Harmonic Spinors for Twisted Dirac Operators

Christian Bär*

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Abstract

We show that for a suitable class of "Dirac-like" operators there holds a Gluing Theorem for connected sums. More precisely, if M_1 and M_2 are closed Riemannian manifolds of dimension $n \geq 3$ together with such operators, then the connected sum $M_1 \# M_2$ can be given a Riemannian metric such that the spectrum of its associated operator is close to the disjoint union of the spectra of the two original operators. As an application, we show that in dimension $n \equiv 3 \bmod 4$ harmonic spinors for the Dirac operator of a spin, spin^c, or spin^h manifold are not topologically obstructed.

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1 Introduction

The interplay between geometric, topological, and analytic invariants of Riemannian manifolds is one of the major topics of current investigations in global Riemannian geometry. A classical example is provided by Hodge-deRham theory. The dimension of the kernel of the Laplace-Beltrami operator acting on p-forms on a closed Riemannian manifold is an important topological invariant, the p-th Betti number. In particular, it does not depend on the Riemannian metric. The question arises whether this is also true for other natural operators like the Dirac operator on a closed Riemannian spin manifold. While there are topological restrictions against harmonic spinors in dimension 2, see [6], Hitchin [10] proved by differential topological methods that this is not the case in dimension $n \equiv 0, \pm 1$ mod 8. In [3] we showed by analytic methods that also in dimension $n \equiv 3$ mod 4 there always exists a Riemannian metric on a closed spin manifold such that the kernel of the corresponding Dirac operator is non-trivial.

In this paper which should be regarded as a sequel to [3] we enlarge the class of operators for which the analogous statement holds. This class contains in

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particular the Dirac operators of spin^c or spin^h manifolds which have regained interest recently because of Seiberg-Witten theory [13]. Some care has to be taken with the statement of the result. In contrast to spin manifolds the Dirac operator of a spin^c manifold is not determined by the Riemannian metric alone but also depends on the choice of a connection on the canonical line bundle. A similar remark holds for spin^h manifolds. The nature of the Seiberg-Witten equations indicates that one should look at modifications of the metric and of the connection on the canonical line bundle separately. It turns out that a modification of the metric is sufficient to produce a non-trivial kernel. Theorem 4.1 applied to spin^c manifolds says

Theorem. *Let M be a closed spin^c manifold of dimension $n \equiv 3 \bmod 4$. Let a connection on the canonical line bundle be fixed. Then there exists a Riemannian metric on M such that there are non-trivial harmonic spinors for the associated Dirac operator.*

The construction shows that this metric can be obtained by deforming any prescribed metric in an arbitrarily small open set. Since the construction is local Theorem 4.1 applies to self-adjopint elliptic operators which look like a twisted Dirac operator in *some* non-empty open subset of the manifold. Outside this set the operator can be anything and will not be modified. The precise statement is as follows

Theorem 4.1. *Let M be a closed Riemannian manifold of dimension $n \equiv 3 \bmod 4$. Let D be an elliptic self-adjoint differential operator over M of order 1. Let $U \subset M$ be a non-empty open subset. Let the restriction of D to U be a twisted Dirac operator.*

Then one can deform the Riemannian metric in U such that the resulting operator $\tilde{D}$ has non-trivial kernel.

The proof relies firstly on the computation of the spectrum of the classical Dirac operator for a certain 1-parameter family of metrics on odd-dimensional spheres, the so-called *Berger metrics*. This computation has been carried out in [3].

Secondly, we prove a Gluing Theorem for such operators on connected sums. Given two closed manifolds of dimension $n \geq 3$ together with operators which look like the classical Dirac operator (or a multiple of it) in some non-empty open subset, then by removing balls in these subsets and gluing one can form the connected sum with a Riemannian metric such that the spectrum of the associated operator on the sum is close to the disjoint union of the spectra of the two original operators.

Theorem 2.1. (*Gluing Theorem*)
Let M_1 and M_2 be n-dimensional closed Riemannian manifolds of dimension $n \geq 3$. Let $U_i \subset M_i$ be open balls, let D_i be self-adjoint elliptic differential operators of first order over M_i of Dirac type over U_i. Let $\Lambda > 0$ such that $\pm\Lambda \notin spec\,(D_1) \cup spec\,(D_2)$. Let $\epsilon > 0$.

Then there exists a Riemannian metric on $X = M_1 \# M_2$ and a self-adjoint elliptic first order differential operator D over X such that X is a disjoint union $X = X_1 \dot\cup X_2 \dot\cup X_3$ where

(i) *X_1 is isometric to $M_1 - U_1$ and D coincides with D_1 over X_1,*

(ii) *X_2 is isometric to $M_2 - U_2$ and D coincides with D_2 over X_2,*

(iii) *X_3 is diffeomorphic to $(0,1) \times S^{n-1}$ and D is of Dirac type over X_3,*

and such that D is (Λ, ϵ)-spectral close to the disjoint union $D_1 \dot\cup D_2$ over $M_1 \dot\cup M_2$.

In [3] we gave a proof of this Gluing Theorem for the classical Dirac operator on odd dimensional spin manifolds, $n \geq 3$. The restriction to odd dimension had technical reasons. We worked with explicit solutions of the eigenspinor equation on Euclidean annuli which can be given by a power series if the dimension is odd. In even dimensions additional logarithmic terms appear. Although it is likely that one can carry over the proof of [3] we chose to avoid the use of explicit solutions in this paper and we work with a-priori estimates instead.

In Section 2 we first describe the class of operators under consideration and then we formulate the Gluing Theorem. The proof is carried out except for the a-priori estimates on the distribution of the L^2-norm of eigenspinors on Euclidean annuli. They are derived in Section 3. In Section 4 we apply the Gluing Theorem and prove existence of metrics with harmonic sections.

An excellent introduction to the Dirac operator of spin and spin^c manifolds is given in [11] or in [7]. Spin^h manifolds are explained in [4], [5], [12]. The variational characterization of eigenvalues is explained in [9], at least for the Laplace operator.

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2 The Gluing Theorem

The aim of this section is to formulate, and partially prove, a theorem which allows us to compare the spectrum of twisted Dirac operators on two closed manifolds M_1 and M_2 with the spectrum of the corresponding operator on the connected sum $M_1 \# M_2$ equipped with a suitable metric.

To start, we describe the class of operators that we will consider. If M is a closed Riemannian spin manifold, then there is a natural self-adjoint elliptic first order differential operator D, the *Dirac operator*. It acts on complex spinor fields and has discrete spectrum. Given an additional complex vector bundle E over M with connection, one can also form the twisted Dirac operator D^E acting on spinors with coefficients in E. See [11] for details. Since most of our considerations will be local in nature we will be able to avoid global topological conditions on M like the spin condition. We make the following

Definition. Let M be a closed Riemannian manifold, let $U \subset M$ be open. A (formally) self-adjoint elliptic first order differential operator D acting on sections of a complex vector bundle over M will be called *of Dirac type over* U, if D coincides over U with a multiple of the Dirac operator acting on spinors, i.e. D coincides over U with the Dirac operator twisted by a trivial flat bundle.

If the manifold is spin, we can take the Dirac operator itself as an example, possibly somehow deformed outside U. Another example is given by the Dirac operator of a spin^c manifold provided the canonical line bundle is flat over U. To get a convenient formulation of the Gluing Theorem we introduce the following terminology.

Definition. Let D_1 and D_2 be two self-adjoint operators with discrete spectrum $\mathrm{spec}\,(D_i)$, let $\Lambda > 0$, and $\epsilon > 0$. We say that D_1 and D_2 are (Λ, ϵ)-*spectral close* iff

(i) $+\Lambda, -\Lambda \notin \mathrm{spec}\,(D_1) \cup \mathrm{spec}\,(D_2)$

(ii) D_1 and D_2 have the same number of eigenvalues in the interval $(-\Lambda, \Lambda)$ (counted with multiplicities). Write

$$\begin{aligned} \mathrm{spec}\,(D_1) \cap (-\Lambda, \Lambda) &= \{\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k\} \\ \mathrm{spec}\,(D_2) \cap (-\Lambda, \Lambda) &= \{\mu_1 \leq \mu_2 \leq \ldots \leq \mu_k\} \end{aligned}$$

(iii) $|\mu_j - \lambda_j| < \epsilon$ for $j = 1, \ldots, k$.

Now we can formulate the main result of this section.

Theorem 2.1. (*Gluing Theorem*)
Let M_1 and M_2 be n-dimensional closed Riemannian manifolds of dimension $n \geq 3$. Let $U_i \subset M_i$ be open balls, let D_i be self-adjoint elliptic differential operators of first order over M_i of Dirac type over U_i. Let $\Lambda > 0$ such that $\pm\Lambda \notin spec\,(D_1) \cup spec\,(D_2)$. Let $\epsilon > 0$.

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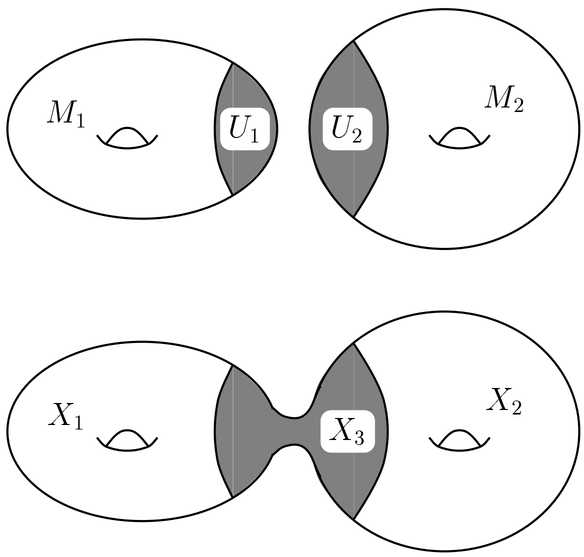

Fig. 1

The operators that we have in mind for applications are locally twisted Dirac operators which are not necessarily globally twisted Dirac operators because the manifold need not be spin. Examples are the Dirac operators for spin^c or spin^h manifolds.

It is an easy exercise to check that the Gluing Theorem does not hold in dimension 1, e.g. for the Dirac operator $D = i \cdot \frac{d}{dt}$ on $S^1_L = \mathbb{R}/L \cdot \mathbb{Z}$, $L > 0$.

The rest of this and the next section are devoted to the proof of the Gluing Theorem. To start, we note that [3, Prop. 7.1] and its proof show the following:

There exists a neighborhood of the Riemannian metric g_i on M_i in the C^1-topology such that for every metric g_i' in this neighborhood which coincides with g_i outside U_i the corresponding operator D_i' (which coincides with D_i outside U_i and is of Dirac type over U_i for the metric g_i') and D_i are $\left(\Lambda, \frac{\epsilon}{2}\right)$-spectral close, say. The Taylor expansion of a Riemannian metric in exponential coordinates shows that one can deform g_i in U_i such that U_i contains a small n-ball with Euclidean metric and this deformation can be made arbitrarily small in the C^1-topology. Hence we may assume w.l.o.g. that that U_1 and U_2 contain small Euclidean n-balls of radius $R > 0$. Denote the centers of these balls by p_i.

Let $0 < t_2 < \min\{R, 2^{-4}\}$. Put $t_{-2} := 2^{-9} \cdot t_2^{16}$. Choose a smooth function $\rho : \mathbb{R} \to \mathbb{R}$ with

a) $\rho(t) = |t|$ for $|t| \geq t_{-2}$

b) $0 < \rho(t) \leq t_{-2}$ for $|t| \leq t_{-2}$

c) $|\dot{\rho}(t)| \leq 1$ for all t.

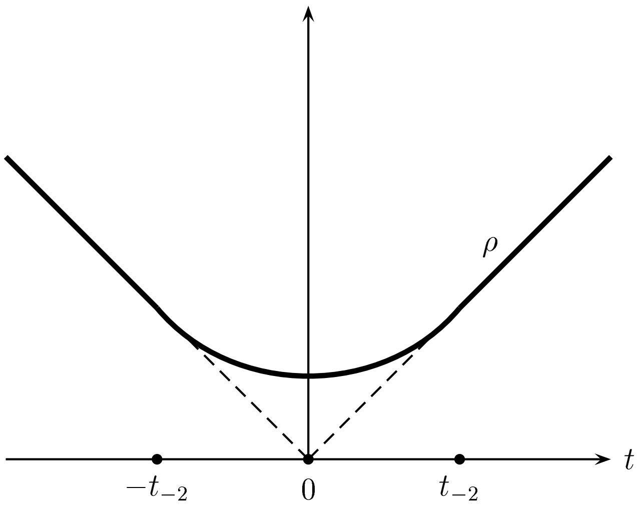

Fig. 2

We construct a metric on $M_1 \# M_2$ as follows. On $M_i - B(p_i, t_{-2})$ take the metric of M_i. Recall that this metric is Euclidean on $B(p_i, R)$. Remove the balls $B(p_i, t_{-2})$ and replace them by

$$[-t_{-2}, t_{-2}] \times S^{n-1}$$

with the warped product metric

$$ds^2 = dt^2 + \rho(t)^2 d\sigma^2$$

where $d\sigma^2$ is the standard metric on S^{n-1} of constant sectional curvature 1. This yields a smooth Riemannian metric g_{t_2} of $M_1 \# M_2$. Set $t_1 := \frac{1}{2} \cdot t_2^4, t_{-1} := \frac{1}{2} t_1$.

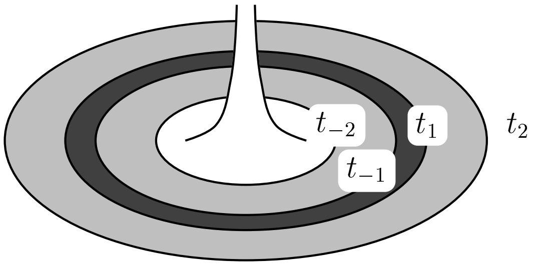

Fig. 3

Choose smooth cut-off functions $\chi_i : M_i \to \mathbb{R}$ such that

a) $0 \leq \chi_i \leq 1$

b) $\chi_i \equiv 1$ on $M_i - B(p_i, t_1)$

c) $\chi_i \equiv 0$ on $B(p_i, t_{-1})$

d) $|\nabla \chi_i| \leq \frac{2}{t_1 - t_{-1}} = \frac{4}{t_1}$.

Let D_{t_2} be the corresponding operator over $X = M_1 \# M_2$ of Dirac type over $[-t_{+2}, t_{+2}] \times S^n$ with respect to the metric g_{t_2}. For an operator D, $a < b$, denote by $E_{[a,b]}(D)$ the sum of all eigenspaces for the eigenvalues $\lambda \in [a, b]$. The Gluing Theorem follows readily from the following

CLAIM: Given $\epsilon > 0$ and $\Lambda > 0$ there exists $\delta = \delta(\epsilon, \Lambda, k) > 0$ where k is the total number of eigenvalues of D_1 and D_2 in the interval $(-\Lambda, \Lambda)$ such that for $\lambda \in (-\Lambda, \Lambda)$ and $0 < t_2 < \min\{\delta, R\}$ we have

$$\dim E_{\{\lambda\}}(D_1) + \dim E_{\{\lambda\}}(D_2) \leq \dim E_{[\lambda-\epsilon, \lambda+\epsilon]}(D_{t_2})$$

$$\leq \dim E_{[\lambda-3\epsilon,\lambda+3\epsilon]}(D_1) + \dim E_{[\lambda-3\epsilon,\lambda+3\epsilon]}(D_2).$$

In fact, it will turn out that $\delta = \min\left\{\frac{1}{100\Lambda^2}, 2^{-4}, \frac{1}{2\Lambda}, \frac{1}{2(k+1)}, 2^{-12}\cdot\frac{\epsilon^2}{(k+1)^2}\right\}$ will do the job.

Proof of inequality

$$\dim E_{\{\lambda\}}(D_1) + \dim E_{\{\lambda\}}(D_2) \leq \dim E_{[\lambda-\epsilon,\lambda+\epsilon]}(D_{t_2}).$$

We define linear maps $E_{\{\lambda\}}(D_i) \to L^2(\Sigma X)$ by $\sigma \to \tilde{\sigma} := \chi_i \cdot \sigma$. We regard $\tilde{\sigma}$ as a spinor over X by extending it by 0 in the obvious manner. By the unique-continuation property for Dirac operators [1] [8] these maps are $1-1$. Since χ_1 and χ_2 have disjoint supports in X, the images $\tilde{E}_{\{\lambda\}}(D_1)$ and $\tilde{E}_{\{\lambda\}}(D_2)$ of $E_{\{\lambda\}}(D_1)$ and $E_{\{\lambda\}}(D_2)$ in $L^2(\Sigma X)$ are L^2-orthogonal. In particular,

$$\dim\left(\tilde{E}_{\{\lambda\}}(D_1) + \tilde{E}_{\{\lambda\}}(D_2)\right) = \dim E_{\{\lambda\}}(D_1) + \dim E_{\{\lambda\}}(D_2).$$

We will show, provided t_2 is small enough, that the Rayleigh quotient of $(D_{t_2}-\lambda)^2$ is bounded by ϵ^2 on $\tilde{E}_{\{\lambda\}}(D_1) \oplus \tilde{E}_{\{\lambda\}}(D_2)$. This obviously implies the inequality in question. Since $\tilde{E}_{\{\lambda\}}(D_1)$ and $\tilde{E}_{\{\lambda\}}(D_2)$ are orthogonal we may look at both spaces separately.

Let $\sigma \in E_{\{\lambda\}}(D_1)$, $\tilde{\sigma} = \chi_1 \cdot \sigma \in \tilde{E}_{\{\lambda\}}(D_1)$. Then

$$\frac{\|(D_{t_2}-\lambda)\tilde{\sigma}\|^2}{\|\tilde{\sigma}\|^2} = \frac{\|\nabla\chi_1\cdot\sigma + \chi_1\cdot D_1\sigma - \lambda\cdot\chi_1\cdot\sigma\|^2}{\|\tilde{\sigma}\|^2} = \frac{\|\nabla\chi_1\cdot\sigma\|^2}{\|\tilde{\sigma}\|^2}. \tag{2.1}$$

By Corollary 2 to Proposition 3.2 (see next section) we know

$$\begin{aligned}
\|\sigma\|^2_{L^2(Z^n_{0,t_1})} &\leq 2^9\cdot t_1^2\cdot t_2\cdot\|\sigma\|^2_{L^2(Z^n_{0,t_2})} \\
&\leq 2^9\cdot 2^{-12}\cdot\|\sigma\|^2_{L^2(M_1)} \qquad \text{since } t_1 \leq t_2 \leq 2^{-4} \\
&\leq \frac{1}{2}\cdot\|\sigma\|^2_{L^2(M_1)}.
\end{aligned}$$

Hence

$$\|\tilde{\sigma}\|^2 \geq \|\sigma\|^2_{L^2(M_1)} - \|\sigma\|^2_{L^2(Z^n_{0,t_1})} \geq \frac{1}{2}\|\sigma\|^2_{L^2(M_1)}. \tag{2.2}$$

Moreover by Corollar 1 to Proposition 3.2 and property d) of χ_1

$$\begin{aligned}
\|\nabla\chi_1\cdot\sigma\|^2 &= \|\nabla\chi_1\cdot\sigma\|^2_{L^2(Z^n_{t_{-1},t_1})} \\
&\leq \|\nabla\chi_1\|^2_{L^\infty}\cdot\|\sigma\|^2_{L^2(Z^n_{t_{-1},t_1})} \\
&\leq \frac{16}{t_1^2}\cdot 2^7\cdot t_1^2\cdot t_2\cdot\|\sigma\|^2_{L^2(Z^n_{t_{-2},t_2})} \\
&\leq 2^{11}\cdot t_2\cdot\|\sigma\|^2_{L^2(M_1)}.
\end{aligned} \tag{2.3}$$

By (2.1), (2.2), and (2.3) we can estimate the Rayleigh quotient

$$\frac{\|(D_{t_2}-\lambda)\tilde{\sigma}\|^2}{\|\tilde{\sigma}\|^2} = \frac{\|\nabla\chi_1\cdot\sigma\|^2}{\|\tilde{\sigma}\|^2} \leq 2^{12}\cdot t_2 \leq \epsilon^2$$

if we assume $t_2 \leq 2^{-12}\cdot\epsilon^2$.

Proof of inequality

$$\dim E_{[\lambda-\epsilon,\lambda+\epsilon]}(D_{t_2}) \leq \dim E_{[\lambda-3\epsilon,\lambda+3\epsilon]}(D_1) + \dim E_{[\lambda-3\epsilon,\lambda+3\epsilon]}(D_2).$$

For $-t_2 \leq \mu < \nu \leq t_2$ denote the connecting cylinder $[\mu,\nu]\times S^{n-1}\subset X$ with metric $ds^2 = dt^2+\rho^2 d\sigma^2$ by $Z_{\mu,\nu}$.

We assume $\Lambda\cdot t_2 \leq \frac{1}{2}$ so that for $|\lambda|\leq\Lambda$:

$$|\lambda|\cdot|\rho| + \frac{1}{2}|\dot{\rho}| \leq \Lambda\cdot t_2 + \frac{1}{2}\cdot 1 \leq 1.$$

Hence we can apply Proposition 3.3 and obtain for any eigenspinor σ on Z_{-t_2,t_2} for the eigenvalue λ

$$\begin{aligned}
\|\sigma\|^2_{L^2(Z_{-t_1,t_1})} &\leq \frac{t_1-(-t_1)}{2(t_2-t_1)}\cdot\|\sigma\|^2_{L^2(Z_{-t_2,t_2})} \\
&= \frac{t_2^4}{2(1-\frac{1}{2}t_2^3)t_2}\cdot\|\sigma\|^2_{L^2(Z_{-t_2,t_2})} \\
&\leq t_2^3\cdot\|\sigma\|^2_{L^2(Z_{-t_2,t_2})}.
\end{aligned}$$

If $\sigma = \Sigma\sigma_j$ is a sum of at most $k+1$ eigenspinors, then we conclude

$$\begin{aligned}
\|\sigma\|_{L^2(Z_{-t_1,t_1})} &\leq \sum_j \|\sigma_j\|_{L^2(Z_{-t_1,t_1})} \\
&\leq t_2^{3/2}\cdot\sum_j \|\sigma_j\|_{L^2(Z_{-t_2,t_2})} \\
&\leq t_2^{3/2}\cdot\sum_j \|\sigma_j\|_{L^2(X)} \\
&\leq t_2\cdot(k+1)\cdot\|\sigma\|_{L^2(X)}.
\end{aligned}$$

Thus if we assume $t_2 \leq \frac{1}{2(k+1)}$, then

$$\|\sigma\|_{L^2(Z_{-t_1,t_1})} \leq \frac{1}{2}\|\sigma\|_{L^2(X)}. \tag{2.4}$$

Put $l := \dim E_{[\lambda-3\epsilon,\lambda+3\epsilon]}(D_1)+\dim E_{[\lambda-3\epsilon,\lambda+3\epsilon]}(D_2)$. We will derive a contradiction from the assumption

$$\dim E_{[\lambda-\epsilon,\lambda+\epsilon]}(D_{t_2}) \geq l+1.$$

Let V be spanned by $l+1$ linearly independent D_{t_2}-eigenspinors for eigenvalues in $[\lambda-\epsilon,\lambda+\epsilon]$. Note that $l+1 \leq k+1$. The linear map

$$\begin{array}{rcl} V & \to & L^2(\Sigma M_1) \oplus L^2(\Sigma M_2) \\ \sigma & \to & (\chi_1 \cdot \sigma, \chi_2 \cdot \sigma) \end{array}$$

is again $1-1$ by the unique-continuation property. We will show that the Rayleigh quotient of $(D_1-\lambda)^2 \oplus (D_2-\lambda)^2$ is bounded by $(3\epsilon)^2$ on the image of V. This would imply that $D_1 \oplus D_2$ has at least $l+1$ eigenvalues in $[\lambda-3\epsilon,\lambda+3\epsilon]$, a contradiction.

$$\begin{array}{rcl} \|(\chi_1\cdot\sigma,\chi_2\cdot\sigma)\|^2 & = & \|\chi_1\cdot\sigma\|^2+\|\chi_2\cdot\sigma\|^2 \\ & \geq & \|\sigma\|^2_{L^2(X)} - \|\sigma\|^2_{L^2(Z_{-t_1,t_1})} \\ & \geq & \frac{1}{2}\|\sigma\|^2_{L^2(X)} \end{array} \qquad (2.5)$$

by (2.4). By Corollary 1 to Proposition 3.2 we know

$$\begin{array}{rcl} \|\nabla\chi_i\cdot\sigma\| & = & \|\nabla\chi_i\cdot\sigma\|_{L^2(Z^n_{t_{-1},t_1})} \\ & \leq & \frac{4}{t_1}\cdot\|\sigma\|_{L^2(Z^n_{t_{-1},t_1})} \\ & \leq & \frac{4}{t_1}\cdot\sqrt{2^7\cdot t_1^2\cdot t_2}\cdot\sum_j\|\sigma_j\|_{L^2(Z^n_{t_{-2},t_2})} \\ & \leq & \sqrt{2^{11}\cdot t_2}\cdot(k+1)\cdot\|\sigma\|_{L^2(X)}. \end{array} \qquad (2.6)$$

Summarizing we get for the Rayleigh quotient

$$\frac{\|(D_1-\lambda)\oplus(D_2-\lambda)(\chi_1\sigma,\chi_2\sigma)\|^2}{\|(\chi_1\sigma,\chi_2\sigma)\|^2} \leq$$

$$\begin{array}{rll} \leq & 2\cdot\frac{\|(D_1-\lambda)(\chi_1\sigma)\|^2+\|(D_2-\lambda)(\chi_2\sigma)\|^2}{\|\sigma\|^2} & \text{by (2.5)} \\ = & 2\cdot\frac{\|\chi_1(D_1-\lambda)\sigma+\nabla\chi_1\cdot\sigma\|^2+\|\chi_2(D_2-\lambda)\sigma+\nabla\chi_2\cdot\sigma\|^2}{\|\sigma\|^2} & \\ \leq & 4\cdot\frac{\|(D_{t_2}-\lambda)\sigma\|^2+\|\nabla\chi_1\cdot\sigma\|^2+\|(D_{t_2}-\lambda)\sigma\|^2+\|\nabla\chi_2\cdot\sigma\|^2}{\|\sigma\|^2} & \\ \leq & 4\cdot\left(\epsilon^2+2^{11}\cdot t_2\cdot(k+1)^2+\epsilon^2+2^{11}\cdot t_2\cdot(k+1)^2\right) & \\ \leq & (3\epsilon)^2 & \text{by (2.6)} \end{array}$$

if $t_2 \leq 2^{-12} \cdot \frac{\epsilon^2}{(k+1)^2}$. This finishes the proof of the Gluing Theorem. $\square$

3 The Estimates

In this section we will derive the a-priori estimates on the distribution of the L^2-norm of eigenspinors on Euclidean annuli which have been used in the previous section to prove the Gluing Theorem. Before we start looking at spinors we prove an estimate which allows us to compare unknown solutions of a linear ordinary differential equation with certain "almost solutions" which can explicitly be written down.

Proposition 3.1. *Let $I \subset \mathbb{R}$ be an interval, $t_0 \in I$, let $A : I \to Mat(n \times n, \mathbb{C})$ be a continuous mapping into the complex $n \times n$-matrices. Let u be a solution of*

$$\dot{u}(t) = A(t)u(t).$$

Moreover, let $v : I \to \mathbb{C}^n$ be a continuously differentiable function such that

(i) $v(t_0) = u(t_0)$

(ii) $|\dot{v}(t) - A(t)v(t)| \leq \delta(t)$

where $\delta : I \to [0, \infty)$ is a continuous function.

Then the following estimate holds

$$|u(t) - v(t)| \leq \left| \int_{t_0}^{t} \delta(s) \cdot e^{\|A\|_\infty \cdot |t-s|} ds \right|$$

where $\|A\|_\infty = \sup\limits_{t \in I} |A(t)|_{\mathrm{Op}}$ and $|A(t)|_{\mathrm{Op}}$ is the operator norm.

Proof. It is sufficient to prove the claim for $t \geq t_0$.
Put $w(t) := u(t) - v(t)$. Then

$$\dot{w}(t) = A(t) \cdot w(t) + \rho(t)$$

where $\rho(t) = A(t) \cdot v(t) - \dot{v}(t)$. Thus

$$\begin{aligned} |w(t)| &= \left| \int_{t_0}^{t} (A(s) \cdot w(s) + \rho(s)) ds \right| \\ &\leq \|A\|_\infty \cdot \int_{t_0}^{t} |w(s)| ds + \int_{t_0}^{t} \delta(s) ds. \end{aligned} \tag{3.1}$$

For $\varphi(t) := \|A\|_\infty \cdot \int_{t_0}^{t} |w(s)| ds + \int_{t_0}^{t} \delta(s) ds$
we have

$$\varphi(t_0) = 0$$

and

$$\dot{\varphi}(t) = \|A\|_\infty |w(t)| + \delta(t). \tag{3.2}$$

Hence by (3.1) and (3.2)

$$\begin{aligned} 0 &\leq \|A\|_\infty \cdot (\varphi(t) - |w(t)|) \\ &= \|A\|_\infty \cdot \varphi(t) - \dot{\varphi}(t) + \delta(t). \end{aligned}$$

Multiplying by $e^{-\|A\|_\infty \cdot (t - t_0)}$ we obtain

$$0 \leq -\left(e^{-\|A\|_\infty \cdot (t-t_0)} \cdot \varphi(t)\right)^{\cdot} + e^{-\|A\|_\infty \cdot (t-t_0)} \cdot \delta(t).$$

Integration yields

$$e^{-\|A\|_\infty \cdot (t-t_0)} \cdot \varphi(t) \leq \int_{t_0}^{t} \left(e^{-\|A\|_\infty \cdot (s-t_0)} \cdot \delta(s)\right) ds$$

which implies again using (3.1)

$$\begin{aligned} |w(t)| &\leq \varphi(t) \\ &\leq \int_{t_0}^{t} e^{\|A\|_\infty (t-s)} \cdot \delta(s) ds. \end{aligned}$$

□

For $0 < a < b$ let $Z^n_{a,b} := \{x \in \mathbb{R}^n \mid a \leq |x| \leq b\}$ be the corresponding *Euclidean annulus*. As a Riemannian manifold we can also describe $Z^n_{a,b}$ as a warped product

$$Z^n_{a,b} = [a, b] \times S^{n-1}$$

with the metric

$$ds^2(t,y) = dt^2 + t^2 d\sigma^2(y)$$

where $d\sigma^2$ is the standard metric on S^{n-1} of constant sectional curvature 1.

Restriction of the spinor bundle $\Sigma\mathbb{R}^n$ of $\mathbb{R}^n$ to S^{n-1} yields the spinor bundle of S^{n-1} if n is odd. If n is even one obtains the direct sum of two copies of the spinor bundle on S^{n-1}. If n is odd let $\tilde{D}$ be the Dirac operator of S^{n-1}, if n is even let $\tilde{D}$ be the sum of the Dirac operator of S^{n-1} and of its negative.

We choose an orthonormal basis of $L^2(\Sigma\mathbb{R}^n|_{S^{n-1}})$ consisting of eigenspinors σ_j of $\tilde{D}$. Clifford multiplication by the normal vector field $\frac{\partial}{\partial t}$ anticommutes with $\tilde{D}$. Thus if σ_j is an eigenspinor for the eigenvalue μ_j, then $\frac{\partial}{\partial t}\cdot\sigma_j$ is an eigenspinor for $-\mu_j$. Hence we may assume $\sigma_{-j} = \frac{\partial}{\partial t}\cdot\sigma_j$, $j \in \mathbb{N}$, $0 < \mu_1 \leq \mu_2 \leq \mu_2 \leq \ldots \to +\infty$. Parallel translation along the t-lines yields smooth sections $(t,y) \mapsto \sigma_j(t,y)$ on $(0,\infty) \times S^{n-1} = \mathbb{R}^n - \{0\}$, $j \in \mathbb{Z}^* = \mathbb{Z} - \{0\} = \mathbb{N} \cup -\mathbb{N}$. If σ is a smooth spinor field on $Z^n_{a,b}$, then we can write

$$\sigma(t,y) = \sum_{j\in\mathbb{Z}^*} \beta_j(t) \cdot t^{-\frac{n-1}{2}} \cdot \sigma_j(t,y)$$

where β_j are smooth functions, $\beta_j : [a,b] \to \mathbb{C}$. If D is the Dirac operator on $Z^n_{a,b}$, then the eigenspinor equation

$$D\sigma = \lambda \cdot \sigma$$

is equivalent to

$$\dot{B}_j(t) = \begin{pmatrix} \frac{\mu_j}{t} & -\lambda \\ \lambda & -\frac{\mu_j}{t} \end{pmatrix} \cdot B_j(t) \tag{3.3}$$

where $B_j(t) = \begin{pmatrix} \beta_{-j}(t) \\ \beta_j(t) \end{pmatrix}$ for all $j \in \mathbb{N}$. The L^2-norm of σ is given by

$$\|\sigma\|^2_{L^2(Z^n_{a,b})} = \sum_{j=1}^{\infty} \int_a^b |B_j(t)|^2 dt.$$

See [3] for the details. Now we come to the main estimate of this section.

Proposition 3.2. *Let $0 < t_{-2} < t_{-1} < t_1 < t_2 \leq 1$, let $\mu \geq 1$, let $\lambda \in \mathbb{R}$. If $|\lambda| t_2^{1/2} \leq \frac{1}{10}$, $t_2 \geq 2t_1$, $t_{-1} \geq 2t_{-2}$, and $t_1^6 \leq t_{-2}$, then every solution B of*

$$\dot{B}(t) = \begin{pmatrix} \frac{\mu}{t} & -\lambda \\ \lambda & \frac{\mu}{t} \end{pmatrix} \cdot B(t)$$

satisfies

$$\frac{\int\limits_{t_{-1}}^{t_1} |B(t)|^2 dt}{\int\limits_{t_{-2}}^{t_2} |B(t)|^2 dt} \le 2^6 \cdot \max\left\{ 3 \cdot \left(\frac{t_1}{t_2}\right)^{2\mu+1}, \frac{t_1}{t_{-1}} \cdot \left(\frac{t_{-2}}{t_{-1}}\right)^{2\mu-1} \right\}$$

Corollary 1. *Let* $0 < t_2 < 2^{-4}$, *let* $\Lambda > 0$. *Put* $t_1 := \frac{1}{2} \cdot t_2^4$, $t_{-1} := \frac{1}{2} t_1$, $t_{-2} := \frac{1}{2} t_{-1}^4$ *Then every eigenspinor* σ *for the eigenvalue* λ *on* $Z^n_{t_{-2},t_2}$ *satisfies*

$$\frac{\|\sigma\|^2_{L^2(Z^n_{t_{-1},t_1})}}{\|\sigma\|^2_{L^2(Z^n_{t_{-2},t_2})}} \le 2^7 \cdot t_1^2 \cdot t_2,$$

provided $|\lambda| \le \Lambda$ *and* $\Lambda \cdot t_2^{1/2} \le \frac{1}{10}$.

Proof of Corollary 1. With our definition of t_1, t_{-1}, and t_{-2} the assumptions on t_j in Proposition 3.2 are trivially satisfied. Moreover, it is well-known that the absolute value of all Dirac eigenvalues of S^{n-1} is at least 1 if $n \ge 3$, compare e.g. [2]. Hence

$$\begin{aligned}
\|\sigma\|^2_{L^2(Z^n_{t_{-1},t_1})} &= \sum_{j=1}^{\infty} \int_{t_{-1}}^{t_1} |B_j(t)|^2 dt \\
&\le 2^6 \cdot \sum_{j=1}^{\infty} \max\left\{ 3 \cdot \left(\frac{1}{2} t_2^3\right)^{2\mu_j+1}, 2 \cdot \left(\frac{1}{2} t_{-1}^3\right)^{2\mu_j-1} \right\} \cdot \int_{t_{-2}}^{t_2} |B_j(t)|^2 dt \\
&\le 2^6 \cdot \max\left\{ 3 \cdot 2^{-3} \cdot t_2^9, \, t_{-1}^3 \right\} \cdot \|\sigma\|^2_{L^2(Z^n_{t_{-2},t_2})} \\
&\le 2^7 \cdot t_1^2 \cdot t_2 \cdot \|\sigma\|^2_{L^2(Z^n_{t_{-2},t_2})}. \qquad \square
\end{aligned}$$

Corollary 2. *Let* $0 < t_2 < 2^{-4}$, *let* $t_1 := \frac{1}{2} t_2^4$, *let* $\Lambda > 0$. *Then every eigenspinor* σ *for the eigenvalue* λ *on the ball* Z^n_{0,t_2} *satisfies*

$$\frac{\|\sigma\|^2_{L^2(Z^n_{0,t_1})}}{\|\sigma\|^2_{L^2(Z^n_{0,t_2})}} \le 2^9 \cdot t_1^2 \cdot t_2$$

if $|\lambda| \le \Lambda$ *and* $\Lambda t_2^{1/2} \le \frac{1}{10}$.

Proof of Corollary 2. We apply Corollary 1 to ($k \in \mathbb{N}_0$)

$$\begin{aligned} t_{2,k} &:= 2^{-k/4} \cdot t_2, \\ t_{1,k} &:= 2^{-k} \cdot t_1, \\ t_{-1,k} &:= \frac{1}{2} t_{1,k} = 2^{-k-1} \cdot t_1, \\ t_{-2,k} &:= \frac{1}{2} t_{-1,k}^4 : \end{aligned}$$

$$\begin{aligned} \|\sigma\|^2_{L^2(Z^n_{t_{-1,k},t_{1,k}})} &\leq 2^7 \cdot t_{1,k}^2 \cdot t_{2,k} \cdot \|\sigma\|^2_{L^2(Z^n_{t_{-2,k},t_{2,k}})} \\ &\leq 2^7 \cdot 2^{-2k} \cdot t_1^2 \cdot t_2 \cdot \|\sigma\|^2_{L^2(Z^n_{0,t_2})}. \end{aligned}$$

Summation over k yields

$$\|\sigma\|^2_{L^2(Z^n_{0,t_1})} \leq 2^7 \cdot \frac{4}{3} \cdot t_1^2 \cdot t_2 \cdot \|\sigma\|^2_{L^2(Z^n_{0,t_2})} \qquad \square$$

Proof of Proposition 3.2. The substitution $t = e^\tau$, $\tilde{B}(\tau) = B(e^\tau)$, translates our differential equation (3.3) into

$$\tilde{B}'(\tau) = \begin{pmatrix} \mu & -\lambda e^\tau \\ \\ \lambda e^\tau & -\mu \end{pmatrix} \cdot \tilde{B}(\tau). \tag{3.4}$$

Set $t_0 := \sqrt{t_1 \cdot t_{-1}}$, $t_k = e^{\tau_k}$, $k = -2, -1, 0, 1, 2$. Write

$$B(t_0) = \tilde{B}(\tau_0) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

We will compare the solution $\tilde{B}$ of (3.4) to the "almost solution"

$$\begin{aligned} \tilde{v}(\tau) &:= \begin{pmatrix} w_1 \cdot e^{\mu(\tau-\tau_0)} \\ w_2 \cdot e^{-\mu(\tau-\tau_0)} \end{pmatrix}, \\ v(t) &:= \begin{pmatrix} w_1 \cdot \left(\frac{t}{t_0}\right)^\mu \\ w_2 \cdot \left(\frac{t}{t_0}\right)^{-\mu} \end{pmatrix}. \end{aligned}$$

Define

$$A(\tau) := \begin{pmatrix} \mu & -\lambda e^\tau \\ \lambda e^\tau & -\mu \end{pmatrix}.$$

Then on $[\tau_{-2}, \tau_2]$ we have $\|A\|_\infty \leq \mu + \frac{1}{10}$ because $|\lambda e^\tau| \leq |\lambda t_2| \leq |\lambda t_2^{1/2}| \leq \frac{1}{10}$. We compute

$$\begin{aligned} |\tilde{v}' - A(\tau) \cdot \tilde{v}(\tau)|^2 &= \left| \lambda \cdot e^\tau \cdot \begin{pmatrix} w_2 \cdot e^{-\mu(\tau-\tau_0)} \\ -w_1 \cdot e^{\mu(\tau-\tau_0)} \end{pmatrix} \right|^2 \\ &= \lambda^2 \cdot e^{2\tau} \cdot \left(|w_1|^2 e^{2\mu(\tau-\tau_0)} + |w_2|^2 e^{-2\mu(\tau-\tau_0)} \right). \end{aligned}$$

Now we estimate the difference between the solution $\tilde{B}$ and the "almost solution" $\tilde{v}$.

Case 1. $\tau \geq \tau_0$.

Put $\delta(\tau) := \lambda \cdot e^\tau \cdot |B(t_0)| \cdot e^{\mu(\tau-\tau_0)}$. Then

$$|\tilde{v}'(\tau) - A(\tau) \cdot \tilde{v}(\tau)| \leq \delta(\tau)$$

and hence by Proposition 3.1

$$\begin{aligned} |\tilde{B}(\tau) - \tilde{v}(\tau)| &\leq \int_{\tau_0}^{\tau} \delta(\sigma) \cdot e^{\|A\|_\infty \cdot (\tau-\sigma)} d\sigma \\ &\leq \lambda \cdot |B(t_0)| \cdot \int_{\tau_0}^{\tau} e^\sigma \cdot e^{\mu(\sigma-\tau_0)} \cdot e^{(\mu+\frac{1}{10})(\tau-\sigma)} d\sigma \\ &= \lambda \cdot e^{\mu(\tau-\tau_0)} \cdot |B(t_0)| \cdot \frac{10}{9} \cdot e^{\frac{\tau}{10}} \cdot \left(e^{\frac{9}{10}\tau} - e^{\frac{9}{10}\tau_0} \right) \\ &\leq \frac{10}{9} \cdot \lambda \cdot e^\tau \cdot |B(t_0)| \cdot e^{\mu(\tau-\tau_0)} \\ &\leq |B(t_0)| \cdot e^{\mu(\tau-\tau_0)}. \end{aligned}$$

Hence

$$|B(t) - v(t)| \leq |B(t_0)| \cdot \left(\frac{t}{t_0} \right)^\mu.$$

Case 2. $\tau \leq \tau_0$.

We set $\delta(\tau) := \lambda \cdot e^\tau \cdot |B(t_0)| \cdot e^{-\mu(\tau-\tau_0)}$. Again we have

$$|\tilde{v}'(\tau) - A(\tau) \cdot \tilde{v}(\tau)| \leq \delta(\tau).$$

Proposition 3.1. yields

$$\begin{aligned}
|\tilde{B}(\tau)-\tilde{v}(\tau)| &\le \int\limits_{\tau}^{\tau_0} \delta(\sigma)\cdot e^{\|A\|_\infty\cdot(\sigma-\tau)}d\sigma \\
&\le \lambda\cdot|B(t_0)|\cdot\int\limits_{\tau}^{\tau_0} e^{\sigma}\cdot e^{-\mu(\sigma-\tau_0)}\cdot e^{(\mu+\frac{1}{10})(\sigma-\tau)}d\sigma \\
&= \lambda\cdot|B(t_0)|\cdot e^{-\mu(\tau-\tau_0)}\cdot e^{-\frac{\tau}{10}}\cdot\int\limits_{\tau}^{\tau_0} e^{\frac{11}{10}\sigma}d\sigma \\
&\le \lambda\cdot|B(t_0)|\cdot e^{-\mu(\tau-\tau_0)}\cdot e^{\tau_0}\cdot e^{\frac{1}{10}(\tau_0-\tau)}.
\end{aligned}$$

Thus

$$\begin{aligned}
|B(t)-v(t)| &\le \lambda\cdot t_0\cdot|B(t_0)|\cdot\left(\frac{t_0}{t_{-2}}\right)^{\frac{1}{10}}\cdot\left(\frac{t}{t_0}\right)^{-\mu} \\
&= \lambda\cdot t_0^{1/2}\cdot|B(t_0)|\cdot\left(\frac{t_0^6}{t_{-2}}\right)^{\frac{1}{10}}\cdot\left(\frac{t}{t_0}\right)^{-\mu} \\
&\le |B(t_0)|\cdot\left(\frac{t}{t_0}\right)^{-\mu}
\end{aligned}$$

because $t_0^6 \le t_1^6 \le t_{-2}$.

Now that we have some control on how much v can differ from the solution B we can look at the growth of B. An upper bound is now easily obtained.

If $t \ge t_0$:

$$|B(t)| \le |v(t)| + |B(t)-v(t)| \le 2\cdot|B(t_0)|\cdot\left(\frac{t}{t_0}\right)^{\mu}$$

and similarly for $t \le t_0$:

$$|B(t)| \le 2\cdot|B(t_0)|\cdot\left(\frac{t}{t_0}\right)^{-\mu}.$$

From this we get

$$\int\limits_{t_{-1}}^{t_1} |B(t)|^2dt = \int\limits_{t_{-1}}^{t_0} |B(t)|^2dt + \int\limits_{t_0}^{t_1} |B(t)|^2dt$$

$$\begin{aligned}
&\leq\ 4\cdot|B(t_0)|^2\cdot\left\{\frac{1}{2\mu-1}\cdot\left[\left(\frac{t_{-1}}{t_0}\right)^{-2\mu}\cdot t_{-1}-t_0\right]\right.\\
&\qquad\left.+\frac{1}{2\mu+1}\left[\left(\frac{t_1}{t_0}\right)^{2\mu}\cdot t_1-t_0\right]\right\}\\
&\leq\ \frac{8}{2\mu-1}\cdot|B(t_0)|^2\cdot\left(\frac{t_1}{t_{-1}}\right)^{\mu}\cdot t_1.
\end{aligned}$$

by the definition of t_0.

To get a lower bound on $|B(t)|$ we have to distinguish two cases.

Case 1. $|w_1|\geq|w_2|$.

We know $|v(t)|^2\geq|w_1|^2\cdot\left(\frac{t}{t_0}\right)^{2\mu}\geq\frac{1}{2}|B(t_0)|^2\cdot\left(\frac{t}{t_0}\right)^{2\mu}$. For $t\in[t_0,t_2]$ we get

$$\begin{aligned}
|B(t)|\ &\geq\ |v(t)|-|B(t)-v(t)|\\
&\geq\ |B(t_0)|\cdot\left(\frac{t}{t_0}\right)^{\mu}\cdot\left\{\frac{1}{\sqrt{2}}-\frac{10}{9}\cdot\lambda\cdot t\right\}\\
&\geq\ \frac{1}{2}\cdot|B(t_0)|\cdot\left(\frac{t}{t_0}\right)^{\mu}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{t_{-2}}^{t_2}|B(t)|^2dt\ &\geq\ \int_{t_0}^{t_2}|B(t)|^2dt\\
&\geq\ \frac{1}{4}\cdot|B(t_0)|^2\cdot\frac{1}{2\mu+1}\cdot\left\{\left(\frac{t_2}{t_0}\right)^{2\mu}\cdot t_2-t_0\right\}\\
&\geq\ \frac{1}{8}\cdot|B(t_0)|^2\cdot\frac{1}{2\mu+1}\cdot\left(\frac{t_2}{t_0}\right)^{2\mu}\cdot t_2
\end{aligned}$$

because $t_2\geq 2\cdot t_0$. This implies

$$\begin{aligned}
\frac{\int_{t_{-1}}^{t_1}|B(t)|^2dt}{\int_{t_{-2}}^{t_2}|B(t)|^2dt}\ &\leq\ \frac{\frac{8}{2\mu-1}\cdot\left(\frac{t_1}{t_2}\right)^{2\mu+1}\cdot\left(\frac{t_1}{t_{-1}}\right)^{\mu}\cdot t_1}{\frac{1}{8}\cdot|B(t_0)|^2\cdot\frac{1}{2\mu+1}\cdot\left(\frac{t_2}{t_0}\right)^{2\mu}\cdot t_1}\\
&\leq\ 2^6\cdot\frac{2\mu+1}{2\mu-1}\cdot\left(\frac{t_1}{t_2}\right)^{2\mu+1}
\end{aligned}$$

$$\leq \quad 2^6 \cdot 3 \cdot \left(\frac{t_1}{t_2}\right)^{2\mu+1}.$$

Case 2. $|w_1| \leq |w_2|$.

This time we have $|v(t)|^2 \geq |w_2|^2 \cdot \left(\frac{t}{t_0}\right)^{-2\mu} \geq \frac{1}{2}|B(t_0)|^2 \cdot \left(\frac{t}{t_0}\right)^{-2\mu}$.
For $t \leq t_0$ we get

$$\begin{aligned}
|B(t)| &\geq |v(t)| - |B(t) - v(t)| \\
&\geq |B(t_0)| \cdot \left(\frac{t}{t_0}\right)^{-\mu} \cdot \left\{\frac{1}{\sqrt{2}} - \lambda \cdot t_0^{1/2}\right\} \\
&\geq \frac{1}{2} \cdot |B(t_0)| \cdot \left(\frac{t}{t_0}\right)^{-\mu}.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_{t_{-2}}^{t_2} |B(t)|^2 dt &\geq \int_{t_{-2}}^{t_0} |B(t)|^2 dt \\
&\geq \frac{1}{4} \cdot |B(t_0)|^2 \cdot \frac{1}{2\mu - 1} \cdot \left\{\left(\frac{t_{-2}}{t_0}\right)^{-2\mu} \cdot t_{-2} - t_0\right\} \\
&\geq \frac{|B(t_0)|^2}{8(2\mu - 1)} \cdot \left(\frac{t_0}{t_{-2}}\right)^{2\mu} \cdot t_{-2}
\end{aligned}$$

because $t_0 \geq t_{-1} \geq 2t_{-2}$ and thus $\left(\frac{t_0}{t_{-2}}\right)^{2\mu} - \left(\frac{t_0}{t_{-2}}\right) \geq \frac{1}{2}\left(\frac{t_0}{t_{-2}}\right)^{2\mu}$.

Hence

$$\frac{\int_{t_{-1}}^{t_1} |B(t)|^2 dt}{\int_{t_{-2}}^{t_2} |B(t)|^2 dt} \leq \frac{\frac{8}{2\mu-1} \cdot |B(t_0)|^2 \cdot \left(\frac{t_1}{t_{-1}}\right)^{\mu} \cdot t_1}{\frac{|B(t_0)|^2}{8(2\mu-1)} \cdot \left(\frac{t_0}{t_{-2}}\right)^{2\mu} \cdot t_{-2}} = 2^6 \cdot \left(\frac{t_{-2}}{t_{-1}}\right)^{2\mu} \cdot \frac{t_1}{t_{-2}}.$$

This finishes the proof of Proposition 3.2. $\square$

To conclude this section we cite the following proposition which has also been used in the proof of the Gluing Theorem.

Proposition 3.3. *Let $n \geq 3$. Let $a, b, c, \lambda \in \mathbb{R}$ such that $a < b$ and $c > 0$. Let $\rho : [a-c, b+c] \to \mathbb{R}$ be a smooth positive function such that*

$$|\lambda| \cdot \|\rho\|_{L^\infty(a-c,b+c)} + \frac{1}{2} \cdot \|\dot{\rho}\|_{L^\infty(a-c,b+c)} \leq 1.$$

Then for every Dirac eigenspinor σ for the eigenvalue λ on $Z_{a-c,b+c} := [a-c, b+c] \times S^{n-1}$ with the warped product metric $ds^2 = dt^2 + \rho(t)^2 d\sigma^2$ the following estimate holds

$$\|\sigma\|^2_{L^2(Z_{a,b})} \leq \frac{b-a}{2c} \left\{ \|\sigma\|^2_{L^2(Z_{b,b+c})} + \|\sigma\|^2_{L^2(Z_{a-c,a})} \right\}.$$

The proof can be found in [3, Prop. 5.1].

4 Harmonic Sections

Our main application of the Gluing Theorem is

Theorem 4.1. *Let M be a closed Riemannian manifold of dimension $n \equiv 3 \bmod 4$. Let D be an elliptic self-adjoint differential operator over M of order 1. Let $U \subset M$ be a non-empty open subset. Let the restriction of D to U be a twisted Dirac operator.*

Then one can deform the Riemannian metric in U such that the resulting operator $\tilde{D}$ has non-trivial kernel.

Note that $\tilde{D}$ coincides with D outside U and that the connection of the coefficient bundle over U is **not** modified. Theorem 4.1 applies for example to the Dirac operators of spin^c or spin^h manifolds. By a suitable local deformation of the Riemannian metric while keeping the connection of the canonical line bundle fixed one can produce a non-trivial kernel for the Dirac operator of a spin^c manifold.

Theorem 4.1 does not apply to the Euler operator $d + \delta$ acting on forms even though it is also a twisted Dirac operator (twisted by spinors). The point is that in this case a change of the Riemannian metric also changes the connection on the coefficient bundle.

Theorem 4.1 is not true in dimension 2. The classical Dirac operator on S^2 has no harmonic spinors no matter which Riemannian metric on S^2 has been chosen.

Proof of Theorem 4.1. Over U our operator D is a twisted Dirac operator, i.e. $D|_U = D^E$ where E is some bundle over U with connection ∇^E. Choose a small n-ball $U_1 \subset\subset U$. By blowing up the metric g in a neighborhood of $\overline{U}_1$ to some metric $\overline{g}$ one can make

$$f = \text{ id} : (U_1, \overline{g}) \to (U_1, g)$$

ϵ-contracting for an arbitrarily small prescribed $\epsilon > 0$.

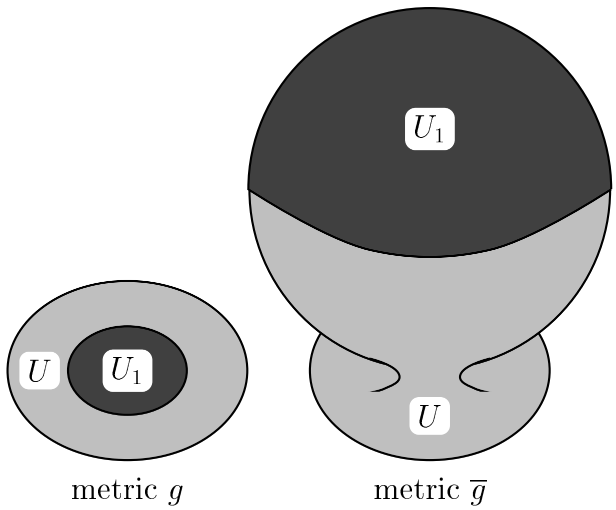

Fig. 4

After trivializing the bundle E over U_1 we can write

$$\nabla^E = \partial + \Gamma$$

where ∂ is a flat coordinate derivative and Γ is an $\mathrm{End}(E)$-valued 1-form given by the Christoffel symbols.

For the pull-back of a connection by a map f one has

$$f^*\nabla^E = \partial + \Gamma \circ df.$$

Since in our case $f =$ id is ϵ-contracting we see that with respect to the new metric $\overline{g}$ we have

$$\|\Gamma\|_{L^\infty(U_1)} < \epsilon'$$

for small $\epsilon' > 0$. Choose a smaller n-ball $U_2 \subset\subset U_1$ and a cut-off function $\chi : M \to \mathbb{R}$ such that

a) $0 \leq \chi \leq 1$

b) $\chi \equiv 0$ on U_2

c) $\chi \equiv 1$ on $M - U_1$.

Define a new connection $\overline{\nabla}$ on E by

$$\overline{\nabla} := \partial + \chi \cdot \Gamma.$$

Then $\overline{\nabla}$ is flat over U_2 and $\overline{\nabla} = \nabla^E$ over $U - U_1$. Denote by D_1 the twisted Dirac operator for the original connection ∇^E and the new metric $\overline{g}$, by D_2 the twisted Dirac operator for $\overline{\nabla}$ and $\overline{g}$. We extend D_1 and D_2 to M such that over $M - U$ we have $D_1 = D_2 = D$. Then the difference $D_1 - D_2$ is an operator of order 0 with support in U_1. Its L^2-operator norm can be estimated by

$$\|D_1 - D_2\|_{L^2,L^2} \leq n \cdot \|\overline{\nabla} - \nabla^E\|_{L^\infty} < n \cdot \epsilon' =: \epsilon''.$$

In particular, D_1 and D_2 are (∞, ϵ'')-spectral close and D_2 is of Dirac type over U_2.

In [3, Section 3] it is shown that in dimension $n \equiv 3 \bmod 4$ there exists a one-parameter family g_T of Riemannian metrics on S^n, $T \in [a, b]$, such that the following holds for the associated Dirac operator D_T:

a) There is $\lambda(T) \in \operatorname{spec}(D_T)$ with $\lambda(a) = -1$, $\lambda(b) = +1$.

b) $\lambda(T)$ depends smoothly (actually linearly) on T.

c) The multiplicity of $\lambda(T)$ is constant in T and can be chosen arbitrarily large.

d) $\lambda(T)$ is the only eigenvalue of D_T in the interval $[-1, 1]$.

We choose the multiplicity of $\lambda(T)$ larger than the total number of eigenvalues of D_2 in the interval $[-1, 1]$. Now we can apply the Gluing Theorem to D_2 and D_T. We obtain a metric $\overline{g}_T$ on $M \# S^n = M$ such that the corresponding operator $\overline{D}_T$ on M is $(1, \epsilon'')$-spectral close to $D_2 \dot{\cup} D_T$.

By construction of $\overline{g}_T$ the identity mapping

$$\text{id} : (U_1, \overline{g}_T) \to (U_1, g)$$

is still ϵ-contracting. Thus $\tilde{D}_T$ and $\overline{D}_T$ are (∞, ϵ'')-spectral close where $\tilde{D}_T$ coincides with D outside U while over U, $\tilde{D}_T$ is the twisted Dirac operator for ∇^E and $\overline{g}_T$. It follows that $\tilde{D}_T$ is $(1, 2\epsilon'')$-spectral close to $D_2 \dot{\cup} D_T$.

Since the multiplicity of $\lambda(T)$ was chosen larger than the total number of eigenvalues of D_2 in $[-1, 1]$, there are more negative eigenvalues of $D_2 \dot{\cup} D_T$ in $[-1, 1]$ for $T = a$ whereas there are more positive ones for $T = b$. If ϵ'' was chosen smaller than $\frac{1}{2}$, then this will remain true for the operator $\tilde{D}_T$. In particular, there must be some $T_0 \in (a, b)$ for which 0 is an eigenvalue. Hence Theorem 4.1 holds with the metric $\overline{g}_{T_0}$ and the corresponding operator $\tilde{D} = \tilde{D}_{T_0}$. $\square$

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Author's address:

Mathematisches Institut
Universität Freiburg
Hebelstr. 29
79104 Freiburg
Germany

e-mail: `baer@arcade.mathematik.uni-freiburg.de`