# INTEGER GROUPS OF COINVARIANTS ASSOCIATED TO OCTAGONAL TILINGS

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Abstract. The integer groups of coinvariants associated to undecorated and decorated octagonal tilings are computed.

## Introduction

A tiling $T$ defines a topological groupoid $\mathcal{R}$ whose continuous groupoid cohomology may be used to characterize the tiling. On the other hand, this groupoid defines a $C^*$-algebra, its reduced groupoid $C^*$-algebra, which coincides with the algebra of observables for a particle moving in the tiling and will be referred to as the algebra associated to the tiling, denoted by $\mathcal{A}_T$. The $K$-group of this algebra is in many cases isomorphic to the continuous cohomology group of $\mathcal{R}$ with coefficients in $\mathbb{Z}$. In fact, such a relation could be proven for tilings which are decorations of $\mathbb{Z}^d$ (i.e. for which $\mathcal{R}$ is isomorphic to a transformation group with group $\mathbb{Z}^d$) [FH95] and then extends to tilings which reduce to such decorations (i.e. for which $\mathcal{R}$ has a regular reduction which is isomorphic to a transformation group with $\mathbb{Z}^d$) [Kel95]. In particular the above relation implies that the cohomology group of highest degree is a direct summand of the even part of the $K$-group, the $K_0$-group. This cohomology group of highest degree is the group of coinvariants of $\mathcal{R}$ with integer coefficients and will be denoted by $H(T)$ to emphasize that it is associated to the tiling $T$. In particular, in two dimensions,

$$K_0(\mathcal{A}_T) = H(T) \oplus \mathbb{Z}. \tag{1}$$

The aim of this article is the computation of the integer groups of coinvariants associated to undecorated and decorated octagonal tilings. These are substitution tilings which reduce to decorations of $\mathbb{Z}^2$ so that the general results of [Kel95] can be applied.

This is not merely of interest for a possible theory of topological invariants for tilings but the relation to $K$-theory brings Bellissard's $K$-theoretical gap labelling into play [Bel82, Bel92, Kel94]. The $K_0$-group of the algebra associated to the tiling contains information about the gap-structure of spectra of Schrödinger operators describing the particles that move in the tiling. Namely the gaps in the spectrum of such an operator may be labelled by elements of a countable ordered group which is the image under a homomorphism of the $K_0$-group of the above algebra. This homomorphism is induced

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by a normalized trace on the algebra and for the computation of its image we rely on a result of van Elst [vE94], which is proven for tilings reducing to decorations of $\mathbb{Z}^d$, $d \leq 3$, stating that all values of its image are already obtained by evaluating it on elements of the integer group of coinvariants.

A detailed exposition of the general theory will not be given here, and the reader is referred to [Kel95] where proofs of the theorems stated below are given. Here we concentrate on the computational aspect. However, we include section 1 to give a short overview on the almost-groupoid structure defined by a tiling which plays an essential role since a substitution is a homomorphism of this almost-groupoid into itself.

## 1. Aspects of the theory for substitution tilings

A tile (in $\mathbb{R}^d$) is a connected bounded subset of $\mathbb{R}^d$ which is the closure of its interior and may be decorated. The decoration, which may consist e.g. of arrows or flags, may serve for the purpose to distinguish tiles which have the same shape. It is used here to break the symmetry of the tiles. A $d$ dimensional tiling is a covering of $\mathbb{R}^d$ by tiles which overlap at most at their boundaries. A pattern of a tiling is a finite collection of its tiles. Although often formulated for a specific tiling, the relevant quantities like the topological groupoid and the almost-groupoid of the local structure depend only on the congruence class of the tiling. A tile- resp. tiling- resp. pattern-class shall here be an equivalence class under translation of a tile resp. tiling resp. pattern, i.e. two such objects belong to the same class if one is a translate of the other. Given such a class we may choose a tile in it and call the class together with this choice pointed. As well we may take two choices, a first and a second, which may coincide (and have to for tile classes) calling the result a doubly pointed class. Doubly pointed pattern- resp. tiling-classes will be sometimes denoted by $M_{xy}$ resp. $S_{xy}$ where $M$ resp. $S$ refers to the pattern- resp. tiling-class, $x$ to the first, and $y$ to the second pointed tile.

On the set of all doubly pointed pattern classes $\mathcal{M}_{\rm II}$ we define a multiplicative structure as follows. First, let $M_{xy} \preceq N_{x'y'}$ if the latter can be obtained by adding tiles to $M_{xy}$ but keeping the choices for the pointed tiles fixed. Then call two doubly pointed pattern classes composable whenever there is an $L_{z_1z_2}$ and a third choice of a tile $z$ of $L$ such that $M_{x_1x_2} \preceq L_{z_1z}$ and $N_{y_1y_2} \preceq L_{zz_2}$. Then define the product of two composable elements $M_{x_1x_2}N_{y_1y_2}$ as the smallest such $L_{z_1z_2}$ with respect to $\preceq$. Finally define the involution ${M_{xy}}^{-1} = M_{yx}$. This is the structure of an almost groupoid [Kel95]. Note that $\mathcal{M}_{\rm II}$ depends only on the local structure of the tiling. In fact, two tilings are said to be locally isomorphic if every pattern class of either tiling can also be found in the other [SS86], which is equivalent to saying that the tilings lead to the same almost-groupoid. Adding a zero element to the almost-groupoid, which shall satisfy $0M_{xy} = M_{xy}0 = 0$, $0^{-1} = 0$, one obtains an inverse semi-group with 0 by extending the multiplication to $M_{x_1x_2}N_{y_1y_2} = 0$ if they are not composable. Indeed, an inverse semi-group is a set with associative multiplication and an involution $c \mapsto c^{-1}$ that satisfies $cc^{-1}c = c$. Inverse semi-groups are well known objects. In particular, the above order may for all inverse semi-groups be defined in the form: $c \preceq c'$ whenever $cc'^{-1} = c'c'^{-1}$ (often written the other way around). Elements of the form $cc^{-1}$ are called units (in groupoid terminology) or idempotents (in inverse semi-group terminology). The set of units of $\mathcal{M}_{\rm II}$ is denoted by $\mathcal{M}_{\rm II}{}^0$, and its elements, which are doubly pointed pattern classes where the first and the second choice coincide, simply regarded as pointed pattern classes and shorter denoted by $M_x$ instead of $M_{xx}$.

The topological groupoid $\mathcal{R}$ associated to the tiling as well as its integer group of coinvariants can be defined without mentioning the almost groupoid but the computation of the integer group of coinvariants requires a substitution, and this is a homomorphism of the almost groupoid into itself. Thus it is more natural to work with the almost groupoid. A homomorphism between almost-groupoids is a map which preserves composability and satisfies

$$\hat\varphi(cc') = \hat\varphi(c)\hat\varphi(c') \quad \text{if } c \text{ and } c' \text{ are composable} \tag{2}$$

and $\hat\varphi(c^{-1}) = \hat\varphi(c)^{-1}$. We call it a pre-homomorphism if (2) is weakened to $\hat\varphi(cc') \succeq \hat\varphi(c)\hat\varphi(c')$ for composable $c$ and $c'$.

A key ingredient which distinguishes almost-groupoids coming from tilings from arbitrary ones is the notion of the radius of a pointed pattern class. Let $\mathrm{rad}(u)$ be the largest $r$ such that all $d$-dimensional balls of radius $r$ having their center inside the pointed tile are covered by $u$. We call a sequence of pointed pattern classes $\{u_\nu\}_\nu$ approximating if $\forall k \geq 1 : \prod_{i=1}^k u_i$ is composable with $u_{k+1}$ and $\mathrm{rad}(u_\nu) \to \infty$. For a pointed tiling $T_x$ we write $M_x \preceq T_x$ to indicate that $M_x$ occurs in $T_x$ at the pointed tile (i.e. the class $M$ occurs in $T$ in such a way that the pointed tile of $M_x$ and $T_x$ coincide). A sequence of pointed pattern classes whose radii diverge and which all occur in $T_x$ at the pointed tile is certainly approximating. Conversely, any approximating sequence defines a pointed tiling as the unique one $\omega$ for which $u_\nu \preceq \omega$ for all $\nu$. We denote it by $\lim u_\nu$. The hull of the tiling class we started with is the set of all tilings obtained from approximating sequences of $\mathcal{M}_{\rm II}$ and it is given the topology which is generated by the sets $U_u := \{\omega \in \Omega | u \preceq \omega\}$. We say that a pre-homomorphism $\hat\varphi$ of $\mathcal{M}_{\rm II}$ satisfies the growth condition (with scaling factor $t > 0$) if the function $t \mapsto |\mathrm{rad}(\hat\varphi(u)) - t\,\mathrm{rad}(u)|$ is bounded on $\mathcal{M}_{\rm II}{}^0$. Thanks to this construction, any pre-homomorphism $\hat\varphi$ of $\mathcal{M}_{\rm II}$ into another tilings almost-groupoid which satisfies the growth condition gives rise to a continuous map $\varphi$ between the hulls by simply setting $\varphi(\lim(u_\nu)) = \lim(\hat\varphi(u_\nu))$. A substitution is a homomorphism satisfying the growth condition. But before coming to its precise definition let us first explain what the integer group of coinvariants is.

A detailed description of the topological groupoid $\mathcal{R}$ defined by the tiling may be found in [Kel94, Kel95] and will not be repeated here. $\mathcal{R}$ can be looked at from several perspectives and we will briefly sketch two.

As a topological space $\mathcal{R}$ consists of all doubly pointed tilings which may be obtained from sequences of doubly pointed pattern classes of the form $\{u_\nu c\}_\nu$ where $\{u_\nu\}_\nu$ is an approximating sequence and $L(c) \preceq u_\nu$, similar to the way pointed tilings where obtained from pointed pattern classes. The limit of such a sequence is the doubly pointed tiling $S_{xy}$ for which for all $\nu : u_\nu c \preceq S_{xy}$, $M_{xy} \preceq S_{xy}$ meaning that $M$ occurs in $S$ such that the pointed tiles coincide. The topology is generated by the sets $\mathcal{U}_c = \{S_{xy} | c \preceq S_{xy}\}$ and a natural extension of multiplication is given by $\lim u_\nu c \lim u'_\nu c' = \lim u_\nu c u'_\nu c'$ provided $\{u_\nu c u'_\nu c^{-1}\}_\nu$ is an approximating sequence (which is equivalent to $\lim u_\nu = \lim c u'_\nu c^{-1}$), or equivalently $S_{xy} S'_{x'y'} = S_{xy'}$ provided $S_y = S'_{x'}$. The inversion is given by $(\lim u_\nu c)^{-1} = \lim (u_\nu c)^{-1}$, or ${S_{xy}}^{-1} = S_{yx}$. A reduction of $\mathcal{R}$ is given by a subset of the form $\{S_{xy} \in \mathcal{R} | S_x, S_y \in U\}$ where $U$ is an open and closed subset of $\Omega$. Such a reduction is regular if any $S_x$ has a translate which lies in $U$. If the tiling is minimal, i.e. if any element of its hull is locally isomorphic to it, then any reduction is regular.

Another way to look at $\mathcal{R}$ is as a generalization of a transformation group. The space is the hull $\Omega$ and the action on it to be considered is the partial action of $\mathcal{M}_{\mathbb{I}\mathbb{I}}$, namely a subset $U_u$ may be shifted to $U_{c^{-1}uc}$ provided $L(c) \preceq u$. This motivates the consideration of the quotient of the group (under point-wise addition) of continuous functions over the hull with values in $\mathbb{Z}$, $C(\Omega, \mathbb{Z})$, modulo the subgroup generated by elements of the form $\chi_{U_u} - \chi_{U_{c^{-1}uc}}$ where $\chi_{U_u}$ is the characteristic function onto $U_u$, $u \in \mathcal{M}^0_{\mathbb{I}\mathbb{I}}$ and $L(c) \preceq u$. In fact, this group is the integer group of coinvariants of the groupoid $\mathcal{R}$ which we denote here by $H(T)$ to emphasize that it is associated to the tiling $T$, in formula

$$H(T) = H(\mathcal{M}_{\mathbb{I}\mathbb{I}}) := \langle \{\chi_{U_u} | u \in {\mathcal{M}_{\mathbb{I}\mathbb{I}}}^0\} \rangle / \langle \{\chi_{U_{L(c)}} - \chi_{U_{R(c)}} | c \in \mathcal{M}_{\mathbb{I}\mathbb{I}}\} \rangle \tag{3}$$

$\langle X \rangle$ denoting the set of all finite sums or differences of elements of $X$. This may be a little bit more simplified. We call a pattern connected if the interior of the set which is covered by it is a connected subset of $\mathbb{R}^d$. This notion carries over to pattern classes and doubly pointed pattern classes. Let $\mathcal{M}_{\mathbb{I}\mathbb{I}2,\neq}$ be the set of connected doubly pointed pattern classes which consists of two tiles and have different choices for the pointed tiles. If $\mathcal{M}_{\mathbb{I}\mathbb{I}2,\neq}$ is finite, which shall be assumed throughout this article, then $\Omega$ is compact. $\mathcal{M}_{\mathbb{I}\mathbb{I}2,\neq}$ generates together with $\mathcal{M}_{\mathbb{I}\mathbb{I}1}$, the (pointed) tile classes, (under almost groupoid multiplication) the set of all connected doubly pointed pattern classes, ${\mathcal{M}_{\mathbb{I}\mathbb{I}}}^c$. Let $M_r(T_x)$ denote the smallest $M_x \preceq T_x$ such that all balls of radius $r$ having their centers inside the pointed tile are covered by the pattern $M$. Then $U_u = \bigcup_{T_x \succeq u} U_{M_r(T_x)}$ and the union can be made finite, by compactness, and disjoint (in fact, $\Omega$ is a compact totally disconnected metric space whose $\epsilon$-neighbourhoods are of the form $U_{M_r(T_x)}$, $\epsilon = e^{-r}$). Therefore is $H(\mathcal{M}_{\mathbb{I}\mathbb{I}}) = H({\mathcal{M}_{\mathbb{I}\mathbb{I}}}^c)$.

We now come to the definition of a substitution. The join of two patterns of a tiling is the smallest pattern which contains both of them. We say that two patterns join without overlap if they overlap at most at their boundaries. A pattern class is a composition without overlap of pattern classes $M_1, \ldots, M_n$ if it has a representative which is the join without overlap of representatives for the $M_i$. Finally let $E^0(u) = \{e \in \mathcal{M}_{\mathbb{I}\mathbb{I}} | e^{-1}e = u\}$ and $x_i : \mathcal{M}_{\mathbb{I}\mathbb{I}} \to \mathcal{M}_{\mathbb{I}\mathbb{I}1}$, $i = 1, 2$, be the map which assigns to a doubly pointed pattern class the class of its $i$th pointed tile.

**Definition 1.** *A substitution of a tiling $T$ (or its class) is a homomorphism of its almost-groupoid into itself,*

$$\hat{\rho} : \mathcal{M}_{\mathrm{II}} \to \mathcal{M}_{\mathrm{II}}, \tag{4}$$

*for which exists a $t > 1$ such that $t \mapsto |\mathrm{rad}(\hat{\rho}(u)) - t\,\mathrm{rad}(u)|$ is a bounded function on ${\mathcal{M}_{\mathrm{II}}}^0$ and for which the pattern class of $\hat{\rho}(c)$ is composed without overlap of the pattern classes of the $\hat{\rho}(x_1(e))$, $e \in E^0(c^{-1}c)$, and the ith pointed tile of $\hat{\rho}(c)$ corresponds to the pointed tile of $\hat{\rho}(x_i(c))$ in the composition.*

As already mentioned such a map induces a continuous map $\rho$ of the hull into itself. A tiling resp. its class is invariant under a substitution if $\rho(T_x)$ is a translate of $T_x$ (this is independent of $x$). It is called substitution tiling if it allows for a substitution such that $\rho(T_x)$ is locally isomorphic to $T_x$ (which is automatic if the tiling is minimal). A substitution is called primitive if for some $n$ and all $a \in \mathcal{M}_{\mathrm{II}1}$ the pattern class of $\hat{\rho}^n(a)$ contains at least one tile of each tile class.

A substitution is called locally invertible if there exists a sub almost-groupoid $\mathcal{N} \subset \mathcal{M}_{\mathrm{II}}$ such that $\forall c \in \mathcal{M}_{\mathrm{II}} \exists c' \in \mathcal{N} : \hat{\rho}(c) \preceq c'$ and a pre-homomorphism $\hat{\psi} : \mathcal{N} \to \mathcal{M}_{\mathrm{II}}$ satisfying the growth condition (with the inverse scaling factor of $\hat{\rho}$) such that $\hat{\rho}(\hat{\psi}(c)) \preceq c$. This is equivalent to saying that the extension of $\rho$ to the topological groupoid $\mathcal{R}$, which is given by $\rho(\lim u_\nu c) = \lim \hat{\rho}(u_\nu c)$, has a left inverse. If a substitution tiling allows for a locally invertible substitution, then the substitution can be used to compute the coinvariants associated to the tiling class [Kel95]. All information that enters in the computation of the integer group of coinvariants is encoded in two combinatorial matrices $\lambda_\upsilon$ and $K^{(l,\upsilon)}$. For clarity we first discuss the case in which the substitution forces its border.

Given a substitution tiling with locally invertible substitution $\hat{\rho}$, an *$n$-substitute* (of $a$ in a pointed tiling $\omega$) is a pointed pattern occurring at the pointed tile of $\omega$ which arose from the $n$-fold application of $\hat{\rho}$ on $a$, i.e. which is congruent to $\hat{\rho}^n(a)$ and $a \preceq \rho^{-n}(\omega)$ (in particular $\omega \in \mathrm{im}\rho^\upsilon$). An *$n$-neighbourhood* (of $a$ in $\omega$) is an $n$-substitute (of $a$ in $\omega$) together with the tiles which have non zero intersection with its boundary. We say that a (locally invertible) substitution forces its border if for some $\upsilon$ there is only one class of $\upsilon$-neighbourhoods per tile. We denote then by $f(a)$ this class together with the pointed tile of $\hat{\rho}^\upsilon(a)$, i.e. $\hat{\rho}^\upsilon(a) \preceq f(a) \in {\mathcal{M}_{\mathrm{II}}}^0$. Let $E^n(a) := \{e \in \mathcal{M}_{\mathrm{II}} | e^{-1}e = \hat{\rho}^n(a)\}$. Then $\lambda_\upsilon = \lambda$ is the substitution matrix, which is the matrix having coefficients

$$\lambda_{a_1 a_2} := |\{e \in E^1(a_1) | x(e) = a_2\}|, \tag{5}$$

$a_i \in \mathcal{M}_{\mathrm{II}1}$, i.e. $\lambda_{a_1 a_2}$ is the number of occurrences of tiles of class $a_2$ in $\hat{\rho}(a_1)$. The other matrix encodes the occurrences of connected 2-patterns in $\upsilon$-neighbourhoods. It has two different indices, $a \in \mathcal{M}_{\mathrm{II}1}$ and $A \in \mathcal{M}_{\mathrm{II}2,\neq}$ and is defined, setting first

$$K^{(\upsilon,\upsilon)}_{aA} := |\{e \in E^\upsilon(a) | L(e^{-1}A) \preceq f(a)\}| \tag{6}$$

where $L(c) = cc^{-1}$, and then

$$\mathcal{L}^{(v,v)}_{aA} := K^{(v,v)}_{aA} - K^{(v,v)}_{aA^{-1}}. \tag{7}$$

In particular $l = v$.

In case the border is not forced by the substitution we have to take the possible different neighbourhoods into account. Let $\mathcal{F}^v(a)$ be the set of all pointed pattern classes of $v$-neighbourhoods of $a$ in some $\omega$ and

$$\mathcal{B}^v_1 := \{(a, f)|a \in \mathcal{M}_{\mathbb{I}1}, f \in \mathcal{F}^v(a)\}. \tag{8}$$

$\mathcal{B}^v_1$ generalizes $\mathcal{M}_{\mathbb{I}1}$, and in place of the first matrix we have to consider the following extended substitution matrix $\lambda_v$ with entries

$$\lambda_{v\,(a_1,f_1)(a_2,f_2)} = |\{e \in E^1(a_1)|x(e) = a_2, f_2 \preceq L(\hat\rho^v(e)\hat\rho(f_1))\}|. \tag{9}$$

The additional label $v$ is a non-negative integer which can be adjusted to the problem. As for the generalization of $\mathcal{M}_{\mathbb{I}2,\neq}$ consider the set of pointed pattern classes of $v$-neighbourhoods of $A \in \mathcal{M}_{\mathbb{I}2,\neq}$, which is $\mathcal{F}^v(A) = \{f_1\hat\rho^v(A)f_2|f_i \preceq \mathcal{F}^v(x_i(A))\}$, and set

$$\mathcal{B}^v_{2\neq} := \{(A, F)|A \in \mathcal{M}_{\mathbb{I}2,\neq}, F \in \mathcal{F}^v(A)\}. \tag{10}$$

The other matrix then generalizes to

$$K^{(l,v)}_{(a,f)(A,F)} := |\{e \in E^l(a)|L(e^{-1}A) \preceq \hat\rho^{l-v}(f), L(\hat\rho^v(e^{-1})F) \preceq \hat\rho^l(f)\}| \tag{11}$$

where $l \geq v$ should be chosen large enough by which we mean that for all $(a, f) \in \mathcal{B}^v_1$, $(A, F) \in \mathcal{B}^v_{2,\neq}$, $a \in E^l(a)$ holds: $L(\hat\rho^v(e^{-1})F) \preceq \hat\rho^l(f)$ whenever $L(\hat\rho^v(e^{-1})F)$ and $\hat\rho^l(f)$ are composable. Such an $l$ does always exist. As above

$$\mathcal{L}^{(l,v)}_{(a,f)(A,F)} := K^{(l,v)}_{(a,f)(A,F)} - K^{(l,v)}_{(a,f)(A^{-1},F^{-1})}. \tag{12}$$

Then the following theorem could be obtained in [Kel95].

**Theorem 1.** *Let $T$ be a substitution tiling with locally invertible substitution and $\lambda_v$ and $\mathcal{L}^{(l,v)}$ as above. If $l$ is large enough in the above sense then, with $k = |\mathcal{B}^v_1|$,*

$$H(T) \cong \{x \in \mathbb{R}^k|\exists n \geq 0 : \lambda^n_v(x) \in \mathbb{Z}^k\}/\{x \in \mathbb{R}^k|\exists n \geq 0 : \lambda^n_v(x) \in \mathrm{im}\mathcal{L}^{(l,v)}\}. \tag{13}$$

For one dimensional substitution tilings (i.e. substitution sequences) this result was obtained earlier by A. Forrest [For94] and by B. Host [Hos93].

The equivalence classes of the elements of the standard base of $\mathbb{R}^{|\mathcal{B}^v_1|}$ in the representation (13) correspond to equivalence classes of characteristic functions on $U_{(a,f)}$, $(a, f) \in \mathcal{B}^v_1$ in representation (3). In many cases, including the octagonal tilings, for all $(a, f) \in \mathcal{B}^v_1$, $a$ is uniquely determined by $f$, i.e. $a = a(f)$. We may then write shorter $f$ for $(a(f), f)$ and $F$ for $(A(F), F)$ and (9,11) simplify to

$$\lambda_{v\,f_1f_2} = |\{e \in E^1(a(f_1))|f_2 \preceq L(\hat\rho^v(e)\hat\rho(f_1))\}| \tag{14}$$

$$K^{(l,v)}_{fF} = |\{e \in E^l(a(f))|L(\hat\rho^v(e^{-1})F) \preceq \hat\rho^l(f)\}|. \tag{15}$$

Note that the latter matrix is well defined for $l \geq 0$.

An invariant measure $\mu$ on the hull of a tiling $T$ is a measure satisfying $\mu(U_{L(c)}) = \mu(U_{L(c^{-1})})$. Any invariant probability measure defines a homomorphism of $H(T)$ into $\mathbb{R}$ (which we also denote by $\mu$) through $\mu([\chi_{U_u}]) = \mu(U_u)$. A particular simple situation arises if the substitution is primitive, because then there is a unique such measure. In that case it is natural to consider the order on $H(T)$ induced by $\mu$, i.e. $[x] > 0$ whenever $\mu([x]) > 0$. This way $\mu$ becomes a state on $H(T)$ as ordered group. Since its image $\mu(H(T))$ is a free group, $H(T)$ with the above order is order isomorphic to $H(T) \cong \mu(H(T)) \oplus \ker \mu$ where the second summands yields the infinitesimal elements (which are neither positive nor negative) and the first summand carries the order induced from $\mathbb{R}$.

The image of $\mu$ can be computed and is important for the $K$-theoretical gap-labelling.

**Theorem 2.** *If the substitution $\hat{\rho}$ of the above theorem is in addition primitive then there is a unique invariant measure on the hull whose image on the group of coinvariants is given by*

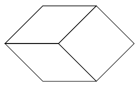
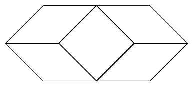
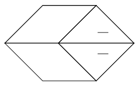
$$\mu(H(T)) = \{\tau^{-n} \sum_i \nu_i n_i | n > 0, n_i \in \mathbb{Z}\} \tag{16}$$

*where $\tau$ is the Perron-Frobenius-eigenvalue and $\nu$ the left Perron-Frobenius-eigenvector of $\lambda_v$ which is normalized to $\sum_i \nu_i = 1$.*

The normalization of the left Perron-Frobenius-eigenvector is fixed through the requirement that $\mu([w]) = 1$ where $[w]$ is the order unit of $H(T)$ which is the class of the constant function 1 on the hull and corresponds in the representation (13) to the class of $w \in \mathbb{R}^k$ for which $w_i = 1$. Below we compute the image of a unique state on the integer group of coinvariants for the undecorated and the decorated octagonal tiling. For that we shall make use of a substitution which is defined for a modified version of the tilings. This will not change the ordered integer group of coinvariants but leads to a different order unit thus affecting the normalization for the left-Perron-Frobenius-eigenvector in (16).

If the tiling reduces to a decoration of $\mathbb{Z}^d$ then $H(T)$ with the above order and order unit embeds into $K_0(\mathcal{A}_T)$ as scaled ordered group. This means that positive elements of $H(T)$ correspond to positive elements of $K_0(\mathcal{A}_T)$, i.e. to the $K_0$-classes coming from projections in the stabilization of $\mathcal{A}_T$ [Bla86], and the order unit of $K_0(\mathcal{A}_T)$, which is the class of the 1, lies in $H(T)$. Moreover, in this case is $K_0(\mathcal{A}_T)$ torsion free [FH95], and hence also $H(T)$. If $d \leq 3$ then the gap labels predicted by $K$-theory are precisely given by (16), because the homomorphism on $K_0(\mathcal{A}_T)$ induced by a trace on $\mathcal{A}_T$ restricts to the state $\mu$ on $H(T)$.

## 2. The integer group of coinvariants for the undecorated and decorated octagonal tiling

An undecorated octagonal tiling is a two dimensional tiling which may be defined by the projection method [DK86] projecting a strip of $\mathbb{Z}^4$ (with standard basis $\{e_i\}_{i=1,\dots,4}$) onto a two dimensional invariant subspace of the eightfold symmetry $e_i \mapsto e_{i+1}$ for $i = 1,2,3$ and $e_4 \mapsto -e_1$ [Soc89]. It consists of squares and $45^0$-rhombi all edges having equal length. Alternatively it may be defined using a substitution. To visualize typical parts of an undecorated octagonal tiling see Figure 5 ignore the bold lines and delete the lines which serve as decorations and the bases of the triangles. A symmetry breaking decoration of the squares may be locally derived from the tiling since all squares occur in hexagon patterns congruent to  but never in patterns congruent to . Therefore, replacing in all hexagon patterns congruent to the above the squares by two decorated triangles so as to obtain  is a well defined local operation which is of course invertible. The tiling made from rhombi and decorated triangles which is obtained in such way will be referred to as the triangle version. The decoration is important for the formulation of a substitution with respect to which the tiling becomes a substitution tiling. As they are defined by the projection method, all undecorated octagonal tilings are locally isomorphic, i.e. they define the same almost-groupoid (in particular they are minimal). Denoting by $\mathcal{T}_u$ any such tiling, $\mathcal{M}_{\mathrm{II}}(\mathcal{T}_u)$ and hence also $H(\mathcal{T}_u)$ do not depend on the specific choice. The same holds true for the triangle version a member of which will be denoted by $\mathcal{T}_{u_3}$. The above local operation defines an injective homomorphism $\hat{\varphi}$ from the sub almost-groupoid of $\mathcal{M}_{\mathrm{II}}(\mathcal{T}_u)$, which contains all doubly pointed pattern classes in which squares occur in hexagons, into $\mathcal{M}_{\mathrm{II}}(\mathcal{T}_{u_3})$. This homomorphism satisfies the growth condition with scaling factor 1 and induces an embedding of the groupoid associated to the original tilings into the one associated to the triangle version. The image is a reduction of $\mathcal{R}_{u_3}$, namely $\varphi(\mathcal{R}_u) = \{S_{xy} \in \mathcal{R}_u | S_x, S_y \in \varphi(\Omega_u)\}$, $\mathcal{R}_u$, $\Omega_u$ resp. $\mathcal{R}_{u_3}$ denoting the groupoid and the hull of $\mathcal{T}_u$ resp. the groupoid of $\mathcal{T}_{u_3}$. Since the tilings are minimal this reduction is regular which implies that $H(\mathcal{T}_u) \cong H(\mathcal{T}_{u_3})$ and that the order unit of $H(\mathcal{T}_u)$ is mapped onto the class of $\chi_{\varphi(\Omega_u)}$, c.f. [Kel95].

A decorated octagonal tiling, which is also called Ammann Beenker tiling and here denoted by $\mathcal{T}_{AB}$, differs from the undecorated one in that the rhombus is decorated with a little flag which breaks its symmetry, see Figure 1. Deleting these decorations one obtains an undecorated octagonal tiling, but in contrast to the case of the squares there is no local operation which inverts this deletion. Stated differently, whereas $H(\mathcal{T}_{AB}) \cong H(\mathcal{T}_{AB_3})$ as ordered groups, $\mathcal{T}_{AB_3}$ denoting the triangle version which may be obtained as above, no isomorphism can be expected between $H(\mathcal{T}_u)$ and $H(\mathcal{T}_{AB})$.

Since an undecorated octagonal tiling may be obtained by the projection method both, the undecorated and the decorated octagonal tilings (as well as their triangle versions), reduce to decorations of $\mathbb{Z}^2$ and their integer group of coinvariants embed into the $K_0$-group of their associated algebra as in (1).

A specific interest in the Ammann Beenker tiling came up since its two congruence classes of tiles form an aperiodic set [GS87]. The decorations may be interpreted as matching conditions for the tiles. These are conditions which determine which tiles may be next to each other. An Ammann Beenker tiling may be defined as an infinite puzzle of tiles which are congruent to the rhombus or the square and which satisfy the matching conditions. The peculiarity is that there are infinitely many possible such tilings none of which is periodic. All such tilings define in fact the same almost-groupoid, or stated differently, they are locally isomorphic. A key ingredient to prove these properties is a substitution with respect to which the tilings become substitution tilings. This allows for an alternative way to define an Ammann Beenker tiling, namely as a tiling which is locally isomorphic to a fixed point under an iterate of the substitution. It is more convenient for us to use the substitution of the triangle version which we denote by $\hat{\rho}_{AB}$. It is given by

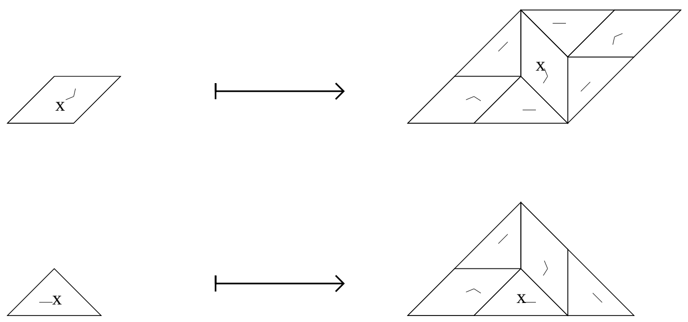



**Figure 1**

and for rotated resp. reflected tiles the result is just rotated resp. reflected. The pointed tile is the one marked with the $x$. If one re-scales a tile $a$ by a factor of $t = 1 + \sqrt{2}$ it covers the same subset as $\hat{\rho}_{AB}(a)$. Therefore, the extension of $\hat{\rho}_{AB}$ to doubly pointed pattern classes with more tiles is straightforwardly defined if one obeys the rule that $\hat{\rho}_{AB}(M_{xy})$ covers the same subset than $M_{xy}$ but scaled up by $t$. The factor $t$ is the scaling factor of $\hat{\rho}_{AB}$. As an example, $\hat{\rho}_{AB}$ acts on a doubly pointed pattern class with two tiles as follows:

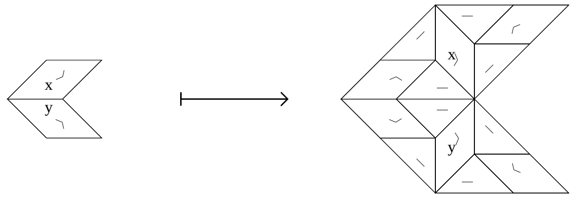



**Figure 2**

Now, a specific Ammann Beenker tiling may be obtained by applying $\hat{\rho}_{AB}^2$ infinitely often on a rhombus. The set of all Ammann Beenker tilings is given by all tilings which are locally isomorphic to such a fixed point, i.e. which define the same almost-groupoid of doubly pointed pattern classes.

A substitution of the undecorated tiling $\hat{\rho}_u$ is obtained by omitting the flags of the rhombi (but not those of the triangles). Note that the undecorated rhombus is partially symmetric so that $\mathcal{T}_{u_3}$ has 20 tile classes whereas $\mathcal{T}_{AB_3}$ has 32. Both substitutions are primitive and locally invertible. In fact, any pattern which is congruent to a substitute is a substitute, i.e. $\hat{\rho}_u(a) \preceq \omega$ implies $a \preceq \rho_u^{-1}(\omega)$.[1] In particular we may take $\mathcal{N} = \hat{\rho}_u(\mathcal{M}_{\mathrm{I\!I}})$ and simply define $\hat{\psi}(\hat{\rho}_u(a)) = a$ (the same holding true for $\hat{\rho}_{AB}$). Hence the requirements for Theorem 1 and 2 are satisfied.

2.1. **The undecorated case.** If the rhombus is not decorated then the boundaries of substitutes are local mirror axes, i.e. all $A \in \mathcal{M}_{\mathrm{I\!I}2,\neq}(\mathcal{T}_{u_3})$ which cross the boundary of $\hat{\rho}_u(a)$ are mirror symmetric. Therefore $\hat{\rho}_u$ forces its border, we can take $v = 2$. The $20 \times 20$ substitution matrix of $\hat{\rho}_u$ is invertible over $\mathbb{Z}$. It has Perron-Frobenius-eigenvalue $t^2$ and left-Perron-Frobenius-eigenvector proportional to

$$\nu \propto (2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}, 2\sqrt{2}, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \tag{17}$$

expressed in a basis in which the first four indices are identified with the rhombi and the remaining sixteen with triangles. Since the substitution was given for the triangle version the order unit of $H(\mathcal{T}_{u_3})$ is the class of $w \in \mathbb{R}^{20}$ with $w_i = 1$. Applying Theorem 2 leads to

$$\mu(H(\mathcal{T}_{u_3})) = \frac{\mathbb{Z} + \sqrt{2}\mathbb{Z}}{16}. \tag{18}$$

The order unit of $H(\mathcal{T}_u)$ is mapped onto the class of the projection $\chi_{\varphi(\Omega_u)}$. In representation (13) this class corresponds to the class of $w$ for which $w_i = 1$ if $i$ corresponds to a rhombus or a triangle which is a rotate of the one in Figure 1, and else $w_i = 0$. Hence, the proper normalization for $\mathcal{T}_u$ is $\sum_{i=1}^{20} \nu_i w_i = \sum_{i=1}^{12} \nu_i = 1$, which yields

$$\mu(H(\mathcal{T}_u)) = \frac{\mathbb{Z} + \sqrt{2}\mathbb{Z}}{8}. \tag{19}$$

[1] In this respect, a octagonal tiling is simpler than a Penrose tiling.

To compute the image of $\mathcal{L}_{aA}^{(2,2)}$ we may restrict to mirror symmetric $A$'s and to one choice out of $A$ and $A^{-1}$. It turns out that $\mathrm{im}\mathcal{L}^{(2,2)}$ is invariant under $\lambda$ and generated by the classes of $\chi_{U_{R^{(n)}}} - \chi_{U_{R^{(n-1)}}}$ and $\chi_{U_{D_+^{(n)}}} - \chi_{U_{D_-^{(n)}}}$ where $R^{(n)}$ resp. $D_+^{(n)}$ is (the undecorated version of) the rhombus resp. triangle at the left of Figure 1 rotated around $\frac{n\pi}{4}$ and $D_-^{(n)}$ is the reflection of $D_+^{(n)}$ along its base. Their span being 11-dimensional one gets

$$H(\mathcal{T}_u) \cong \mathbb{Z}^9. \tag{20}$$

The generators may be assigned to the 8 decorated squares and the congruence class under rotation of the rhombus.

2.2. **The decorated case.** Due to the decoration of the rhombus the substitution $\hat{\rho}_{AB}$ does not force its border. A check of the pattern classes appearing yields that the 2-neighborhoods are determined by the substitution up to the choice of three orientations. Here we mean that a rhombus is $+$-oriented if its flag is left bended and a triangle is $+$-oriented if it is a rotate of the triangle at the lower left of Figure 1. Once the orientations of the three numbered tiles in Figure 3 are chosen their neighbourhood is completely fixed. This follows from the fact that the pattern classes of the stars (which are formed be eight rhombi) carry decorations which are such that all flags point outwards and any two rhombi which stand in opposition have opposite orientation.

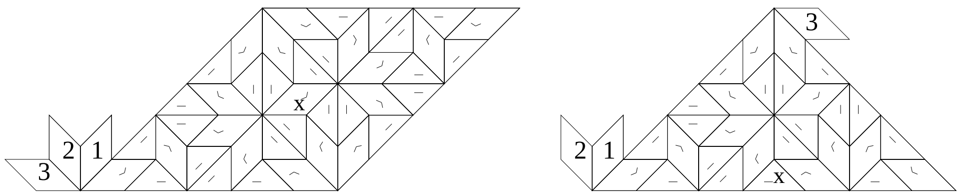



**Figure 3**

Therefore we choose an extension of the substitution matrix with $\upsilon = 2$. Clearly $\hat{\rho}_{AB}$ is injective so that given a pair $(a, f) \in \mathcal{B}_1^2$ the $a$ is already determined by the $f$. A class of a 2-neighbourhood of $a$, $(a, f)$ may be parameterized as follows: $R_{+,\epsilon_1,\epsilon_2,\epsilon_3}^{(n)}$ resp. $D_{+,\epsilon_1,\epsilon_2,\epsilon_3}^{(0)}$ is the left resp. right pointed pattern class of Figure 3 rotated around $\frac{n\pi}{4}$, where $\epsilon_i$ is the orientation of the $i$th tile, and $R_{-,-\epsilon_1,-\epsilon_2,-\epsilon_3}^{(n-1)}$ resp. $D_{-,-\epsilon_1,-\epsilon_2,-\epsilon_3}^{(n)}$ correspond to their reflection along their base lines; the base lines in Figure 3 being along the $x$-axis. It turns out that not all possible orientations occur in an Ammann Beenker tiling, in fact we have only $a_{\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3}^{(n)}$, with $\epsilon_0\epsilon_1\epsilon_2\epsilon_3 \in \{++-+, +-++, +-+-, +--+, --+-, -+--, -+-+, -++-\}$ for $a = R$, and $\epsilon_0\epsilon_1\epsilon_2\epsilon_3 \in \{+-$

$--,+--+,++-+,++--,+-++,-+++,-++-,--+-,--++,-+--\}$ for $a=D$. In particular $\mathcal{B}_1^2$ has 144 elements. Figure 5 shows patterns which are uniquely determined by $\hat\rho_{AB}(R^{(0)}_{+,e,f,g})$ resp. $\hat\rho_{AB}(D^{(0)}_{+,e,f,g})$ and placed on top of it in bold lines 2-neighbourhoods in which, for clarity reasons, the part given by $\hat\rho^2(a)$ is replaced by $a$ re-scaled by a factor of $t^2$. From this then may be read off that $\hat\rho_{AB}(a^{(n)}_{\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3})$ contains for $a=R$ the patterns

(21)

$$R^{(n)}_{-\epsilon_0,\epsilon_3,\epsilon_2,\epsilon_1},\ R^{(n-2\epsilon_0)}_{-\epsilon_0,\epsilon_0,-\epsilon_0,\epsilon_0},\ R^{(n-4)}_{\epsilon_0,\epsilon_0,-\epsilon_0,\epsilon_0},\ D^{(n-4)}_{-,+,+,\epsilon_0},\ D^{(n-3)}_{+,-,-,\epsilon_0},\ D^{(n+1)}_{+,+,-,-\epsilon_0},\ D^{(n)}_{-,-,+,-\epsilon_0}$$

and for $a=D$

$$R^{(n+\frac{\epsilon_0-1}{2})}_{-\epsilon_0,\epsilon_0,\epsilon_2,\epsilon_1},\ R^{(n-\frac{3\epsilon_0+1}{2})}_{-\epsilon_0,\epsilon_0,-\epsilon_0,\epsilon_3},\ D^{(n+4)}_{-\epsilon_0,\epsilon_0,\epsilon_0,\epsilon_0},\ D^{(n-3\epsilon_0)}_{\epsilon_0,-\epsilon_0,-\epsilon_3,\epsilon_0},\ D^{(n+3\epsilon_0)}_{\epsilon_0,-\epsilon_0,\epsilon_0,\epsilon_0} \tag{22}$$

at the required positions, i.e. for any $e\in E^1(a)$ there is precisely one element $f$ of (21) resp. (22) for $a=R$ resp. $a=D$ such that $L(\hat\rho_{AB}(e)f)\preceq\hat\rho_{AB}(a^{(n)}_{\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3})$. This yields, according to an ordering of the elements of $\mathcal{B}_1^2$ given by $a^{(n)}_\epsilon < a'^{(n')}_{\epsilon'}$ if, either $a=R$ and $a'=D$, or $a=a'$ and $\epsilon$ stands to the left of $\epsilon'$ in the above set, or $a=a'$ and $\epsilon=\epsilon'$ and $n<n'$,

$$\lambda_2=\begin{pmatrix}
s^4&0&0&0&0&0&1+s^6&0&0&s^5&0&s&0&s^4&0&1&0&0\\
s^4&0&0&0&0&0&s^6&1&0&s^5&0&s&0&s^4&0&1&0&0\\
s^4&0&0&0&1&0&s^6&0&0&s^5&0&s&0&s^4&0&1&0&0\\
s^4&0&0&0&0&1&s^6&0&0&s^5&0&s&0&s^4&0&1&0&0\\
0&0&1+s^2&0&s^4&0&0&0&s^5&0&s&0&0&0&s^4&0&1&0\\
0&0&s^2&1&s^4&0&0&0&s^5&0&s&0&0&0&s^4&0&1&0\\
1&0&s^2&0&s^4&0&0&0&s^5&0&s&0&0&0&s^4&0&1&0\\
0&1&s^2&0&s^4&0&0&0&s^5&0&s&0&0&0&s^4&0&1&0\\
0&0&0&0&0&1+s^6&0&0&0&0&0&0&s^3+s^5&s^4&0&0&0&0\\
0&0&0&0&0&1&s^6&0&0&s^5&0&0&s^3&s^4&0&0&0&0\\
0&0&0&0&0&0&1+s^6&0&0&s^5&0&0&s^3&s^4&0&0&0&0\\
0&0&0&0&0&s^6&1&0&0&0&0&0&s^3+s^5&s^4&0&0&0&0\\
0&0&0&0&0&0&s^6&1&0&s^5&0&0&s^3&s^4&0&0&0&0\\
0&s+s^7&0&0&0&0&0&0&s^4&0&0&0&0&0&0&0&0&s^3+s^5\\
0&s^7&s&0&0&0&0&0&s^4&0&0&0&0&0&s^3&0&0&s^5\\
0&0&s+s^7&0&0&0&0&0&s^4&0&0&0&0&0&s^3&0&0&s^5\\
0&s&s^7&0&0&0&0&0&s^4&0&0&0&0&0&0&0&0&s^3+s^5\\
0&0&s&s^7&0&0&0&0&s^4&0&0&0&0&0&s^3&0&0&s^5
\end{pmatrix} \tag{23}$$

where $s$ is the $8\times 8$ matrix with coefficients $s_{ij}=\delta^{mod\,8}_{i-j,1}$. Of course, also $\lambda_2$ has Perron-Frobenius-eigenvalue $t^2=3+2\sqrt{2}$. A left-Perron-Frobenius-eigenvector is proportional to $\nu = \nu' \otimes (1,1,1,1,1,1,1,1)$ with

$$\begin{aligned}(24)\qquad \nu' \;\propto\; & (1,1,1+\frac{1}{\sqrt{2}},\sqrt{2}-1,1,1,1+\frac{1}{\sqrt{2}},\sqrt{2}-1,\\ & 1+\frac{1}{\sqrt{2}},2-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\sqrt{2},1+\frac{1}{\sqrt{2}},2-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\sqrt{2}).\end{aligned}$$

Taking into account the normalization for $\mathcal{T}_{AB_3}$, which is $8\sum_{i=1}^{18}\nu'_i = 1$, we get

$$(25)\qquad \mu(H(\mathcal{T}_{AB_3})) = \frac{\mathbb{Z}+\sqrt{2}\mathbb{Z}}{16}.$$

Similar to the undecorated case the order unit of $H(\mathcal{T}_{AB})$ is given by the class of $w$ for which $w_i = 1$ except if $i$ corresponds to $D^{(n)}_{-,\epsilon_1,\epsilon_2,\epsilon_3}$ in which case $w_i = 0$. The normalization for $\mathcal{T}_{AB}$ is therefore $8\sum_{i=1}^{13}\nu'_i = 1$ so that, for the original Ammann Beenker tiling,

$$(26)\qquad \mu(H(\mathcal{T}_{AB})) = \frac{\mathbb{Z}+\sqrt{2}\mathbb{Z}}{8}.$$

The coincidence with the undecorated case seems to be accidental.

The elements of $\mathcal{B}^2_{2\neq}$ which enter into the computation of $\mathrm{im}\mathcal{L}^{(l,2)}$ are those which are classes of 2-neighbourhoods of mirror symmetric $A \in \mathcal{M}_{\mathbb{I}2,\neq}$, and again we only have to consider half of them. They may be taken to be the rotates of the patterns of the below Figure 4, where we choose the upper 2-substitute to carry the first pointed tile (indicated by $x$) and the lower to carry the second (indicated by $y$).

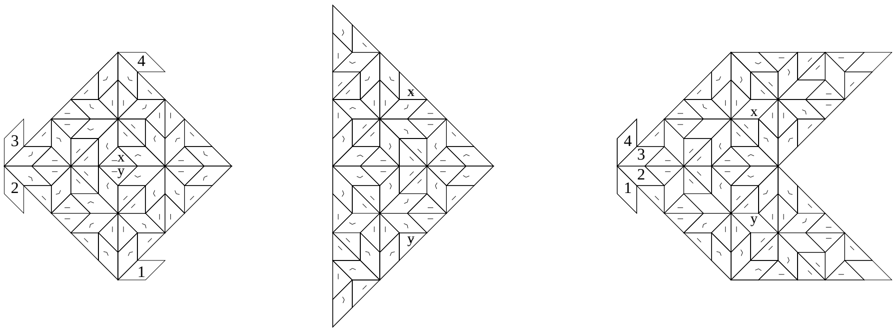



**Figure 4**

Once the orientations of the four numbered tiles in the left figure $F_1$ and the right figure $F_3$ are chosen the patterns are determined. The tiles at the boundary of the middle figure $F_2$ are already uniquely fixed. It turns out that there are altogether 136 patterns to consider, namely $F^{(n)}_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4}$, where $F = F_i$, $i = 1,2,3$, $n$ denotes its rotate, and $\epsilon_i$ the orientation of the $i$th numbered tile, which is restricted to $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 \in \{{-}{-}{-}{+}, {-}{+}{-}{-}, {-}{+}{-}{+}, {-}{+}{+}{-}, {-}{+}{+}{+}, {+}{-}{-}{+}, {+}{+}{-}{-}, {+}{+}{-}{+}\}$ for $F = F_1$, and $\epsilon_1\epsilon_2\epsilon_3\epsilon_4 \in \{{-}{-}{+}{-}, {-}{+}{-}{-}, {-}{+}{-}{+}, {-}{+}{+}{-}, {+}{-}{-}{+}, {+}{-}{+}{-}, {+}{-}{+}{+}, {+}{+}{-}{+}\}$

for $F = F_3$ and one choice only for $F = F_2$. As may be seen from figure 5 we have for all $f \in \mathcal{B}_1^2$, $F \in \mathcal{B}_{2,\neq}^2$, and $e \in E^1(a(f))$ that $L(\hat\rho_{AB}^2(e^{-1})F) \preceq \hat\rho_{AB}(f)$ whenever $L(\hat\rho_{AB}^2(e^{-1})F)$ and $\hat\rho_{AB}(f)$ are composable. Therefore $\mathcal{L}^{(2,2)} = \lambda_2\mathcal{L}^{(1,2)}$ may be used in Theorem 1. For the coefficients $\mathcal{L}_{fF}^{(1,2)}$ of the matrix $\mathcal{L}^{(1,2)}$ with $F$ restricted to the above set one obtains

(27)
$$\begin{pmatrix}
-1 & 0 & 0 & s & 0 & 0 & -s^4 & s^5 & 0 & 0 & 1 & 0 & -s^5 & -s & 0 & s^4 & 0 \\
-1 & 0 & s^5 & s & 0 & 0 & -s^4 & 0 & 0 & 0 & 0 & 0 & -s^5 & 0 & -s & s^4 & 1 \\
-1 & 0 & s^5 & s & 0 & 0 & 0 & -s^4 & 0 & -s & 0 & 0 & -s^5 & 1 & 0 & s^4 & 0 \\
-1 & 0 & s^5 & s & 0 & 0 & -s^4 & 0 & 0 & 0 & 0 & 1 & -s^5 & 0 & 0 & -s+s^4 & 0 \\
0 & -s^4 & 0 & 0 & s & -1 & s^5 & 0 & 0 & -s^5 & 0 & 0 & 1 & s^4 & 0 & 0 & -s \\
0 & 0 & -s^4 & 0 & s & -1 & s^5 & 0 & 0 & -s^5 & -s & 0 & 0 & s^4 & 1 & 0 & 0 \\
0 & s^5 & -s^4 & 0 & s & -1 & 0 & 0 & 0 & -s^5 & 0 & 0 & -s & s^4 & 0 & 1 & 0 \\
0 & 0 & -s^4 & 0 & s & -1 & s^5 & 0 & 0 & 1-s^5 & 0 & -s & 0 & s^4 & 0 & 0 & 0 \\
s^3+s^5 & 0 & 0 & 0 & 0 & 0 & -s^4 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -s-s^7 & 0 \\
0 & 0 & s^5 & 0 & 0 & s^3 & -s^4 & 0 & 1 & 0 & 0 & 1 & 0 & -s^7 & 0 & -s & 0 \\
0 & 0 & 0 & 0 & 0 & s^3 & -s^4 & s^5 & 1 & 0 & 1 & 0 & 0 & -s-s^7 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & s^3 & -s^4 & s^5 & 1 & 0 & 1 & 0 & 0 & -s-s^7 & 0 & 0 & 0 \\
0 & 0 & s^5 & 0 & 0 & s^3 & -s^4 & 0 & 1 & 0 & 0 & 0 & 0 & -s^7 & -s & 0 & 1 \\
0 & 0 & 0 & 0 & -s^3-s^5 & 0 & s^4 & 0 & -1 & s+s^7 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -s^3 & -s^5 & 0 & 0 & s^4 & 0 & -1 & s^7 & 0 & -1 & s & 0 & 0 & 0 & 0 \\
0 & -s^3 & 0 & -s^5 & 0 & 0 & s^4 & 0 & -1 & 0 & 0 & 0 & s+s^7 & 0 & 0 & 0 & -1 \\
0 & -s^3 & 0 & -s^5 & 0 & 0 & s^4 & 0 & -1 & 0 & 0 & 0 & s+s^7 & 0 & 0 & 0 & -1 \\
0 & 0 & -s^3 & -s^5 & 0 & 0 & s^4 & 0 & -1 & 0 & -1 & 0 & s & 0 & s^7 & 0 & 0
\end{pmatrix}$$

where we used the same principle of ordering as for $\mathcal{B}_1^2$ but $F_1 < F_2 < F_3$. It turns out that $\mathrm{im}\mathcal{L}^{(2,2)} \subset \mathrm{im}\lambda_2^2$ and that $\lambda_2^2$ restricts to an automorphism on its image. Therefore

$$H(\mathcal{T}_{AB}) = \mathrm{im}\lambda_2^2/\mathrm{im}\mathcal{L}^{(2,2)}. \tag{28}$$

Since $H(\mathcal{T}_{AB})$ is torsion free, all we have to do is to determine the dimensions of the images. The dimension of the first being 80 and that of the second 57 implies

$$H(\mathcal{T}_{AB}) = \mathbb{Z}^{23}. \tag{29}$$

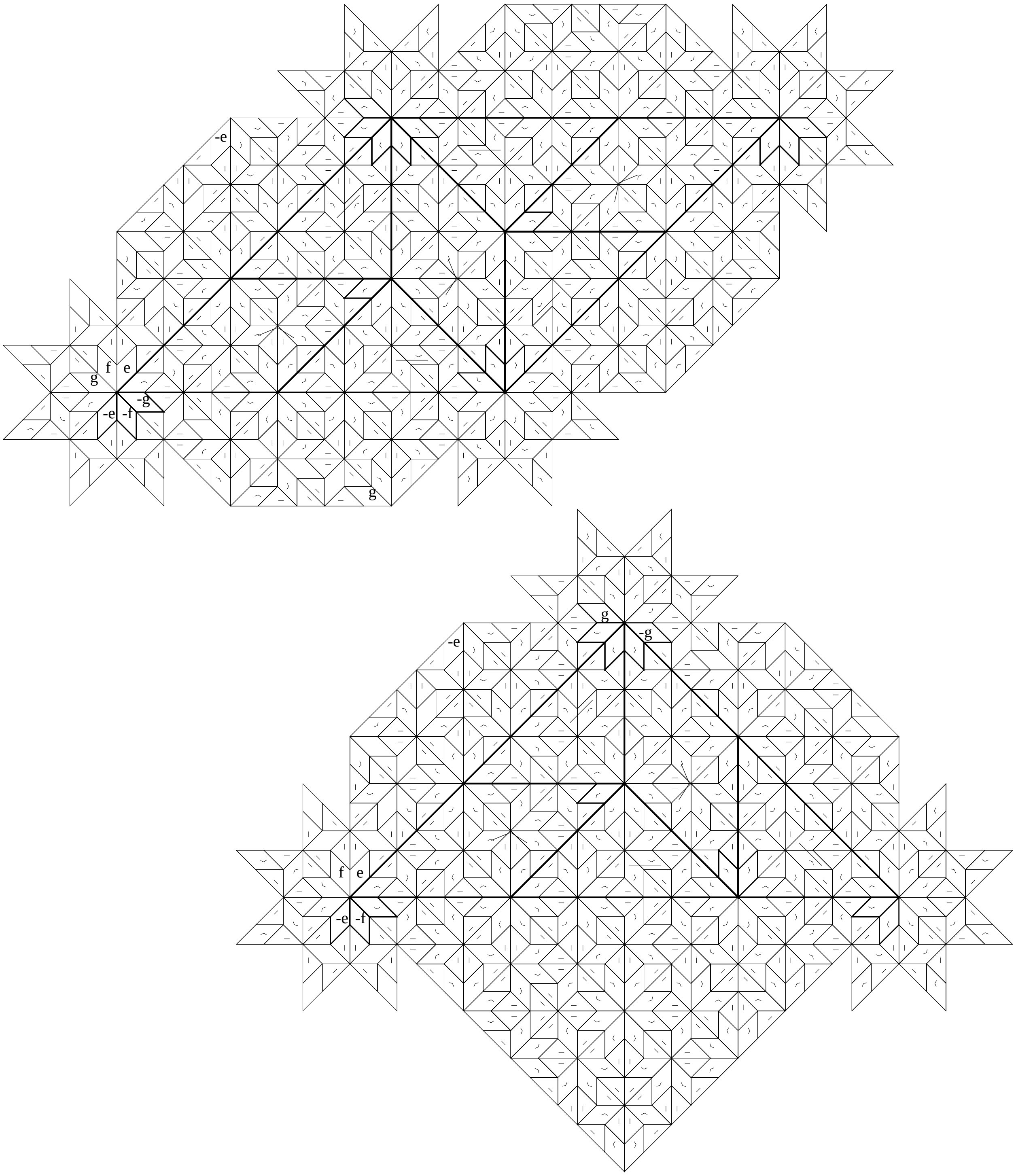



**Figure 5:** Patterns determined by $\hat{\rho}(R^{(0)}_{+,e,f,g})$ and $\hat{\rho}(D^{(0)}_{+,e,f,g})$. The bolder lines indicate the possible positions of 2-neighbourhoods.

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