)})\psi_{(5)}) \otimes \beta_\mu(\psi_{(3)}) \\
&= \mathbf{1} \otimes \beta_\mu(\psi)
\end{aligned}$$

where in the second line we have used (4.36) and in the third line (4.14). *Q.e.d.*

A Finite dimensional C*-Hopf algebras

There is an extended literature on Hopf algebra theory the nomenclature of which, however, is by far not unanimous [BaSk,Dr,E,ES,Sw,W]. Therefore we summarize in this appendix some standard notions in order to fix our conventions and notations.

A linear space B over $\mathbb{C}$ together with linear maps

$$\begin{array}{rclllrcll} m: B\otimes B & \to & B & \text{(multiplication)}, & & \Delta: B & \to & B\otimes B & \text{(comultiplication)}, \\ \iota: \mathbb{C} & \to & B & \text{(unit)}, & & \varepsilon: B & \to & \mathbb{C} & \text{(counit)} \end{array}$$

is called a *bialgebra* and denoted by $B(m,\iota,\Delta,\varepsilon)$ if the following axioms hold:

$$\begin{array}{rclrcl} m\circ(m\otimes \mathrm{id}\,) & = & m\circ(\mathrm{id}\,\otimes m)\,, & (\Delta\otimes\mathrm{id}\,)\circ\Delta & = & (\mathrm{id}\,\otimes\Delta)\circ\Delta \\ m\circ(\iota\otimes\mathrm{id}\,) & = & m\circ(\mathrm{id}\,\otimes\iota)=\mathrm{id}\;, & (\varepsilon\otimes\mathrm{id}\,)\circ\Delta & = & (\mathrm{id}\,\otimes\varepsilon)\circ\Delta=\mathrm{id} \\ \varepsilon\circ m & = & \varepsilon\otimes\varepsilon\,, & \Delta\circ\iota=\iota\otimes\iota & & \end{array}$$
$$\Delta\circ m=(m\otimes m)\circ\tau_{23}\circ(\Delta\otimes\Delta)$$

where τ_{23} denotes the permutation of the tensor factors 2 and 3. We use Sweedler's notation $\Delta(x)=x_{(1)}\otimes x_{(2)}$, where the right hand side is understood as a sum $\sum_i x^i_{(1)}\otimes x^i_{(2)}\in B\otimes B$. For iterated coproducts we write $x_{(1)}\otimes x_{(2)}\otimes x_{(3)} := \Delta(x_{(1)})\otimes x_{(2)} \equiv x_{(1)}\otimes\Delta(x_{(2)})$, etc. The image under ι of the number $1\in\mathbb{C}$ is the unit element of B denoted by $\mathbf{1}$. The linear dual $\hat{B}$ becomes also a bialgebra by transposing the structural maps $m,\iota,\Delta,\varepsilon$ by means of the canonical pairing $\langle\ ,\ \rangle: \hat{B}\times B\to\mathbb{C}$.

A bialgebra $H(m,\iota,\Delta,\varepsilon)$ is called a *Hopf algebra* $H(m,\iota,S,\Delta,\varepsilon)$ if there exists an antipode $S: H\to H$, i.e. a linear map satisfying

$$m\circ(S\otimes\mathrm{id}\,)\circ\Delta=m\circ(\mathrm{id}\,\otimes S)\circ\Delta=\iota\circ\varepsilon \tag{A.1}$$

Using the above notation equ. (A1) takes the form $S(x_{(1)})x_{(2)}=x_{(1)}S(x_{(2)})=\varepsilon(x)\mathbf{1}$, which in connection with the coassociativity of Δ is often applied in formulas involving iterated coproducts like, e.g., $x_{(1)}\otimes x_{(4)}S(x_{(2)})x_{(3)}=x_{(1)}\otimes x_{(2)}$. All other properties of the antipode, i.e. $S(xy)=S(y)S(x)$, $\Delta\circ S=(S\otimes S)\circ\Delta_{op}$ and $\varepsilon\circ S=\varepsilon$, as well as the uniqueness of S are all consequences of the axiom (A.1) [Sw]. The dual bialgebra $\hat{H}$ of H is also a Hopf algebra with the antipode defined by

$$\langle S(\varphi),x\rangle := \langle\varphi,S(x)\rangle \quad \varphi\in\hat{H},\ x\in H\,. \tag{A.2}$$

A $*$-Hopf algebra $H(m,\iota,S,\Delta,\varepsilon,*)$ is a Hopf algebra $H(m,\iota,S,\Delta,\varepsilon)$ together with an antilinear involution $^*: H\to H$ such that $H(m,\iota,*)$ is a $*$-algebra and Δ and ε are *-algebra maps. It follows that $\overline{S} := *\circ S\circ *$ is the antipode in the Hopf algebra H_{op} (i.e. with opposite muliplication) and therefore $\overline{S}=S^{-1}$ [Sw]. The dual of a $*$-Hopf algebra is also a $*$-Hopf algebra with *-operation defined by $\varphi^* := S(\varphi_*)$, where $\varphi\to\varphi_*$ is the antilinear involutive algebra automorphism given by

$$\langle\varphi_*,x\rangle := \overline{\langle\varphi,x^*\rangle}\,. \tag{A.3}$$

Let $\mathcal{A}$ be a $*$-algebra and let H be a $*$-Hopf algebra. A (Hopf module) left action of H on $\mathcal{A}$ is a linear map $\gamma: H\otimes\mathcal{A}\to\mathcal{A}$ satisfying the following axioms: For $A,B\in\mathcal{A}$, $x,y\in H$

$$\begin{array}{rcl} \gamma_x\circ\gamma_y(A) & = & \gamma_{xy}(A) \\ \gamma_x(AB) & = & \gamma_{x_{(1)}}(A)\gamma_{x_{(2)}}(B) \\ \gamma_x(A)^* & = & \gamma_{x_*}(A^*) \end{array} \tag{A.4}$$

where as above $x_* = S^{-1}(x^*)$. A right action of H is a left action of H_{op}. Important examples are the action of H on $\hat{H}$ and that of $\hat{H}$ on H given by the Sweedler's arrows:

$$\gamma_x(\varphi) = x \rightarrow \varphi \quad := \quad \varphi_{(1)}\langle x, \varphi_{(2)}\rangle \tag{A.5a}$$
$$\gamma_\varphi(x) = \varphi \rightarrow x \quad := \quad x_{(1)}\langle \varphi, x_{(2)}\rangle \tag{A.5b}$$

A left action is called inner if there exists a *-algebra map $i : H \rightarrow \mathcal{A}$ such that $\gamma_x(A) = i(x_{(1)})\, A\, i(S(x_{(2)}))$. Left H-actions γ are in one-to-one corespondence with right $\hat{H}$*-coactions* (often denoted by the same symbol) $\gamma : \mathcal{A} \rightarrow \mathcal{A} \otimes \hat{H}$ defined by

$$\gamma(A) := \gamma_{b_i}(A) \otimes \xi^i, \quad A \in \mathcal{A}$$

where $\{b_i\}$ is a basis in H and $\{\xi^i\}$ is the dual basis in $\hat{H}$ and where for simplicity we assume from now on H to be finite dimensional. Conversely, we have $\gamma_x = (\mathrm{id}_{\mathcal{A}} \otimes x) \circ \gamma$. The defining properties of a coaction are given in equs. (3.11a-e).

Given a left H-action (right $\hat{H}$-coaction) γ one defines the *crossed product* $\mathcal{A} \rtimes_\gamma H$ as the $\mathbb{C}$-vector space $\mathcal{A} \otimes H$ with $*$-algebra structure

$$(A \otimes x)(B \otimes y) \quad := \quad A\gamma_{x_{(1)}}(B) \otimes x_{(2)} y \tag{A.6a}$$
$$(A \otimes x)^* \quad := \quad (\mathbf{1}_{\mathcal{A}} \otimes x^*)(A^* \otimes \mathbf{1}_H) \tag{A.6b}$$

An important example is the "Weyl algebra" $\mathcal{W}(\hat{H}) := \hat{H} \rtimes H$, where the crossed product is taken with respect to the natural left action (A.5a). We have $\mathcal{W}(\hat{H}) \cong \mathrm{End}\, \hat{H}$ where the isomorphism is given by (see [N] for a review)

$$w: \ \psi \otimes x \mapsto Q^+(\psi) P^+(x)\ . \tag{A.7}$$

Here we have introduced $Q^+(\psi),\ \psi \in \hat{H}$ and $P^+(x),\ x \in H$ as operators in $\mathrm{End}\, \hat{H}$ defined on $\xi \in \hat{H}$ by

$$Q^+(\psi)\xi \quad := \quad \psi\xi$$
$$P^+(x)\xi \quad := \quad x \rightarrow \xi$$

Any right H-coaction $\beta : \mathcal{A} \rightarrow \mathcal{A} \otimes H$ gives rise to a natural left H-action γ on $\mathcal{A} \rtimes_\beta \hat{H}$

$$\gamma_x(A \otimes \psi) := A \otimes (x \rightarrow \psi) \tag{A.8}$$

The resulting double crossed product $(\mathcal{A} \rtimes_\beta \hat{H}) \rtimes_\gamma H$ contains $\mathcal{W}(\hat{H}) \cong \mathrm{End}\, \hat{H}$ as the subalgebra given by $\mathbf{1}_{\mathcal{A}} \otimes \psi \otimes x \cong Q^+(\psi)P^+(x),\ \psi \in \hat{H},\ x \in H$. Moreover, by the Takesaki duality theorem [Ta,NaTa] the double crossed product $(\mathcal{A} \rtimes_\beta \hat{H}) \rtimes_\gamma H$ is canonically isomorphic to $\mathcal{A} \otimes \mathrm{End}\, \hat{H}$. In fact, defining the representation $L : H \rightarrow \mathrm{End}\, \hat{H}$ by

$$L(x)\xi := \xi \leftarrow S^{-1}(x) \equiv \langle \xi_{(1)}\, ,\, S^{-1}(x)\rangle \xi_{(2)} \tag{A.9}$$

one easily verifies that $\mathcal{T} : (\mathcal{A} \rtimes_\beta \hat{H}) \rtimes_\gamma H \rightarrow \mathcal{A} \otimes \mathrm{End}\, \hat{H}$

$$\mathcal{T}(A \otimes \mathbf{1}_{\hat{H}} \otimes \mathbf{1}_H) \quad := \quad (\mathrm{id}_{\mathcal{A}} \otimes L)(\beta(A)) \tag{A.10a}$$
$$\mathcal{T}(\mathbf{1}_{\mathcal{A}} \otimes \psi \otimes x) \quad := \quad \mathbf{1}_{\mathcal{A}} \otimes Q^+(\psi) P^+(x) \tag{A.10b}$$

defines a $*$-algebra map. $\mathcal{T}$ is surjective since w is surjective and therefore $\mathbf{1}_{\mathcal{A}} \otimes \mathrm{End}\,\hat{H} \in \mathrm{Im}\,\mathcal{T}$ and

$$\begin{aligned} A \otimes \mathbf{1}_{\mathrm{End}\,\hat{H}} &\equiv A_{(0)} \otimes L(A_{(1)} S(A_{(2)})) \\ &= \mathcal{T}(A_{(0)} \otimes \mathbf{1}_{\hat{H}} \otimes \mathbf{1}_H)(\mathbf{1}_{\mathcal{A}} \otimes L(S(A_{(1)}))) \\ &\in \mathrm{Im}\,\mathcal{T} \end{aligned}$$

for all $A \in \mathcal{A}$. Here we have used the notation $A_{(0)} \otimes A_{(1)} = \beta(A)$,

$$A_{(0)} \otimes A_{(1)} \otimes A_{(2)} = (\beta \otimes \mathrm{id}_H)(\beta(A)) \equiv (\mathrm{id}_{\mathcal{A}} \otimes \Delta)(\beta(A))$$

(including a summation convention) and the identity $(\mathrm{id}_{\mathcal{A}} \otimes \varepsilon) \circ \beta = \mathrm{id}_{\mathcal{A}}$, see equs. (3.11d,e). The inverse of $\mathcal{T}$ is given by

$$\mathcal{T}^{-1}(\mathbf{1}_A \otimes W) = \mathbf{1}_{\mathcal{A}} \otimes w^{-1}(W) \tag{A.11a}$$

$$\mathcal{T}^{-1}(A \otimes \mathbf{1}_{\mathrm{End}\,\hat{H}}) = A_{(0)} \otimes w^{-1}(L(S(A_{(1)}))) \tag{A.11b}$$

for $W \in \mathrm{End}\,\hat{H}$ and $A \in \mathcal{A}$.

A *left(right) integral* in $\hat{H}$ is an element $\chi^L(\chi^R) \in \hat{H}$ satisfying

$$\varphi\chi^L = \chi^L \cdot \varepsilon(\varphi) \qquad \chi^R\varphi = \varepsilon(\varphi) \cdot \chi^R \tag{A.12a}$$

for all $\varphi \in \hat{H}$ or equivalently

$$\chi^L \rightharpoonup x = \langle \chi^L, x \rangle \mathbf{1}\,, \qquad x \leftharpoonup \chi^R = \langle \chi^R, x \rangle \mathbf{1} \tag{$A.12b$}$$

for all $x \in H$. Similarly one defines left(right) integrals in H.

If H is finite dimensional and semisimple then so is $\hat{H}$ [LaRa] and in this case they are both *unimodular*, i.e. left and right integrals coincide and are all given as scalar multiples of a unique one dimensional central projection

$$e_\varepsilon = e_\varepsilon^* = e_\varepsilon^2 = S(e_\varepsilon) \tag{A.13}$$

which is then called the *Haar integral.*

For $\varphi, \psi \in \hat{H}$ and $h \equiv e_\varepsilon \in H$ the Haar integral define the hermitian form

$$\langle \varphi | \psi \rangle := \langle \varphi^* \psi, h \rangle \tag{A.14}$$

Then $\langle \cdot | \cdot \rangle$ is nondegenerate [LaSw] and it is positve definite — i.e. the Haar integral h provides a positive state (*the Haar "measure"*) on $\hat{H}$ — if and only if $\hat{H}$ is a *C^*-Hopf algebra.* These are the "finite matrix pseudogroups" of [W]. They also satisfy $S^2 = \mathrm{id}$ and $\Delta(h) = \Delta_{op}(h)$ [W]. If $\hat{H}$ is a finite dimensional C^*-Hopf algebra then so is H, since $H \ni x \to P^+(x) \in \mathrm{End}\,\hat{H}$ defines a faithful $*$-representation on the Hilbert space $\mathcal{H} \equiv L^2(\hat{H}, h)$. Hence finite dimensional C^*-Hopf algebras always come in dual pairs. Any such pair serves as a building block for our Hopf spin model.

B The Drinfeld Double

Here we list the basic properties of the Drinfeld double $\mathcal{D}(H)$ (also called quantum double) of a finite dimensional $*$-Hopf algebra H [Dr,Maj1]. Although most of them are well known in the literature, the presentation (B.1) by generators and relations given below seems to be new.

As a $*$-algebra $\mathcal{D}(H)$ is generated by elements $\mathcal{D}(a)$, $a \in H$ and $\mathcal{D}(\varphi)$, $\varphi \in \hat{H}$ subjected to the following relations:

$$\begin{aligned} \mathcal{D}(a)\mathcal{D}(b) &= \mathcal{D}(ab) && \text{(B.1a)} \\ \mathcal{D}(\varphi)\mathcal{D}(\psi) &= \mathcal{D}(\varphi\psi) && \text{(B.1b)} \\ \mathcal{D}(a_{(1)})\,\langle a_{(2)}, \varphi_{(1)}\rangle\,\mathcal{D}(\varphi_{(2)}) &= \mathcal{D}(\varphi_{(1)})\,\langle \varphi_{(2)}, a_{(1)}\rangle\,\mathcal{D}(a_{(2)}) && \text{(B.1c)} \\ \mathcal{D}(a)^* = \mathcal{D}(a^*) \quad &, \quad \mathcal{D}(\varphi)^* = \mathcal{D}(\varphi^*) && \text{(B.1d)} \end{aligned}$$

The relation (B.1c) is equivalent to any one of the following two relations

$$\begin{aligned} \mathcal{D}(a)\mathcal{D}(\varphi) &= \mathcal{D}(\varphi_{(2)})\mathcal{D}(a_{(2)})\,\langle a_{(1)}, \varphi_{(3)}\rangle\langle S^{-1}(a_{(3)}), \varphi_{(1)}\rangle && \text{(B.2a)} \\ \mathcal{D}(\varphi)\mathcal{D}(a) &= \mathcal{D}(a_{(2)})\mathcal{D}(\varphi_{(2)})\,\langle \varphi_{(1)}, a_{(3)}\rangle\langle S^{-1}(\varphi_{(3)}), a_{(1)}\rangle && \text{(B.2b)} \end{aligned}$$

These imply that as a linear space $\mathcal{D}(H) \cong H \otimes \hat{H}$ and also that as a $*$-algebra $\mathcal{D}(H)$ and $\mathcal{D}(\hat{H})$ are isomorphic. This $*$-algebra will be denoted by $\mathcal{G}$.

The Hopf algebraic structure of $\mathcal{D}(H)$ is given by the following coproduct, counit, and antipode:

$$\begin{aligned} \Delta_{\mathcal{D}}(\mathcal{D}(a)) &= \mathcal{D}(a_{(1)}) \otimes \mathcal{D}(a_{(2)}) & \Delta_{\mathcal{D}}(\mathcal{D}(\varphi)) &= \mathcal{D}(\varphi_{(2)}) \otimes \mathcal{D}(\varphi_{(1)}) && \text{(B.3a)} \\ \varepsilon_{\mathcal{D}}(\mathcal{D}(a)) &= \varepsilon(a) & \varepsilon_{\mathcal{D}}(\mathcal{D}(\varphi)) &= \varepsilon(\varphi) && \text{(B.3b)} \\ S_{\mathcal{D}}(\mathcal{D}(a)) &= \mathcal{D}(S(a)) & S_{\mathcal{D}}(\mathcal{D}(\varphi)) &= \mathcal{D}(S^{-1}(\varphi)) && \text{(B.3c)} \end{aligned}$$

It is straightforward to check that equs. (B.3) provide a $*$-Hopf algebra structure on $\mathcal{D}(H)$. Moreover, $\mathcal{D}(\hat{H}) = (\mathcal{D}(H))_{cop}$ (i.e. with opposite coproduct) by (B.3a).

If H and $\hat{H}$ are C^*-Hopf algebras then so is $\mathcal{D}(H)$. To see this one may use the faithful $*$-representations of $\mathcal{D}(H)$ on the Hilbert spaces $\mathcal{H}_{n,m}$ in Lemma 2.2. Alternatively, it is not difficult to see that

$$\mathcal{D}(h)\mathcal{D}(\chi) = \mathcal{D}(\chi)\mathcal{D}(h) =: h_{\mathcal{D}} \tag{B.4}$$

provides the Haar integral in $\mathcal{D}(H)$ and that the positivity of the Haar states $h \in H$ and $\chi \in \hat{H}$ implies the positvity of the state $h_{\mathcal{D}}$ on $\widehat{\mathcal{D}(H)}$.

The dual $\widehat{\mathcal{D}(H)}$ of $\mathcal{D}(H)$ has been studied by [PoWo]. As a coalgebra it is $\hat{\mathcal{G}}$ and coincides with the coalgebra $\widehat{\mathcal{D}(\hat{H})}$. The latter one, however, as an algebra differs from $\widehat{\mathcal{D}(H)}$ in that the multiplication is replaced by the opposite multiplication.

The remarkable property of the double construction is that it always yields a *quasitriangular* Hopf algebra [Dr]. By definition this means that there exists a unitary $R \in \mathcal{D}(H) \otimes \mathcal{D}(H)$ satisfying the hexagonal identities $R^{13}R^{12} = (\mathrm{id} \otimes \Delta)(R)$, $R^{13}R^{23} = (\Delta \otimes \mathrm{id})(R)$, and the intertwining property $R\Delta(x) = \Delta^{op}(x)R$, $x \in \mathcal{D}(H)$, where $\Delta^{op}\colon x \mapsto x_{(2)} \otimes x_{(1)}$.

If $\{b_A\}$ and $\{\beta^A\}$ denote bases of H and $\hat{H}$, respectively, that are dual to each other, $\langle \beta^A, b_B\rangle = \delta^A_B$, then

$$R \equiv R_1 \otimes R_2 := \sum_A \mathcal{D}(b_A) \otimes \mathcal{D}(\beta^A) \tag{B.5}$$

is independent of the choice of the bases and satisfies the above identities.

An important theorem proven by Drinfeld [Dr2] claims that in a quasitriangular Hopf algebra $\mathcal{G}(m, u, S, \Delta, \varepsilon, R)$ there exists a canonically chosen element $s \in \mathcal{G}$ implementing the square of the antipode, namely $s = S(R_2)R_1$. Its coproduct is related to the R-matrix by the equation

$$\Delta(s) = (R^{op}R)^{-1}(s \otimes s) = (s \otimes s)(R^{op}R)^{-1} \tag{B.6}$$

which turns out to mean that s defines a universal balancing element in the category of representations of $\mathcal{G}$.

The universal balancing element s of $\mathcal{D}(H)$ takes the form

$$s := S_{\mathcal{D}}(R_2)R_1 \equiv \mathcal{D}(S^{-1}(\beta^A))\mathcal{D}(b_A) \tag{B.7}$$

and if H (and therefore $\mathcal{D}(H)$) is a C^*-Hopf algebra then s is a central unitary of $\mathcal{D}(H)$. Its inverse can be written simply as

$$s^{-1} = R_1R_2 = R_2R_1\,. \tag{B.8}$$

The existence of s satisfying (B.6) is needed in Section 4.1 to prove that in the Hopf spin model the two-point amplimorphisms (and therefore, by Lemma 3.16, *all* universal amplimorphisms) are strictly translation covariant.

References

[AFS] A. Alekseev, L. Faddev, M. Semenov-Tian-Shanski, Commun.Math.Phys. **149**, 335 (1992).

[BS] S. Baaj, G. Skandalis, Ann.Sci.ENS **26**, 425 (1993)

[DHR] S. Doplicher, R. Haag and J.E. Roberts, Commun.Math.Phys. **13**, 1 (1969); ibid **15**, 173 (1969); ibid **23**, 199 (1971); ibid **35**, 49 (1974)

[DR1] S. Doplicher and J.E. Roberts, Ann.Math. **130**, 75 (1989)

[DR2] S. Doplicher and J.E. Roberts, Comm.Math.Phys. **131**, 51 (1990)

[Dr1] V.G. Drinfeld, Quantum groups, *In:* Proc. Int. Cong. Math., Berkeley, 1986, p.798

[Dr2] V.G. Drinfeld, Leningrad Math. J. **1** 321, (1990)

[E] M. Enock, Kac algebras and crossed products, Colloques Internationaux C.N.R.S., No 274 – Algebres d'operateurs et leurs applications en physique mathematique

[ES] M. Enock, J.M. Schwartz, "Kac algebras and duality of locally compact groups", Springer 1992

[FNW] M. Fannes, B. Nachtergaele, R.F. Werner, Quantum spin chains with quantum group symmetry, preprint – KUL-TF-94/8

[F] K. Fredenhagen, Generalizations of the theory of superselection sectors, *In:* Algebraic Theory of Superselection Sectors, ed. D. Kastler, World Scientific 1990

[FRS] K. Fredenhagen, K.-H. Rehren and B. Schroer, Commun.Math.Phys. **125**, 201 (1989); Rev.Math.Phys. Special Issue 113 (1992)

[FröGab] J. Fröhlich and F. Gabbiani, Braid statistics in local quantum theory, Rev.Math.Phys. **2**, 251 (1990).

[H] R. Haag, "Local Quantum Physics", Springer 1992

[J] V.F.R. Jones, Invent.Math. **72**, 1 (1983)

[LaRa] R.G. Larson, D.E. Radford, Finite dimensional cosemisimple Hopf algebras in characteristic 0 are semisimple, J.Algebra **117**, 267 (1988).

[LaSw] R.G. Larson, M.E. Seedler, An associative orthogonal bilinear form for Hopf algebas, Amer.J.Math. **91**, 75 (1969).

[Maj1] S. Majid, Quasitriangular Hopf Algebras and Yang Baxter equations, Int.J.Mod.Phys. **A5**, 1 (1990).

[Maj2] S. Majid, Tannaka-Krein theorem for quasi-Hopf algebras and other results, *In:* Contemp.Math. **134**, 219 (1992)

[MS1] G. Mack and V. Schomerus, Commun.Math.Phys.**134**, 139 (1990)

[MS2] G. Mack and V. Schomerus, Nucl.Phys.**B370**, 185 (1992)

[Mu] G.J. Murphy, "C^*-algebras and operator theory", Academic Press 1990

[NaTa] Y. Nakagami, M. Takesaki, "Duality for Crossed Products of von Neumann Algebras", Lecture Notes in Mathematics 731, Springer 1979

[N] F. Nill, Weyl algebras, Fourier transformations, and integrals on finite dimensional Hopf algebras, Rev.Math.Phys. **6**, 149 (1994).

[NSz1] F. Nill, K. Szlachányi, Quantum chains of Hopf algebras and order-disorder fields with quantum double symmetry, preprint hep-th9507174.

[NSz2] F. Nill, K. Szlachányi, Construction and classification of field algebras from finite universal cosymmetries, Preprint to appear.

[P] V. Pasquier, Commun.Math.Phys. **118**, 355 (1988); Nucl.Phys. **B295** [FS21], 491 (1988)

[PS] V. Pasquier, H. Saleur, XXZ chain and quantum $SU(2)$, *In:* Fields, Strings, and Critical Phenomena, Les Houches 1988, eds. E. Brezin, J. Zinn-Justin, Elsevier 1989

[PoWo] P. Podleś, S.L. Woronowicz, Quantum deformation of the Lorentz group, Comm.Math.Phys. **130**, 381 (1990).

[R1] K.-H. Rehren, Braid Group Statistics and their Superselection Rules, *In:* Algebraic Theory of Superselection Sectors, ed. D. Kastler, World Scientific 1990

[R2] K.-H. Rehren, Commun.Math.Phys.**145**, 123 (1992)

[S] V. Schomerus, Quantum symmetry in quantum theory, DESY 93-018 preprint .

[SzV] K. Szlachányi, P. Vecsernyés, Quantum symmetry and braid group statistics in G-spin models, Commun.Math.Phys.**156**, 127 (1993)

[Sw] M.E. Sweedler, "Hopf algebras", Benjamin 1969

[Ta] M. Takesaki, Duality and von Neumann algebras, *In:* "Lectures on opertor algebras", eds. A. Dold and B. Eckmann, Lecture Notes in Mathematics **247**, Springer, Berlin 1972.

[U] K.H. Ulbrich, Israel J.Math. **72**, 252 (1990)

[W] S.L. Woronovich, Commun.Math.Phys. **111**, 613 (1987)