Kählerian E-spinors

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November 8, 1995

Abstract

If M^{2m} is a spin Kähler manifold with complex structure J, scalar curvature $R \not\equiv 0$, and spinor bundle S, then, for $r \in \{0, 1, \ldots, m\}$ and $\psi \in \Gamma(S)$, the differential equations of second order

$$\nabla_X D_\pm \psi + \frac{R}{8(2r-1)}(X \mp iJX) \cdot \psi = 0 \qquad (E^r_\pm)$$

are considered. They must be satisfied for each real vector field X. The equations E^r_+ and E^r_- are equivalent in the sense that there is a canonical antilinear isomorphism between the corresponding $\mathbb{C}$-vector spaces of solutions $\mathcal{E}^r_+$ and $\mathcal{E}^r_-$. The non-trivial elements of these spaces are called Kählerian E-spinors of type $(r, +)$ and $(r, -)$, respectively. There is a basic relation between Kählerian E-spinors and Kählerian twistor spinors that is essential for the proofs of all results of the paper.

Introduction

In 1988 A. Lichnerowicz introduced, on a Riemannian spin manifold M^n, a canonical equation for a spinor field ψ, which he called the equation E (see [11]). Let D be the Dirac operator and $R \not\equiv 0$ the scalar curvature, then this equation is of the form

$$\nabla_X D\psi + \frac{R}{4n-4} X \cdot \psi = 0 \qquad (E)$$

and it must be satisfied for each vector field X on M^n. There are some interesting results concerning this equation. For example, if M^n is connected of dimension $n \geq 3$ and there is a non-trivial solution of equation E, then R is constant. Moreover, if M^n is connected and $R \neq 0$ is constant, then the space of solutions of equation E is isomorphic to the space of twistor spinors of M^n (see [1], Section 2.6). On the other hand, it is known that in case where M^n is Kähler of dimension $n \geq 4$ and scalar with curvature $R \not\equiv 0$, each twistor spinor is zero (see [7], Theorem18). Hence, in the Kähler case with $R \not\equiv 0$ and $n \neq 2$, the equation E has the trivial solution only. Thus we see that the equation E is not a suitable one if M^n is a spin Kähler manifold. In this paper we introduce equations being the Kähler counterpart to the equation E of Lichnerowicz. To write down these equations we consider a spin Kähler manifold M^{2m} of complex dimension m with complex structure J, Kähler metric g, scalar curvature $R \not\equiv 0$, and spinor bundle S. Moreover, let D_+ and D_- be the operators defined by $D_\pm := \frac{1}{2}(D \mp i\tilde{D})$, where D is the Dirac operator and $\tilde{D}$ the Kähler twist of D locally given by $\tilde{D}\psi := J(X^k) \cdot \nabla_{X_k}\psi$. Further, for any real vector or vector field X, we use the notations $p(X) := \frac{1}{2}(X - iJX), \bar{p}(X) := \frac{1}{2}(X + iJX)$. Let $\psi \in \Gamma(S)$, then, for each $r \in \{0, 1, \cdots, m\}$, one has two types of Kählerian E-equations. The Kählerian E-equation of type $(r, +)$ is of the form

$$\nabla_X D_+\psi + \frac{R}{8r - 4} p(X) \cdot \psi = 0 \qquad (E^r_+)$$

and the E-equation of type $(r, -)$ reads

$$\nabla_X D_-\psi + \frac{R}{8r - 4} \bar{p}(X) \cdot \psi = 0, \qquad (E^r_-)$$

where these equations must be satisfied for each real vector field X. A detailed motivation of the structure of these equations is given in Section 2. Like the Kählerian twistor equations the Kählerian E-equations are a further canonical kind of field equations on a spin Kähler manifold. A non-trivial solution ψ of the equation $E^r_\pm$ is called a Kählerian E-spinor of type $(r, \pm)$. By $\mathcal{E}^r_+$ and $\mathcal{E}^r_-$ we denote the $\mathbb{C}$-vector spaces of all solutions of E^r_+ and E^r_-, respectively. The equations E^r_+ and E^r_- are compatible with the canonical splitting

$$S = S_\circ \oplus S_1 \oplus \cdots \oplus S_m$$

of the spinor bundle ($S_r \cong S_\circ \oplus \wedge^{0,r}$), i. e., we have the decompositions

$$\mathcal{E}^r_\pm = \oplus^m_{s=0} \mathcal{E}^{(r,s)}_\pm \quad (r = 0, 1, \cdots, m)$$

with $\mathcal{E}_{\pm}^{(r,s)} := \mathcal{E}_{\pm}^{r} \cap \Gamma(S_s)$. The non-trivial elements of the subspace $\mathcal{E}_{\pm}^{(r,s)} \subseteq \mathcal{E}_{\pm}^{r}$ are called E-spinors of type $(r, s, \pm)$.
Let $j : S \longrightarrow S$ be the canonical anti-unitary bundle map having the properties

$$j^2 = (-1)^{m(m+1)/2}, \quad jS_r = S_{m-r} \quad (r = 0, 1, \cdots, m).$$

Moreover, j is parallel ($\nabla j = 0$) and commutes with the Clifford multiplication by real vectors ($Xj = jX$). Thus, multiplying the equation $E_{\pm}^{r}$ by j we immediately obtain the relation

$$j\mathcal{E}_{\pm}^{(r,s)} = \mathcal{E}_{\mp}^{(r,m-s)} \quad (r, s = 0, 1, \cdots, m).$$

Hence, it suffices to consider one of the sets of equations E_{+}^{r} or E_{-}^{r} only. In this paper we restrict ourselves to the equations E_{-}^{r}.
We recall that, for $\psi \in \Gamma(S_r)$, we have $D_{\pm}\psi \in \Gamma(S_{r\pm1})$ and $p(X) \cdot \psi \in \Gamma(S_{r+1}), \bar{p}(X) \cdot \psi \in \Gamma(S_{r-1})$. This implies $\mathcal{E}_{-}^{(r,0)} = \Gamma(S_o)$, since the equations E_{-}^{r} are satisfied trivially for all C^{∞}-sections of the line bundle S_o. Thus, only the reduced E-spaces $\tilde{\mathcal{E}}_{-}^{r} := \oplus_{s=1}^{m} \mathcal{E}_{-}^{(r,s)}$ are of interest.
In Section 3 we show that Kählerian E-spinors are closely related to Kählerian twistor spinors. For all $r = 0, 1, \cdots, m$ and $s = 1, 2, \cdots, m$ we have the fundamental inclusion

$$D_{-}\mathcal{E}_{-}^{(r,s)} \subseteq ker\mathcal{D}^{(s,+)}, \tag{$*$}$$

where $ker\mathcal{D}^{(s,+)}$ is the space of twistor spinors of type $(s, +)$. Moreover, the kernel of the map

$$\psi \in \mathcal{E}_{-}^{(r,s)} \longmapsto D_{-}\psi \in ker\mathcal{D}^{(s,+)}$$

is zero (Proposition 6). In case $\mathcal{E}_{-}^{(r,s)} \neq 0$ with $s \geq 2$, this map is an isomorphism and the scalar curvature R does not possess any zero (Proposition 7). By the relation $(*)$ it is possible to apply all the results of the theory of Kählerian twistor spinors to the investigation of Kählerian E-spinors. (The basic properties of Kählerian twistor spinors are summarized in Section 1.) In this way we obtain first integrals (conservation laws) for the Kählerian E-equations (Corollary 6.4) and obstructions for E-spinors. We prove a first basic vanishing theorem (Corollary 7.1), which states that all E-spaces $\mathcal{E}_-^{(r,s)}$ with $r \neq s \geq 2$ vanish if R has a critical point and, hence, if M^{2m} is closed for example. Thus, in the closed case one has to consider the spaces $\mathcal{E}_-^{(r,1)}(r = 0, 1 \cdots, m)$ and $\mathcal{E}_-^{(r,r)}(r = 2, 3, \cdots, m)$ only. If M^{2m} is closed and if there is an $r \geq 2$ with $\mathcal{E}_-^{(r,r)} \neq 0$, then R is positive (Corollary 7.2). Moreover, if M^{2m} is closed and $R > 0$, then, for $r \geq 2$, the spaces $\mathcal{E}_-^{(r,r)}$ and $\ker \mathcal{D}^{(r,+)}$ are isomorphic (Theorem 9). This implies that each twistor spinor of type $(r, +)$ with $r \geq 2$ possesses a uniquely determined potential in $\mathcal{E}_-^{(r,r)}$, i.e., for each $\varphi \in \ker \mathcal{D}^{(r,+)}$ there is a unique $\psi \in \mathcal{E}_-^{(r,r)}$ such that $\varphi = D_-\psi$. This statement is valid more generally (Proposition 10).

The Kählerian E-equations (twistor equations) are essentially overdetermined differential equations of second (first) order. However, in case M^{2m} is closed, Kählerian E-spinors and Kählerian twistor spinors can be characterized by simpler equations of analogous structure. By Theorem 2 in Section 1, we have $\varphi \in ker\mathcal{D}^{(r,+)}$ in the closed case iff $\varphi \in \Gamma(S_{r-1})$ and φ satisfies the equations

$$D_-D_+\varphi = \frac{rR}{4r-2}\varphi \quad , \quad (\partial R) \cdot \varphi = 0.$$

In Section 3 we prove that $\psi \in \mathcal{E}_-^{(r,r)}$ holds iff $\psi \in \Gamma(S_r)$ and if there are the equations

$$D_+D_-\psi = \frac{rR}{4r-2}\psi \quad , \quad (\partial R) \cdot \psi = 0$$

(Theorem 11). We recall that, by Theorem 26 in [8], the elements of the space $ker\mathcal{D}^{(r,+)}$ are the critical points of a variational principle if M^{2m} is closed. The corresponding question for Kählerian E-spinors is still open if R ist not constant.

In Section 4 we investigate the relation between Kählerian E-spinors and the Einstein condition. By Theorem 13.B, Corollary 13.1 and Theorem 14, we have the following: If M^{2m} is Einstein with $R \neq 0$ and $m = 2r - 1 \geq 3$, then $\tilde{\mathcal{E}}_+^r = \mathcal{E}_+^{(r,r-1)} = ker\mathcal{D}^{(r,+)} \cong \mathcal{K}_+(M^{2m})$, where $K_+(M^{2m})$ is the space of Kählerian Killing spinors to the Killing number $\sqrt{R/4m(m+1)}$. Moreover, if M^{2m} is closed with $m = 2r - 1 \geq 3$ and $\mathcal{E}_-^{(r,r)} \neq 0$, then M^{2m} is

Einstein.
In particular, we see that Kählerian Killing spinors yield examples of Kählerian E-spinors (in [9],[10],[15],[16] one finds examples of Kählerian Killing spinors, classification results and charaterizations of Kähler manifolds admitting Kählerian Killing spinors).
In Section 5 we use results of Section 3 to derive a set of vanishing theorems for Kählerian E-spinors. Here we only mention a result that does not obviously follow from the theory of Kählerian twistor spinors. It states that all spaces $\mathcal{E}_-^{(r,s)}$ with $r \neq s \geq 2$ and $(r,s) \neq (1,2)$ vanish (Theorem 15). Comparing this with Corollary 7.1 we see that only the space $\mathcal{E}_-^{(1,2)}$ can appear addidtionally if we omit the supposition that R has a critical point.
In Section 6 we consider the case of constant scalar curvature $R \neq 0$. In this case all E-spinors of the spaces $\tilde{\mathcal{E}}_-^r$ are closed, i.e., we have $D_+\tilde{\mathcal{E}}_-^r = 0$. Moreover, it holds that $\tilde{\mathcal{E}}_-^0 = 0, \tilde{\mathcal{E}}_-^1 = \mathcal{E}_-^{(1,1)} \cong \bar{H}^0(M^{2m}, S_0)$ and there is at most an $r \geq 2$ with $\tilde{\mathcal{E}}_-^r \neq 0$. If $\tilde{\mathcal{E}}_-^r \neq 0$ for an $r \geq 2$, then $\tilde{\mathcal{E}}_-^r = \mathcal{E}_-^{(r,r)} \cong \ker \mathcal{D}^{(r,+)}$ and there is the equation $|Ric|^2 = R^2/(4r-2)$ (Theorem 22). By Theorem 23 and Corollary 23.1, we obtain a relation between E-spinors and the eigenvalue problem of the Dirac operator for M^{2m} closed.
Spin Kähler manifolds M^{2m} with the property that there is an $r \neq 1$ with $\mathcal{E}_-^{(r,1)} \cap ker D_+ \neq 0$ are investigated in Section 7. Our main result in this direction is Theorem 27, which states that locally the theory of such manifolds is equivalent to the theory of Kähler structures for which R has no critical point and which satisfy the differential equation

$$\rho + 2i\frac{r-1}{2r-1} d\left(\frac{R^2}{|dR|^2}\bar{\partial}R\right) = 0,$$

where ρ is the Ricci form. Particularly, there is at most one $r \neq 1$ with $\mathcal{E}_-^{(r,1)} \cap \ker D_+ \neq 0$. Further, we prove $\dim_{\mathbb{C}}\left(\mathcal{E}_-^{(r,1)} \cap \ker D_+\right) \leq 1$ (Theorem 25).

1 The basic properties of Kählerian twistor spinors

Let (M^{2m}, J, g, S) be a spin Kähler manifold of complex dimension m with complex structure J, Kähler metric g and spinor bundle S. Then we have the canonical decomposition

$$S = S_0 \oplus S_1 \oplus \cdots \oplus S_m \quad , \tag{1.1}$$

where the $S_r (r = 1, \ldots, m)$ are holomorphic subbundles of S with $\mathrm{rank}_{\mathbb{C}} S_r = \binom{m}{r}$. For $r \in \{1, \cdots, m\}$, let

$$\mathcal{D}^{(r,+)}, \mathcal{D}^{(r,-)} : \Gamma(S) \longrightarrow \Gamma\left(TM^{2m} \otimes S\right)$$

be the operators locally defined by

$$\mathcal{D}^{(r,\pm)}\psi := X^k \otimes \left(\nabla_{X_k}\psi + \frac{1}{4r}\left(X_k \pm JX_k\right) \cdot D_{\pm}\psi\right),$$

where $(X_1, \cdots, X_{2m})$ is any local frame of vector fields. The operators $\mathcal{D}^{(r,+)}$ and $\mathcal{D}^{(r,-)}$ are called the Kählerian twistor operators of type $(r,+)$ and $(r,-)$, respectively, and the non-parallel elements of the $\mathbb{C}$-vector spaces $\ker \mathcal{D}^{(r,+)}$ and $\ker \mathcal{D}^{(r,-)}$ are called Kählerian twistor spinors of type $(r,+)$ and $(r,-)$, respectively. There is the relation

$$j \cdot \ker \mathcal{D}^{(r,\pm)} = \ker \mathcal{D}^{(r,\mp)} \quad (r = 1, \cdots, m),$$

where $j : S \longrightarrow S$ is the canonical anti-unitary bundle map. Hence, it is sufficient to consider the spaces $\ker \mathcal{D}^{(r,+)}$. By definition, we have $\psi \in \ker \mathcal{D}^{(r,+)}$ iff ψ satisfies the differential equation of first order

$$\nabla_X \psi + \frac{1}{2r}\bar{p}\,(X) \cdot D_+\psi = 0 \tag{1.2}$$

for each real vector field X on M^{2m}. This equation is called the Kählerian twistor equation of type $(r,+)$. Equation (1.2) implies immediately that ψ is antiholomorhic, i.e., for each real vector field X, ψ satisfies the equation

$$\nabla_{p(X)}\psi = 0. \tag{1.3}$$

Moreover, from (1.2) one can derive the equations

$$D_-\psi = 0, \tag{1.4}$$

$$D_- D_+ \psi = \frac{rR}{4r-2}\psi, \tag{1.5}$$

$$(\partial R) \cdot \psi = 0, \tag{1.6}$$

$$\mathrm{grad}|\psi|^2 = -\frac{1}{2r} J\left(s_\psi\right), \tag{1.7}$$

$$\triangle|\psi|^2 = \frac{R}{4r-2}|\psi|^2 - \frac{1}{r}|D\psi|^2, \tag{1.8}$$

$$\bar{p}\left(\mathrm{Ric}\left(X\right) - \frac{R}{4r-2}X\right) \cdot \psi = 0, \tag{1.9}$$

$$\rho \cdot \psi = i\frac{R}{4r-2}\psi, \tag{1.10}$$

$$\nabla_{p(X)} D_+\psi + \frac{1}{2}p\left(\mathrm{Ric}\left(X\right)\right) \cdot \psi = 0, \tag{1.11}$$

$$(r-1)\, \nabla_{\bar{p}(X)} D_+\psi = 0, \tag{1.12}$$

where Ric is the Ricci tensor and ρ the Ricci form defined by $\rho(X,Y) = g\left(\mathrm{Ric}\left(X\right), JY\right)$. The vector field s_ψ in equation (1.7) is given by $s_\psi := \left(D\psi\right)\psi + \psi\left(D\psi\right)$. We recall that, for any $\psi, \varphi \in \Gamma(S)$, the complex vector field $\psi\varphi \in \Gamma(T^cM^{2m})$ is defined by $g(X, \psi\varphi) = -i\langle X \cdot \psi, \varphi\rangle$. Equation (1.12) shows that $D_+\psi$ is holomorphic for $r \geq 2$. In case $r \geq 2$, the Kählerian twistor equation (1.2) implies the additional equations

$$\mathrm{grad}\ |D\psi|^2 = \frac{1}{2}\ \mathrm{Ric}\ \left(J(s_\psi)\right) \quad , \tag{1.13}$$

$$\bar{p}\left(\ \mathrm{Ric}\ (X) - \frac{R}{4r-2}X\right) \cdot D_+\psi = -\frac{r}{2r-1}\left(\bar{\partial}R\right)(X) \cdot \psi \quad , \tag{1.14}$$

$$\rho \cdot D_+\psi = -i\frac{R}{4r-2} \cdot D_+\psi - i\frac{r}{2r-1}\left(\bar{\partial}R\right) \cdot \psi \quad , \tag{1.15}$$

$$(\partial R) \cdot D_+\psi = \frac{r}{2r-2}G_r \cdot \psi \quad , \tag{1.16}$$

$$\left(\partial\bar{\partial}R\right)\cdot\psi=\frac{1}{2}\left(\triangle R-G_r\right)\cdot\psi \quad , \tag{1.17}$$

where the function G_r is defined by

$$G_r:=\triangle R-\frac{2r-2}{r}\left(|\mathrm{Ric}|^2-\frac{R^2}{4r-2}\right).$$

It is known that G_r is an obstruction for Kählerian twistor spinors of type $(r,+)$ with $r\geq 2$ in case the scalar curvature R has a critical point (e.g., if M^{2m} is closed.). Moreover, for each $\psi\in\ker\mathcal{D}^{(r,+)}$ with $r\geq 2$, the functions

$$C_\psi:=\frac{rR}{4r-2}|\psi|^2+|D\psi|^2 \text{ and } Q_\psi:=G_r|\psi|^{2r}$$

are constant and, hence, first integrals of the twistor equation (1.2).
Concerning the equations (1.6), (1.10) and (1.15) - (1.17) we recall that for each q-form ω on M^{2m} and each $\varphi\in\Gamma(S)$ the Clifford product $\omega\cdot\varphi\in\Gamma(S)$ is defined and that the general formula

$$D\left(\omega\cdot\varphi\right)=\left((d+d^*)\omega\right)\cdot\varphi+(-1)^q\omega\cdot D\varphi-2\left(X^k\lrcorner\,\omega\right)\cdot\nabla_{X_k}\varphi \tag{1.18}$$

is valid, where $(X_1,\ldots,X_{2m})$ is any local frame of vector fields. In this sense each differential form can be considered as an endomorphism of the spinor bundle. For example, let Ω be the Kähler form defined by $\Omega\left(X,Y\right)=g\left(X,JY\right)$ and let $\varphi=\varphi_\circ+\cdots+\varphi_m$ be the decomposition of any $\varphi\in\Gamma(S)$ according to the splitting (1.1). Then we have

$$\Omega\cdot\varphi=\sum_{s=0}^{m}i(m-2s)\varphi_s \quad . \tag{1.19}$$

Moreover, in the sense of endomorphisms, there are the relations

$$\bar{p}\left(X^k\right)\cdot p\left(X_k\right)=-m+i\Omega \quad , \tag{1.20}$$

$$p\left(X^k\right)\cdot\bar{p}\left(X_k\right)=-m-i\Omega \quad , \tag{1.21}$$

$$X^k\cdot\mathrm{Ric}\ \left(X_k\right)=\ \mathrm{Ric}\ \left(X_k\right)\cdot X^k=-R \quad , \tag{1.22}$$

$$\mathrm{Ric}\ \left(JX^k\right)\cdot X_k=-X_k\cdot\ \mathrm{Ric}\ \left(JX^k\right)=2\rho \quad , \tag{1.23}$$

Furthermore, for any $\varphi \in \Gamma(S)$, there are the formulas (see [8], Section 6)

$$|\nabla\varphi|^2 = \text{ Re } \langle D^2\varphi, \varphi\rangle - \frac{R}{4}|\varphi|^2 - \frac{1}{2} \triangle |\varphi|^2 \quad , \tag{1.24}$$

$$|D_+\varphi|^2 = \text{ Re } \langle D_- D_+\varphi, \varphi\rangle - \frac{1}{2} \text{ div } \left(J(s_\varphi^+)\right) \tag{1.25}$$

with $s_\varphi^+ := (D_+\varphi)\varphi + \varphi(D_+\varphi)$. Finally, we recall the identities of operators

$$D_+^2 = 0 \qquad D_-^2 = 0 \tag{1.26}$$

$$D^2 = D_+D_- + D_-D_+ \qquad \tilde{D}D = i\,(D_+D_- - D_-D_+) = -D\tilde{D}. \tag{1.27}$$

The proofs of the basic equation (1.3) - (1.17) and the other facts can be found in [6], [7], [8].

In the second part of this section we derive some other characterizations of Kählerian twistor spinors of type $(r,+)$. First of all, we remark that $\psi \in \ker \mathcal{D}^{(r,+)}$ holds iff each of the components appearing in the decomposition of ψ according to (1.1) satisfies the equation (1.2). Hence, to find the general solution of the Kählerian twistor equation it is sufficient to consider this equation for sections of the various subbundles $S_s (s = 0, \cdots, m)$ of the spinor bundle S.

Lemma 1.1 *Let $\psi \in \ker \mathcal{D}^{(r,+)} \cap \Gamma(S_s)$ with $s \neq r-1$, then ψ is parallel.*

Proof: Using (1.19) and (1.21) we have by equation (1.2)

$$\begin{aligned}
0 &= X^k \cdot \left(\nabla_{X_k}\psi + \frac{1}{2r}\bar{p}\,(X_k) \cdot D_+\psi\right) = \\
&= X^k \cdot \nabla_{X_k}\psi + \frac{1}{2r} X^k \cdot \bar{p}(X_k) \cdot D_+\psi = \\
&= D\psi + \frac{1}{2r} p\left(X^k\right) \cdot \bar{p}\,(X_k) \cdot D_+\psi = \\
&= D\psi - \frac{1}{2r}\,(m + i\Omega)\, D_+\psi = \\
&= D\psi - \frac{1}{2r}\,(m - (m - 2\,(s+1)))\, D_+\psi = \\
&= D\psi - \frac{s+1}{r} D_+\psi = D_-\psi + \frac{r-1-s}{r} D_+\psi.
\end{aligned}$$

This implies $D_-\psi = 0$ and $D_+\psi = 0$ and, hence, $\nabla\psi = 0$ by equation (1.2), q.e.d.

Lemma 1.2 *Let $\psi \in \Gamma(S_s)$ with $s \in \{0, \cdots, m\}$, then there is the identity*

$$\left|\mathcal{D}^{(r,+)}\psi\right|^2 = \frac{1}{2}\ div\left(grad\quad |\psi|^2 - \frac{s+1-r}{2r^2}J\left(s_\psi^+\right)\right) + \tag{1.28}$$

$$+\ Re\ \langle D^2\psi + \frac{s+1-2r}{2r^2}D_-D_+\psi - \frac{R}{4}\psi, \psi\rangle$$

for each $r \in \{1, \cdots, m\}$.

Proof: Let $\psi \in \Gamma(S_s)$ and let $(X_1, \cdots, X_{2m})$ be any local frame of vector fields. Using (1.19) and (1.21) we calculate

$$\left|\mathcal{D}^{(r,+)}\psi\right|^2 = g^{kl}\left\langle \nabla_{X_k}\psi + \frac{1}{2r}\bar{p}\left(X_k\right)D_+\psi, \nabla_{X_l}\psi + \frac{1}{2r}\bar{p}(X_l)D_+\psi\right\rangle =$$

$$= g^{kl}\left\langle \nabla_{X_k}\psi, \nabla_{X_l}\psi\right\rangle - \frac{1}{2r}\left(\left\langle p\left(X^k\right)\nabla_{X_k}\psi, D_+\psi\right\rangle + \left\langle D_+\psi, p\left(X^k\right)\nabla_{X_k}\psi\right\rangle\right) -$$

$$-\frac{1}{4r^2}\left\langle p\left(X^k\right)\bar{p}(X_k)D_+\psi, D_+\psi\right\rangle$$

$$= |\nabla\psi|^2 - \frac{1}{r}|D_+\psi|^2 + \frac{1}{4_{r^2}}\left\langle (m + i\Omega)\, D_+\psi, D_+\psi\right\rangle =$$

$$= |\nabla\psi|^2 - \frac{1}{r}|D_+\psi|^2 + \frac{s+1}{2r^2}|D_+\psi|^2$$

and obtain

$$\left|\mathcal{D}^{(r,+)}\psi\right|^2 = |\nabla\psi|^2 + \frac{s+1-2_r}{2r^2}|D_+\psi|^2,$$

By (1.24) and (1.25) this yields (1.28), q.e.d.

Proposition 1. *Let M^{2m} be a spin Kähler manifold and let $s \in \{0, 1, \cdots, m\}$, then the following holds:*
(i) For $\psi \in \Gamma(S_s)$, the Kählerian twistor equation of type $(r, +)$ is equivalent to the two equations

$$div\left(grad|\psi|^2 - \frac{s+1-2r}{2r^2}J\left(s_\psi^+\right)\right) = 0 \tag{1.29}$$

$$D^2\psi + \frac{s+1-2r}{2r^2} D_- D_+ \psi = \frac{R}{4}\psi. \tag{1.30}$$

Moreover, if M^{2m} is closed, then, for $\psi \in \Gamma(S_s)$, the equation $\mathcal{D}^{(r,+)}\psi = 0$ is equivalent to equation (1.30).
(ii) If M^{2m} is not Ricci-flat, then, for $\psi \in \Gamma(S_s)$ and $s \neq r-1$, the system of equations (1.29), (1.30) has the trivial solution only. Moreover, if M^{2m} is closed and not Ricci-flat, then equation (1.30) can only be solved trivially with $\psi \in \Gamma(S_s)$ and $s \neq r-1$.

Proof: By Lemma 1.2, $\mathcal{D}^{(r,+)}\psi = 0$ follows if $\psi \in \Gamma(S_s)$ satisfies the equations (1.29) and (1.30). Conversely, let $0 \neq \psi \in \Gamma(S_s) \cap \ker \mathcal{D}^{(r,+)}$. In case $s \neq r-1$, $\nabla\psi = 0$ follows by Lemma 1.1. Now, it is well-known that the existence of a non-trivial parallel spinor implies the Ricci-flatness $Ric = 0$ and, hence, $R = 0$. Thus, in case $s \neq r-1$, the equations (1.29), (1.30) are satisfied trivially. For $s = r-1$, equation (1.30) holds by (1.4), (1,5) and (1.27). Moreover, we have $s_\psi = s_\psi^+$ by (1.4) and, hence, (1.29) by (1.7). Integrating equation (1.28) we prove the corresponding assertion for the closed case by the same arguments as before.
The part (ii) of Proposition 1 follows immediately from part (i) and Lemma 1.1, q.e.d.

Remark *By Lemma 1.1 Kählerian twistor spinors of type $(r,+)$ can only be sections of S_{r-1}. (This is already known by Proposition 5 in [6].)*

Theorem 2 *Let M^{2m} be a closed spin Kähler manifold with scalar curvature $R \not\equiv 0$ and let $r \in \{1, 2, \cdots, m\}$. Then, for $0 \neq \psi \in \Gamma(S)$, the following conditions are equivalent:*
(i) ψ is a Kählerian twistor spinor of type $(r,+)$.
(ii) $\psi \in \Gamma(S_{r-1})$ and ψ satisfies the equation

$$D^2\psi = \frac{1}{2r} D_- D_+ \psi + \frac{R}{4}\psi. \tag{1.31}$$

(iii) $\psi \in \Gamma(S_{r-1})$ and the equations

$$D^2\psi = \frac{rR}{4r-2}\psi, \quad D_-\psi = 0 \tag{1.32}$$

are valid.
(iv) $\psi \in \Gamma(S_{r-1})$ *and the equations*

$$D_-D_+\psi = \frac{rR}{4r-2}\psi \quad ,(\partial R)\cdot\psi = 0 \tag{1.33}$$

are satisfied.

Proof: By Proposition 1, the conditions (i) and (ii) are equivalent. Moreover, we see from (1.4) – (1.6), (1.27) that (i) implies (iii) and (iv). Further, equation (1.31) can be written in the form

$$\frac{2r}{2r-1}D_+D_-\psi + D_-D_+\psi = \frac{rR}{4r-2}\psi$$

and the system (1.32) is equivalent to

$$D_-D_+\psi = \frac{rR}{4r-2}\psi \quad ,D_-\psi = 0.$$

Thus, (iii) implies (ii). Finally, applying the operator D_- to the first equation of the system (1.33), by the second equation of (1.33) and the identities (1.18) and (1.26), we have

$$0 = D_-(R\cdot\psi) = (\partial R)\cdot\psi + R\cdot D_-\psi = R\cdot D_-\psi$$

and, hence, the equation $R\cdot D_-\psi = 0$ providing $D_-\psi = 0$ since $R \not\equiv 0$ by supposition. Thus, (iv) implies (iii), q.e.d.

A spinor field $\psi \in \Gamma(S)$ is called D_--closed (D_+-closed) iff there is the equation $D_-\psi = 0(D_+\psi = 0)$. If ψ is D_--closed (D_+-closed), then ψ is called D_--exact (D_+-exact) iff there is a spinor field φ such that $\psi = D_-\varphi(\psi = D_+\varphi)$. In both cases such a φ is called a potential of ψ.
For example, each Kählerian twistor spinor of type $(r,+)$ is D_--closed by (1.4).

2 The structure of the Kählerian E-equations

In this section we derive the possible kinds of E-equations in the Kähler case. Let (M^{2m}, J, g, S) be a (connected) spin Kähler manifold with scalar curvature $R \not\equiv 0$. To find the Kähler counterpart of the equation E of Lichnerowicz we start with the general equation

$$\nabla_X D\psi + \frac{R}{2}(aX + ibJX)\cdot\psi = 0 \quad , \tag{2.1}$$

where $(0,0) \neq (a,b) \in \mathbb{C}^2$ must be determined in a suitable way.

Proposition 3 *Let M^{2m} be closed with $m \geq 3$. Moreover, let $R \not\equiv 0$ and $R \geq 0$. Then, for $a^2 \neq b^2$, the equation (2.1) has the trivial solution only.*

Proof: Let $(X_1, \cdots, X_{2m})$ be any local frame of vector fields. Then we have by (2.1)

$$\begin{aligned} 0 &= X^k \cdot \left(\nabla_{X_k} D\psi + \frac{R}{2}(aX_k + ibJX_k)\,\psi\right) = \\ &= D^2\psi + \frac{R}{2}(-2ma - 2ib\Omega)\,\psi = \\ &= D^2\psi - R\,(ma + ib\Omega)\,\psi \end{aligned}$$

and, hence,

$$D^2\psi = R\,(ma + ib\Omega)\,\psi. \tag{2.2}$$

On the other hand, it follows

$$\begin{aligned} 0 &= J\left(X^k\right)\left(\nabla_{X_k} D\psi + \frac{R}{2}(aX_k + ibJX_k)\,\psi\right) = \\ &= \tilde{D}D\psi + \frac{R}{2}(2a\Omega - 2imb)\,\psi = \\ &= \tilde{D}D\psi - R\,(imb - a\Omega)\,. \end{aligned}$$

This yields the equation

$$\tilde{D}D\psi = R(imb - a\Omega)\psi. \tag{2.3}$$

By (1.27), the equations (2.2) and (2.3) are equivalent to the equations

$$D_+D_-\psi = \frac{a+b}{2}R(m + i\Omega)\psi \tag{2.4}$$

$$D_-D_+\psi = \frac{a-b}{2}R(m - i\Omega)\psi. \tag{2.5}$$

Now, let $\psi = \psi_0 + \psi_1 + \cdots + \psi_m$ be the decomposition of ψ according to (1.1). We see that ψ satisfies the equations (2.4) and (2.5) iff each component ψ_r of ψ satisfies these equations. Hence, by (1.19) the system of equations (2.4), (2.5) is equivalent to

$$\begin{aligned} D_+D_-\psi_r &= r(a+b)R\psi_r \qquad (2.6)\\ &\qquad\qquad (r = 0, 1, \cdots, m)\\ D_-D_+\psi_r &= (m-r)(a-b)R\psi_r. \qquad (2.7) \end{aligned}$$

By (1.26) and $a^2 \neq b^2$, this implies $D\,(R\psi_r) = 0$ for $r \neq 0, m$. But it is well-known that in case M^{2m} is closed with $R \not\equiv 0$, and $R \geq 0$ each harmonic spinor is zero. Thus, it follows that $R\psi_r = 0$ and, hence, $\psi_r = 0$ for $r \neq 0, m$. Hence, we have $\psi = \psi_0 + \psi_m$. Inserting this into equation (2.1) we obtain

$$\begin{aligned} 0 &= \nabla_X D\,(\psi_0 + \psi_m) + \frac{R}{2}\,(aX + ibJX)\,(\psi_0 + \psi_m) = \\ &= \nabla_X D_+\psi_0 + \nabla_X D_-\psi_m + \frac{a-b}{2}Rp(X)\psi_0 + \frac{a+b}{2}R\bar{p}(X)\psi_m. \end{aligned}$$

Since $\nabla_X D_+\psi_0 + \frac{a-b}{2}Rp(X)\psi_0 \in \Gamma(S_1)$ and $\nabla_X D_-\psi_m + \frac{a+b}{2}R\bar{p}(X)\psi_m \in \Gamma(S_{m-1})$, the equations

$$\nabla_X D_+\psi_0 + \frac{a-b}{2}Rp(X)\psi_0 = 0, \qquad (*)$$

$$\nabla_X D_-\psi_m + \frac{a+b}{2}R\bar{p}(X)\psi_m = 0 \qquad (2*)$$

follow in case $m \neq 2$. Moreover, by (2.6) and (2.7), for $r = 0$ and $r = m$, we have

$$D_+D_-\psi_m = m(a+b)R\psi_m, \qquad (3*)$$

$$D_-D_+\psi_0 = m(a-b)R\psi_0. \qquad (4*)$$

Inserting (4*) into (*) and (3*) into (2*) we obtain

$$\nabla_X D_+\psi + \frac{1}{2m} p(X) D_- D_+ \psi_0 = 0 \tag{5*}$$

$$\nabla_X D_-\psi_m + \frac{1}{2m} \bar{p}(X) D_+ D_- \psi_m = 0, \tag{6*}$$

i.e., $D_+\psi_0$ and $D_-\psi_m$ are Kählerian twistor spinors of type m. By a vanishing theorem (see [7], Th. 17) this implies $D_+\psi_0 = 0$ and $D_-\psi_m = 0$, i.e., $D\psi_0 = 0$ and $D\psi_m = 0$. Hence, it follows that $\psi_0 = 0, \psi_m = 0$ since there are trivial harmonic spinors only in the case considered here, q.e.d.

Proposition 3 shows that equation (2.1) is a suitable candidate for a Kählerian E-equation in the cases $b = a$ and $b = -a$ only. Thus, we obtain two possible types of Kählerian E-equations, namely

$$\nabla_X D\psi + aR\bar{p}(X) \cdot \psi = 0 \tag{2.8}$$

and

$$\nabla_X D\psi + aRp(X) \cdot \psi = 0, \tag{2.9}$$

where $0 \neq a \in \mathbb{C}$ is a parameter which will be determined in the following. Let $\mathcal{E}_a^-$ and $\mathcal{E}_a^+$ denote the $\mathbb{C}$-vector spaces of solutions of (2.8) and (2.9), respectively. Multiplying the equations (2.8), (2.9) by j we obtain the relations

$$j\mathcal{E}_a^- = \mathcal{E}_{\bar{a}}^+, \qquad j\mathcal{E}_a^+ = \mathcal{E}_{\bar{a}}^- \tag{2.10}$$

Hence, it is sufficient to consider equation (2.8).

Proposition 4 *Let M^{2m} be closed. Then the following holds:*
(i) equation (2.8) is equivalent to the two equations

$$\nabla_X D_-\psi + aR\bar{p}(X) \cdot \psi = 0 \quad , \tag{2.11}$$

$$D_+\psi = 0. \tag{2.12}$$

(ii) If $\psi = \psi_0 = \psi_1 + \cdots + \psi_m$ satisfies equation (2.8), then $D_-\psi_r \in$ $\ker \mathcal{D}^{(r,+)}$ for $r = 1, 2, \cdots, m$.

Proof: The equations (2.11) and (2.12) imply equation (2.8) in general. Conversely, let us suppose that $\psi = \psi_0 + \psi_1 + \cdots + \psi_m$ satisfies equation (2.8). Then, in case $a = b$, the equations (2.6) and (2.7) have the form

$$\begin{aligned} D_+D_-\psi_r &= 2raR\psi_r && (2.13) \\ & \qquad\qquad (r = 0, 1, \cdots, m) \\ D_-D_+\psi_r &= 0. && (2.14) \end{aligned}$$

By the latter equation we have

$$0 = \int_{M^{2m}} \langle D_-D_+\psi_r, \psi_r\rangle = \int_{M^{2m}} |D_+\psi_r|^2 = ||D_+\psi_r||^2$$

and, hence, $D_+\psi_r = 0$ for $r = 0, 1, \cdots, m$, i.e., $D_+\psi = 0$. Thus, ψ satisfies equation (2.11) since it is a solution of (2.8) and (2.12). This proves the assertion (i). Moreover, we see that the equations (2.11) and (2.12) are equivalent to the equations

$$\begin{aligned} \nabla_X D_-\psi_r &+ aR\bar{p}(X)\cdot\psi_r = 0 && (2.15) \\ & \qquad\qquad (r = 0, 1, \cdots, m) \\ D_+\psi_r &= 0, && (2.16) \end{aligned}$$

where $\psi_0, \cdots, \psi_m$ are the components of ψ. For $r \geq 1$, the equations (2.13) and (2.15) provide

$$\nabla_X D_-\psi_r + \frac{1}{2r}\bar{p}(X)\cdot D_+D_-\psi_r = 0 \qquad (r = 1, 2, \cdots, m), \quad (2.17)$$

i.e., we have $D_-\psi_r \in \ker \mathcal{D}^{(r,+)}$, q.e.d.

Let $\mathcal{E}_a^{(r,-)}$ and $\mathcal{E}_a^{(r,+)}$ be the $\mathbb{C}$-vector spaces defined by

$$\mathcal{E}_a^{(r,\pm)} := \Gamma(S_r) \cap \mathcal{E}_a^{\pm} \quad (r = 0, 1, \cdots, m).$$

Now, from Proposition 4, (i) we immediately obtain the

Corollary 4.1 *If M^{2m} is closed, then*

$$\mathcal{E}_a^{\pm} = \oplus_{r=0}^{m}\mathcal{E}_a^{(r,\pm)}. \qquad (2.18)$$

Thus, we see that in the closed case the space $\mathcal{E}_a^-$ can be determined solving the system of equations (2.11), (2.12) on each subbundle S_r of S.

Proposition 5 *Let M^{2m} be a closed spin Kähler manifold with scalar curvature $R \not\equiv 0$ and $R \geq 0$. Then the following holds:*
(i) The space $\mathcal{E}_a^{(0,-)}$ is zero for all $0 \neq a \in \mathbb{C}$.
(ii) If $m \geq 2$, then $\mathcal{E}_a^{(m,-)} = 0$ for all $0 \neq a \in \mathbb{C}$.
(iii) If there is an $r \geq 2$ such that $\mathcal{E}_a^{(r,-)} \neq 0$, then R is constant and, moreover,

$$a = \frac{1}{8r-4} \quad , \tag{2.19}$$

$$\mathcal{E}_a^- = \mathcal{E}_a^{(r,-)} \quad , \tag{2.20}$$

$$D_-\mathcal{E}_a^{(r,-)} = \ker \mathcal{D}^{(r,+)}. \tag{2.21}$$

Proof: Let $\psi_0 \in \mathcal{E}_a^{(0,-)}$. Then $0 = D_+\psi_0 = D\psi_0$ follows by (2.16) and, hence, $\psi_0 = 0$ since there are no harmonic spinors in the closed case with $R \not\equiv 0, R \geq 0$. If $\psi_m \in \mathcal{E}_a^{(m,-)}$, then it follwos from Proposition 4, (ii) that $D_-\psi_m$ is a Kählerian twistor spinor of type m. By Proposition 11 and Theorem 18 in [7] (vanishing theorem), this yields $0 = D_-\psi_m = D\psi_m$ and, hence, $\psi_m = 0$. Thus, we have (i) and (ii). Now, let us suppose that there is an $r \geq 2$ with $\mathcal{E}_a^{(r,-)} \neq 0$. We choose an element $0 \neq \psi_r \in \mathcal{E}_a^{(r,-)}$. By (2.13) and Proposition 4, (ii), ψ_r yields an element $0 \neq D_-\psi_r \in \ker \mathcal{D}^{(r,+)}$. Since in the closed case the function G_r is an obstruction for Kählerian twistor spinors of type $(r,+)$, this implies $G_r = 0$ and thus

$$(\partial R)\psi_r = 0 \tag{*}$$

by (1.16), (2.13) and $R > 0$. Moreover, applying the operator D_+ to equation (2.13) we obtain by (1.26) and (2.16)

$$0 \;=\; D_+^2 D_-\psi_r = 2raD_+(R\psi_r) =$$

$$
\begin{aligned}
&= 2ra\left((\bar\partial R)\cdot\psi_r + RD_+\psi_r\right) = \\
&= 2ra(\bar\partial R)\cdot\psi_r,
\end{aligned}
$$

i.e., the equation

$$(\bar\partial R)\cdot\psi_r = 0. \tag{2*}$$

The equations (*), (2*) imply that R is constant. Thus, applying the operator D_- to equation (2.13) we derive the identity

$$D_-D_+D_-\psi_r = 2raRD_-\psi_r.$$

Comparing this with (1.5) for $\psi = D_-\psi_r$ we obtain equation (2.19), which by (i) immediately implies

$$\mathcal{E}_a^{(s,-)} = 0 \qquad (s \neq 1, r).$$

Thus, to prove (2.20) it remains to show that $\mathcal{E}_a^{(1,-)} = 0$ holds. Let $\psi_1 \in \mathcal{E}_a^{(1,-)}$. Then, by (2.13) and (2.19), ψ_1 satisfies the equation

$$D_+D_-\psi_1 = \frac{R}{4r-2}\psi_1, \tag{3*}$$

which immediately implies

$$D_-D_+D_-\psi_1 = \frac{R}{4r-2}D_-\psi_1 \tag{4*}$$

since R is constant. On the other hand, we have $D_-\psi_1 \in \ker\mathcal{D}^{(1,+)}$ by Proposition 4, (ii). By equation (1.5) this yields

$$D_-D_+D_-\psi_1 = \frac{R}{2}D_-\psi_1. \tag{5*}$$

Comparing (4*) and (5*) we obtain $D_-\psi_1 = 0$ since $R \geq 2$ and, hence, $\psi_1 = 0$ by (3*).
To prove (2.21) it remains to check that each $\psi \in \ker\mathcal{D}^{(r,+)}$ has the form $\psi = D_-\varphi$ with $\varphi \in \mathcal{E}_a^{(r,-)}$. Let $\psi \in \ker\mathcal{D}^{(r,+)}$. Then we define

$$\varphi := \frac{4r-2}{rR} D_+\psi \tag{6*}$$

and from $dR = 0$ and (1.5) we see easily that $D_-\varphi = \psi$ follows. Thus, by (1.2) and (6*), we have

$$\begin{aligned} 0 &= \nabla_X \psi + \frac{1}{2r}\bar{p}(X) \cdot D_+\psi = \\ &= \nabla_X D_-\varphi + \frac{R}{8r-4}\bar{p}(X) \cdot \varphi = \\ &= \nabla_X D_-\varphi + aR\bar{p}(X) \cdot \varphi. \end{aligned}$$

This yields $\varphi \in \mathcal{E}_a^{(r,-)}$ since the equation $D_+\varphi = 0$ is satisfied trivially, q.e.d.

The Propositions 4 and 5 give us the motive for the following basic definitions:
Let M^{2m} be a spin Kähler manifold with scalar curvature $R \not\equiv 0$ and let $r \in \{0, 1. \cdots, m\}$. Then, for $\psi \in \Gamma(S)$, we call the equations

$$\nabla_X D_-\psi + \frac{R}{8r-4}\bar{p}(X) \cdot \psi = 0 \tag{2.22}$$

and

$$\nabla_X D_+\psi + \frac{R}{8r-4}p(X) \cdot \psi = 0 \tag{2.23}$$

the Kählerian E-equations of type $(r,-)$ and $(r,+)$, respectively, and by $\mathcal{E}_-^r$ and $\mathcal{E}_+^r$ we denote the corresponding $\mathbb{C}$-vector spaces of solutions ψ of these equations. Each spinor field $0 \neq \psi \in \mathcal{E}_-^r$ $(0 \neq \psi \in \mathcal{E}_+^r)$ we call a Kählerian E-spinor of type $(r,-)$ (of type $(r,+)$). Moreover, we say that $\psi \in \mathcal{E}_-^r$ $(\psi \in \mathcal{E}_+^r)$ is closed iff ψ has the additional property $D_+\psi = 0$ $(D_-\psi = 0)$.
We remark that the E-equations (2.22), (2.23) as well as the equations for closed E-spinors are compatible with the splitting (1.1) of the spinor bundle, i.e. $\psi \in \Gamma(S)$ satisfies the E-equation (2.22) or (2.23), respectively, iff each component of ψ has this property. The equations (2.8) and (2.9) are not compatible with the splitting (1.1) in general since, for example,

the component of equation (2.8) in the subbundle $S_s (s = 0, 1, \cdots, m)$ has the form

$$\nabla_X D_+ \psi_{s-1} + \nabla_X D_- \psi_{s+1} + aR\bar{p}(X) \cdot \psi_{s+1} = 0, \tag{2.24}$$

where $\psi = \psi_0 + \psi_1 + \cdots + \psi_m$ is the corresponding decomposition of ψ. The compatiblility implies the essential fact that $D_- \psi_s \in \ker \mathcal{D}^{(s,+)}$ for $s = 1, 2, \cdots, m$ providing the possibility to apply the theory of twistor spinors. On the other hand, we see that, in general, (2.24) implies $D_- \psi_s \in \ker \mathcal{D}^{(s,+)}$ for $s = 1$ only.

3 The relation between E-spinors and twistor spinors

In the following we always suppose that the scalar curvature R of the underlying spin Kähler manifold M^{2m} does not vanish identically.
Moreover, we use the notations

$$\mathcal{E}_{\pm}^{(r,s)} := \mathcal{E}_{\pm}^{r} \cap \Gamma(S_s)$$

for $r, s \in \{0, 1, \cdots, m\}$. The elements of $\mathcal{E}_{+}^{(r,s)}$ and $\mathcal{E}_{-}^{(r,s)}$ are called E-spinors of type $(r, s, +)$ and $(r, s, -)$, respectively. By the compatibility of the Kählerian E-equations with the splitting (1.1) we have the decompositions

$$\mathcal{E}_{\pm}^{r} = \oplus_{s=0}^{m} \mathcal{E}_{\pm}^{(r,s)} \tag{3.1}$$

as well as the identities

$$\begin{aligned} j \cdot \mathcal{E}_{\pm}^{(r,s)} &= \mathcal{E}_{\pm}^{(r,m-s)}, & (3.2) \\ \mathcal{E}_{-}^{(r,0)} &= \Gamma(S_0), & (3.3) \\ \mathcal{E}_{+}^{(r,m)} &= \Gamma(S_m). & (3.4) \end{aligned}$$

Thus, we see that it is sufficient to consider the spaces $\mathcal{E}_{-}^{(r,s)}$ for $r = 0, 1, \cdots, m$ and $s = 1, 2, \cdots, m$. Further, in case of closed E-spinors there are the relations

$$\mathcal{E}_{\pm}^{r} \cap \ker D_{\mp} = \oplus_{s=0}^{m} \left(\mathcal{E}_{\pm}^{(r,s)} \cap \ker D_{\mp} \right). \tag{3.5}$$

Proposition 6 *Let $r \in \{0, 1, \cdots, m\}$ and $s \in \{1, 2, \cdots, m\}$. Then the following holds:*

(i) Each $\psi \in \mathcal{E}_-^{(r,s)}$ satisfies the equations

$$D_+ D_- \psi = \frac{sR}{4r-2}\psi, \tag{3.6}$$

$$\left(\bar{\partial} R\right) \cdot \psi = -R D_+ \psi \tag{3.7}$$

$$(\partial R) \cdot \psi = 2\frac{r-s}{2s-1} R D_- \psi. \tag{3.8}$$

(ii) There is the inclusion

$$D_- \mathcal{E}_-^{(r,s)} \subseteq \ker \mathcal{D}^{(s,+)}$$

and the kernel of the map

$$\psi \in \mathcal{E}_-^{(r,s)} \longmapsto D_- \psi \in \ker \mathcal{D}^{(s,+)}$$

is zero.
(iii) We have the identity of spaces

$$\mathcal{E}_-^{(r,s)} = \ker \left(D_+ D_- - \frac{sR}{4r-2} \right) \cap D_-^{-1} \left(\ker \mathcal{D}^{(s,+)} \right).$$

Proof: Let $\psi \in \mathcal{E}_-^{(r,s)}$, i.e., $\psi \in \Gamma(S_s)$ and ψ satisfies equation (2.22). Thus, with respect to any local frame $(X_1, \cdots, X_{2m})$ we have

$$\begin{aligned} 0 &= X^k \cdot \left(\nabla_{X_k} D_- \psi + \frac{R}{8r-4} \bar{p}\,(X_k) \cdot \psi \right) = \\ &= D D_- \psi - \frac{R}{8r-4} (m + i\Omega)\, \psi = \\ &= D_+ D_- \psi - \frac{sR}{4r-2} \psi \end{aligned}$$

and, hence, equation (3.6). Applying D_+ to (3.6) we obtain the equation

$$D_+ (R\psi) = 0, \tag{3.9}$$

which is equivalent to (3.7). Inserting (3.6) into (2.22) we find

$$\nabla_X D_-\psi + \frac{1}{2s}\bar{p}\,(X)\cdot D_+D_-\psi = 0. \tag{3.10}$$

Thus, we see that $D_-\psi \in \ker\mathcal{D}^{(s,+)}$. Moreover, $\psi \neq 0$ and $R \not\equiv 0$ implies $D_-\psi \neq 0$ by (3.6). This proves the assertion (ii). Since $D_-\psi \in \ker\mathcal{D}^{(s,+)}$, we have the equation

$$D_-D_+D_-\psi = \frac{sR}{4s-2}D_-\psi \tag{3.11}$$

by (1.5). Applying D_- to (3.6) and comparing the result with (3.11) we obtain the equation (3.8). This completes the proof of assertion (i). To prove (iii) we remark that the intersection contains $\mathcal{E}_-^{(r,s)}$ by (3.6), and (3.10). Conversely, let us suppose that $\psi \in \Gamma(S)$ satisfies the equations (3.6) and (3.10). Inserting (3.6) into (3.10) we see that ψ is a solution of equation (2.22). It remains to show that $\psi \in \Gamma\,(S_s)$. As we have seen in the proof of equation (3.6) equation (2.22) implies

$$D_+D_-\psi = \frac{R}{8r-4}(m+i\Omega)\psi.$$

Comparing this equation with equation (3.6) and using the supposition $R \not\equiv 0$ we obtain

$$\Omega\psi = i(m-2s)\psi,$$

i.e., $\psi \in \Gamma\,(S_s)$, q.e.d.

From equation (3.8) we immediately derive

Corollary 6.1 *For each $r \in \{0,1,\cdots,m\}$ and each $\psi \in \mathcal{E}_-^{(r,r)}$, the equation*

$$(\partial R)\cdot\psi = 0 \tag{3.12}$$

is valid.

Corollary 6.2 *Let $r \in \{0,1,\cdots,m\}$ and $s \in \{1,2,\cdots,m\}$. If $\psi \in \mathcal{E}_-^{(r,s)}$, then $\varphi := D_-\psi \in \ker\mathcal{D}^{(s,+)}$ satisfies the equation*

$$(\partial R)\cdot D_+\varphi = \frac{s(r-s)}{(2s-1)(2r-1)}R^2\varphi. \tag{3.13}$$

Proof: Multiplying equation (3.8) by $\frac{sR}{4r-2}$ we obtain equation (3.13) using (3.6), q.e.d.

The assertion (ii) of Proposition 6 immediately yields

Corollary 6.3 *If* $\ker \mathcal{D}^{(s,+)} = 0$ *for any* $s \in \{1, 2, \cdots, m\}$, *then* $\mathcal{E}_-^{(r,s)} = 0$ *for all* $r \in \{0, 1, \cdots, m\}$.

Corollary 6.4 *Let* $\psi \in \mathcal{E}_-^{(r,s)}$ *with* $s \geq 2$, *then the functions*

$$\begin{aligned} E_\psi \,:\, &= \frac{sR}{4s-2}|D_-\psi|^2 + \left(\frac{sR}{4r-2}\right)^2 \cdot |\psi|^2, \\ P_\psi \,:\, &= \left(\triangle R + \frac{R^2}{2s} - \frac{2s-1}{s}|Ric|^2\right) \cdot |D_-\psi|^{2s} \end{aligned}$$

are constant.

Proof: $\psi \in \mathcal{E}_-^{(r,s)}$ implies $D_-\psi \in \ker \mathcal{D}^{(s,+)}$ by Proposition 6, (ii). Hence, for $s \geq 2$, the functions $C_{D_-\psi}$ and $Q_{D_-\psi}$ are constant (see Section 1). By definition, we have $P_\psi = Q_{D_-\psi}$ and equation (3.6) yields $E_\psi = C_{D_-\psi}$, q.e.d.

The constants E_ψ and P_ψ are the basic first integrals of the Kählerian E-equations.

Proposition 7 *Suppose that* $\mathcal{E}_-^{(r,s)} \neq 0$ *with* $s \geq 2$. *Then the following holds:*
(i) The scalar curvature R *does not possess any zeros.*
(ii) There is the equation

$$G_s = 2\frac{(r-s)(s-1)}{(2r-1)(2s-1)}R^2. \tag{3.14}$$

Moreover, the equation

$$|dR|^2 = \frac{s(r-s)}{2(2r-1)(2s-1)}R^3 \tag{3.15}$$

is valid if $r \neq s$.
(iii) There is the identity

$$D_-\mathcal{E}_-^{(r,s)} = \ker \mathcal{D}^{(s,+)},$$

i.e., the spaces $\mathcal{E}_-^{(r,s)}$ and $\ker \mathcal{D}^{(s,+)}$ are isomorphic.

Proof: Let $0 \neq \psi \in \mathcal{E}_-^{(r,s)}$ with $s \geq 2$ and let us assume that $P \in M^{2m}$ is a zero of R. Then we have $E_\psi = 0$ by Corollary 6.4 and, hence,

$$\frac{s}{4s-2}|D_-\psi|^2 + \left(\frac{s}{4r-2}\right) R|\psi|^2 = 0$$

since $R \not\equiv 0$. Because of $R(P) = 0$ this equation and equation (3.6) yield

$$(D_-\psi)_P = 0 \quad , \quad (D_+D_-\psi)_P = 0. \tag{*}$$

Since $D_-\psi \in \ker \mathcal{D}^{(s,+)}$ by Proposition 6, (ii), ψ satisfies equation (3.10) and, furthemore the equation

$$\nabla_X D_+D_-\psi + \frac{1}{2}p(Ric(X)) \cdot D_-\psi = 0 \tag{3.16}$$

by (1.11) and (1.12). Now, on the subbundle $S_{s-1} \oplus S_s$ of S we consider the covariant derivative $\nabla^{(s)}$ defined by

$$\nabla_X^{(s)}(\psi_{s-1} + \psi_s) = \nabla_X(\psi_{s-1} + \psi_s) + \frac{1}{2s}\bar{p}(X) \cdot \psi_s + \frac{1}{2}p(Ric(X)) \cdot \psi_{s-1}.$$

From (3.10) and (3.16) we see that

$$\nabla_X^{(s)}(D_-\psi + D_+D_-\psi) = 0,$$

i.e., the spinor field $D_-\psi + D_+D_-\psi$ is parallel with respect to $\nabla^{(s)}$. By (*) this implies $D_-\psi + D_+D_-\psi = 0$, i.e., $D_-\psi = 0$ and $D_+D_-\psi = 0$. Thus, we obtain the contradiction $\psi = 0$ by Proposition 6, (ii). This proves our assertion (i). Further, if $0 \neq \psi \in \mathcal{E}_-^{(r,s)}$, then $0 \neq \varphi := D_-\psi$ satisfies equation (3.13) by Corollary 6.2. Moreover, since $\varphi \in \ker \mathcal{D}^{(s,+)}$ with $s \geq 2$, it follows from (1.16) that the equation

$$(\partial R)\cdot D_+\varphi = \frac{s}{2s-2} G_s\cdot\varphi \tag{2*}$$

is valid. Comparing the equations (3.13) and (2*) we obtain

$$\left(G_s - 2\frac{(r-s)(s-1)}{(2r-1)(2s-1)}R^2\right)\cdot\varphi = 0$$

and, hence, (3.14) since $\varphi\neq 0$. Furthermore, by Theorem 6 in [8], we have

$$g(\mathrm{grad}(G_s),\mathrm{grad}(R)) = \frac{s}{2(s-1)}G_s^2. \tag{3*}$$

Equation (3.14) implies

$$\mathrm{grad}\ (G_s) = 4\frac{(r-s)(s-1)}{(2r-1)(2s-1)}R\ \mathrm{grad}\ (R).$$

This equation yields

$$g(\mathrm{grad}(G_s),\ \mathrm{grad}\ (R)) = 4\frac{(r-s)(s-1)}{(2r-1)(2s-1)}R|dR|^2. \tag{4*}$$

Comparing the equations (3*) and (4*) we obtain (3.15) using (3.14). To prove the assertion (iii) we have to prove that

$$D_-\mathcal{E}_-^{(r,s)} = \ker\mathcal{D}^{(s,+)}$$

is valid. For any given $\varphi\in\ker\mathcal{D}^{(s,+)}$, we can define the spinor

$$\psi := \frac{4r-2}{s}R^{-1}D_+\varphi \tag{5*}$$

since R has no zero according to (i). Using the equations (1.5), (1.16) and (3.14) we calculate

$$\begin{aligned}
D_-\psi &= \frac{4r-2}{s}D_-(R^{-1}D_+\varphi) = \\
&= \frac{4r-2}{s}(-R^{-2}(\partial R)\cdot D_+\varphi + R^{-1}D_-D_+\varphi) = \\
&= \frac{4r-2}{s}\left(-R^{-2}\frac{s}{2s-2}G_s\cdot\varphi + \frac{s}{4s-2}\varphi\right) = \\
&= \frac{4r-2}{s}\left(-\frac{s(r-s)}{(2r-1)(2s-1)} + \frac{s}{4s-2}\right)\varphi = \varphi.
\end{aligned}$$

Thus, the result of our calculation is

$$D_-\psi = \varphi. \tag{6*}$$

It remains to show that $\psi \in \mathcal{E}_-^{(r,s)}$. Since $\varphi \in \ker \mathcal{D}^{(s,+)}$, we have the equation

$$\nabla_X\varphi + \frac{1}{2s}\bar{p}(X)\cdot D_+\varphi = 0. \tag{7*}$$

Inserting (5*) and (6*) into (7*) we obtain

$$\nabla_X D_-\psi + \frac{R}{8r-4}\bar{p}(X)\cdot\psi = 0$$

and, hence, $\psi \in \mathcal{E}_-^{(r,s)}$ because $\psi \in \Gamma(S_s)$ by Definition (5*), q.e.d.

The identities (3.3) and (3.4) motivate the introduction of the reduced E-spaces

$$\tilde{\mathcal{E}}_-^r := \bigoplus_{s=1}^{m} \mathcal{E}_-^{(r,s)} \quad , \quad \tilde{\mathcal{E}}_+^r := \bigoplus_{s=0}^{m-1} \mathcal{E}_+^{(r,s)}.$$

Corollary 7.1 *If R has a critical point, then $\mathcal{E}_-^{(r,s)} = 0$ for $r \neq s \geq 2$, i.e.,*

$$\begin{aligned} \tilde{\mathcal{E}}_-^0 &= \mathcal{E}_-^{(0,1)} \quad , \quad \tilde{\mathcal{E}}_-^1 = \mathcal{E}_-^{(1,1)}, \\ \tilde{\mathcal{E}}_-^r &= \mathcal{E}_-^{(r,1)} \oplus \mathcal{E}_-^{(r,r)} \quad (r = 2, 3, ..., m). \end{aligned} \tag{3.17}$$

Proof: Assume that there is a space $\mathcal{E}_-^{(r,s)} \neq 0$ with $r \neq s \geq 2$. Then equation (3.15) shows that each critical point of R is a zero of R. Thus, R has a zero. But this contradicts assertion (i) of Proposition 7, q.e.d.

By Corollary 7.1 we see in particular that, if M^{2m} is closed, only the spaces $\mathcal{E}_-^{(r,1)} (r = 0, 1, ..., m)$ and $\mathcal{E}_-^{(r,r)}$ with $r \geq 2$ are of interest .

Corollary 7.2 *Suppose that M^{2m} is closed. If there is an $r \geq 2$ with $\mathcal{E}_-^{(r,r)} \neq 0$, then R is positive.*

Proof: By Proposition 7, (i) there are the possibilities $R > 0$ and $R < 0$ only. Assume that $R < 0$. Then, for any $0 \neq \psi \in \mathcal{E}_-^{(r,r)}$, we obtain by Proposition 6, (ii) and equation (3.6) the contradiction

$$0 < ||D_-\psi||^2 = \int_{M^{2m}} \langle D_+D_-\psi, \psi\rangle = \frac{r}{4r-2}\int_{M^{2m}} R|\psi|^2 < 0.$$

This implies $R > 0, q.e.d.$

From equation (3.14) we immediately obtain

Corollary 7.3 *Suppose that $\mathcal{E}_-^{(r,r)} \neq 0$ for any $r \geq 2$. Then $G_r \equiv 0$.*

By Corollary 7.3 the function G_r is a geometrical obstruction for the spaces $\mathcal{E}_-^{(r,r)}$ with $r \geq 2$. The assertions (i) and (ii) of Proposition 7 imply

Corollary 7.4 *Let $\mathcal{E}_-^{(r,s)} \neq 0$ with $r \neq s \geq 2$. Then the functions G_s and R do not possess any zero or any critical point. G_s and R are positive (negative) iff $r > s$ $(r < s)$.*

Corollary 7.5 *If $\psi \in \mathcal{E}_-^{(r,s)}$ with $s \geq 2$, then*

$$P_\psi = 2\frac{(r-s)(s-1)}{(2r-1)(2s-1)}R^2|D_-\psi|^{2s}. \tag{3.18}$$

Proof: Equation (3.18) is satisfied trivially if $\mathcal{E}_-^{(r,s)} = 0$. In case $\mathcal{E}_-^{(r,s)} \neq 0$, equation (3.14) yields (3.18), q.e.d.

Corollary 7.6 *Let $\mathcal{E}_-^{(r,s)} \neq 0$ with $r \neq s \geq 2$. Then each $\psi \in \mathcal{E}_-^{(r,s)}$ satisfies the equations*

$$\begin{aligned} (\bar{\partial} R) \cdot D_- \psi &= -\frac{s}{8r-4} R^2 \psi, && (3.19) \\ (\bar{\partial} R) \cdot \psi &= 0, && (3.20) \\ D_+ \psi &= 0. && (3.21) \end{aligned}$$

Proof: If $\psi \in \mathcal{E}_-^{(r,s)}$, then $D_-\psi \in \ker \mathcal{D}^{(s,r)}$ by Proposition 6, (ii). Thus, because of $s \geq 2$ we have

$$(\bar{\partial} G_s) \cdot D_- \psi = -G_s D_+ D_- \psi$$

by Proposition 5 in [8]. Using the equations (3.6) and (3.14) this equation yields (3.19). Multiplying (3.19) by $\bar{\partial} R$ we obtain (3.20). Finally, equation (3.20) implies (3.21) by (3.7), q.e.d.

By equation (3.21) we see that the E-spinors of type $(r, s, -)$ with $r \neq s \geq 2$ are closed.
We remark that the proof of Proposition 7, (iii) contains the proof of

Proposition 8 *Let M^{2m} be a spin Kähler manifold such that the scalar curvature R has no zero and equation (3.14) is satisfied for any $r \in \{0, 1, ..., m\}$ and any $s \in \{2, 3, ..., m\}$. Then the map*

$$\varphi \in \ker \mathcal{D}^{(s,+)} \mapsto \frac{4r-2}{s} R^{-1} D_+ \varphi \in \mathcal{E}_-^{(r,s)}$$

is an isomorphism of the spaces $\ker \mathcal{D}^{(s,+)}$ and $\mathcal{E}_-^{(r,s)}$, where the corresponding inverse isomorphism is given by

$$\psi \in \mathcal{E}_-^{(r,s)} \mapsto D_- \psi \in \ker \mathcal{D}^{(s,+)}.$$

Theorem 9 *Let M^{2m} be a closed spin Kähler manifold of positive scalar curvature and let $r \in \{2, 3, ..., m\}$. Then the spaces $\mathcal{E}_-^{(r,r)}$ and $\ker \mathcal{D}^{(r,+)}$ are isomorphic.*

Proof: If $\ker \mathcal{D}^{(r,+)} = 0$, then $\mathcal{E}_-^{(r,r)} = 0$ by Proposition 6, (ii). The case $\ker \mathcal{D}^{(r,r)} \neq 0$ implies $G_r \equiv 0$ since the function G_r is an obstruction for Kählerian twistor spinors of type $(r,+)$ with $r \geq 2$ if M^{2m} is closed. Thus, equation (3.14) is satisfied trivially for $r = s \geq 2$ and, hence, for the suppositions of Proposition 8, q.e.d.

Moreover, we immediately obtain

Proposition 10 *Under the assumptions of Proposition 8, each Kählerian twistor spinor of type* $(s,+)$ *possesses a uniquely determined potential in* $\mathcal{E}_-^{(r,s)}$*, i.e., for each* $\varphi \in \ker \mathcal{D}^{(s,+)}$*, there is a unique* $\psi \in \mathcal{E}_-^{(r,s)}$ *such that* $\varphi = D_-\psi$*.*

The Kählerian E-equations are essentially overdetermined. However, in the closed case the spaces $\mathcal{E}_-^{(r,r)}$ in the decomposition (3.17) can be described by simpler equations. This is shown by

Theorem 11 *Let* M^{2m} *be closed and let* $r \in \{0, 1, ..., m\}$*. Then we have* $\psi \in \mathcal{E}_-^{(r,r)}$ *iff* $\psi \in \Gamma(S_r)$ *and* ψ *satisfies the two equations*

$$D_+D_-\psi = \frac{rR}{4r-2}\psi \quad , \quad (\partial R) \cdot \psi = 0. \tag{3.22}$$

Proof: The assertion is trivial for $r = 0$. Thus, we have to consider the case $r \in \{1, 2, ..., m\}$. Let $\psi \in \mathcal{E}_-^{(r,r)}$, then ψ is a solution of the equations (3.22) by Theorem 6, (i). Conversely, let us suppose that $\psi \in \Gamma(S_r)$ satisfies the equations (3.22). Applying the operator D_- to the first equation and using the second one we obtain

$$D_-D_+D_-\psi = \frac{rR}{4r-2}D_-\psi,$$

which immediately implies

$$(\partial R) \cdot D_-\psi = 0.$$

Thus, we see that $D_-\psi \in \Gamma(S_{r-1})$ satisfies the equations (1.33). This implies $D_-\psi \in \ker \mathcal{D}^{(r,+)}$ by Theorem 2 and, hence, $\psi \in \mathcal{E}_-^{(r,r)}$ by Proposition

6, (iii), q.e.d.

Corollary 11.1 *Let M^{2m} be a closed spin Kähler manifold with $R \not\equiv 0$ and let $r \in \{2, 3, ..., m\}$. If the equations (3.22) have a non-trivial solution, then $R > 0$ and there is the equation*

$$\triangle R + \frac{1}{2r} R^2 = \frac{2r-1}{r} | \text{ Ric } |^2.$$

Proof: The proof of the assertion is given by Theorem 11 and the Corollaries 7.2, 7.3, q.e.d.

By the Propositions 7 and 8 we have a complete description of the spaces $\mathcal{E}_-^{(r,s)}$ with $s \geq 2$ in terms of Kählerian twistor spinors of type $(s, +)$. The following theorem deals with the structure of the exceptional spaces $\mathcal{E}_-^{(r,1)}$.

Theorem 12 *Let M^{2m} be a spin Kähler manifold with scalar curvature $R \not\equiv 0$. Then there is the inclusion*

$$D_- \mathcal{E}_-^{(r,1)} \subseteq \bar{H}^0(M^{2m}, S_0) \cap \ker(\nabla_{grad(R)} + \frac{r-1}{4r-2} R^2).$$

Moreover, if R has no zero, this inclusion is an equality.

Proof: By Proposition 6, (ii), we have the relation

$$D_- \mathcal{E}_-^{(r,1)} \subseteq \ker \mathcal{D}^{(1,+)},$$

and from Theorem 12 in [7] and Proposition 1 in [8] we know that the space $\ker \mathcal{D}^{(1,+)}$ coincides with the space $\bar{H}^0(M^{2m}, S_0)$ of global antiholomorphic sections of the line bundle S_0. Thus, to prove the inclusion above it remains to show that, for each $\psi \in \mathcal{E}_-^{(r,1)}$, the equation

$$\nabla_{\text{grad}(R)} D_- \psi + \frac{r-1}{4r-2} R^2 D_- \psi = 0 \qquad (3.23)$$

is valid. Since $\psi \in \mathcal{E}_-^{(r,1)}$,

$$\nabla_X D_- \psi + \frac{R}{8r-4} \bar{p}(X) \cdot \psi = 0$$

holds for all real vector fields X on M^{2m}. In particular, for $X = \mathrm{grad}(R)$, this yields the equation

$$\nabla_{\mathrm{grad}(R)} D_-\psi + \frac{R}{8r-4}(\partial R)\cdot\psi = 0. \tag{*}$$

Further, in case $s = 1$, equation (3.8) is of the form

$$(\partial R)\cdot\psi = 2(r-1)RD_-\psi. \tag{3.24}$$

Inserting this into equation (*) we obtain (3.23). Now we suppose that R has no zero. We have to show that the injective map

$$\psi \in \mathcal{E}_-^{(r,1)} \longmapsto D_-\psi \in \bar{H}^0(M^{2m}, S_0) \cap \ker\left(\nabla_{\mathrm{grad}(R)} + \frac{r-1}{4r-2}R^2\right)$$

is surjective, too. Let ψ_0 be an element of the intersection, then ψ_0 satisfies the equations

$$\nabla_X\psi_0 + \frac{1}{2}\bar{p}(X)\cdot D_+\psi_0 = 0, \tag{3.25}$$

$$\nabla_{\mathrm{grad}(R)}\psi_0 + \frac{r-1}{4r-2}R^2\psi_0 = 0. \tag{3.26}$$

These equations imply immediately

$$(\partial R)\cdot D_+\psi_0 = \frac{r-1}{2r-1}R^2\psi_0. \tag{3.27}$$

Moreover, we know that

$$D_-D_+\psi_0 = \frac{R}{2}\psi_0 \tag{3.28}$$

follows from (3.25). Now we define

$$\psi := \frac{4r-2}{R}D_+\psi_0. \tag{3.29}$$

Using (3.27) and (3.28) we find by a straight-forward calculation

$$D_-\psi = \psi_0. \tag{3.30}$$

Inserting (3.29) and (3.30) into (3.25) we see that ψ satisfies equation (2.22). Thus, we have $\psi \in \mathcal{E}_-^{(r,1)}$, q.e.d.

4 Kählerian E-spinors and the Einstein condition

The following theorem is the Kähler counterpart to Theorem 13 in [1].

Theorem 13 *Let M^{2m} be a spin Kähler manifold of scalar curvature $R \not\equiv 0$.*

A) If $m \geq 2$ and if there are integers $r \in \{0, 1, ..., m\}, s \in \{1, 2, ..., m-1\}$ with $\mathcal{E}^r_+ \cap \ker \mathcal{D}^{(s,+)} \neq 0$, then $r = s, m = 2r - 1$ and M^{2m} is Einstein.

B) If $m = 2r - 1 \geq 3$ and M^{2m} is Einstein, then $\tilde{\mathcal{E}}^r_+ = \ker \mathcal{D}^{(r,+)}$.

Proof: Let $0 \neq \psi \in \mathcal{E}^r_+ \cap \ker \mathcal{D}^{(s,+)}$ with $r \in \{0, 1, ..., m\}$ and $s \in \{1, 2, ..., m-1\}$. Since $\psi \in \mathcal{E}^r_+$, we have

$$\nabla_X D_+\psi + \frac{R}{8r-4} p(X) \cdot \psi = 0. \tag{*}$$

Particularly, this equation implies

$$\nabla_{\bar{p}(X)} D_+\psi = 0. \tag{2*}$$

Moreover, since $\psi \in \ker \mathcal{D}^{(s,+)}$, by (1.11) and (2*) ψ satisfies the equation

$$\nabla_X D_+\psi + \frac{1}{2} p(Ric(X)) \cdot \psi = 0 \; . \tag{3*}$$

Comparing the equation (*) and (3*) we obtain

$$p(Ric(X)) - \frac{R}{4r-2} X) \cdot \psi = 0 \; . \tag{4*}$$

Multiplying the equation (*) by j we find

$$\nabla_X D_- j\psi + \frac{R}{8r-4} \bar{p}(X) j\psi = 0, \tag{5*}$$

i.e., $j\psi \in \mathcal{E}^{(r,m-s+1)}_-$. By (3.6) this yields

$$D_+D_-j\psi = \frac{m-s+1}{4r-2}Rj\psi, \tag{6*}$$

or the equivalent equation

$$D_-D_+\psi = \frac{m-s+1}{4r-2}R\psi. \tag{7*}$$

On the other hand, because of $\psi \in \ker \mathcal{D}^{(s,+)}$, we have

$$D_-D_+\psi = \frac{sR}{4s-2}\psi \tag{8*}$$

by (1.5). Comparing (7*) and (8*) we obtain

$$r = m - s + 2 - \frac{m+1}{2s}\ . \tag{9*}$$

Now we show that

$$r = m - s + 1 \tag{10*}$$

holds. Assume that $r \neq m-s+1$. Then we have $r \neq m-s+1 \geq 2$ since $s \leq m-1$ by supposition. Thus, by $0 \neq j\psi \in \mathcal{E}_-^{(r,m-s+1)}$ we see that the suppositions of Corollary 7.6 are satisfied. This yields $0 = D_+j\psi = jD_-\psi$ by equation (3.21) and, hence, the contradiction $D_-\psi = 0$ by Proposition 6, (ii). Thus, we have (10*).
The equations (9*), (10*) are equivalent to

$$m = 2r - 1 \qquad (11^*) \qquad\qquad r = s. \qquad (12^*)$$

Equation (12*) implies $\psi \in \ker \mathcal{D}^{(r,+)}$ and, consequently,

$$\bar{p}\left(Ric(X) - \frac{R}{4r-2}X\right)\cdot\psi = 0 \tag{13*}$$

by (1.9). By (4*) and (13*) we obtain

$$\left(Ric - \frac{R}{4r-2}\right)(X)\cdot\psi = 0$$

and, hence ,

$$\left(Ric - \frac{R}{4r-2}\right)(X) = 0$$

since $\psi \neq 0$. This provides

$$Ric = \frac{R}{2m} \tag{14*}$$

by (11*), i.e., M^{2m} is Einstein. To prove the assertion B we remark that the supposition $m = 2r-1 \geq 3$ implies $r \geq 2$. Let $\psi \in \ker \mathcal{D}^{(r,+)}$, then ψ satisfies the equation (3*) by (1.11) and (1.12). Since M^{2m} is Einstein, equation (3*) is equivalent to the equation

$$\nabla_X D_+\psi + \frac{R}{4m} p(X) \cdot \psi = 0, \tag{15*}$$

which shows that $\psi \in \mathcal{E}_+^{(r,r-1)}$ since $4m = 8r-4$. Conversely, let $\psi \in \mathcal{E}_+^{(r,r-1)}$. Then $\psi \in \Gamma(S_{r-1})$ and ψ satisfies equation (15*). Since $\psi \in \Gamma(S_{r-1})$ and $m = 2r-1$, equation (15*) implies

$$D_-D_+\psi = \frac{r}{2m} R\psi \ . \tag{16*}$$

Inserting (16*) into (15*) and multiplying the equation by j we obtain

$$\nabla_X D_- j\psi + \frac{1}{2r}\bar{p}(X) D_+ D_- j\psi = 0, \tag{17*}$$

which shows that $D_- j\psi \in \ker \mathcal{D}^{(r,+)}$. Since $r \geq 2$, by (1.11) and (1.12) this provides

$$\nabla_X D_+ D_- j\psi + \frac{1}{2} p(Ric(X)) \cdot D_- j\psi = 0,$$

or the equivalent equation

$$\nabla_X D_- D_+\psi + \frac{1}{2}\bar{p}(Ric(X)) \cdot D_+\psi = 0. \tag{18*}$$

Inserting (16*) into (18*) and using that M^{2m} is Einstein we obtain

$$
\begin{aligned}
0 &= \frac{rR}{2m}\nabla_X\psi + \frac{R}{4m}\bar{p}(X)\cdot D_+\psi = \\
&= \frac{rR}{2m}\Big(\nabla_X\psi + \frac{1}{2r}\bar{p}(X)\cdot D_+\psi\Big)
\end{aligned}
$$

and thus

$$
\nabla_X\psi + \frac{1}{2r}\bar{p}(X)\cdot D_+\psi = 0,
$$

i.e., $\psi \in \ker \mathcal{D}^{(r,+)}$. Thus, the identity of spaces is shown

$$
\mathcal{E}_+^{(r,r-1)} = \ker \mathcal{D}^{(r,+)}. \tag{19*}
$$

By (3.2), Corollary 7.1 and (19*) we have

$$
\begin{aligned}
\tilde{\mathcal{E}}_+^r &= j\tilde{\mathcal{E}}_-^r = j\mathcal{E}_-^{(r,1)} \oplus j\mathcal{E}_-^{(r,r)} = \\
&= j\mathcal{E}_-^{(r,1)} \oplus \mathcal{E}_+^{(r,r-1)} = \\
&= j\mathcal{E}_-^{(r,1)} \oplus \ker \mathcal{D}^{(r,+)}.
\end{aligned}
$$

Finally, we see from Proposition 6, (ii) and Theorem 12 that $\mathcal{E}_-^{(r,1)} = 0$ if R is constant and $r \geq 2$, q.e.d.

We remark that the supposition $s \neq m$ of Theorem 13.A is not a restriction since $\ker \mathcal{D}^{(m,+)} = 0$ for $m \geq 2$ and $R \not\equiv 0$ (see [7], Prop. 11 and Th. 18). Moreover, by Theorem 6 in [9] our Theorem 13.B yields

Corollary 13.1 *Under the suppositions of Theorem 13.B, the space* $\tilde{\mathcal{E}}_+^{\frac{m+1}{2}}$ *is isomorphic to the space* $\mathcal{K}_+(M^{2m})$ *of Kählerian Killing spinors.*

By Corollary 13.1, we obtain first examples of spin Kähler manifolds admitting Kählerian E-spinors. It was shown in [9] that the complex projective spaces $\mathbb{C}P^m$ and the complex hyperbolic spaces $\mathbb{C}H^m$ of odd complex dimension m admit Kählerian Killing spinors and, hence, Kählerian E-spinors by Corollary 13.1. Further examples of Kählerian Killing spinors on closed manifolds and certain characterizations of closed spin Kähler manifolds admitting Kählerian spinors are given in [10], [15], [16]. Moreover, by Theorem 5 in [9], there is also a relation between E-spinors and the eigenvalue problem of the Dirac operator if M^{2m} is closed. By Theorem 12 in [8] and Proposition 7, (iii), we immediately obtain

Theorem 14 *Let M^{2m} be a closed spin Kähler manifold of odd complex dimension $m = 2r-1 \geq 3$ and scalar curvature $R \not\equiv 0$. If $\mathcal{E}_-^{(r,r)} \neq 0$, then M^{2m} is Einstein.*

5 Vanishing theorems for Kählerian E-spinors

By Corollary 7.1 we already have a main vanishing theorem. This result leads to the question what one can say in case the supposition that the scalar curvature has a critical point is omitted. In the first part of this section we investigate the problem which of the spaces $\mathcal{E}^{(r,s)}$ with $r \neq s$ can be different from zero.

Lemma 15.1 *Let $r \in \{0, 1, ..., m\}, s \in \{1, 2, ..., m\}$ and let $\psi \in \mathcal{E}_-^{(r,s)}$ be closed. Then*

$$(\bar{\partial} R) \cdot \psi = 0, \tag{5.1}$$

$$4(r-s) grad |D_-\psi|^2 + \frac{2s-1}{2r-1} |\psi|^2 \ grad \ (R) = 0. \tag{5.2}$$

Proof: Let $\psi \in \mathcal{E}_-^{(r,s)} \cap \ker D_+$. Then equation (5.1) follows immediately from (3.7) since $D_+\psi = 0$. Thus, by (3.8) and (5.1), we obtain the equation

$$(dR) \cdot \psi = 2 \cdot \frac{r-s}{2s-1} R \cdot D_-\psi. \tag{5.3}$$

Moreover, since $D_-\psi \in \ker calD^{(s,+)}$ by (1.7) and (3.6), we have

$$\begin{aligned} \text{grad}|D_-\psi|^2 &= -\frac{1}{2s} J(s_{D_-\psi}) = \\ &= -\frac{1}{2s} J((D_+D_-\psi)(D_-\psi) + (D_-\psi)(D_+D_-\psi)) = \end{aligned}$$

$$= \; -\frac{R}{8r-4}J(\psi(D_-\psi) + (D_-\psi)\psi) =$$
$$= \; -i\frac{R}{8r-4}(\psi(D_-\psi) - (D_-\psi)\psi)$$

and, hence,

$$\mathrm{grad}|D_-\psi|^2 = -i\frac{R}{8r-4}(\psi(D_-\psi) - (D_-\psi)\psi).$$

Multiplying this equation by $4(r-s)$ and using equation (5.3) we obtain

$$4(r-s)\mathrm{grad}|D_-\psi|^2 = -i\frac{2s-1}{4r-2}(\psi((dR)\cdot\psi) - ((dR)\cdot\psi)\psi),$$

which implies (5.2) by the general formula

$$\psi((dR)\cdot\psi) - ((dR)\cdot\psi)\psi = -2i|\psi|^2\mathrm{grad}(R),$$

q.e.d.

The part A of the following result is our basic vanishing theorem concerning Kählerian E-spinors.

Theorem 15 *Let M^{2m} be a spin Kähler manifold with scalar curvature $R \not\equiv 0$.*

A) Each of the spaces $\mathcal{E}_-^{(r,s)}$ with $r \neq s \geq 2$ and $(r,s) \neq (1,2)$ vanishes.

B) If $\mathcal{E}_-^{(1,2)} \neq 0$, then there are the equations

$$Ric(grad(R)) = 0, \tag{5.4}$$

$$\triangle R + \frac{1}{6}R^2 = 0, \tag{5.5}$$

$$|Ric|^2 = \frac{1}{2}R^2, \tag{5.6}$$

$$|dR|^2 + \frac{1}{3}R^3 = 0. \tag{5.7}$$

Proof: Assume that $\mathcal{E}_-^{(r,s)} \neq 0$ with $r \neq s \geq 2$ and let $0 \neq \psi \in \mathcal{E}_-^{(r,s)}$. Then (5.3) implies

$$|dR|^2|\psi|^2 = 4\left(\frac{r-s}{2s-1}\right)^2 R^2|D_-\psi|^2.$$

By (3.15) this provides

$$|D_-\psi|^2 = \frac{s(2s-1)}{8(r-s)(2r-1)}R|\psi|^2. \tag{5.8}$$

Moreover, since $D_-\psi \in \ker \mathcal{D}^{(s,+)}$ by Proposition 6, (ii) and $D_+\psi = 0$ by Corollary 7.6, we find by (1.7) and equation (5.2) of Lemma 15.1

$$2(r-s)s_{D_-\psi} = -s\frac{2s-1}{2r-1}|\psi|^2 J(\mathrm{grad}(R)). \tag{5.9}$$

On the other hand, by Prop. 3 in [8], it holds that

$$Ric(s_{D_-\psi}) = \frac{R}{4s-2}s_{D_-\psi} + \frac{s}{2s-1}|D_-\psi|^2 J(\mathrm{grad}(R)). \tag{5.10}$$

The equations (5.8) - (5.10) imply

$$Ric(\text{ grad } (R)) = -\frac{s-2}{2s-1}\frac{R}{4}\mathrm{grad}(R). \tag{*}$$

Now we show that only $s = 2$ is possible. First of all, we remark that, by Proposition 7, (i) and Corollary 7.5, the twistor spinor $D_-\psi$ has the property $Q_{D_-\psi} = P_\psi \neq 0$. Hence, by Theorem 17 in [8], the constant

$$a_{(r,s)} := |P_\psi|^{\frac{1}{s}}E_\psi = |Q_{D_-\psi}|^{-\frac{1}{s}}C_{D_-\psi}$$

is independent on the choice of $0 \neq \psi \in \mathcal{E}_-^{(r,s)}$ and it holds that

$$a_{(r,s)} = \frac{sR}{4s-2}e^{2u_s} + 4s^2|de^{u_s}|^2 \tag{5.11}$$

with $u_s := -\frac{1}{2s}\log|G_s|$. By (3.14) and (3.15), we have

$$e^{2u_s} = |\frac{(2r-1)(2s-1)}{2(r-s)(s-1)}|^{\frac{1}{s}} \cdot R^{-\frac{2}{s}}, \tag{2*}$$

$$|de^{u_s}|^2 = |\frac{(2r-1)(2s-1)}{2(r-s)(s-1)}|^{\frac{1}{s}} \cdot \frac{r-s}{2s(2r-1)(2s-1)} R^{1-\frac{2}{s}}. \tag{3*}$$

Inserting (2*) and (3*) into (5.11) we obtain

$$a_{(r,s)} = |\frac{(2r-1)(2s-1)}{2(r-s)(s-1)}|^{\frac{1}{s}} \cdot \frac{s(6r-4r-1)}{2(2r-1)(2s-1)} R^{1-\frac{2}{s}}. \tag{4*}$$

Since $a_{(r,s)}$ is a constant and R cannot be constant in this case, equation (4*) shows that $s=2$ holds. In particular, $s=2$ implies (5.4) by (*). Moreover, for $s=2$ the equations (2*) and (4*) are of the form

$$e^{2u_2} = \sqrt{\frac{3}{2} \cdot \frac{2r-1}{|r-2|}} \cdot \frac{1}{|R|}. \tag{5*}$$

$$a_{(r,2)} = \frac{|2r-3|}{\sqrt{\frac{2}{3}|r-2|(2r-1)}}. \tag{6*}$$

Here we have used the relation $|2r-3||R| = (2r-3)R$, which is valid by Corollary 7.4 since the sign of $r-2$ coincides with that of $2r-3$ for $r \neq 2$. In case $s=2$, equation (3.15) yields

$$|dR|^2 = \frac{1}{3} \cdot \frac{r-2}{2r-1} R^3. \tag{7*}$$

Using the equations (5*) and (7*) we calculate

$$\begin{aligned} \triangle u_2 &= -div(\text{ grad } (u_2)) = -div(-\frac{1}{2} \text{ grad } (\log|R|)) = \\ &= \frac{1}{2} div(\frac{1}{R}\text{grad}(R)) = -\frac{1}{2R}(\triangle R + \frac{1}{R}|dR|^2) = \\ &= -\frac{1}{2R}(\triangle R + \frac{1}{3} \cdot \frac{r-2}{2r-1} R^2), \end{aligned}$$

and we obtain

$$\triangle u_2 = -\frac{1}{2R}(\triangle R + \frac{1}{3}\cdot\frac{r-2}{2r-1}R^2). \tag{8*}$$

On the other hand, by Theorem 18 in [8], there is the equation

$$\triangle u_2 + \frac{1}{8}a_{(r,2)}e^{-2u_2} = \frac{R}{8}.$$

Thus, using the equations (5*), (6*), (8*) we have

$$\begin{aligned} 0 &= \triangle u_2 + \frac{1}{8}a_{(r,2)}e^{-2u_2} - \frac{R}{8} = \\ &= -\frac{1}{2R}(\triangle R + \frac{1}{3}\cdot\frac{r-2}{2r-1}R^2) + \frac{1}{8}\frac{|2r-3||R|}{\sqrt{\frac{2}{3}\cdot|r-2|(2r-1)}}\cdot\sqrt{\frac{2}{3}\cdot\frac{|r-2|}{2r-1}} - \frac{R}{8} \\ &= -\frac{1}{2R}(\triangle R + \frac{1}{3}\cdot\frac{r-2}{2r-1}R^2 - \frac{2r-3}{2r-1}\cdot\frac{R^2}{4} + \frac{R^2}{4}) = \\ &= -\frac{1}{2R}(\triangle R + \frac{R^2}{6}) \end{aligned}$$

and, hence, equation (5.5). Further, for $s = 2$, equation (3.14) provides

$$G_2 = \frac{2}{3}\cdot\frac{r-2}{2r-1}R^2.$$

By the definition of G_2 and (5.5) this yields

$$\begin{aligned} 0 &= G_2 - \frac{2}{3}\cdot\frac{r-2}{2r-1}R^2 = \\ &= \triangle R + \frac{1}{4}R^2 - \frac{3}{2}|Ric|^2 - \frac{2}{3}\cdot\frac{r-2}{2r-1}R^2 = \\ &= \frac{1}{12}R^2 - \frac{3}{2}|Ric|^2 - \frac{2}{3}\cdot\frac{r-2}{2r-1}R^2 = \\ &= \frac{15-6r}{2r-1}\cdot\frac{R^2}{12} - \frac{3}{2}|Ric|^2. \end{aligned}$$

Thus, we find the equation

$$|Ric|^2 = \frac{15-6r}{2r-1}\cdot\frac{R^2}{18},$$

which implies $r = 1$ and (5.6). Finally, for $r = 1$, equation (7*) provides (5.7), q.e.d.

Our Theorem 15.A implies immediately

Corollary 15.1 *There are the identities of spaces*

$$\tilde{\mathcal{E}}^0_- = \mathcal{E}^{(0,1)}_-, \tag{5.12}$$

$$\tilde{\mathcal{E}}^1_- = \mathcal{E}^{(1,1)}_- \oplus \mathcal{E}^{(1,2)}_-, \tag{5.13}$$

$$\tilde{\mathcal{E}}^r_- = \mathcal{E}^{(r,1)}_- \oplus \mathcal{E}^{(r,r)}_- \quad (r \geq 2). \tag{5.14}$$

Corollary 15.2 *If $\mathcal{E}^{(1,2)}_- \neq 0$, then $\mathcal{E}^{(r,r)}_- = 0$ for all $r \geq 2$. Moreover, if there is an $r \geq 2$ with $\mathcal{E}^{(r,r)}_- \neq 0$, then $\mathcal{E}^{(1,2)}_- = 0$.*

Proof: Assume that $\mathcal{E}^{(1,2)}_- \neq 0$ and that $\mathcal{E}^{(r,r)}_- \neq 0$ for an $r \geq 2$. Then, by Corollary 7.3 and (5.5), (5.6), we obtain

$$\begin{aligned} 0 &= G_r = \triangle R + \frac{R^2}{2r} - \frac{2r-1}{r}|Ric|^2 = \\ &= -\frac{R^2}{6} + \frac{R^2}{2r} - \frac{2r-1}{r}\frac{R^2}{2} = \frac{6-7r}{6r}R^2 \end{aligned}$$

and, hence, the contradiction $R \equiv 0$, q.e.d.

By Corollary 7.3, the proofs of the following three theorems are quite analogous to the corresponding proofs of the Theorems 12, 13 and 14 in [8].

Theorem 16 *Let M^{2m} be a closed spin Kähler manifold with scalar curvature $R \not\equiv 0$. Then each space $\mathcal{E}^{(r,r)}_-$ with $r > (m+1)/2$ vanishes.*

Theorem 17 *Let M^{2m} be a non-closed spin Kähler manifold whose scalar curvature has a non-vanishing local minimum. Then $\mathcal{E}^{(r,r)}_- = 0$ holds for $r > (m+1)/2$.*

Theorem 18 *Let* $\mathcal{E}_-^{(r,r)} \neq 0$ *for an* $r > 2$. *Then* $\mathcal{E}_-^{(s,s)} = 0$ *for all* s *with* $s \geq 2$ *and* $s \neq r$.

The assertion (i) of Proposition 7 implies immediately

Theorem 19 *Let* $R \not\equiv 0$. *If* R *has any zero, then each space* $\mathcal{E}_-^{(r,s)}$ *with* $s \geq 2$ *vanishes.*

Further, since the holomorphic line bundle S_o is isomorphic to the square root $\sqrt{\Lambda^{m,0}}$ of the canonical bundle $\Lambda^{m,0}$, Proposition 6 , (ii) and Theorem 12 yield

Theorem 20 *If* $\bar{H}^0(M^{2m}, \sqrt{\Lambda^{m,0}}) = 0$, *then* $\mathcal{E}_-^{(r,1)} = 0$ *for all* $r = 0, 1, ..., m$.

Finally, by Proposition 11 and Theorem 18 in [7], the assertion (iii) of Proposition 7 provides

Theorem 21 *The space* $\mathcal{E}_-^{(m,m)}$ *vanishes for any complex dimension* $m \geq 2$.

6 The case of constant scalar curvature and closed E-spinors

Up to now, all known examples of Kählerian twistor spinors and, hence, all known examples of Kählerian E-spinors have been of constant scalar curvature. Thus the case of $dR = 0$ is very important.

Theorem 22 *Suppose that* R *is constant but not zero. Then we have the following:*

(i) Each E-spinor of any reduced E-space is closed, i.e., for all r,

$$D_+ \tilde{\mathcal{E}}_-^r = 0.$$

(ii) It holds that $\tilde{\mathcal{E}}_-^0 = 0$ *and* $\tilde{\mathcal{E}}_-^1 = \mathcal{E}_-^{(1,1)} \cong \bar{H}^0(M^{2m}, S_0)$.

(iii) There is at most an $r \geq 2$ with $\tilde{\mathcal{E}}^r_- \neq 0$. If $\tilde{\mathcal{E}}^r_- \neq 0$ for an $r \geq 2$, then $\tilde{\mathcal{E}}^r_- = \mathcal{E}^{(r,r)}_- \cong \ker \mathcal{D}^{(r,+)}$ and

$$|Ric|^2 = \frac{R^2}{4r-2} \tag{6.1}$$

holds.

Proof: The assertion (i) is implied by equation (3.7). By Theorem 12 we see that $\mathcal{E}^{(r,1)}_- = 0$ for $r \neq 1$ and, moreover, $D_-\mathcal{E}^{(1,1)}_- = \bar{H}^0(M^{2m}, S_0)$. Further, Theorem 15.B yields $\mathcal{E}^{(1,2)}_- = 0$ if $dR = 0$. By Corollary 7.3, Corollary 15.1 and Theorem 18, this proves the assertions (ii) and (iii), q.e.d.

Corollary 22.1 *Let M^{2m} be of constant scalar curvature $R \neq 0$ and let $m = 2r-1 \geq 3$. If $\tilde{\mathcal{E}}^r_- \neq 0$, then M^{2m} is Einstein.*

Proof: Since $\tilde{\mathcal{E}}^r_- \neq 0$ with $r \geq 2$, Theorem 22, (iv) yields

$$|Ric|^2 = \frac{R^2}{2m} \quad . \tag{*}$$

Moreover, in general we have the equation

$$|Ric - \frac{R}{2m}|^2 = |Ric|^2 - \frac{R^2}{2m}$$

implying

$$Ric = \frac{R}{2m}$$

by (*). Thus, M^{2m} is Einstein, q.e.d.

From Theorem 11 and Corollary 11.1 we immediately obtain

Theorem 23 *Let M^{2m} be closed of constant scalar curvature $R \neq 0$. Then, for any $r \in \{0, 1, ..., m\}$, we have $\psi \in \tilde{\mathcal{E}}^{(r,r)}_-$ iff $\psi \in \Gamma(S_r)$ and ψ satisfies the equation*

$$D_+D_-\psi = \frac{rR}{4r-2}\psi.$$

Moreover, if this equation has a non-trivial solution for any $r \geq 2$, then the equation (6.1) holds.

Corollary 23.1 *Let M^{2m} be closed of constant scalar curvature $R > 0$ and let $r \in \{1, 2, ..., [\frac{m+1}{m}]\}$. Then $\tilde{\mathcal{E}}_-^{(r,r)} \neq 0$ iff $\lambda_+ := \sqrt{rR/(4r-2)}$ and $\lambda_- := -\lambda_+$ are eigenvalues of the Dirac operator D.*

Proof: Let $0 \neq \psi \in \mathcal{E}_-^{(r,r)}$. Then we consider the spinors ψ_+ and ψ_- defined by

$$\psi_\pm := \psi + \frac{1}{\lambda_\pm} D_- \psi$$

Now, by $D_+\psi = 0$ and (6.2), we have

$$\begin{aligned} D\psi_\pm &= D_-\psi + \frac{1}{\lambda_\pm} D_+ D_- \psi = D_-\psi + \lambda_\pm \psi = \\ &= \lambda_\pm (\psi + \frac{1}{\lambda_\pm} D_- \psi) = \lambda_\pm \psi_\pm . \end{aligned}$$

Thus, λ_+ and λ_- are eigenvalues of D. Conversely, suppose that λ_+ and λ_- are eigenvalues. Then, by Theorem 4 in [6], we have $0 \neq D_-\psi \in \ker \mathcal{D}^{(r,+)}$ for each eigenspinor $\varphi \in E_r^{\lambda_+}(D)$. Hence, by (1.5), it holds

$$D_- D_+ D_- \varphi = \frac{rR}{4r-2} D_- \varphi. \qquad (*)$$

We apply D_+ to this equation and obtain

$$D_+ D_- (D_+ D_- \varphi) = \frac{rR}{4r-2} D_+ D_- \varphi,$$

which implies $D_+ D_- \varphi \in \mathcal{E}_-^{(r,r)}$ by Theorem 23. Moreover, by $R > 0$ and $D_-\varphi \neq 0$, the equation (*) provides $D_+ D_- \varphi \neq 0$. Thus, we have $\mathcal{E}_-^{(r,r)} \neq 0$, q.e.d.

Corollary 23.1 shows the relation between E-spinors and the eigenvalue problem of the Dirac operator (see [5], [6]).
If we look out for closed E-spinors, by the Corollaries 7.6 and 15.1, we have to consider the spaces $\mathcal{E}_-^{(r,r)} \cap \ker D_+ \, (r = 1, 2, ..., m)$, the spaces $\mathcal{E}_-^{(r,1)} \cap \ker D_+ \, (r \neq 1)$ and the space $\mathcal{E}_-^{(1,2)}$. (We omit here the space $\Gamma(S_0) \cap \ker D_+$.) In the following we investigate these three possibilities. The equation (5.2) of Lemma 15.1 implies

Theorem 24 *Suppose that $\mathcal{E}_-^{(r,r)} \cap \ker D_+ \neq 0$ for an $r \geq 1$. Then the scalar curvature R is constant.*

Thus, we see that the first possibility leads to the case of constant scalar curvature. In the following we consider the remaining two cases where closed E-spinors can appear.

7 Closed E-spinors of type $(r, 1, -)$ with $r \neq 1$

By Theorem 12 and Theorem 15.B, the scalar curvature R can not be constant if $\mathcal{E}_-^{(r,1)} \cap \ker D_+ \neq 0 \, (r \neq 1)$ or if $\mathcal{E}_-^{(1,2)} \neq 0$.

Lemma 25.1 *Suppose that there is any closed E-spinor $\psi \in \mathcal{E}_-^{(r,1)}$ with $r \neq 1$. Then the following holds:*

(i) Each zero of $D_-\psi$ is a critical point of R.

(ii) Outside the critical points of R, there is the equation

$$d \log |D_-\psi|^2 = -\frac{r-1}{2r-1} \, \frac{R^2}{|dR|^2} dR. \tag{7.1}$$

Proof: Let $0 \neq \psi \in \mathcal{E}_-^{(r,1)}$ be closed. Then, using the notation $\psi_0 := D_-\psi$, by equation (5.2) of Lemma 13.1 we obtain

$$d|\psi_0|^2 = -\frac{|\psi|^2}{(r-1)(8r-4)} dR \tag{7.2}$$

Since ψ_0 is antiholomorphic, on the set $M^{2m}-$ zero (ψ_0), there is the equation

$$D_+\psi_0 = (\bar{\partial}\log|\psi_0|^2)\cdot\psi_0. \tag{7.3}$$

Moreover, by Corollary 6.2, it holds

$$(\partial R)\cdot D_+\psi_0 = \frac{r-1}{2r-1}R^2\psi_0. \tag{7.4}$$

Using the equations (7.3) and (7.4) we have

$$\frac{r-1}{2r-1}R^2\psi_0 = (\partial R)(\bar{\partial}\log|\psi_0|^2)\cdot\psi_0$$

$$= \bar{p}(\mathrm{grad}(R))\cdot p(\mathrm{grad}\log|\psi_0|^2)\cdot\psi_0$$

$$= -2g(p(\mathrm{grad}\log|\psi_0|^2), \bar{p}\,(grad(R)))\cdot\psi_0$$

and, hence,

$$\frac{r-1}{2r-1}R^2 = -2g(p(\mathrm{grad}\log|\psi_0|^2), \bar{p}\,(grad(R))) \quad .$$

The real part of this equation yields

$$g(\mathrm{grad}|\psi_0|^2, grad(R)) = -\frac{r-1}{2r-1}R^2|\psi_0|^2. \tag{*}$$

On the other hand, equation (7.2) implies

$$g(\mathrm{grad}|\psi_0|^2,\ \mathrm{grad}\ (R)) = -\frac{|\psi|^2}{(r-1)(8r-4)}|dR|^2. \tag{2*}$$

Comparing (*) and (2*) we obtain

$$4(r-1)^2R^2|\psi_0|^2 = |dR|^2|\psi|^2. \tag{7.5}$$

Thus, on $M^{2m}-$ zero (dR) we have

$$|\psi|^2 = 4(r-1)^2\frac{R^2}{|dR|^2}|\psi_0|^2.$$

Inserting this into (7.2) we obtain

$$\frac{d|\psi_0|^2}{|\psi_0|^2} = -\frac{r-1}{2r-1}\frac{R^2}{|dR|^2}dR \tag{3*}$$

on $M^{2m}-(\mathrm{zero}(\psi_0) \cup \mathrm{zero}(dR))$. But the 1-form on the right-hand side of (3*) is defined on the set $M^{2m} - \mathrm{zero}(dR)$. Thus, the 1-form on the left-hand side can be extended on $M^{2m}-$ zero (dR). This implies $(M^{2m}-$ zero $(dR)) \cap \mathrm{zero}(\psi_0) = \emptyset$ and, hence, the assertion (i) providing (ii) by (3*) , q.e.d.

Theorem 25 *Let $\mathcal{E}_-^{(r,1)} \cap \ker D_+ \neq 0$ for an $r \neq 1$. Then it holds*

$$\dim_{\mathbb{C}}(\mathcal{E}_-^{(r,1)} \cap \ker D_+) = 1. \tag{7.6}$$

Moreover, on $M^{2m}-$ zero (dR), the 1-form $dR/|dR|^2$ is closed and there is the equation

$$\rho = -2i\frac{r-1}{2r-1}\partial\left(\frac{R^2}{|dR|^2}\bar{\partial}R\right). \tag{7.7}$$

Proof: Let $0 \neq \psi \in \mathcal{E}_-^{(r,1)} \cap \ker D_+$ and let $\psi_0 := D_-\psi$. Then, by assertion (ii) of Proposition 6, we have $\psi_0 \in \ker \mathcal{D}^{(1,+)}$. Thus, ψ_0 satisfies the equation

$$\nabla_X\psi_0 + \frac{1}{2}\bar{p}(X) \cdot D_+\psi_0 = 0. \tag{*}$$

On the other hand, by (7.1) and (7.3), there is the equation

$$D_+\psi_0 = -\frac{r-1}{2r-1}\frac{R^2}{|dR|^2}(\bar{\partial}R) \cdot \psi_0 \tag{7.8}$$

on $M^{2m}-$ zero (dR). Inserting (7.8) into (*) we obtain

$$\nabla_X\psi_0 + \frac{r-1}{2r-1}\frac{R^2}{|dR|^2}(\bar{\partial}R)(X)\psi_0 = 0. \tag{7.9}$$

Thus, outside the critical points of R, ψ_0 is parallel with respect to the covariant derivative $\nabla^{(r,0)}$ defined on $S_0\,|_{M^{2m}-\mathrm{zero}(dR)}$ by

$$\nabla^{(r,0)}_X \varphi := \nabla_X \varphi + \frac{r-1}{2r-1}\frac{R^2}{|dR|^2}(\bar\partial R)(X)\cdot\varphi. \tag{7.10}$$

Since $\mathrm{rank}_{\mathbb{C}} S_0 = 1$, this implies that the restriction of the space $D_-(\mathcal{E}^{(r,1)}_- \cap \ker D_+)$ to each connected component of the set $M^{2m}-$ zero(dR) is of dimension 1. Thus, we obtain $\dim_{\mathbb{C}}(D_-(\mathcal{E}^{(r,1)}_- \cap \ker D_+)) = 1$ and, hence, (7.6) by Proposition 6, (ii).
Moreover, by equation (7.1), we have

$$\begin{aligned} 0 &= d\left(\frac{R^2}{|dR|^2}dR\right) = 2RdR\wedge\frac{dR}{|dR|^2} + R^2 d\left(\frac{dR}{|dR|^2}\right) = \\ &= R^2 d\left(\frac{dR}{|dR|^2}\right). \end{aligned}$$

This provides the equation

$$d\left(\frac{dR}{|dR|^2}\right) = 0. \tag{7.11}$$

Finally, it is well-known that the curvature form of S_0 is equal to $-\frac{i}{2}\rho$. Thus, with respect to any local antiholomorphic section ψ_0 of S_0, the Ricci form ρ is represented by

$$\rho = 2i\partial\bar\partial \log|\psi_0|^2. \tag{7.12}$$

By equation (7.1) of Lemma 25.1, this yields the representation (7.7) outside the critical points of R, q.e.d.

Corollary 25.1 *The 2-forms $d\left(\frac{dR}{|dR|^2}\right)$ and $\rho + 2i\frac{r-1}{2r-1}\partial\left(\frac{R^2}{|dR|^2}\bar\partial R\right)$ defined on $M^{2m}-$zero(dR) are obstructions for the space $\mathcal{E}^{(r,1)}_- \cap \ker D_+$ with $r \in \{0,2,3,...,m\}$.*

Further, by equation (7.7), we also obtain immediately

Corollary 25.2 *There is at most one $r \in \{0, 2, 3, ..., m\}$ such that $\mathcal{E}_-^{(r,1)}$ contains closed E-spinors.*

Theorem 26 *Let zero$(dR) = \emptyset$ and $r \in \{0, 2, 3, ..., m\}$. Then the spaces $\mathcal{E}_-^{(r,1)} \cap \ker D_+$ and $\ker \nabla^{(r,0)}$ are isomorphic and there is the relation*

$$D_-(\mathcal{E}_-^{(r,1)} \cap \ker D_+) = \ker \nabla^{(r,0)}. \tag{7.13}$$

Proof: Since zero $(dR) = \emptyset$, the covariant derivative $\nabla^{(r,0)}$ (see (7.10)) is well defined on S_0. The curvature form $\omega^{(r,0)}$ of $\nabla^{(r,0)}$ is given by

$$\omega^{(r,0)} = -\frac{i}{2}\rho + \frac{r-1}{2r-1} d\left(\frac{R^2}{|dR|^2}\bar{\partial}R\right). \tag{7.14}$$

From the proof of Theorem 25 we already know that there is the inclusion $D_-(\mathcal{E}_-^{(r,1)} \cap \ker D_+) \subset \ker \nabla^{(r,0)}$. Now, let $0 \neq \psi_0 \in \ker \nabla^{(r,0)}$, i.e., $0 \neq \psi_0 \in \Gamma(S_0)$ and ψ_0 satisfies equation (7.9), which implies $\psi_0 \in \bar{H}^0(M^{2m}, S_0)$ and, moreover, the equations

$$\nabla_{\mathrm{grad}(R)}\psi_0 = -\frac{r-1}{4r-2} R^2 \psi_0, \tag{*}$$

$$D_+\psi_0 = -\frac{r-1}{2r-1}\frac{R^2}{|dR|^2}\left(\bar{\partial}R\right) \cdot \psi_0, \tag{2*}$$

$$D_+\psi_0 = (\bar{\partial} \log |\psi_0|^2) \cdot \psi_0 \ . \tag{3*}$$

We remark that zero$(\psi_0) = \emptyset$ since $\nabla^{(r,0)}\psi_0 = 0$. Comparing the equations (2*) and (3*) we obtain

$$d \log |\psi_0|^2 = -\frac{r-1}{2r-1}\frac{R^2}{|dR|^2} dR \tag{4*}$$

and, hence,

$$d\left(\frac{dR}{|dR|^2}\right) = 0. \tag{5*}$$

Moreover, the J-trace of equation (7.12) yields

$$\triangle \log |\psi_0|^2 = \frac{R}{2} \tag{7.15}$$

Let us consider the spinor $\psi \in \Gamma(S_1)$ defined by

$$\psi := -2(r-1)\frac{R^2}{|dR|^2}(dR)\cdot\psi_0. \tag{7.16}$$

Using formula (1.18) and the equations (*), (2*), (5*) we find

$$D\psi = -2(r-1)d^*\left(\frac{R^2}{|dR|^2}dR\right)\cdot\psi_0. \tag{6*}$$

Now, by the equations (4*) and (7.15), on the set $M^{2m} - \mathrm{zero}(R)$ we have

$$\begin{aligned}
d^*\left(\frac{R^2}{|dR|^2}dR\right) &= -\mathrm{div}\left(\frac{1}{R}\left(\frac{R^2}{|dR|^2}\mathrm{grad}(R)\right)\right) = \\
&= -\left(\frac{R^2}{|dR|^2}\mathrm{grad}(R)(\frac{1}{R}) + \frac{1}{R}\mathrm{div}\left(\frac{R^2}{|dR|^2}\mathrm{grad}(R)\right)\right) = \\
&= -\left(\frac{R^2}{|dR|^2}\left(-\frac{|dR|^2}{R^2}\right) - \frac{1}{R}d^*\left(\frac{R^2}{|dR|^2}dR\right)\right) = \\
&= 1 - \frac{2r-1}{r-1}\frac{1}{R}d^*d\log|\psi_0|^2 = 1 - \frac{2r-1}{r-1}\frac{1}{R}\triangle\log|\psi_0|^2 \\
&= 1 - \frac{2r-1}{2(r-1)} = -\frac{1}{2r-2}.
\end{aligned}$$

Thus, by reasons of continuity, the equation

$$d^*\left(\frac{R^2}{|dR|^2}dR\right) = -\frac{1}{2r-2} \tag{7.17}$$

is valid on M^{2m}. The equations (6*) and (7.17) imply the relation $D\psi = \psi_0$, which is equivalent to the two equations

$$D_-\psi = \psi_0 \quad (7^*)\ , \qquad\qquad D_+\psi = 0. \quad (8^*)$$

Using the equations (7.9), (7.16) and (7*) we see that $\psi \in \mathcal{E}_-^{(r,1)}$ and, hence, $\psi \in \mathcal{E}_-^{(r,1)} \cap \ker D_+$ by (8*). This proves equation (7.13), q.e.d.

Theorem 27 *Let M^{2m} be a simply connected spin Kähler manifold and suppose that zero$(dR) = \emptyset$. Then the equations (7.7) and (7.11) are necessary and sufficient conditions for $\mathcal{E}_-^{(r,1)} \cap \ker D_+ \neq 0$.*

Proof: The equations (7.7) and (7.11) are equivalent to the equation

$$\rho = -2i\frac{r-1}{2r-1}d\left(\frac{R^2}{|dR|^2}\bar{\partial}R\right). \tag{7.18}$$

By Theorem 25 we know that this is a necessary condition. Moreover, equation (7.18) is equivalent to

$$\omega^{(r,0)} = 0 \tag{7.19}$$

by (7.14). Thus, we have $\ker \nabla^{(r,0)} \neq 0$ since M^{2m} is simply connected. This implies $\mathcal{E}_-^{(r,1)} \cap \ker D_+ \neq 0$ by Theorem 26, q.e.d.

According to Theorem 27 the local theory of closed E-spinors of type $(r, 1, -)$ with $r \in \{0, 2, 3, ..., m\}$ is equivalent to the problem of investigating of the Kähler structures (M^{2m}, J, g) satisfying the differential equation (7.18). In case such a solution possesses a continuation $(\bar{M}^{2m}, \bar{J}, \bar{g})$ with $\mathrm{zero}(dR) \neq \emptyset$, one has to check the possibility of a corresponding continuation of the E-spinor.

Theorem 28 *Let $\mathcal{E}_-^{(1,2)} \neq 0$. Then $\mathcal{E}_-^{(r,1)} \cap \ker D_+ = 0$ for all r.*

Proof: The supposition $\mathcal{E}_-^{(1,2)} \neq 0$ implies $\mathrm{zero}(R) = \emptyset$ and $\mathrm{zero}(dR) = \emptyset$ by Corollary 7.4. Let us assume that $\mathcal{E}_-^{(r,1)} \cap \ker D_- \neq 0$ for an $r \in \{0, 2, 3, ..., m\}$ and let $0 \neq \psi \in \mathcal{E}_-^{(r,1)} \cap \ker D_+$. Then the equations (5.6) and (7.1) yield

$$d\log|D_-\psi|^2 = 3\frac{r-1}{2r-1}\cdot\frac{dR}{R}.$$

Applying d^* to this equation and using the equations (5.5), (5.6) and (7.15) we have

$$\begin{aligned}
\frac{R}{2} &= \triangle\log|D_-\psi|^2 = d^*d\log|D_-\psi|^2 = 3\frac{r-1}{2r-1}d^*\left(\frac{dR}{R}\right) \\
&= -3\frac{r-1}{2r-1}\mathrm{div}\left(\frac{1}{R}\mathrm{grad}(R)\right) = \\
&= -3\frac{r-1}{2r-1}\left(-\frac{|dR|^2}{R^2} - \frac{1}{R}\triangle R\right) = \\
&= -\frac{r-1}{2r-1}\left(\frac{R}{3} + \frac{R}{6}\right) = -3\frac{r-1}{2r-1}\cdot\frac{R}{2}
\end{aligned}$$

and, hence, the contradiction $R \equiv 0$. Finally, in case $\mathcal{E}_-^{(1,1)} \cap \ker D_+ \neq 0$, the scalar curvature is constant by Theorem 24, q.e.d.

8 Concluding remarks

At present, the main problem of the theory of Kählerian E-spinors is the construction of examples in case the scalar curvature R is not constant. The possibility that there are solutions of some type of Kählerian E-equations for non constant R is an essential difference to the E-equation of Lichnerowicz. As we have seen, each such example would yield an example of a Kählerian twistor spinor with $R \neq$ constant. We recall that R is constant for all examples of Kählerian twistor spinors known until now (each of them appears as the limiting case of an inequality for the first eigenvalue of the Dirac operator. See [3], [5] and [6].). Moreover, if M^{2m} admits an E-spinor of type (1,2,-) or a closed E-spinor of type (r, 1, -) with $r \neq 1$, it is even impossible that R is constant. In these two cases there are strong geometrical restrictions (Theorem 15.B, Theorem 27) and one can try to classify spin Kähler manifolds with E-spinors of such type. In case $\mathcal{E}_-^{(1,2)} \neq 0$, the space $\mathcal{E}_-^{(1,2)}$ is isomorphic to the space $\ker \mathcal{D}^{(2,+)}$ (Proposition 7) and each twistor spinor $\psi \in \ker \mathcal{D}^{(2,+)}$ has the properties $C_\psi \neq 0$ and $Q_\psi \neq 0$ (This follows from the proof of Theorem 15.). Thus, we have a situation that was treated more generally in Section 4 of the paper [8]. Using these results one can derive further geometrical restrictions.

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