Harmonic Inverse Mean Curvature Surfaces and Painlevé Equations*

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Abstract

In this paper we study surfaces immersed in $\mathbb{R}^3$ such that the mean curvature function H satisfies the equation $\Delta(1/H) = 0$, where Δ is the Laplace operator of the induced metric. We call them HIMC surfaces. All HIMC surfaces of revolution are classified in terms of the third Painlevé transcendent. In the general class of HIMC surfaces we distinguish a subclass of θ-isothermic surfaces, which is a generalization of the isothermic HIMC surfaces, and classify all the θ-isothermic HIMC surfaces in terms of the solutions of the fifth and sixth Painlevé transcendents.

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1 Introduction

In [1] a new class of surfaces was introduced, defined by the property

$$\Delta\left(\frac{1}{H}\right)=0, \tag{1}$$

where H is the mean curvature of the surface and Δ is the Laplace operator of the induced metric. We call these surfaces the harmonic inverse mean curvature surfaces (HIMC surfaces).

HIMC surfaces can be considered as a natural generalization of surfaces with constant mean curvature (CMC surfaces). The latter class has been intensively studied (see [2], [3], [4]) by the methods of the theory of integrable equations (the solition theory). The starting point of this theory is a representation of the nonlinear differential equation (in differential geometry these are the Gauss-Codazzi equations) in a form of compatibility condition (Lax or Zakharov-Shabat representation in the theory of solitons)

$$U_y(\lambda)-V_x(\lambda)+[U(\lambda),V(\lambda)]=0. \tag{2}$$

The parameter λ, which is called a spectral parameter in the theory of solitons, describes some special one-parametric deformation of surfaces, which is called an associated family in geometry. The representation (2) for HIMC surfaces was found in [1], where the associated families of HIMC surfaces were described as one-parametric conformal deformations preserving the ratio of the principle curvatures. The conformal factor is the modulus of a holomorphic function.
The mean curvature function of a HIMC surface is given by

$$H(w,\bar{w})=\frac{1}{w+\bar{w}},$$

where $w(z)$ is a holomorphic function of the complex coordinate z on the surface. We assume that w itself is a local complex coordinate. In this coordinate one can characterize HIMC surfaces via the solutions of the non-linear system of partial differential equations for the Hopf differential $Q\,(d\,w)^2$:

$$\begin{array}{rcl}\left(\dfrac{Q_{w\bar{w}}}{Q_{\bar{w}}}\right)_{\bar{w}}-Q_{\bar{w}} & = & \dfrac{1}{(w+\bar{w})^2}\left(2-\dfrac{|Q|^2}{\bar{Q}_w}\right), \\ Q_{\bar{w}} & = & \bar{Q}_w .\end{array} \tag{3}$$

Note, that system (3) is integrable from the point of view of the soliton theory. By the substitution $Q=f_w$ this system can be reduced to one equation for one real-valued function $f(w,\bar{w})$:

$$\Delta^2 f-\frac{|\nabla\Delta f|^2}{\Delta f}-(\Delta f)^2=\frac{4}{(w+\bar{w})^2}\left(2\,\Delta f-|\nabla f|^2\right),$$

where $|\nabla f|=2\,|f_w|$, $\Delta=4\,\partial_w\,\partial_{\bar{w}}$.
Another confirmation of the fact that HIMC surfaces are natural generalization of CMC surfaces was obtained in [5], where it was shown that only these two classes admit a Lie-point group of transformations of a certain type.

The integrability allows us to study HIMC surfaces by analytical methods of the soliton theory. By using one of these techniques, namely, the so-called dressing procedure, a Bäcklund transformation for HIMC surfaces was constructed in [6]. In spite of the mentioned analytical results no concrete examples of non-trivial (not CMC) HIMC surfaces are constructed up to nowadays. This is the problem we address in the present publication. We exclude the CMC case from our consideration and assume $H \neq const.$

Following [1] we use the quaternionic representation of surfaces in $\mathbb{R}^3$, which is explained in Section **2**.
The notion of the associated family and the corresponding Lax representation play an important role in our study. Therefore we recall the results concerning them in Section **3**.
Since there are no harmonic functions without zeros on a compact Riemann surface, there are no compact non-trivial HIMC immersions. The simplest case to study are the surfaces of revolution. Surfaces of revolution are isothermic, i.e. they allow conformal curvature line parametrizations. The duals of isothermic HIMC surfaces (for the definition of the dual isothermic surface see for example [1]) are the Bonnet surfaces, i.e. the surfaces admitting isometries preserving the mean curvature function. In [7] all Bonnet surfaces are described in terms of the third and sixth Painlevé transcendents. Hence, all isothermic HIMC surfaces can be also described in terms of the same Painlevé transcendents. In particular, in Section **4** we classify all HIMC surfaces of revolution in terms of the third Painlevé transcendent and employ its asymptotic properties to analyze their embeddedness.
Isothermic surfaces admit isothermic coordinates, i.e conformal coordinates in which $\mathrm{Im}(Q) = 0$. A generalization of the isothermic surfaces is suggested in Section **5**. In this section we consider the surfaces which allow a conformal parametrization, such that the Hopf differential has constant imaginary part

$$\mathrm{Im}(Q) = \theta. \tag{4}$$

We call these surfaces **θ-isothermic** and classify their associated families in terms of the solutions of special third order ordinary differential equation, which can be obtained by imposing (4) into (3). These equations can be viewed as generalizations of the Hazzidakis equations [8]. We find the following geometrical property of θ-isothermic HIMC surfaces: the associated family of each θ-isothermic HIMC surface acts on it just by scaling. We conjecture that this self-similarity property is a characterization of the θ-isothermic HIMC surfaces.
In Section **6** and **7** the general solutions of the generalized Hazzidakis equations corresponding to θ-isothermic HIMC surfaces in terms of the fifth and sixth Painlevé transcendents are obtained. These Painlevé equations are more general than those in the isothermic case. For instance, for $\mathrm{Im}(Q) \neq 0$ the fifth Painlevé equation can not be reduced to the third Painlevé equation as it is in the isothermic case. Note, that the solutions of the Painlevé equations are recognized as non-linear special functions [9], [10], and their properties are rather well-known. Therefore the θ-isothermic HIMC surfaces can be investigated as well as the isothermic ones.
In Section **8** we study the exceptional families of HIMC surfaces, which are not related with the solutions of the Painlevé equations. We call them Cartan cones, since Cartan studied these surfaces but from a different point of view: as the class of applicable Bonnet surfaces [11].
In the Appendix we establish the relation between the Hamiltonian functions for the Painlevé equations and the Hopf differentials of θ-isothermic HIMC surfaces.

2 Quaternionic description of surfaces in Euclidean 3-space

For analytical researches of surfaces in $\mathbb{R}^3$ it is convenient to describe them in terms of 2×2 matrices (for more details see [1]). In Sections **6** and **7** this description allows us to identify the equations for the moving frame of HIMC surfaces, which satisfy some special symmetry reduction, with the zero-curvature representation for the Painlevé equations.

Let $F : \mathcal{R} \to I\!R^3$ be a conformal parametrization of an orientable surface $\mathcal{F}$:

$$< F_z, F_z >=< F_{\bar z}, F_{\bar z} >= 0, \qquad < F_z, F_{\bar z} >= \frac{1}{2} e^u.$$

Here $\mathcal{R}$ is a Riemann surface with the induced complex structure,

$$< v, w >= v_1 w_1 + v_2 w_2 + v_3 w_3 \,,$$

z is a complex coordinate, and

$$F_z = \frac{\partial F}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) F \,, \qquad F_{\bar z} = \frac{\partial F}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) F.$$

The vectors F_z, $F_{\bar z}$, and N define a moving frame on the surface. The fundamental forms are as follows:

$$\begin{array}{rcl} < d\,F, d\,F > & = & e^u \, dz \, d\bar z, \\ - < d\,F, d\,N > & = & Q\, dz^2 + H\, e^u\, dz\, d\bar z + \bar Q\, d\bar z^2, \end{array}$$

where $Q =< F_{zz}, N >$ is the Hopf differential and H the mean curvature function on $\mathcal{F}$.

The compatibility conditions of the moving frame equations are:

$$\begin{array}{lrcl} \text{Gauss-equation} & u_{z\bar z} + \frac{H^2}{2} e^u - 2Q\bar Q e^{-u} & = & 0\,, \\ \text{1. Codazzi-equation} & Q_{\bar z} & = & \frac{H_z}{2} e^u\,, \\ \text{2. Codazzi-equation} & \bar Q_z & = & \frac{H_{\bar z}}{2} e^u\,. \end{array} \tag{5}$$

Let us denote the algebra of quaternions by $I\!H$ and the standard basis by $\{\mathbb{1}, \mathbb{i}, \mathbb{j}, \mathbb{k}\}$:

$$\mathbb{i}\,\mathbb{j} = -\mathbb{j}\,\mathbb{i}\,, \qquad \mathbb{j}\,\mathbb{k} = -\mathbb{k}\,\mathbb{j}\,, \qquad \mathbb{k}\,\mathbb{i} = -\mathbb{i}\,\mathbb{k}\,.$$

We will use the standard matrix representation of $I\!H$:

$$\mathbb{i} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad \mathbb{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbb{k} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \qquad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6}$$

We identify 3-dimensional Euclidean space with the space of imaginary quaternions

$$\mathbf{Im}\, I\!H = \mathrm{su}(2) = \mathrm{span}(\mathbb{i}, \mathbb{j}, \mathbb{k})$$

by

$$X = (x_1, x_2, x_3)^t = x_1 \mathbb{i} + x_2 \mathbb{j} + x_3\, \mathbb{k} \in su(2). \tag{7}$$

The scalar product of vectors in terms of matrices is then

$$< X, Y >= -\frac{1}{2} \mathrm{tr}(XY).$$

Let us take $\Phi \in \mathrm{SU}(2)$ which transforms the basis $\mathbb{i}$, $\mathbb{j}$, $\mathbb{k}$ into the frame F_x, F_y, N:

$$F_x = e^{u/2} \Phi^{-1} \mathbb{i} \Phi, \quad F_y = e^{u/2} \Phi^{-1} \mathbb{j} \Phi, \quad N = \Phi^{-1} \mathbb{k} \Phi$$

One can prove [1] that Φ satisfies the following linear system:

$$\Phi_z \Phi^{-1} = \begin{pmatrix} \frac{u_z}{4} & -Q e^{-u/2} \\ \frac{H}{2} e^{u/2} & -\frac{u_z}{4} \end{pmatrix} \qquad \Phi_{\bar z} \Phi^{-1} = \begin{pmatrix} -\frac{u_{\bar z}}{4} & -\frac{H}{2} e^{u/2} \\ \bar Q e^{-u/2} & \frac{u_{\bar z}}{4} \end{pmatrix}. \tag{8}$$

3 Associated Family of HIMC Surfaces

Since, to the best of our knowledge, the attention to HIMC surfaces were payed only in the recent works [1], [5], [6] not too many results about them are known. The most important properties can be summarized as follows:
1. Such surfaces are included in associated family with conformal equivalent induced metric.
2. There is a *Sym*-formula, which gives the surface as a left-translation of the tangent vector field on the SU(2)-representation of the moving frame of the associated family.

Let us start with the Lax-pair (8) for an arbitrary surface with the special condition

$$H(z,\bar z) = \frac{1}{h(z)+\bar h(\bar z)}, \qquad \text{where } h(z) \text{ is an arbitrary holomorphic function,} \tag{9}$$

and z is a local conformal coordinate. Note, that $h(z)$ is not supposed to be a local coordinate rather then w in (3).
The main results concerning HIMC surfaces are formulated in the following

Lemma 1 *Let $\mathcal{F}$ be a surface with local conformal parametrization $F(z,\bar z)$ such that condition (9) is valid. Denote e^u ($u = u(z,\bar z)$) its induced metric and $Q = Q(z,\bar z)$ its Hopf differential. There exists a one-parametric ($\tau \in I\!R$) family of surfaces $\mathcal{F}_\tau$ (called the* **associated family** *of $\mathcal{F} = \mathcal{F}_0$) which have the property (1). Each $\mathcal{F}_\tau$ is given locally by the conformal parametrization (**Sym-formula**)*

$$F(\tau,z,\bar z) = 2\,\Psi(\tau,z,\bar z)^{-1}\,\frac{\partial}{\partial\,\tau}\Psi(\tau,z,\bar z), \tag{10}$$

where $\Psi = \Psi(\tau,z,\bar z) \in \mathrm{SU}(2)$ solves the following system:

$$\Psi_z\Psi^{-1} = \begin{pmatrix} \dfrac{u_z}{4} & -Qe^{-u/2} \\ \Lambda(\tau,z,\bar z)\,\dfrac{H}{2}e^{u/2} & -\dfrac{u_z}{4} \end{pmatrix}, \quad \Psi_{\bar z}\Psi^{-1} = \begin{pmatrix} -\dfrac{u_{\bar z}}{4} & -\dfrac{H}{2\,\Lambda(\tau,z,\bar z)}e^{u/2} \\ \bar Q e^{-u/2} & \dfrac{u_{\bar z}}{4} \end{pmatrix}. \tag{11}$$

Here

$$\Lambda = \frac{1-\mathrm{i}\,\tau\,\bar h(\bar z)}{1+\mathrm{i}\,\tau\,h(z)} \tag{12}$$

and $F(z,\bar z) = F(0,z,\bar z)$. The fundamental functions of the associated family F_τ in parametrization $F(\tau,z,\bar z)$ are as follows:

$$\begin{aligned} e^{u_\tau(z,\bar z)} &= \frac{e^{u(z,\bar z)}}{|1+\mathrm{i}\,\tau\,h(z)|^4}, \\ Q_\tau(z,\bar z) &= \frac{Q(z,\bar z)}{(1+\mathrm{i}\,\tau\,h(z))^2}, \\ H(\tau,z,\bar z) &= H(z,\bar z)|1+\mathrm{i}\,\tau\,h(z)|^2. \end{aligned} \tag{13}$$

Proof : For the proof see [1].

4 All HIMC surfaces of revolution

The goal of this section is to derive explicit formulas for all analytical HIMC surfaces of revolution. The properties of analytical surfaces of revolution, which we need here, are summarized in

Lemma 2 *Let $\mathcal{F}$ be a surface of revolution different from a plane. Then there exist local coordinates x, y such that the parametrization*

$$F(x,y) = \frac{1}{a}\begin{pmatrix} e^{u(x)/2}\,\cos(a\,y) \\ e^{u(x)/2}\,\sin(a\,y) \\ c(x)\end{pmatrix}, \qquad a \in I\!R\setminus\{0\}, \tag{14}$$

of $\mathcal{F}$ is isothermic, i.e.

$$1 = \frac{1}{a^2}\left(\left(\frac{u'(x)}{2}\right)^2 + c'(x)^2\,e^{-u(x)}\right). \tag{15}$$

Moreover, the mean curvature function $H = H(x)$ is given by the following equation:

$$H(x) = \frac{4\,a^2 - u'(x)^2 - 2\,u''(x)}{8\,c'(x)} \tag{16}$$

and

$$c''(x) = H(x)\,u'(x)\,e^{u(x)}. \tag{17}$$

Proof : The equation (14) is just a curvature line parametrization of $\mathcal{F}$. The coordinates are isothermic iff they are conformal curvature line coordinates. The conformity condition for the above representations yields (15). Thus the isothermic coordinates $(\tilde{x},\tilde{y})$ read

$$\tilde{x}(x) = \frac{1}{a}\int_{x_0}^{x}\sqrt{\left(\frac{u'(\xi)}{2}\right)^2 + c'(\xi)^2\,e^{-u(\xi)}}\,d\,\xi, \quad \tilde{y} = y.$$

To derive (16) one uses the definition of the mean curvature in terms of the functions $u(x)$ and $c(x)$, the formula (15), and that c' does not identically vanish in some neighbourhood of x. Finally, differentiating (15) and using (16) we arrive at (17).

Corollary 1 *For non-CMC HIMC surfaces of revolution there exist isothermic coordinates (x,y), such that the mean curvature function is of the form*

$$H(x) = \frac{1}{x}.$$

Proof : According Lemma **2** the mean curvature is a function of x, $H = H(x)$. Since $H(x)$ is a solution of (1) and we are considering non-CMC surfaces $H(x) = 1/x$.

Theorem 1 *Let $\mathcal{F}$ be a HIMC surface of revolution with (14) as a local parametrization with $a = 2$. Then there exists a real-valued smooth function $\phi(x)$ such that*

$$e^{u(x)} = \frac{x^2\,(\phi'(x) + 2\,\sin(\phi(x)))^2}{4} \tag{18}$$

$$c(x) = -\frac{x^2}{4}\,(\phi'(x)^2 - 4\,\sin^2(\phi(x)). \tag{19}$$

Moreover, ϕ solves the Painlevé III equation (in trigonometric form),

$$x\,\left(\phi''(x) - 2\,\sin(2\,\phi(x)\right) + \phi'(x) + 2\,\sin(\phi(x)) = 0. \tag{20}$$

Inversely, let $\phi(x)$ be any arbitrary solution of (20) with $\phi'(x) + 2\,\sin(\phi(x) \not\equiv 0$. Then with $u(x)$ and $c(x)$ defined by (18) and (19) in (14) we define an isothermic parametrization of a HIMC surface of revolution.

Proof : Using (15) we can introduce a function ϕ defined as follows:

$$\sin(\phi(x)) = \frac{1}{2}\,c'(x)\,e^{-u(x)/2}, \qquad \cos(\phi(x)) = \frac{1}{4}\,u'(x). \tag{21}$$

Putting these equations into (16) we get (18). Taking the logarithm of (18), differentiating it with respect to x, and substituting the result into the second equation of (21), we find (20). Finally, integration of (17) gives us

$$c(x) = x\,c'(x) - e^{u(x)} + C_0,$$

where C_0 can be chosen to be zero. Inserting $e^{u(x)}$ (18) and $c'(x)$ (16), in which $u'(x)$ and $u''(x)$ are defined by the second equation of (21) we get (19).
To prove the inverse statement one should just substitute (18) and (19), in which $\phi(x)$ solves (20), into (15) and (16) and find that these equations are valid identically.

It is interesting to notice that the functions important for the geometrical description of HIMC surfaces of revolution can be presented in a lucid form by using a formulation of the Painlevé III equation as a time-dependent Hamiltonian system. Actually, by introducing the mechanical notations:

$$\tau \;\text{(time)} = x, \quad q \;\text{(coordinate)} = \phi, \quad p \;\text{(impulse)} = \frac{x}{2}\,(\phi'(x) + 2\sin(\phi(x))),$$

we arrive at the following

Proposition 1 *The Painlevé III equation (20) can be considered as the Hamiltonian system*

$$\dot{p} = \{\mathcal{H}, p\}, \qquad \dot{q} = \{\mathcal{H}, q\},$$

with the canonical Poisson structure

$$\{p,p\} = \{q,q\} = 0, \qquad \{p,q\} = 1,$$

and the time-dependent Hamiltonian

$$\mathcal{H}(\tau,p,q) = \frac{p^2}{\tau} - 2\,p\,\sin(q).$$

The immersion function in these notations reads as

$$F(x,y) = \frac{1}{2}\begin{pmatrix} p\,\cos(2\,y) \\ p\,\sin(2\,y) \\ -\tau\,\mathcal{H}(\tau,p,q)\end{pmatrix}.$$

The fundamental functions are given by

$$e^{u(x)} = p^2 = -\tau\,\mathcal{L}(\tau,p,q), \qquad H(x) = \frac{1}{\tau}, \qquad Q(x) = \frac{1}{2}\,\mathcal{H}(\tau,p,q),$$

where e^u denotes the induced metric, H is the mean curvature function, and Q is the Hopf differential. Moreover, $\mathcal{L}$ is the Lagrangian function defined via the Legendre transformation

$$\mathcal{L}(\tau, p, q) = \mathcal{H}(\tau, p, q) - p \frac{d}{d\,\tau} q.$$

Proof : The proof is just a direct verification.

Lemma 3 *Let ϕ be a solution of (20), then:*
(i) *If there is $x_0 \in (0, \infty)$ with*

$$\phi'(x_0) + 2\,\sin(\phi(x_0)) \neq 0 \tag{22}$$

then inequality (22) is valid for all $x \in I\!R^+$.
(ii) *The solution ϕ is a real analytical function for $x \in I\!R^+$.*

Proof : To prove (**i**) one should notice that any solution of

$$\phi'(x) + 2\,\sin(\phi(x)) = 0 \tag{23}$$

solves (20) as well. So, from the uniqueness theorem for solutions of ordinary differential equations the first part of the lemma follows. The statement (**ii**) is a result of the Painlevé property for $p(x) = e^{\mathrm{i}\,\phi(x)}$. The function $p(x)$ solves the Painlevé III equation in the canonical form and, therefore, its possible singularities for $x \neq 0, \infty$ are poles of the first order [12] (the Painlevé property). That is why the real solution (20), $\phi(x)$, an analytical function without singularities for $x \in I\!R^+$.

Definition 1 *The solutions of equation (20), which are different of that defined by (23), are said to be transcendental.*

Remark 1 *Actually, while all solutions of (23) are elementary functions defined by the equation*

$$\cos(\phi(x)) = \frac{1 - C\,e^{-4\,x}}{1 + C\,e^{-4\,x}}, \qquad or \qquad \phi(x) \equiv \pi\,k \quad k \in \mathbb{Z}, \tag{24}$$

one can show that the transcendental solutions, in the above sense, can not be "expressed in a certain way" in terms of elementary functions. See the work [13] for the mathematical details. So, from the Theorem **1***, we see that it is the Painlevé transcendent which is responsible for the geometry of HIMC surfaces of revolution.*

Corollary 2 *Any analytically immersed HIMC surface of revolution can be globally parametrized by*

$$F(x, y) = \frac{x\,(\phi'(x) + 2\,\sin(\phi(x)))}{8} \begin{pmatrix} 2\,\cos(2\,y) \\ 2\,\sin(2\,y) \\ -x\,(\phi'(x) - 2\,\sin(\phi(x))) \end{pmatrix}, \tag{25}$$

where $\phi(x)$ is a transcendental solution of (20).

Proof : The statement is a result of Lemma **2** and Theorem **1**.

Remark 2 *Since $H(x)$ is a function on the surface any such surface is singular for $x = 0$. Moreover, all the functions involved in (25) are analytical (on $\mathbb{R}^+$), and so are the surfaces defined in that way.*

For the further discussion of the geometry of HIMC surfaces of revolution we need some asymptotic results for the solutions of (20).

Lemma 4 *Solutions of (20) are transcendental, iff they are not holomorphic functions in a neighborhood of $x = 0$.*

Proof : Let $\phi_0(x)$ be a solution of (20) holomorphic in the neighborhood of $x = 0$. Substituting the corresponding Taylor expansion for $\phi_0(x)$ into (20) one gets that there exists only one one-parameter family of such solutions of (20). On the other hand a one-parameter family of (20) is given by (24). Comparing the Taylor expansions of $\phi_0(x)$ and $\phi(x)$, one finishes the proof.

Remark 3 *As follows from Theorem* **2** *(see below) non-transcendental solutions of (20) can be specified also by their asymptotic at the other singular point of equation (20), $x = \infty$, as*

$$\phi(x) \underset{x\to+\infty}{\longrightarrow} \pi\, k, \qquad k \in \mathbb{Z}.$$

Now we are ready to formulate

Theorem 2 *For each real transcendental solution of (20) there exist real numbers k_0, p_0, σ_0 and β_0, and $n \in \mathbb{Z}$, such that the following asymptotic expansions are valid:*

$$\begin{aligned}\pm\phi(x) \underset{x\to+\infty}{=} & \; 2\,\pi\, n + \frac{\pi}{2} + \frac{2\,k_0}{\sqrt{2\,x}}\,\sin(\varphi_\infty) - \frac{1}{2\,x} + \\ & \frac{1}{(2\,x)^{3/2}}\left(k_0^3\,(\sin(\varphi_\infty) + \frac{1}{6}\,\sin(3\,\varphi_\infty)) + \frac{3}{2}\,k_0\,(\frac{1}{2} + k_0^4)\,\cos(\varphi_\infty)\right) + O(\frac{1}{x^2}),\end{aligned} \tag{26}$$

$$\begin{aligned}\phi(x) \underset{x\to+0}{=} & \; \varphi_0 - \frac{2\,x}{1+\sigma_0^2}\,\sin(\varphi_0 + \kappa) + \\ & \left(\frac{x}{1+\sigma_0^2}\right)^2\left(-\sigma_0 + \sqrt{1 + \frac{\sigma_0^4}{4}}\,\sin(2\,\varphi_0 + 2\,\kappa + 2\,\kappa_2)\right) + O(x^3),\end{aligned} \tag{27}$$

where

$$\varphi_\infty = 2\,x - k_0^2\,\log(2\,x) + p_0, \qquad \varphi_0 = \sigma_0\,\log(x) + \beta_0,$$

and

$$\cos(\kappa) = \frac{1-\sigma_0^2}{1+\sigma_0^2}, \quad \sin(\kappa) = -\frac{2\,\sigma_0}{1+\sigma_0^2}, \quad \cos(2\,\kappa_2) = \frac{2-\sigma_0^2}{\sqrt{4+\sigma_0^4}}, \quad \sin(2\,\kappa_2) = \frac{2\sigma_0}{\sqrt{4+\sigma_0^4}}.$$

The asymptotic expansions (26) and (27) are differentiable, i.e. the asymptotic of $\phi'(x)$ are just the derivations of (26) and (27) <u>*but with the same error estimations*</u> *as in (26) and (27).*
Vice versa, an arbitrary pair of real numbers can be taken as the pair $(k_0,\, p_0)$ or $(\sigma_0,\, \beta_0)$, in the last case $\sigma_0 \neq 0$, to define, via asymptotic expansions (26) or correspondingly (27), a unique real transcendental solution of (20).

Proof : For the proof see the papers [14], [15], [16], [17], [18].

Remark 4 *Under the notation* $g(x) \underset{x\to a}{=} O(f(x))$ *we mean that* $|g(x)/f(x)| < C$ *for some* $C > 0$ *as* $x \to a$.

Since the case, when $k_0 = 0$ in equation (26) plays an important role in our classification of HIMC surfaces of revolution, we formulate the following

Theorem 3 *There exists a transcendental solution of (20),* $\phi(x)$ *, which has the asymptotic expansion*

$$\pm\phi(x) \underset{x\to+\infty}{=} 2\,\pi\,n + \frac{\pi}{2} - \frac{1}{2\,x} + \frac{5}{6\,(2\,x)^3} + \sum_{m=2}^{\infty} \frac{a_m}{(2\,x)^{2m+1}} + O(x^{-\infty}). \tag{28}$$

The coefficients a_m *are real numbers, which can be obtained recursively, by substitution of expansion (28) into (20).*

Proof : It is given in [18].

Remark 5 *Note, that the asymptotic expansion (28) does not contain any free parameter. According [18] there exist at least one (at most one one-parametric family of solutions, where the family parameter is possibly hidden in the term* $O(x^{-\infty})$*) up to the sign* $\pm$ *and the shift* $\phi \to \phi + 2\,\pi\,n$ *solution of (20) with the asymptotic expansion (28). The corresponding small* x *behavior for this solution(s) is given by (27) with*

$$\sigma_0 = \frac{1}{\pi}\,\log(3+\sqrt{8}) = 0.5610998.....$$

Our numerical calculations show that

$$\sigma_0 = 0.56108 \pm 0.00005, \qquad \beta_0 = 0.908 \pm 0.005.$$

Corollary 3 *The asymptotics of the immersion function is given by (14) (a = 2), where:*

$$e^{u/2} \underset{x\to+\infty}{=} x + k_0\,\sqrt{2\,x}\,\cos(\varphi_\infty) - k_0^2\,\sin^2(\varphi_\infty) + \tag{29}$$
$$\frac{1}{\sqrt{2\,x}}\left(\left(\frac{k_0}{8} - \frac{3}{4}\,k_0^5\right)\,\sin(\varphi_\infty) - \frac{k_0^3}{2}\,\cos(\varphi_\infty) + \frac{k_0^3}{4}\,\cos(3\,\varphi_\infty)\right) + O\left(\frac{1}{x}\right),$$

$$e^{u/2} \underset{x\to+0}{=} \frac{\sigma_0}{2} + \frac{\sigma_0\,x}{\sqrt{1+\sigma_0^2}}\,\cos(\varphi_0 + \frac{\kappa}{2}) + \tag{30}$$
$$\frac{\sigma_0\,x^2}{(1+\sigma_0^2)^2}\left(1 - \frac{\sigma_0\,\sqrt{1+\sigma_0^2}}{2}\,\sin(2\,\varphi_0 + \frac{3}{2}\,\kappa)\right) + O(x^3),$$

$$c(x) \underset{x\to+\infty}{=} x^2 - 2\,k_0^2\,x + k_0\,\sqrt{2\,x}\,\sin(\varphi_\infty) - \frac{1}{4} + \frac{k_0^4}{2} + \frac{k_0^2}{2}\,\sin(2\,\varphi_\infty) + O\left(\frac{1}{\sqrt{x}}\right), \tag{31}$$

$$c(x) \underset{x\to+0}{=} -\frac{\sigma_0^2}{4} + \frac{\sigma_0\,x}{\sqrt{1+\sigma_0^2}}\,\sin(\varphi_0 + \frac{\kappa}{2}) + \frac{\sigma_0^2\,(3+\sigma_0^2)\,x^2}{2\,(1+\sigma_0^2)^2} + \tag{32}$$
$$\frac{\sigma_0^2\,x^2}{2\,(1+\sigma_0^2)^{3/2}}\,\cos(2\,\varphi_0 + \frac{3}{2}\,\kappa) + O(x^3),$$

where the notations are the same as in Theorem **2** *and*

$$\cos(\frac{\kappa}{2}) = \frac{1}{\sqrt{1+\sigma_0^2}}, \quad \sin(\frac{\kappa}{2}) = -\frac{\sigma_0}{\sqrt{1+\sigma_0^2}}.$$

Proof : The formulas (29) to (32) are resulted from a substitution of (26) and (27) into (18) and (19) correspondingly.

Corollary 4 *For any HIMC surface of revolution there exists $x_0 > 0$, such that equation (25) for $0 < x < x_0$ and $\frac{1}{x_0} < x$ is an embedding.*

Proof : As it follows from (31) and Theorem **2** $c'(x)$ can not vanish for large x. This proves embeddedness of the HIMC surfaces of revolution for large x.
To prove the embeddedness for small x, consider the logarithmic spiral defined via the first two terms of (30) and (32). This spiral is an embedding. For $x \to +0$ the curve $(e^{u(x)/2}, c(x))$ can be interpreted as a deformation of the order $O(x^2)$ of this spiral. But it is easy to see, that such a deformation of the spiral can not destroy its property of embeddedness.

Lemma 5 *For arbitrary $x_0 > 0$ the meridian curve $(e^{u(x)/2}, c(x))$ for $x_0 < x < 1/x_0$ has a finite number of points of self-intersection.*

Proof : Suppose that the number of intersections is not finite. Then using the formula for the immersion (25) and the properties of the function ϕ stated in Lemma **3**, one proves that there exists a point $x^* \neq 0, \infty$, such that holds $\left(\frac{d}{dx}\right)^k \phi(x^*) = 0$ for any integer k. Now the real analyticity of ϕ implies $\phi(x) = \text{const}$ for all x.

Corollary 5 *Iff the point with the coordinates $(\sigma_0/2, -\sigma_0^2/4)$ does not belong to the meridian curve $(0 < x < \infty)$, then the latter has a finite number of self-intersections.*

Proof : The statement about the finite number of self-intersections ensue from Corollary **4** and Lemma **5**. If the point $(\sigma_0/2, -\sigma_0^2/4)$ belongs to the meridian curve, then the infinite number of self-intersections occur according the asymptotics given in Corollary **3**.

Proposition 2 *Intersection of every HIMC surface of revolution $\mathcal{F}$ in parametrization (25) with:*
1. P^+, the paraboloid of revolution (with respect to the same axis as for $\mathcal{F}$) of the curve $F_3 = 2\,F_1^2$, is an infinite set of circles (we exclude those HIMC surfaces of revolution, which correspond to the special solutions of Theorem **3** *and Remark* **5***). A point belongs to one of these circles, iff it corresponds to an extremum value ($\phi'(x) = 0$) for the related Painlevé function (20).*
2. P^-, the paraboloid of revolution of the curve $F_3 = -2\,F_1^2$, is an infinite set of circles. A point belongs to one of these circles, iff it corresponds to a solution of the equation $\sin(\phi(x)) = 0$.
3. the plane $F_3 = 0$ is an umbilic circle. All umbilic points of $\mathcal{F}$ belong to this circle.
A point belongs to the intersection of $\mathcal{F}$ with the paraboloids P^+ , P^-, iff the Gaussian curvature K of $\mathcal{F}$ vanishes there.
The part of $\mathcal{F}$ lying above the umbilic circle is embedded.

Proof : The Gaussian curvature of $\mathcal{F}$ parametrized by (25) is given by the following equation

$$K = \frac{8\,\sin(\phi(x))\,\phi'(x)}{x^2\,(\phi'(x) + 2\sin(\phi(x)))^2}.$$

To prove the assertions use the immersion formula (25) and the properties of the function $\phi(x)$, established in Lemma **3** and Proposition **1**.

Remark 6 *The analytical description of the global embeddedness of HIMC surfaces of revolution in terms of the solutions of (20) seams to be a complicated problem. Obviously, there are two possibilities. According our numerical investigations both of them are realized. Moreover the existence of HIMC surfaces, which meridian curves have infinitely many self-intersections, is quite probable. The results of our numerical simulations are presented on the Figure* **2**.

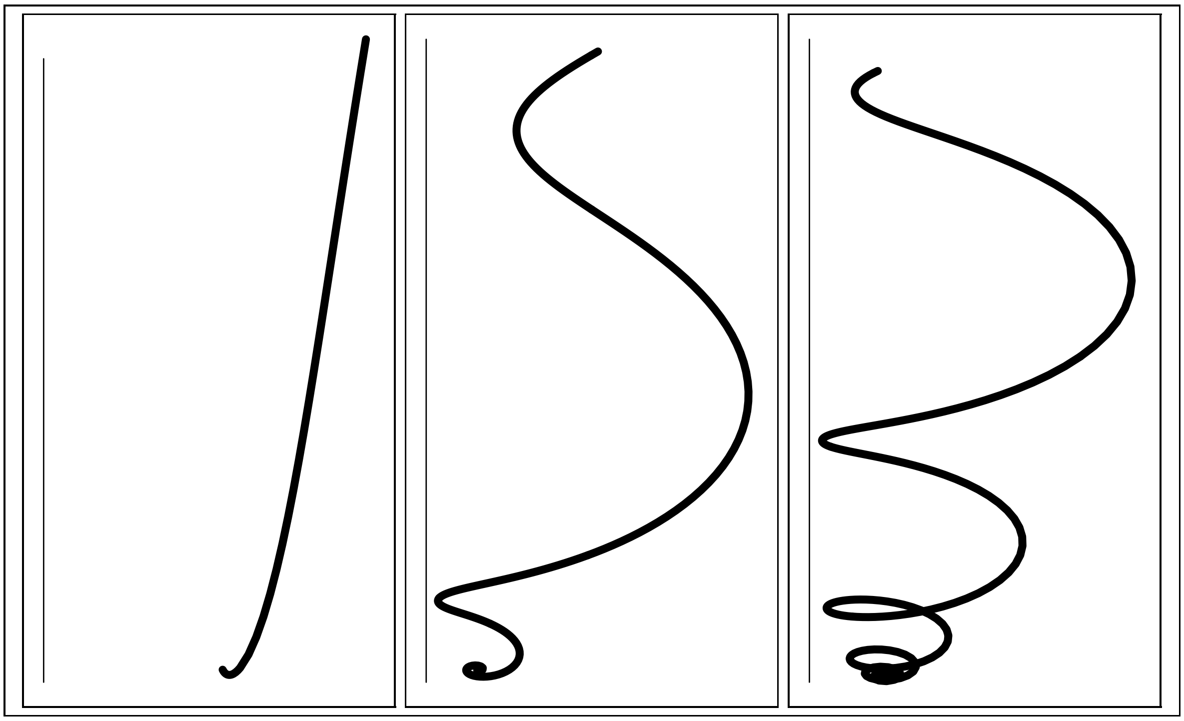

Figure 2: Types of Possible Meridian Curves With Their Axis of Rotation

The first plot in Figure **2** is the meridian curve of a HIMC surface of revolution described by the special solution of Theorem **3** and Remark **5**. The second plot is numerically generated by the solution of (20) with $\sigma_0 = 3.0$, $\beta_0 = -1.41$ and the third with $\sigma_0 = 10$, $\beta_0 = 0$. Evidently, the third curve describes a not embedded HIMC surface, while the first two should correspond to the embedded ones. Considering our asymptotic results and numerical simulations we come to the

Conjecture 1 *All HIMC surfaces of revolution for σ_0, β_0 with $|\sigma_0| > 4$ are not embedded, while for rather small modulus of σ_0 ($|\sigma_0| \leq 3$) all of them are embedded.*

5 Special Reductions to Ordinary Differential equations

Let us assume a (local) conformal parametrization of a surface with $\left(\frac{1}{H}\right)_{z\bar{z}} = 0$, which means (9). Now, by inserting (9) into the Codazzi-equations (5), we get

$$h_z(z)\,\bar{Q}_z(z,\bar{z}) = h_{\bar{z}}(\bar{z})\,Q_{\bar{z}}(z,\bar{z}).$$

Then solving the first Codazzi-equation with respect to u and putting it into the Gauss-equation we arrive at the following third order partial differential equation for Q:

$$h_z(z)\left(\frac{Q_{z\bar{z}}}{Q_{\bar{z}}}\right)_{\bar{z}} - Q_{\bar{z}} = \frac{|h_z(z)|^2}{(h(z)+\bar{h}(\bar{z}))^2}\left(2 - \frac{|Q|^2}{\bar{Q}_z}\right).$$

As long as $h'(z) \neq 0$ we can assume $w = h(z)$ as a proper coordinate. Thus,we obtain system (3). In the following part of this section we study some properties of (3).

Lemma 6 *There is no geometrical solution of (3) with the property that the right hand side (as well as the left hand side) vanishes identically.*

Proof : In our case the second Codazzi-equation reads

$$e^u = -2\,\bar{Q}_w\,(w+\bar{w})^2.$$

As a consequence one gets, that $\bar{Q}_w$ is strictly negative, which finishes the proof.

Lemma 7 *Let $Q(w,\bar{w})$ be any solution of (3). Then*

$$Q_\xi(z,\bar{z}) = \frac{1}{(\xi\,z-1)^2}\,Q(w(z),\bar{w}(\bar{z})), \quad \textit{where} \quad w(z) = -\frac{z}{\xi\,z-1} \quad \textit{and} \quad \xi \in \mathrm{i}\mathbb{R},$$

solves equation (3) but with z, $\bar{z}$ instead of w and $\bar{w}$.

Proof : The proof is either a direct calculation or the result of the transformation law for the Hopf-differential for changing of conformal parameter together with Lemma **1** .

Remark 7 *In the following it will be sometimes convenient for us to use w as the original local coordinate on the surface and $z = z(w)$ for its local reparametrization. We hope it will not confuse the reader.*

It seems to be a rather complicated problem, to study the general solution of system (3). Actually, in the previous section we studied some special real solutions of (3) for $w = z/2$. In general, we can formulate the following

Lemma 8 *Any isothermic HIMC surface is dual to a Bonnet surface.*

Proof : We recall, that for an isothermic surface $Q(w,\bar{w}) = f(w)\,q(w,\bar{w})$, where $q(w,\bar{w}) \in \mathbb{R}$ and $f(w)$ is holomorphic. If w is an isothermic coordinate, then $f(w) \equiv 1$. For this coordinate the fundamental functions of the dual surface (given with *) reads as follows:

$$e^{u^*(w,\bar{w})} = e^{-u(w,\bar{w})} \qquad H^*(w,\bar{w}) = q(w,\bar{w}) \qquad Q^*(w,\bar{w}) = \frac{1}{2}\,H(w,\bar{w}). \tag{33}$$

So from the last equation, we get that the Hopf-differential of the dual surface is real and $(1/Q^*)_{w\bar{w}} = 0$, which is an equivalent characterization for Bonnet surfaces as was found first by Graustein [19].

Corollary 6 *All isothermic HIMC surfaces are describable in terms of Painlevé III and VI equations with the same set of coefficients as for Bonnet surfaces.*

Proof : As it follows from (33) any solution for H^* generates a solution for Q and vice versa.

The next natural step to continue the studying of system (3) is to consider the following generalization of the isothermic property (given in the proof of Lemma **8**). In any conformal parametrization the Hopf differential reads as follows:

$$Q(w,\bar{w}) = f(w)\,(q(w,\bar{w}) + \mathrm{i}\,\theta), \tag{34}$$

where $f(w)$ is holomorphic, $q(w,\bar{w})$ a real-valued function, and θ a real constant. It is convenient for us to make the following

Definition 2 *A HIMC surface is said to be θ-isothermic, if there exists a local conformal coordinate z such that the Hopf differential of the surface is of the form (34).*

Remark 8 *Note, that of course θ is not an invariant of the surface, as it is dependent on f and on the choice of the conformal coordinate. If $\theta \neq 0$ then it can always be put equal to 1. But we find convenient in our study to keep θ as a free parameter.*

Proposition 3 *A θ-isothermic surface with $\theta \neq 0$ is isothermic, iff it is a Bonnet or CMC surface.*

Proof : If a θ-isothermic surface with $\theta \neq 0$ is isothermic then the following equation holds:

$$\tilde{f}(z)\,\tilde{q}(z,\bar{z}) = f(z)\,(q(z,\bar{z}) + \mathrm{i}\,\theta), \tag{35}$$

where $\tilde{f}(z)$, $f(z)$ are holomorphic functions and $\tilde{q}(z,\bar{z})$, $q(z,\bar{z})$ are real-valued functions. Putting $h(z) = -\dfrac{\mathrm{i}\,\tilde{f}(z)}{2\,\theta\,f(z)}$ one finds that

$$\tilde{q}(z,\bar{z}) = \frac{1}{h(z) + \bar{h}(\bar{z})}, \tag{36}$$

which means that for $h(z) \equiv const$ the surface is a CMC surface and otherwise it is a Bonnet surface [19].
Inversely, if the surface is a CMC or a Bonnet surface, then for any conformal coordinate z $Q = \tilde{f}(z)\,\tilde{q}(z,\bar{z})$, where $\tilde{q}(z,\bar{z})$ is of the form (36). Define

$$q(z,\bar{z}) = -\theta\,\frac{\mathrm{Im}(h(z))}{\mathrm{Re}(h(z))}, \quad f(z) = \frac{\tilde{f}(z)}{2\,\theta\,\mathrm{i}\,h(z)},$$

to find that (35) holds.

Lemma 9 *Let Q be given by (34). With respect to the new variable*

$$z = \int f(w)\, dw \tag{37}$$

equation (3) reduces to the ordinary differential equation

$$\left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = s(t)\left(2 - \frac{q^2(t)+\theta^2}{q'(t)}\right), \tag{38}$$

where $t = z + \bar{z}$ and $s(t)$ is a real analytical function of $t \in I\!R \setminus D$, where D is a discrete set.

Proof : Let us insert the ansatz (34) into the second equation of (3), then we get

$$\bar{f}(\bar{w})\, q_w(w,\bar{w}) = f(w)\, q_{\bar{w}}(w,\bar{w}),$$

which implies, that for z defined in (37) we have $q_z = q_{\bar{z}}$. As a consequence of it, we get that $q(z,\bar{z}) \equiv q(t)$. Inserting it into the first equation of (3) and using Lemma **6** we see that we can define the function

$$s(t) = \frac{1}{|f \circ w(z)|^2\,(w(z)+\bar{w}(\bar{z}))^2}, \tag{39}$$

which has the properties stated in the lemma.

Using the coordinate (37), we can rewrite the fundamental functions as follows:

$$\begin{aligned} Q(z,\bar{z}) &= \frac{q(t)+\mathrm{i}\,\theta}{f \circ w(z)}, \\ e^{u(z,\bar{z})} &= -2\,q'(t)\,(w(z)+\bar{w}(\bar{z}))^2, \\ H(z,\bar{z}) &= \frac{1}{w(z)+\bar{w}(\bar{z})}. \end{aligned} \tag{40}$$

Corollary 7 *Let $f(w)$ be as in the Lemma* **9***, then*

$$f(w) = \frac{1}{a\,w^2 + \mathrm{i}\,b\,w + c} \tag{41}$$

with $a, b, c \in I\!R$.

Proof : Denote $g(w) = 1/f(w)$, then differentiate $s(t)$ (39) with respect to z and then to $\bar{z}$ to get $s_z(t) = s_{\bar{z}}(t)$, which, by means of (37), is equivalent to

$$2\,g(w) - g_w(w)\,(w+\bar{w}) = 2\,\bar{g}(\bar{w}) - \bar{g}_{\bar{w}}(\bar{w})\,(w+\bar{w}). \tag{42}$$

Differentiation of (42) with respect to w and $\bar{w}$ gives us

$$g_{ww}(w) = \bar{g}_{\bar{w}\bar{w}}(\bar{w}) \equiv 2\,a \in I\!R. \tag{43}$$

The solution of (43) is a polynomial of second degree. Substitution of it into (42) yields the denominator of (41).

Lemma 10 *Let $\mathcal{F}$ be a θ-isothermic HIMC surface, then, up to scaling and analytical reparametrization, its fundamental functions are given by (40), where $f(w)$ and $w(z)$ is of one of the following forms:*

$$
\begin{array}{llllllll}
\mathbf{A}) & f(w) & = & \dfrac{1}{4\,\mathrm{i}\,w}, & w(\tilde z) & = & -\mathrm{i}\,e^{4\,\mathrm{i}\,\tilde z}, & \tilde z = z + \dfrac{\pi}{8}, \\
\mathbf{B}) & f(w) & = & \dfrac{1}{-w^2+4}, & w(z) & = & 2\coth(2\,z), & \\
\mathbf{C}) & f(w) & = & 2, & w(z) & = & \dfrac{1}{2}\,z, & \\
\mathbf{D}) & f(w) & = & \dfrac{1}{2\,w^2}, & w(z) & = & -\dfrac{1}{2\,z}, & \\
\mathbf{E}) & f(w) & = & \dfrac{1}{w^2+4}, & w(z) & = & -2\cot(2\,z). &
\end{array}
$$

Proof : The result of this lemma is a careful handling of $f(w)$ given in (41) by making use of shifting, scaling of w, and scaling of the surface as a whole. To get the corresponding formulas for $w(z)$ one should just invert the result of integration of $f(w)$ (37). This integration defines z.

Remark 9 *It will be convenient for us, to introduce the new variable $\tilde z$ in the* **A** *case and denote it further z.*

Corollary 8 *There are five 3-real parameters sets of associated families of θ-isothermic HIMC surfaces. Their fundamental and immersion functions are given by the equations (10) - (13) in Lemma* **1***, in which one should substitute $w(z)$ for $h(z)$ and (40).*

Proof : Actually, we can choose five different pairs for $w(z)$ and $f(w)$ involved in the equation for the immersion function (see formulas (10) and (11)) for the associated families. Moreover, these equations are parametrized via a general solution of the third order ordinary differential equation (38).

Lemma 11 $\mathbf{A} \equiv \mathbf{E}$*,* $\mathbf{C} \equiv \mathbf{D}$*.*

Proof : Calculating the fundamental functions for the cases **A** and **C** (correspondingly for **D** and **E**) we get, that these functions are the same (up to scaling), if one assumes the following relationships between the family parameters ($\tau := \tau_A,\ \tau_B, \tau_C \tau_D, \tau_E$, see Lemma **1**):

$$\tau_A = \frac{2\,\tau_E + 1}{2\,\tau_E - 1}, \qquad \tau_C = \frac{4}{\tau_D}. \tag{44}$$

Each surface of a family of the **A** set (of the set **C**) corresponds to a surface of the set **E** (of the set **D**) as follows:

$$F_A(\tau_A) = \frac{(1 - 2\,\tau_E)^2}{4}\,F_E(\tau_E), \qquad -\tau_D^2\,F_C(\tau_C) = F_D(\tau_D),$$

where F_ν denotes a parametrization of a surface in the set ν, $\nu = \mathbf{A}, \mathbf{C}, \mathbf{D}, \mathbf{E}$.

Remark 10 *Actually, we see that in each family from the sets* **E** *and* **C** *one surface is not included in the corresponding family from the sets* **A** *and* **D**. *Since we see that equations (44) have singular points at* $\tau_E = 1/2$ *and* $\tau_C = 0$. *To include these special surfaces of the* **E**- *and* **C**-*families into the* **A**- *and* **D**-*families, we have to enlarge the domain of* τ *by adding to it the point* $\tau = \infty$. *So, in the Lemma* **11**, *one should understand the family parameter* τ *as an element of* $\mathbb{R}P^1$.

Now, we are ready to formulate the main result of this section.

Theorem 4 *There exist only three sets (namely:* **A**, **B**, *and* **C**) *of associated families of* θ-*isothermic HIMC surfaces. The immersion function of each family is given by the equations (10) and (11), and the fundamental functions are presented in Table* **1**, *where* $q(t)$ *is an arbitrary real solution of (38) with* $s(t)$ *specified in Table* **1** *and* $q'(t) < 0$.

Proof : The first statement follows from the Lemmas **10** and **11**. The statement about the immersion function is the result of the Corollary **8**. The fundamental functions given in Table **1** were calculated via the inserting of $f(w)$ and $w(z)$, presented in Lemma **10**, into the formulas (40) and using Lemma **1**. To compute $s(t)$ one substitutes the results of Lemma **10** into (39). The properties of $q(t)$ are consequences of Lemma **9** and the positiveness of the metrics (see Table **1**).

Remark 11 *The equation (38), with* $s(t)$ *given by in Table* **1**, *is a generalization of the* Hazzidakis *equation [8].*

	A	B	C
$e^{u(\tau,z,\bar z)}$	$-\dfrac{q'(t)\,\sin^2(2t)}{2\,\tau^2\,\lvert\sin(2(z-\zeta))\rvert^4}$	$-\dfrac{8\,q'(t)\,\sinh^2(2t)}{(1+4\tau^2)^2\lvert\sinh(2(z-\zeta))\rvert^4}$	$-\dfrac{q'(t)\,t^2\,\zeta^4}{2\lvert z-\zeta\rvert^4}$
$Q(\tau,z,\bar z)$	$\dfrac{(q(t)+\mathrm{i}\,\theta)}{\tau\,\sin^2(2(z-\zeta))}$	$\dfrac{4(q(t)+\mathrm{i}\,\theta)}{(1+4\tau^2)\,\sinh^2(2(z-\zeta))}$	$\dfrac{\zeta^2(q(t)+\mathrm{i}\,\theta)}{2(z-\zeta)^2}$
$H(\tau,z,\bar z)$	$-\dfrac{2\,\tau\,\lvert\sin(2(z-\zeta))\rvert^2}{\sin(2\,t)}$	$-\dfrac{(1+4\tau^2)\lvert\sinh(2(z-\zeta))\rvert^2}{2\,\sinh(2\,t)}$	$-\dfrac{2\lvert z-\zeta\rvert^2}{\zeta^2\,t}$
$\zeta=\zeta(\tau)$	$\dfrac{\mathrm{i}}{2}\log\left(\sqrt{-\tau}\right)$	$-\dfrac{1}{2}\,\log\left(\sqrt{\dfrac{1+2\,\mathrm{i}\,\tau}{1-2\,\mathrm{i}\,\tau}}\right)$	$-\dfrac{2\,\mathrm{i}}{\tau}$
$s(t)$	$\dfrac{4}{\sin^2(2\,t)}$	$\dfrac{4}{\sinh^2(2\,t)}$	$\dfrac{1}{t^2}$
$h(z)$	$-\mathrm{i}\,e^{4\,\mathrm{i}\,z}$	$2\,\coth(2\,z)$	$\dfrac{z}{2}$

Table 1: Table of Fundamental Functions of the Associated Family

Theorem 5 *Up to scaling any associated family of the set* **A** *contains 4 different surfaces, any family of the set* **B** *contains 1 surface and any family of the set* **C** *contains 2 different surfaces. The corresponding associated families are described by scaling of these surfaces.*

Proof : To define the different (up to scaling) surfaces in each family one have to consider the following values of the family parameter τ:

$$\tau_A = -1, 0, 1, \infty, \qquad \tau_B = 0, \qquad \tau_C = 0, \infty.$$

It is easy to see from Table **1**, that these surfaces generate via scaling the whole corresponding associated family.

Remark 12 *In the cases* **A** *and* **C** *one needs to consider also the asymptotic limits as $\tau \to 0, \infty$ of the fundamental functions to cover the whole family. To handle the limits $\tau \to 0, \infty$ in the case* **A**, *one has to use the* Euler *formula for* $\sin(.)$, *the formula for $\zeta(\tau)$ given in the Table* **1**, *and, for the case $\tau \to \infty$, to scale the surface by a factor of the order $O(\tau^2)$.*

Remark 13 *The surfaces, which belong to the associated* **A**-, **B**- ,**C**-*families of Theorem* **4** *in the isothermic case ($\theta = 0$) are dual to the* **A**-, **B**-, **C**-*families of Bonnet surfaces, introduced first by E. Cartan [11].*

Remark 14 *From the viewpoint of the above classification the HIMC surfaces of revolution are included in case* **C** *for $\theta = \tau = 0$, because they are included in* **D**-*families for $1/\theta = \tau = \infty$ and according Lemma* **11** *and Remarks* **10**, **12** *these* **C** *and* **D** *cases are equivalent. Note, that the surfaces of revolution do not cover the whole* **C** *case for $\theta = \tau = 0$. To get the whole* **C** *case for $\theta = \tau = 0$, we discuss a special one-parametric family of surfaces in Section* **8**. *These surfaces are yielded by the explicit solution $q(t) = \frac{\alpha}{t}$, where $\alpha > 0$ is a parameter, of equation (38) with $s(t)$ as in the* **C** *case. To get the whole* **C** *set, one has to substitute the fundamental functions of the surfaces of revolution into (13) and integrate the moving frame equations (11). According to the Theorem* **4**, *we describe the whole* **C** *set for $\theta = 0$ or, equivalently, (see Lemma* **11***) the* **D** *set for $\theta = 0$.*

We finish this section by the following

Conjecture 2 *Iff the scaling of a HIMC surface generates the associated family, then this surface is θ-isothermic.*

6 A- and B-Families and the Painlevé VI Equation

In this section we solve the following generalization of the Hazzidakis equation (see Remark **11**):

$$\left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = \frac{4}{\sin^2(2t)}\left(2 - \frac{q^2(t)+\theta^2}{q'(t)}\right), \qquad t = z + \bar{z}, \tag{45}$$

in terms of the sixth Painlevé equation [12]:

$$\begin{aligned} y''(s) \;&=\; \frac{1}{2}\left(\frac{1}{y(s)} + \frac{1}{y(s)-1} + \frac{1}{y(s)-s}\right) y'^2(s) - \left(\frac{1}{s} + \frac{1}{s-1} + \frac{1}{y(s)-s}\right) y'(s) + \\ &\frac{y(s)(y(s)-1)(y(s)-s)}{s^2(s-1)^2}\left(\alpha + \beta\,\frac{s}{y^2(s)} + \gamma\,\frac{(s-1)}{(y(s)-1)^2} + \delta\,\frac{s(s-1)}{(y(s)-s)^2}\right), \end{aligned} \tag{46}$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$.

Lemma 12 *The equation (45) has the first integral*

$$2\left(\frac{q''}{4\,q'}+\cot(2\,t)\right)^2-\frac{1}{q'}\left(\theta^2\left(\cot^2(2\,t)+1\right)+q^2+\left(q\,\cot(2t)+\frac{q'}{2}\right)^2\right)=\mu. \tag{47}$$

Remark 15 *One can easily prove Lemma* **12** *by checking, that the derivative of the left hand side of (47) is a product of the difference between the left and right hand sides of (45) with a factor, which is finite for the solutions of (45). But we choose another way to prove this lemma, since we need later the notations, which we introduce in its proof, and we want to explain, how to get this first integral via the Lax representation for the differential equation (45).*

Proof of Lemma **12**: Let us introduce the similarity variables

$$s=e^{4\,\mathrm{i}\,t},\quad \lambda=-\tau\,e^{4\,\mathrm{i}\,z},\quad \Psi(\lambda,s)=\Psi(\tau,z,\bar z). \tag{48}$$

Inserting them into the moving frame equations (11), in which the fundamental functions are taken from Table **1** for $\tau=-1$, we get the following system:

$$\begin{aligned}\Psi_\lambda\,\Psi^{-1} &= \frac{1}{\lambda}\,A_0(s)+\frac{1}{\lambda-1}\,A_1(s)+\frac{1}{\lambda-s}\,A_s(s),\\ \Psi_s\,\Psi^{-1} &= \frac{1}{s-\lambda}\,A_s(s)+B(s),\end{aligned} \tag{49}$$

where

$$A_0(s)=\begin{pmatrix}a(s) & \varphi(s)\\ -\varphi^*(s) & -a(s)\end{pmatrix},\quad A_1(s)=-\frac{\sigma(s)}{s}\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad A_s(s)=\sigma(s)\begin{pmatrix}0&1\\0&0\end{pmatrix},$$

$$B(s)=\begin{pmatrix}-\dfrac{a(s)}{2\,s}+\dfrac{1}{4\,s} & -\dfrac{\sigma(s)}{(s-1)}\\[2ex] \dfrac{\varphi^*(s)}{s}+\dfrac{\sigma(s)}{s\,(s-1)} & \dfrac{a(s)}{2\,s}-\dfrac{1}{4\,s}\end{pmatrix},$$

and

$$\begin{aligned}a(s) &= -\frac{\mathrm{i}}{2}\left(\frac{q''(t)}{4\,q'(t)}+\cot(2\,t)\right), & \varphi(s) &= -\frac{\mathrm{i}\sqrt{-2\,q'(t)}}{8}+\frac{\mathrm{i}\,e^{2\,\mathrm{i}\,t}(q(t)+\mathrm{i}\,\theta)}{2\,\sin(2\,t)\sqrt{-2\,q'(t)}},\\ \sigma(s) &= \frac{e^{2\,\mathrm{i}\,t}}{4}\,\sin(2\,t)\sqrt{-2\,q'(t)}, & \varphi^*(s) &= \frac{\mathrm{i}\sqrt{-2\,q'(t)}}{8}-\frac{\mathrm{i}\,e^{-2\,\mathrm{i}\,t}(q(t)-\mathrm{i}\,\theta)}{2\,\sin(2\,t)\sqrt{-2\,q'(t)}}.\end{aligned} \tag{50}$$

Define the matrix $A_\infty=-A_0-A_1-A_s$. Denote the eigenvalues of A_ν as $\pm\theta_\nu/2$ ($\nu=0,1,s,\infty$). The compatibility condition for the system (49) yields that the θ_ν's are constants. Since $\det(A_\nu)=-\theta_\nu^2/4$, we get $\theta_1=\theta_s=0$ and

$$\left(\frac{\theta_0}{2}\right)^2=a(s)^2-\varphi(s)\,\varphi^*(s),\quad \left(\frac{\theta_\infty}{2}\right)^2=a(s)^2-(\varphi(s)+\sigma(s))(\varphi^*(s)+\frac{\sigma(s)}{s}). \tag{51}$$

Putting (50) into (51) one finds the first integral (47), where

$$\mu=-\theta_0^2-\theta_\infty^2,\qquad \theta=\theta_\infty^2-\theta_0^2.$$

Remark 16 *Although we start from the geometrical situation, in which $q'(t) < 0$, and get the Lax representation for (45) by using the moving frame equations, this representation (49) is valid for general complex solutions of (45). For the most part of this section we consider the relation between the general solutions of (45) and the Painlevé VI functions, since we were not able to find a suitable description of geometrical solutions in terms of Painlevé VI functions.*

Lemma 13 *Let $q(t)$ be a solution of (45) with the first integral (47) and $\theta_\infty = \sqrt{(\theta-\mu)/2} \neq 0$. Then*

$$y(s) = 1 + \frac{(s-1)}{1 + s^2 \left(\dfrac{a - \theta_\infty/2}{\sigma + \varphi}\right)^2}, \tag{52}$$

where $a(s)$, $\varphi(s)$ and $\sigma(s)$ are defined in (50), solves the Painlevé VI equation (46) with

$$\alpha = \frac{1}{2}(\theta_\infty - 1)^2, \quad \beta = -\frac{\mu+\theta}{4}, \quad \gamma = 0, \quad \delta = \frac{1}{2}. \tag{53}$$

If $\theta \in I\!R$ and $q'(t) < 0$, then $\mu > max(0, \theta)$.
Inversely, let $y(s)$ be an arbitrary solution of (46) with

$$\alpha, \beta \in \mathbb{C}, \quad \gamma = 0, \quad \delta = \frac{1}{2}. \tag{54}$$

Define

$$\theta_\infty = (1 + \sqrt{2\,\alpha}), \quad \theta = (1+\sqrt{2\,\alpha})^2 + 2\,\beta, \quad \mu = 2\,\beta - (1 + \sqrt{2\,\alpha})^2, \tag{55}$$

where $\sqrt{2\,\alpha} = 1$, if $\alpha = 1/2$. Then

$$q(t) = \mathrm{i}\,\theta\,\frac{s\,(y-1) + y - s}{y\,(s-1)} + 2\,\mathrm{i}\,\frac{\theta_\infty^2\,(y-s)^2\,(y-1)^2 - (s(s-1)y' - y(y-1))^2}{(s-1)y(y-1)(y-s)}, \tag{56}$$

with $s = e^{4\,\mathrm{i}\,t}$, is a solution of (45), with the first integral (47). Furthermore, if $\mathrm{i}\,(1+\sqrt{2\,\alpha}) \in I\!R$, $\beta \geq 0$, and

$$q'(t) = -8\,\frac{(\theta_\infty(y-s)(y-1) - y(y-1) + s(s-1)y')^2}{(s-1)^2(y-1)(y-s)} < 0, \tag{57}$$

then according Theorem **4**, *$q(t)$ defines an* **A**-*family of HIMC surfaces, which fundamental functions are given in Table* **1**.

Proof : Since $\theta_\infty \neq 0$, one proves that $a(s) \pm \theta_\infty/2 \neq 0$. Thus by defining $\Phi = \Omega\Psi$, where

$$\Omega = \mathrm{i}\sqrt{\frac{a}{\theta_\infty} + \frac{1}{2}}\begin{pmatrix} \dfrac{\sigma + s\,\varphi^*}{s\,(a + \theta_\infty/2)} & 1 \\ 1 & \dfrac{\sigma+\varphi}{a + \theta_\infty/2} \end{pmatrix},$$

one finds that the function Φ solves the system of the form (49), but with other matrices $A_\nu \to \tilde{A}_\nu$ $(\nu, 0, 1, s)$ and $B \to \tilde{B}$, where $\tilde{B}$ is a traceless diagonal matrix. In particular, for the matrix elements we find:

$$\frac{\tilde{A}_1^{12}}{\tilde{A}_s^{12}} = \frac{1}{s}\left(\frac{\sigma+\varphi}{a - \theta_\infty/2}\right)^2, \quad \tilde{A}_0^{11} = \frac{\theta}{4\,\theta_\infty} - \frac{\theta_\infty}{2} - \frac{\sigma^2}{s\,\theta_\infty}, \tag{58}$$

$$\tilde{A}_1^{11} = -\frac{\theta}{4\,\theta_\infty} - \frac{\sigma\,\varphi^*}{\theta_\infty}\ , \quad \tilde{A}_s^{11} = -\frac{\theta}{4\theta_\infty} - \frac{\sigma}{s\,\theta_\infty\,\varphi}. \tag{59}$$

Now, to prove (52) and (53) we use the parametrization of $\tilde{A}_\nu^{ik}$ in terms of Painlevé VI function presented in [20]. If $\theta \in I\!\!R$ and $q'(t) < 0$, then $\varphi^* = \bar{\varphi}$ and a is pure imaginary. Therefore (51) yields $\theta_0, \theta_\infty \in \mathrm{i}I\!\!R$, which proves that $\mu > max(0, \theta)$.
Let $y(s)$ be a solution of (46) with coefficients given by (54). Then we define the parameters $\theta_\infty \neq 0$, μ and θ according (55). Since $\theta_\infty \neq 0$, we can use the parametrization of the system (49) ($A_\nu \to \tilde{A}_\nu$) in terms of Painlevé VI functions [20]. We substitute this parametrization into the left hand side of (59) and equations (50) into the right hand side for $\sigma\,\varphi^*$ and σ/φ to get a linear algebraic system for $q(t)$ and $q'(t)$. Solving this system we arrive at the formulas for $q(t)$ (56) and $q'(t)$ (57). Differentiating (56) and applying to it (46) we prove the correctness of this definition. Now using the compatibility conditions for (49) in terms of a, σ, φ, φ^* we get the formula for $a(s)$ (50) and that the function $q(t)$ satisfies (45).

Theorem 6 *If $q(t) \neq \pm\theta\,\cot(2\,t)$ is a solution of (45), then it satifies (47). The function $y(s)$ given by (52) with $\theta_\infty = \sqrt{(\theta-\mu)/2}$ solves the Painlevé VI equation (46) with the coefficients (53). Moreover, if $\theta \in I\!\!R$ and $q'(t) < 0$, then $\mu > max(0, \theta)$.*
Inversely, if $y(s)$ is a solution of the Painlevé VI equation (46) with the coefficients (54), then $q(t)$ given by (56) and (55) is a solution of (45), with the first integral (47).Furthermore, if $\mathrm{i}\,(1+\sqrt{2\,\alpha}) \in I\!\!R$, $\beta \geq 0$, and the inequality (57) is valid, then $q(t)$ defines an **A**-*family of HIMC surfaces according Theorem* **4**.

Proof : Since the case $\theta_\infty \neq 0$ is considered in Lemma **13**, we must handle only the case $\theta_\infty = 0$. If $\theta_\infty = 0$ and $q(t) \neq \theta\,\cot(2\,t)$, then the formulas (52) and (56) are well-defined for the solutions of (45) and (46) respectively. One proves the statement of the theorem in this case by a direct calculation.

Remark 17 *The function $q(t) = \theta\,\cot(2\,t)$ is a solution of (45). For $\theta > 0$ it is a geometrical solution, which is investigated in Section* **8**.

In the rest of this section we consider the solution of another generalization of the Hazzidakis equation (38) (case **B**)

$$\left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = \frac{4}{\sinh^2(2\,t)}\left(2 - \frac{q^2(t)+\theta^2}{q'(t)}\right), \qquad \text{with} \qquad t = w + \bar{w}, \tag{60}$$

in terms of Painlevé VI equations. Since, as well as in the **A** case, we did not find a proper description of the geometrical solution in terms of Painlevé VI functions, we present here only a result for the general complex solutions of (60). An explicit formula for the case **B** can be obtained as a direct corollary of Theorem **6** and the following

Lemma 14 *Let $q_A(t_A)$ be an arbitrary solution of (45) with $\theta = \theta_A \in \mathbb{C}$. Then*

$$q_B(t_B) = -\mathrm{i}\,q_A(t_A) \qquad \textit{where} \qquad t_B = \mathrm{i}\,t_A \tag{61}$$

is a solution of (60) with $\theta = \pm\mathrm{i}\,\theta_A$.

Proof : The proof is a straightforward verification.

Remark 18 *One finds an explicit formula for $q_B(t_B)$ in terms of Painlevé VI function by substituting $\theta \to \mathrm{i}\,\theta$, $q(t) \to \mathrm{i}\,q(t)$, and $s \to e^{4t}$ into (56). This formula yields all solutions of (60) except $q(t) = \pm\,\mathrm{i}\,\theta\,\coth(2\,t)$.*

7 C-Families and the Painlevé V Equation

In this section we are considering the solution of the following generalization of the Hazzidakis equation (see Remark **11**):

$$\left(\frac{q''(t)}{q'(t)}\right)' - q'(t) = \frac{1}{t^2}\left(2 - \frac{q^2(t)+\theta^2}{q'(t)}\right) \tag{62}$$

in terms of the fifth Painlevé equation [12]:

$$y'' = \left(\frac{1}{2\,y} + \frac{1}{y-1}\right) y'^2 - \frac{y'}{t} + \frac{(y-1)^2}{t^2}\left(\alpha\, y + \beta\,\frac{1}{y}\right) + \gamma\,\frac{y}{t} + \delta\,\frac{y\,(y+1)}{y-1}, \tag{63}$$

where $\alpha, \beta, \gamma, \delta \in I\!R$. Let us start with the following

Lemma 15 *The equation (62) has the first integral*

$$\left(\frac{q''(t)}{2q'(t)} + \frac{1}{t}\right)^2 - \frac{q^2(t)+\theta^2}{2\,q'(t)\,t^2} - \frac{q'(t)}{2} - \frac{q(t)}{t} = \frac{\mu^2}{4}. \tag{64}$$

Proof : To prove this lemma we use the way more convinient for us rather than short. The reasons are analogous to those explained in Remark **15**). We introduce the similarity variables:

$$\lambda = \frac{z+\zeta}{z+\bar{z}}, \qquad t = z+\bar{z}, \qquad \Psi(\lambda,t) = \Psi(\tau,z,\bar{z}), \tag{65}$$

where ζ is given in Table **1**. By inserting them into the moving frame equations (11), in which the fundamental functions are taken from the Table **1** for $\zeta = \infty$, we get the following system:

$$\begin{aligned} \Psi_\lambda\,\Psi^{-1} &= t\begin{pmatrix} a(t) & \varphi(t) \\ \bar{\varphi}(t) & -a(t)\end{pmatrix} + \frac{e^{u(t)/2}}{2}\begin{pmatrix} 0 & \frac{1}{\lambda-1} \\ -\frac{1}{\lambda} & 0\end{pmatrix}, \\ \Psi_t\,\Psi^{-1} &= \lambda\begin{pmatrix} a(t) & \varphi(t) \\ \bar{\varphi}(t) & -a(t)\end{pmatrix} + \begin{pmatrix} -\frac{a(t)}{2} & 0 \\ -\bar{\varphi}(t) & \frac{a(t)}{2}\end{pmatrix}, \end{aligned} \tag{66}$$

where

$$a(t) = \frac{u'(t)}{2} = \frac{q''(t)}{2\,q'(t)} + \frac{1}{t}, \qquad \varphi(t) = -\frac{q(t)+\mathrm{i}\,\theta}{\sqrt{-2\,q'(t)}\,t} + \sqrt{\frac{-q'(t)}{2}}. \tag{67}$$

It is a consequence of the compatibility condition of the system (66), that the determinant of the first matrix is independent of λ and t. Thus we get the first integral of (62):

$$a^2(t) + |\varphi(t)|^2 = \frac{\mu^2}{4}, \tag{68}$$

which can be rewritten due to (67) as (64).

Remark 19 *Note, that the case $\mu = 0$ for real solutions of (62) is possible, iff $\theta = 0$, because otherwise there must exists a real-valued function q which solves*

$$q' = -\frac{1}{t}(q + i\theta). \tag{69}$$

*The case $\mu = \theta = 0$ is investigated in Section **8**, and for the rest of this section $\mu \neq 0$.*

Theorem 7 *If $q(t) \neq C/t$ ($C \in \mathbb{C}$) is a solution of (62), then it satisfies (64), and the functions*

$$z(t) = -\frac{1}{2\,\mu}\,\varphi(t)\,e^{\mathbf{u}(t)/2}, \qquad y(t) = \frac{2\,a(t) + \mu}{2\,a(t) - \mu}, \tag{70}$$

where $\varphi(t)$ and $a(t)$ are given by (67), solve the following system:

$$\begin{aligned} t\,z' &= z\left(-\mathrm{i}\,\frac{\theta}{\mu} + z\right)\left(y - \frac{1}{y}\right), \\ t\,y' &= \mu\,t\,y - 2\,z\,(y-1)^2 + \mathrm{i}\,\frac{\theta}{\mu}\,(y-1)^2. \end{aligned} \tag{71}$$

Moreover, if $q'(t) < 0$, then y is a negative solution of (63) with the coefficients

$$\alpha = -\beta = -\frac{\theta^2}{4\,\mu^2}, \qquad \gamma = \mu, \qquad \delta = -\frac{\mu^2}{2}. \tag{72}$$

Inversely, if y is a solution of the Painlevé V (63) with the coefficients (72),where θ and $\mu \neq 0$ are arbitrary numbers, then

$$q(t) = \frac{t(\mu^2\,y^2 - y'^2)}{2\,y\,(y-1)^2} - \theta^2\,\frac{(y-1)^2}{2\,\mu^2\,t\,y} \tag{73}$$

is a solution of (62) with the first integral μ (64). Moreover, if $y < 0$ and does not solve

$$y' = \mu\,y \pm \mathrm{i}\,\frac{\theta}{\mu\,t}\,(y-1)^2,$$

then $q(t)$ given by (73) is real and defines a HIMC surface (see Theorem **4**) *with the metric function*

$$e^{\mathbf{u}(t)} = -\theta^2\,\frac{(y-1)^2}{\mu^2\,y} - \frac{t^2\,(\mu\,y - y')^2}{y\,(y-1)^2}.$$

Proof : The existence of the first integral (64) is the result of Lemma **15**. To find an explicit representation of $q(t)$ in terms of the fifth Painlevé functions, let us consider the following gauge transformation $\Phi = \Omega(t)\,\Psi$, where

$$\Omega(t) = \frac{v(t)^{\mathrm{i}\mathbb{k}}}{\sqrt{\mu\,\varphi(t)}} \begin{pmatrix} a(t) + \frac{\mu}{2} & \varphi(t) \\ a(t) - \frac{\mu}{2} & \varphi(t) \end{pmatrix},$$

where $\mathbb{k}$ is defined in (6) and $v(t)$ is a solution of the differential equation

$$v'(t) = v(t)\,\frac{\varphi(t)}{\mu}\,\left(\frac{a(t)}{\varphi(t)}\right)' + \frac{\mu}{4}.$$

Remark 20 *Note, that for $\mu \neq 0$, the function $\varphi(t)$ has no zeros. Since, any zero of $\varphi(t)$ solves (69), there are no such zeros for $\theta \neq 0$. Moreover, for $\theta = 0$, any solution $\varphi(t)$, having at least one zero , is identical zero. This is only possible for $\mu = 0$ (see equation (68)).*

Remark 21 *The gauge transformation is only an analytical trick rather than a geometrical transformation of the moving frame, because $\Omega(t) \notin SU(2)$.*

Now, for Φ we get the following system:

$$\begin{aligned}
\Phi_\lambda\,\Phi^{-1} &= \frac{\mu\,t}{2}\,\sigma_3 + \frac{1}{\lambda}\,\frac{\varphi(t)\,e^{u(t)/2}}{2\,\mu}\begin{pmatrix} -1 & v(t)^2 \\ -\dfrac{1}{v(t)^2} & 1\end{pmatrix} \\
&\quad + \frac{1}{\lambda-1}\,\frac{e^{u(t)/2}}{2\,\mu\,\varphi(t)}\begin{pmatrix} |\varphi(t)|^2 & (a(t)+\dfrac{\mu}{2})^2\,v(t)^2 \\ -\dfrac{(2\,a(t)-\mu)^2}{4\,v(t)^2} & -|\varphi(t)|^2\end{pmatrix} \qquad (74) \\
\Phi_t\,\Phi^{-1} &= \frac{\mu\,\lambda}{2}\,\sigma_3 + \begin{pmatrix} -\dfrac{\mu}{4} & \dfrac{v(t)^2}{2}\,(a(t)+\dfrac{\mu}{2}) \\ \dfrac{1}{v(t)^2}\,(a(t)-\dfrac{\mu}{2}) & \dfrac{\mu}{4}\end{pmatrix} + \Omega_t\,\Omega^{-1}.
\end{aligned}$$

From one side the coefficients $a(t)$ and $\varphi(t)$ of the system (74) are expressed in terms of the function $q(t)$ via (67). On the other side, it is a result of work the [20], that these coefficients can be presented in terms of a Painlevé V function (63), (72). Comparing (74) with the parametrization given in [20] we get the following system:

$$\begin{aligned}
a(t) &= \frac{\mu}{2}\,\frac{y(t)+1}{y(t)-1} &&= \frac{q''}{2\,q'} + \frac{1}{t}, \\
\varphi(t)\,e^{u(t)/2} &= \mu\,\frac{t\,(y'-\mu\,y)}{(y-1)^2} - \mathrm{i}\,\theta &&= -q'\,t - q - \mathrm{i}\,\theta, \qquad (75) \\
\bar\varphi(t)\,e^{-u(t)/2} &= \frac{\mu^2\,y}{\mu\,t\,(\mu\,y-y') + \mathrm{i}\,\theta(y-1)^2} &&= \frac{1}{2\,t} + \frac{q-i\theta}{2\,t^2\,q'},
\end{aligned}$$

and that the functions $z(t)$ and $y(t)$ defined by (70) satisfy the system (71).
If $q'(t) < 0$, then $q(t)$ is real and hence $a(t)$ is real. Since $q(t)$ is real $q(t) \neq C/t$, it is not a solution of (69) and, thus $\varphi(t) \neq 0$. Now, from (68) and (70) we get that $y(t) < 0$.
Inversely, let $y(t)$ be a solution of (63) with (72). Then, we define the function $q(t)$ by using the system (75). The consistency condition for this system gives us (64). Differentiating (64) and taking into consideration that $(q''/q' + 2/t) \neq 0$, since $y(t) = -1$ is not a solution of (63),(72), one arrives at (62). Solving the last two equations in (75), one finds an explicit expression for $q(t)$ (73) and

$$q'(t) = \frac{1}{y(t)}\left(\theta^2\,\frac{(y(t)-1)^2}{2\,\mu^2\,t^2} + \frac{(y'(t)-\mu\,y(t))^2}{2\,(y(t)-1)^2}\right).$$

If $y(t) < 0$, then $q(t)$ is real and $q'(t) < 0$. Now, to finish the proof, we use Theorem **4** and the formula for $e^{u(t)}$ form Table **1**.

Remark 22 *In the case $\theta = 0$ one puts*

$$y(t) = -\cot^2(\phi(x)),$$

into the fifth Painlevé equation (63) with the coefficients (72), to get a third Painlevé equation in trigonometric form (20). The solutions of this Painlevé III equation are in one-to-one correspondence with the HIMC surfaces of revolution, as investigated in Section **4**.

Remark 23 *The parameter μ is not an essential parameter in our description of HIMC surfaces, since it can be removed from the Painlevé V (63) with (72), via the scaling $t \to \mu\, t$. This scaling is just a reparametrization of the surfaces. So, in the case $\mu \neq 0$, one may fix $\mu = 1$.*

Remark 24 *The scaling property of the associated families stated in Theorem* **5**, *which we establish by calculation of the fundamental functions, can be derived in another way: by using the Sym-formula (10) and the λ-equations (49), (66) obtained in Section* **6** *and here. Actually, by means of (49), (66) we can rewrite the Sym-formula as follows:*

$$F(\tau, z, \bar{z}) = \frac{\partial \lambda}{\partial \tau} \Psi^{-1}(\lambda, t)\, A(\lambda, t)\, \Psi(\lambda, t), \tag{76}$$

where $A(\lambda, t)\, \Psi(\lambda, t)$ denotes the right hand side of (49) or (66). The derivative $\dfrac{\partial \lambda}{\partial \tau}$ (see (48), (65)) yields the proper scaling factors and all other terms depend of τ only through the variable λ. This dependence can be removed by shifting (48), (65) of the conformal coordinate z, which is just a reparametrization of the surface.

8 Special Families of HIMC Surfaces and Cartan-Families

It is proved that in the **A** case there are two (and only two) special solutions of the generalized Hazzidakis equation (45), $q(t) = \pm\theta \cot(2\,t)$, which are not related with the Painlevé VI equation. Since we are considering $t > 0$, only one of these solutions,

$$q(t) = \theta \cot(2\,t), \quad \theta > 0, \tag{77}$$

generates a HIMC **A**-family. In section **7**, where we analyzed the **C**-case, we obtained, that for $\theta = \mu = 0$ there is a one-parameter family of solution

$$q(t) = \frac{\alpha}{t}, \quad \alpha > 0 \text{ for } t > 0, \tag{78}$$

which is not related with the fifth Painlevé equation, but generates a HIMC **C**-family. The **A**-families produced by the solutions (77) seem to be at first glance non-isothermic since $\theta \neq 0$, while the **C**-families are of course isothermic ones. Unexpectedly, both solutions (77) and (78) define the same families.

Proposition 4 *The* **A**-*families defined by the solutions (77) coincide with the* **C**-*families generated by the solutions (78) if $\theta = 2\,\alpha$.*

Proof : The following equations are an explicit mapping from the special **A**-family into the **C** one:

$$\alpha = \frac{\theta}{2}, \quad \tau_C = \sqrt{2\,|\tau_A|}, \quad z_C = \mathrm{sign}(\tau_A)\,\mathrm{i}\,\left(e^{4\,\mathrm{i}\,z_A} - 1\right),$$

where τ_A and τ_C are the family parameters and z_A and z_C are local coordinates.

Definition 3 *The special* **C**-*families of HIMC surfaces corresponding to the solution (78) are called* **Cartan-families**.

The motivation for this definition might be clarified by the following

Proposition 5 *Iff an isothermic HIMC surface is dual to itself and is not a CMC surface, then it belongs to a Cartan-family.*

Proof : If a non-CMC HIMC surface is dual to itself then $1/H$ and $1/Q$ are non-constant harmonic functions. By using the Codazzi equations (5) one proves that there is a local conformal parametrization of this surface, in which the fundamental functions read as follows:

$$Q(z,\bar{z}) = b\,H(z,\bar{z}) = \frac{a}{z+\bar{z}}, \quad e^{u(z,\bar{z})} = 2\,\frac{a^2}{b}, \quad |a|b > 0,$$

which means that it is an applicable Bonnet (and HIMC) surface. Now, scaling its local coordinate $z \to a/(2b)z$ we get exactly the same fundamental functions as for the special HIMC surface defined by the solution (78) with $\alpha = a^2/b$ and $\tau \to 0$ (see Table **1** and Theorem **4**).

Remark 25 *Due to this proposition the special* **C**-*families (78) of HIMC surfaces are simultaneously special* **C**-*families of Bonnet surfaces. Cartan was the first, who distinguished these special* **C**-*families as being the only kind of families of applicable Bonnet surfaces [7],[11].*

Corollary 9 *All θ-isothermic HIMC surfaces with $\theta \neq 0$ which do not belong to the Cartan-families are non-isothermic.*

Proof : This statement is a consequence of Propositions **3** and **5**.

Proposition 6 *The immersion function for the Cartan-families is as follows:*

$$F(\tau,z,\bar{z}) = \frac{1}{t}\begin{pmatrix} \mathrm{Re}\left(\dfrac{\omega^2}{1-\lambda}\,G^2(\lambda) - \lambda\,G'^2(\lambda)\right) \\ \mathrm{Im}\left(\dfrac{\omega^2}{1-\lambda}\,G^2(\lambda) - \lambda\,G'^2(\lambda)\right) \\ 2\,\omega\,\mathrm{Re}\,(G'(\lambda)\,G(1-\lambda)) \end{pmatrix}, \tag{79}$$

where

$$G(\lambda) = C_1\,{}_2F_1(\mathrm{i}\,\omega, -\mathrm{i}\,\omega, 1; \lambda) + C_2\,(1-\lambda)\,{}_2F_1(1+\mathrm{i}\,\omega, 1-\mathrm{i}\,\omega, 2; 1-\lambda),$$

$C_1, C_2 \in I\!R$, *and*

$$\lambda = \frac{\tau\,\bar{z} + 2\mathrm{i}}{\tau(z+\bar{z})}, \qquad t = z+\bar{z}, \qquad \omega = \frac{1}{2}\sqrt{\frac{\alpha}{2}}.$$

Here ${}_2F_1(a,b,c;t)$ is the standard notation for the Gaussian hypergeometric functions [21].

Proof : To produce the formula (79) one substitutes into the *Sym*-formula (10) $\Psi_\tau(\tau,z,\bar{z}) = \lambda_\tau \Psi_\lambda(\lambda,t)$, in which $\Psi_\lambda(\lambda,t)$ is given by first equation of (66) with λ changed to $1-\lambda$. Since in our case $a(t) = \varphi(t) = \theta = \mu = 0$, (66) reads as follows:

$$\Psi_\lambda(\lambda,t)\,\Psi^{-1}(\lambda,t) = \omega\begin{pmatrix} 0 & \dfrac{1}{\lambda} \\ \dfrac{1}{1-\lambda} & 0 \end{pmatrix}.$$

The general solution of this equation is given by

$$\Psi(\lambda,t)=\begin{pmatrix} G(\lambda) & -\dfrac{1}{\omega}\,(1-\lambda)\,G'(1-\lambda) \\ \dfrac{1}{\omega}\,\lambda\,G'(\lambda) & G(1-\lambda) \end{pmatrix},$$

where G solves the hypergeometric differential equation

$$\lambda\,(1-\lambda)\,G''(\lambda)+(1-\lambda)\,G'(\lambda)-\omega^2\,G(\lambda)=0.$$

Since $\Psi(\lambda,t)\in\mathrm{SU}(2)$, $C_1,C_2\in\mathbb{R}$. One uses the Sym-formula (10), to get the immersion function $F(\tau,z,\bar{z})$ in the quaternionic representation. Since the Cartan-surfaces are cones, one simplifies $F(\tau,z,\bar{z})$ by scaling to get after identification (7) the final result (79).

Remark 26 *The variation of the constants C_1 and C_2 means just a rotation of the surface as a whole. Therefore they can be fixed arbitrary but not both equal to zero.*

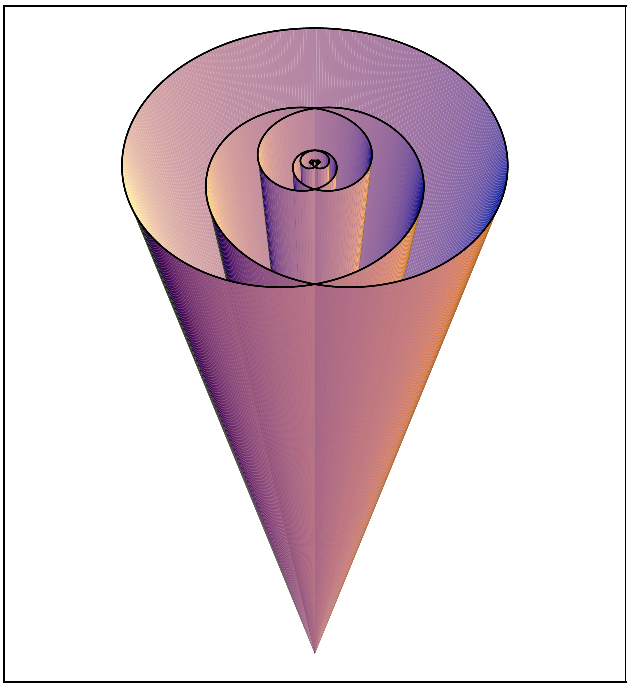

Figure 3 ($\alpha = 32$)

According Proposition **6** each Cartan family consists of two different surfaces: a cone ($\tau \neq 0$) and a cylinder ($\tau = 0$). Actually, from the point of view of the projective geometry, there is only one surface in each family. We call it the Cartan cone.

In the Figures **3** - **5** we present parts of different Cartan cones inside spheres centered at their tops. The black curves are the sections of the cones by the spheres. In all pictures these spherical curves, wind around two centers, which positions depends on the parameter α in equation (78). Thereby α regulates the number of self-intersections of the cone. Since the Cartan cones smoothly depend on α, there should be a discrete set of values of α, such that the corresponding Cartan cones have infinitly many lines of self-intersections: when the centers of the spherical curves belong to them.

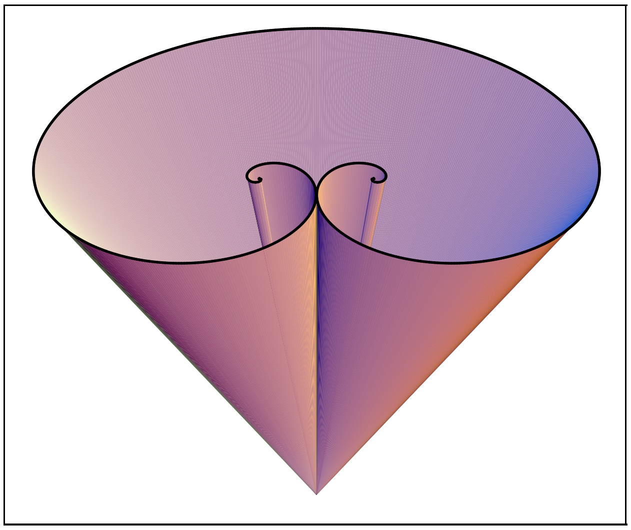

Figure 4 ($\alpha = 8$)

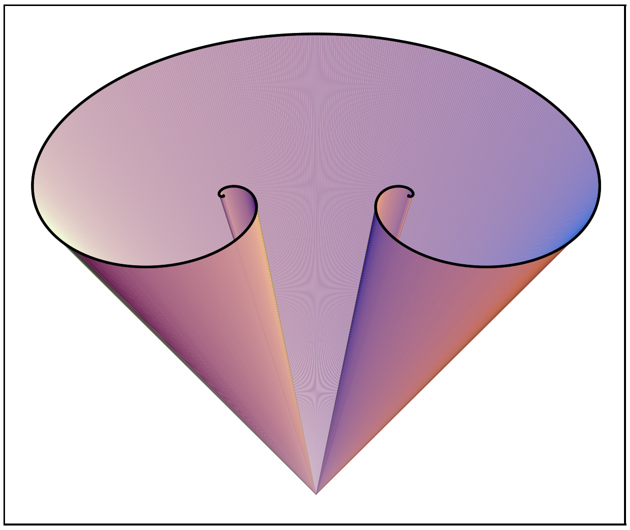

Figure 5 ($\alpha = 2$)

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Appendix
The Hopf Differential as a Hamiltonian for the Painlevé Equations

It is well known that the Painlevé equations admit Hamiltonian formulations [20], [22], [23]. Since, as it was shown in Section **4**, Proposition **1**, geometrical objects acquire a simple formulation in Hamiltonian notations, we consider here Hamiltonian structures associated with the Painlevé equations arising in our study of HIMC surfaces. In the **A** and **B** cases a reference to the Hamiltonian structure allow us to get another representation of solutions of the generalized Hazzidakis equations (45), (60) in terms of sixth Painlevé functions, then that obtained in Section **6**.

Proposition 7 *Let $q(t)$ be a solution of (45) with the first integral (47), and $q(t) \neq \pm\theta\,\cot(2\,t)$. Denote $\{i,j,k,l\}$ a permutation of the set $\{1,2,3,4\}$ and put*

$$b_k = b_l = 0, \quad b_{i,j} = \frac{1}{2}\sqrt{-\mu \pm \sqrt{\mu^2-\theta^2}}.$$

Then

$$H_6(s) = \frac{1}{s(s-1)}\left(\frac{q(t)}{8\,\mathrm{i}} - (b_1b_3 + b_1b_4 + b_3b_4)s + \frac{1}{2}\sum_{1\le m<n\le 4} b_m\,b_n\right), \tag{80}$$

where $s = 1/2 - \mathrm{i}/2\,\cot(2\,t)$, is the Hamiltonian function corresponding to the Hamiltonian $\mathcal{H}_6(q_6,p_6,s)$: $H_6(s) = \mathcal{H}_6(q_6(s),p_6(s),s)$,

$$\begin{aligned}\mathcal{H}_6(q_6,p_6,s) \quad = \quad & \frac{1}{s(s-1)}\Big(q_6(q_6-1)(q_6-s)p_6^2 - ((b_1+b_2)\,(q_6-1)\,(q_6-s) + \\ & (b_1-b_2)\,(q_6-s)\,q_6 + (b_3+b_4)\,(q_6-1)\,q_6)\,p_6 + (b_1+b_3)\,(b_1+b_4)\,(q_6-s)\Big),\end{aligned}$$

where $q_6(s)$, $p_6(s)$ is a solution of the Hamiltonian system

$$\frac{d\,q_6}{d\,s} = \frac{\partial\,\mathcal{H}_6(q_6,p_6,s)}{\partial\,p_6}, \quad \frac{d\,p_6}{d\,s} = -\frac{\partial\,\mathcal{H}_6(q_6,p_6,s)}{\partial\,q_6}. \tag{81}$$

The function $y(s) = q_6(s)$ solves the Painlevé VI equation (46) with the coefficients

$$\alpha = \frac{1}{2}\,(b_3-b_4)^2,\ \beta = -\frac{1}{2}\,(b_1+b_2)^2,\ \gamma = \frac{1}{2}\,(b_1-b_2)^2,\ \delta = \frac{1}{2}\,(1-(b_3+b_4+1)^2). \tag{82}$$

Inversely, if $y(s) = q_6(s)$ is a solution of (46) with the coefficients (82), then one finds from the first equation of (81) the function $p_6(s)$ and thereby, from equation (80), the function $q(t) \neq \theta\,\cot(2\,t)$.

Proof : The Hamiltonian formulation for the Painlevé VI equation is due to Okamato [22]. He also proved that the Hamiltonian function for Painlevé VI equation is the general solution of a second order differential equation, quadratic with respect to the second derivative. We just notice, that the first integral (47) can be transformed into the Okamoto equation by the change of variables presented in the proposition.

Corollary 10 *Let $Q_A(\tau, z, \bar{z})$ be a Hopf differential in parametrization given in Theorem* **4** *for* **A**-*families of HIMC surfaces defined by solutions of the sixth Painlevé equation (46) with the coefficients*

$$\alpha = 0, \quad \beta = \frac{\mu - \theta}{4}, \quad \gamma = -\frac{\mu + \theta}{4}, \quad \delta = 0.$$

Then

$$Q_A(\tau, z, \bar{z}) = \frac{8\,\mathrm{i}\, s(s-1)\, H_6(s)}{\tau\, \sin^2(2\,(z - \zeta(\tau)))}, \tag{83}$$

where $H_6(s)$ is defined in Proposition **7** *for*

$$b_3 = b_4 = 0, \quad b_{1,2} = \frac{1}{2}\sqrt{-\mu \pm \sqrt{\mu^2 - \theta^2}}.$$

Proof : It is a direct consequence of Theorem **4** and Proposition **7**.

Corollary 11 *Let $q(t)$ be a solution of (60), then it has the first integral*

$$2\left(\frac{q''}{4\,q'} + \coth(2\,t)\right)^2 - \frac{1}{q'}\left(\theta^2\left(\coth^2(2\,t) - 1\right) - q^2 + \left(q\,\coth(2\,t) + \frac{q'}{2}\right)^2\right) = \mu.$$

Let $\{i, j, k, l\}$ be a permutation of the set $\{1, 2, 3, 4\}$. Define

$$b_k = b_l = 0, \quad b_{i,j} = \frac{1}{2}\sqrt{\mu \pm \sqrt{\mu^2 + \theta^2}},$$

then for $q(t) \neq \pm\mathrm{i}\,\theta\,\coth(2\,t)$ the function

$$H_6(s) = \frac{1}{s(s-1)}\left(\frac{q(t)}{8} - (b_1 b_3 + b_1 b_4 + b_3 b_4)s + \frac{1}{2}\sum_{1 \le m < n \le 4} b_m\, b_n\right),$$

where $s = 1/2(1 + \coth(2\,t))$, is a Hamiltonian function corresponding to the Hamiltonian $\mathcal{H}_6(q_6, p_6, s)$ defined in Proposition **7**.

For $\{i, j, k, l\} = \{3, 4, 2, 1\}$ and $q'(t) < 0$ the Hopf differential is as follows:

$$Q_B(\tau, z, \bar{z}) = \frac{32\, s\,(s-1)\, H_6(s)}{(1 + 4\tau^2)\, \sinh^2(2(z - \zeta(\tau)))}. \tag{84}$$

The function $y(s) = q_6(s)$ solves a Painlevé VI equation with the coefficients

$$\alpha = 0, \quad \beta = -\frac{\mu + \mathrm{i}\,\theta}{2}, \quad \gamma = \frac{\mu - \mathrm{i}\,\theta}{2}, \quad \delta = 0.$$

Proof : This statement is an implication of Theorem **4**, Lemma **14**, and Proposition **7**.

Proposition 8 *Let $y(t)$ be a solution of the fifth Painlevé equation (63) with the coefficients (72), then the system (71) with respect to the canonical variables $q_5 = y$, $p_5 = z/y$ gain a Hamiltonian form*

$$\frac{d\,q_5}{d\,t} = \frac{\partial\,\mathcal{H}_5(q_5, p_5, t)}{\partial\,p_5}, \quad \frac{d\,p_5}{d\,t} = -\frac{\partial\,\mathcal{H}_5(q_5, p_5, t)}{\partial\,q_5},$$

where the Hamiltonian

$$\mathcal{H}_5(q_5, p_5, t) = -\frac{1}{t}\,\left((q_5-1)^2\, q_5\, p_5^2 - \mu\, t\, p_5\, q_5 - \mathrm{i}\,\frac{\theta}{\mu}\,(q_5-1)^2\, p_5\right).$$

The Hopf differential $Q_{\mathbf{C}}(\tau, z, \bar{z})$ for the non-Cartan **C***-families (see Theorem* **4** *and Appendix* **A***) is, up to a holomorphic function, equal to the Hamiltonian function. In the parametrization of Theorem* **4**

$$Q_{\mathbf{C}}(\tau, z, \bar{z}) = \frac{\zeta^2(\tau)}{(z-\zeta(\tau))^2}\,\mathcal{H}_5\left(y(t), \frac{z(t)}{y(t)}, t\right). \tag{85}$$

Proof : For the Hamiltonian formulation of the fifth Painlevé equation see the works [20] and [23]. The proof follows from the fact that

$$\mathrm{Res}_{\lambda=1}\left(\mathrm{tr}\left(\Psi_\lambda\,\Psi^{-1}\right)^2\right) = \mathrm{Res}_{\lambda=1}\left(\mathrm{tr}\left(\Phi_\lambda\,\Phi^{-1}\right)^2\right),$$

where Ψ and $\Phi = \Omega(t)\,\Psi$ are the solutions of (66) and (74) respectively, and the definition of Hamiltonian in the framework of the Isomonodromy Deformation Method [20].

Remark 27 *As it is obvious from (85), $Q_{\mathbf{C}}(\tau, w, \bar{w}) = H_5(t)$ for a proper local conformal reparametrization of the surfaces in this case. In the* **A** *and* **B** *cases one can not find such reparametrization for (83) and (84), since the factor $s(s-1)$, which is not a holomorphic function. But possibly the richness of the group of the canonical transformations for the sixth Painlevé Hamiltonian system [22], may allow one to find instead of (83) and (84) relations between the Hopf differentials and Hamiltonians for these cases with holomorphic factors.*