Cohomological properties of the canonical globalizations of Harish-Chandra modules

Consequences of Theorems of Kashiwara-Schmid, Casselman, and Schneider-Stuhler

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Contents

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1 Introduction

Let $G_{\mathbf{C}}$ be a connected, reductive algebraic group and $G = G_{\mathbf{R}}$ be a real form of $G_{\mathbf{C}}$. Let $K \subset G$ be a maximal compact subgroup. By $Mod(\mathbf{g}, K)$ we denote the abelian category of $(\mathbf{g}, K)$-modules and by $\mathcal{HC}(\mathbf{g}, K)$ the full subcategory of Harish-Chandra modules of $(\mathbf{g}, K)$. Let $C^\infty(G)_K \in Mod(\mathbf{g}, K)$ be the space of smooth functions on G which are right K-finite. The left $\mathbf{g}$ and G-module structures of $C^\infty(G)_K$ induce corresponding structures on $\mathrm{Hom}_{(\mathbf{g},K)}(V, C^\infty(G)_K)$ for any $V \in Mod(\mathbf{g}, K)$. Let $R\mathrm{Hom}_{(\mathbf{g},K)}(., C^\infty(G)_K)$ be the derived functor of $\mathrm{Hom}_{(\mathbf{g},K)}(., C^\infty(G)_K)$ from $Mod(\mathbf{g}, K)$ to complex vector spaces. The following theorem is due to Kashiwara-Schmid [10].

Theorem 1.1 *If $V_K \in \mathcal{HC}(\mathbf{g}, K)$, then*

$$R\mathrm{Hom}_{(\mathbf{g},K)}(V_K, C^\infty(G)_K) \cong MG(\widetilde{V_K}) \ , \tag{1}$$

where we view the right-hand side of (1) as the complex which is zero except at the zeroth place where it is the maximal globalization $MG(\widetilde{V_K})$ (defined in [12], [10]) of the dual Harish-Chandra module $\widetilde{V_K}$. The isomorphism is compatible with the natural G- and $\mathbf{g}$-module structures on both sides.

In fact Kashiwara-Schmid define a natural topology on the left-hand side such that (1) becomes an isomorphism in a certain derived category of topological vector spaces (see [10], Sec. 3).

We say that a complex of Fréchet representations of G has property MG, if its differentials have closed range and its cohomology groups are, as topological representations of G, isomorphic to the maximal globalizations of their underlying Harish-Chandra modules (compare with [10]).

In order to simplify the notation we introduce the abbreviation

$$F(V_K) := \mathrm{Hom}_{(\mathbf{g},K)}(V_K, C^\infty(G)_K) \ .$$

Let $V_K \in Mod(\mathbf{g}, K)$ be finitely generated (as $\mathcal{U}(\mathbf{g})$-module). Then we can define a natural Fréchet topology on $F(V_K)$ as follows. Let $W \subset V_K$ be a finite-dimensional K-invariant generating subspace of V_K. Then we have a quotient

$$\mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{k})} W \to V_K \to 0$$

in $Mod(\mathbf{g}, K)$, and applying F we obtain the embedding

$$0 \to F(V_K) \to F(\mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{k})} W) = C^\infty(G \times_K \tilde{W}) \ .$$

This embedding induces a Fréchet topology on $F(V_K)$, which is independent of the choice of W (see [12]).

Lemma 1.2 *Let*

$$\mathcal{P} : \ldots \to P_{-2} \to P_{-1} \to P_0 \to 0$$

be a complex of finitely generated, F-acyclic modules in $Mod(\mathbf{g}, K)$, such that $H^*(\mathcal{P}) \in \mathcal{HC}(\mathbf{g}, K)$. Then $F(\mathcal{P})$ has property MG and

$$H^*(F(\mathcal{P})) = F(H^*(\mathcal{P})) \ . \tag{2}$$

Proof. We decompose $\mathcal{P}$ into short exact sequences

$$0 \to Z_i \to P_i \stackrel{d}{\to} B_{i+1} \to 0 \tag{3}$$
$$0 \to B_i \to Z_i \ \to H_i \to 0 \ . \tag{4}$$

Since $\mathcal{U}(\mathbf{g})$ is Noether the modules Z_i are finitely generated. Applying the functor F to these short exact sequences we obtain the long exact sequences

$$0 \to F(B_{i+1}) \stackrel{F(d)}{\to} F(P_i) \to F(Z_i) \to R^1F(B_{i+1}) \ldots \tag{5}$$
$$0 \to F(H_i) \to F(Z_i) \to F(B_i) \to R^1F(H_i) \ldots \ . \tag{6}$$

Note that P_i (by assumption) and H_i (by Theorem 1.1) are F-acyclic. Since $Z_0 = P_0$, we conclude $R^qF(B_0) = R^qF(Z_0) = 0$ for $q \geq 1$ using (6). Now by (5) we obtain $R^qF(P_{-1}) = R^qF(Z_{-1}) = 0$ and inductively

$$R^qF(Z_i) = R^qF(P_i) = 0$$

for all i and $q \geq 1$. From (3) we obtain

$$0 \to F(B_{i+1}) \stackrel{F(d)}{\to} F(P_i) \to F(Z_i) \to 0 \tag{7}$$
$$F(Z_i) = F(P_i)/F(B_{i+1}) \tag{8}$$

and from (4)

$$0 \to F(H_i) \to F(Z_i) \to F(B_i) \to 0 \tag{9}$$
$$0 \to F(H_i) \to F(P_i)/F(B_{i+1}) \to F(B_i) \to 0 \ . \tag{10}$$

Thus $F(H^i)$ is the middle cohomology of the sequence

$$0 \to F(B_{i+1}) \stackrel{F(d)}{\to} F(P_i) \stackrel{\alpha}{\to} F(B_i) \to 0 \ . \tag{11}$$

Since $\ker(\alpha) = \ker(F(d) : F(P_i) \to F(P_{i-1})$ we have identified $H^*(F(\mathcal{P}))$ with $F(H^*(\mathcal{P}))$.

The complex (7) is an exact complex of Fréchet spaces and thus (8) represents $F(Z_i)$ as a topological quotient. Moreover, (9) and (10) are exact complexes of Fréchet spaces, and the differential of (11) has closed range. Since $F(H_i) = MG(F(H_i)_K)$, the complex

(11) has property MG. We conclude that $F(\mathcal{P})$ has property MG, too. $\square$

In this note we derive the following two consequences of Theorem 1.1 and its strengthening Lemma 1.2.

Let $P = M_P A_P N_P$ be a Langlands decomposition of a parabolic subgroup $P \subset G$ such that $\mathbf{k} \perp \mathbf{a}_P$, and $K_P := M_P \cap K$, where $\mathbf{m}_P, \mathbf{a}_P, \mathbf{n}_P$ denote the Lie algebras of M_P, A_P, N_P. By Hecht-Schmid [8], if $V_K \in \mathcal{HC}(\mathbf{g}, K)$, then the Lie algebra cohomology

$$H^*(\mathbf{n}_P, V_K) \in \mathcal{HC}(\mathbf{m}_P \oplus \mathbf{a}_P, K_P).$$

If V is a representation of $\mathbf{n}_P$ on a topological vector space, then we equip $H^*(\mathbf{n}_P, V)$ with the topology induced from the standard $\mathbf{n}_P$-cohomology complex. For $V_K \in \mathcal{HC}(\mathbf{g}, K)$ let $mg(V_K)$ denote the minimal globalization of V_K.

Theorem 1.3 *For $V_K \in \mathcal{HC}(\mathbf{g}, K)$ we have topological isomorphisms*

$$\begin{aligned} MG(\widetilde{H_*(\mathbf{n}_P, \widetilde{V_K})}) &= H^*(\mathbf{n}_P, MG(V_K)) && (12)\\ mg(H^*(\mathbf{n}_P, V_K)) &= H^*(\mathbf{n}_P, mg(V_K)) \ . && (13)\end{aligned}$$

In particular, the differentials of the standard $\mathbf{n}_P$-cohomology complexes of $mg(V_K)$ and $MG(V_K)$ have closed range.

As far as we know this theorem was previously announced by Hecht-Taylor and Bratten. Apparently these authors use the technique of analytic localization (see [9], [3]).

Let $\Gamma \subset G$ be a discrete, torsion-free, and cocompact subgroup. Then

$$L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \hat{G}_u}^{Hilbert} N_\Gamma(\pi) V_\pi,$$

where $\hat{G}_u$ is the unitary dual of G and $N_\Gamma(\pi) \in \mathbf{N}_0$ is the multiplicity of π in $L^2(\Gamma \backslash G)$. Let $V_{\pi,K} \in \mathcal{HC}(\mathbf{g}, K)$ be the underlying Harish-Chandra module of V_π and $\mathrm{Ann}(V_{\pi,K}) \subset \mathcal{Z}(\mathbf{g})$ be the annihilator of $V_{\pi,K}$ in the center $\mathcal{Z}(\mathbf{g})$ of the universal enveloping algebra $\mathcal{U}(\mathbf{g})$ of $\mathbf{g}$.

Theorem 1.4 *For $V_K \in \mathcal{HC}(\mathbf{g}, K)$ and a discrete, cocompact, torsion-free subgroup $\Gamma \subset G$ we have*

$$\begin{aligned} H^*(\Gamma, MG(V_K)) &= \bigoplus N_\Gamma(\pi) \mathrm{Ext}^*_{(\mathbf{g},K)}(\widetilde{V_K}, V_{\pi,K}) \ , && (14)\\ \dim H^*(\Gamma, MG(V_K)) &< \infty \ , && (15)\\ H^*(\Gamma, mg(V_K))^* &= H^{\dim(G/K)-*}(\Gamma, MG(V_K)), && (16)\end{aligned}$$

where the sum in (14) is over the finite set of $\pi \in \hat{G}_u$ *with* $\mathrm{Ann}(V_K) \subset \mathrm{Ann}(V_{\pi,K})$.

The proofs of Theorems 1.3 and 1.4 are based on similar ideas and are the contents of Sections 2 and 3.

Besides the maximal and minimal globalizations there are further canonical globalizations $V_{\pm\infty}$ of a Harish-Chandra module V_K. Here V_∞ is the space of smooth vectors in any Banach globalization of V_K and $V_{-\infty}$ is the space of distribution vectors (see [15], Ch.11). As above let $P \subset G$ be a parabolic subgroup. The following theorem was shown by Casselman.

Theorem 1.5 *The differentials of the standard* $\mathbf{n}_P$*-homology complex of* V_∞ *have closed range and we have topological isomorphisms*

$$H_*(\mathbf{n}_P, V_K)_\infty = H_*(\mathbf{n}_P, V_\infty)\ .$$

This implies

$$H^*(\mathbf{n}_P, V_K)_\infty = H^*(\mathbf{n}_P, V_\infty) \tag{17}$$

$$(H_*(\widetilde{\mathbf{n}_P, V_K}))_{-\infty} = H^*(\mathbf{n}_P, V_{-\infty}). \tag{18}$$

In the present paper we show the following consequences of Theorem 1.5.

Theorem 1.6 *Let* $\Gamma \subset G$ *be a discrete, torsion-free, cocompact subgroup and* $V_K \in \mathcal{HC}(\mathbf{g}, K)$. *Then* $\dim\, H^*(\Gamma, V_{\pm\infty}) < \infty$ *and Poincaré duality holds:*

$$H^*(\Gamma, V_\infty)^* = H^{\dim(G/K)-*}(\Gamma, V_{-\infty})\ . \tag{19}$$

Moreover we have $H^*(\Gamma, V_\infty) \cong H^*(\Gamma, mg(V_K))$ *and* $H^*(\Gamma, V_{-\infty}) \cong H^*(\Gamma, MG(V_K))$.

Theorem 1.7 *Let* $W_{K_P} \in \mathcal{HC}(\mathbf{m}_P \oplus \mathbf{a}_P, K_P)$ *and* W_∞ *its smooth globalization. Then there is a spectral sequence*

$$E_2^{p,q} = \bigoplus N_\Gamma(\pi) H^p(\mathbf{m}_P \oplus \mathbf{a}_P, K_P, H^q(\mathbf{n}_P, V_{\pi,\infty} \otimes W_\infty))$$

converging to $H^{p+q}(\Gamma, \mathrm{Ind}_{\mathbf{p},K_P}^{\mathbf{g},K}(W_{K_P})_\infty)$. *Here the sum is over the finite set of* $\pi \in \hat{G}_u$ *with* $\gamma^{-1}(\mathrm{Ann}(W_{K_P})) \subset \mathrm{Ann}(V_{\pi,K})$, *where* $\gamma : \mathcal{Z}(\mathbf{g}) \to \mathcal{Z}(\mathbf{m}_P \oplus \mathbf{a}_P)$ *is the Harish-Chandra homomorphism, and* $\otimes$ *is the injective topological tensor product.*

Now we change the notation slightly. Let G be a reductive, connected algebraic group defined over $\mathbf{Q}$. Moreover, let S be a finite set of places of $\mathbf{Q}$ including the infinite place ω. We combine the results of Kashiwara-Schmid und Schneider-Stuhler [13] in order to prove finiteness results for the cohomology of S-arithmetic groups with coefficients in infinite-dimensional representations. A special case is the following theorem.

Theorem 1.8 *Assume that* $\mathrm{rank}_{\mathbf{Q}}(G) = 0$. *If* $\Gamma \subset G(\mathbf{Q})$ *is* S*-arithmetic and* $V_K \in \mathcal{HC}(\mathbf{g}, K)$, *then* $H^*(\Gamma, MG(V_K))$ *is finite-dimensional.*

2 The comparison theorem for $\mathbf{n}_P$-cohomology

We use the notation introduced in Section 1. We consider $C^\infty(G)_K$ as a $(\mathbf{n}_P, (\mathbf{g}, K))$-bimodule. Then $\mathrm{Hom}_{(\mathbf{g},K)}(., C^\infty(G)_K)$ is a functor from $Mod(\mathbf{g}, K)$ with values in the category $\mathbf{n}_P$-mod of $\mathbf{n}_P$-modules. If I is an injective $(\mathbf{n}_P, (\mathbf{g}, K))$-bimodule, then $\mathrm{Hom}_{(\mathbf{g},K)}(V_K, I)$ is an injective $\mathbf{n}_P$-module for any $V_K \in Mod(\mathbf{g}, K)$. Similarly, the functor $\mathbf{n}_P$–inv of taking $\mathbf{n}_P$-invariants carries (injective) $(\mathbf{n}_P, (\mathbf{g}, K))$-bimodules to (injective) $(\mathbf{g}, K)$-modules. Let $R\mathbf{n}_P$–inv be the right-derived functor of $\mathbf{n}_P$–inv. Then we have a natural isomorphism

$$\begin{aligned} & R\mathbf{n}_P\text{–inv} \circ R\mathrm{Hom}_{(\mathbf{g},K)}(., C^\infty(G)_K) \\ & \qquad = R(\mathbf{n}_P\text{–inv} \circ \mathrm{Hom}_{(\mathbf{g},K)}(., C^\infty(G)_K)). \end{aligned} \tag{20}$$

Since $G \to N_P\backslash G$ is a fibration with fibres which are isomorphic to an Euclidean space we have $R\mathbf{n}_P\text{–inv}(C^\infty(G)_K) = C^\infty(N_P\backslash G)_K$. Inserting this into (20) we obtain

$$\begin{aligned} & R\mathbf{n}_P\text{–inv} \circ R\mathrm{Hom}_{(\mathbf{g},K)}(., C^\infty(G)_K) \\ & \qquad = R\mathrm{Hom}_{(\mathbf{g},K)}(., C^\infty(N_P\backslash G)_K). \end{aligned}$$

Let now $V_K \in \mathcal{HC}(\mathbf{g}, K)$. Then by Theorem 1.1 we have

$$\begin{aligned} R\mathbf{n}_P\text{–inv}(MG(V_K)) &= R\mathbf{n}_P\text{–inv} \circ R\mathrm{Hom}_{(\mathbf{g},K)}(\widetilde{V_K}, C^\infty(G)_K) \\ &= R\mathrm{Hom}_{(\mathbf{g},K)}(\widetilde{V_K}, C^\infty(N_P\backslash G)_K) \\ &= R\mathrm{Hom}_{(\mathbf{g},K)}(\widetilde{V_K}, \mathrm{Ind}^{\mathbf{g},K}_{\mathbf{p},K_P}(C^\infty(M_PA_P)_{K_P})) \\ &= R\mathrm{Hom}_{(\mathbf{p},K_P)}(\mathcal{F}^{\mathbf{g},K}_{\mathbf{p},K_P}(\widetilde{V_K}), C^\infty(M_PA_P)_{K_P}), \end{aligned} \tag{21}$$

where $\mathcal{F}^{\mathbf{g},K}_{\mathbf{p},K_P}$ is the forgetful functor, $\mathbf{p}$ is the Lie algebra of P, and we applied Shapiro's Lemma (see [11], Thm. 7.4.2), in order to obtain the last line. $\mathbf{n}_P \subset \mathbf{p}$ acts trivially on $C^\infty(M_PA_P)_{K_P-finite}$. Let $\mathbf{n}_P\text{–coinv}(V_K) := V_K/\mathbf{n}_PV_K$ be the functor taking $\mathbf{n}_P$-coinvariants and $L\mathbf{n}_P$–coinv be its left-derived functor. Since $\mathbf{n}_P$–coinv maps (finitely generated, projective) $(\mathbf{p}, K_P)$-modules to (finitely generated, projective) $(\mathbf{m}_P \oplus \mathbf{a}_P)$-modules we have for $W_{K_P} \in Mod(\mathbf{p}, K_P)$

$$\begin{aligned} &R\mathrm{Hom}_{(\mathbf{p},K_P)}(W_{K_P}, C^\infty(M_PA_P)_{K_P}) \\ &\qquad = \; R\mathrm{Hom}_{(\mathbf{m}_P\oplus\mathbf{a}_P,K_P)}(L\mathbf{n}_P\text{–coinv}(W_{K_P}), C^\infty(M_PA_P)_{K_P}). \end{aligned} \tag{22}$$

Put $W_K := \mathcal{F}^{\mathbf{g},K}_{\mathbf{p},K_P}(\widetilde{V_K})$. There is a projective resolution

$$\mathcal{P} : \ldots \to P_{-2} \to P_{-1} \to P_0 \to \widetilde{V_K} \to 0$$

such that $P_i = \mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{k})} E_i$, where E_i are finite-dimensional representations of K. Then $L\mathbf{n}_P\text{–coinv}(\widetilde{V_K}) = \mathbf{n}_P\text{–coinv}(\mathcal{P})$. Since $\mathbf{n}_P\text{–coinv}(P_i) \in Mod(\mathbf{m}_P \oplus \mathbf{a}_P, K_P)$ are projective and finitely generated, and $H^*(\mathbf{n}_P\text{–coinv}(\mathcal{P})) = H_{-*}(\mathbf{n}_P, V_K) \in \mathcal{HC}(\mathbf{m}_P \oplus \mathbf{a}_P, K_P)$ the complex $\mathbf{n}_P\text{–coinv}(\mathcal{P})$ satisfies the assumption of Lemma 1.2 (for M_PA_P in the place of G). We conclude that

$$R\mathbf{n}_P\text{–inv}(MG(V_K)) = R\mathrm{Hom}_{(\mathbf{m}_P\oplus\mathbf{a}_P,K_P)}(\mathbf{n}_P\text{–coinv}(\mathcal{P}), C^\infty(M_PA_P)_{K_P})$$

and the right-hand side has property MG. We also obtain from (2)

$$\begin{aligned} H^*(\mathbf{n}_P, MG(V_K)) &= H^*R\mathrm{Hom}_{(\mathbf{m}_P\oplus\mathbf{a}_P,K_P)}(\mathbf{n}_P\text{–coinv}(\mathcal{P}), C^\infty(M_PA_P)_{K_P}) \\ &= \mathrm{Hom}_{(\mathbf{m}_P\oplus\mathbf{a}_P,K_P)}(H^{-*}(\mathbf{n}_P\text{–coinv}(\mathcal{P}), C^\infty(M_PA_P)_{K_P}) \\ &= MG(H_*(\widetilde{\mathbf{n}_P, \widetilde{V_K}})) \end{aligned}$$

and thus (12).

As in [4], Lemma 4.3, we conclude that the differentials of the standard $\mathbf{n}_P$-cohomology complex of $MG(\widetilde{V_K})$

$$0 \to MG(\widetilde{V_K}) \to MG(\widetilde{V_K}) \otimes \Lambda^1\mathbf{n}_P^* \to MG(\widetilde{V_K}) \otimes \Lambda^2\mathbf{n}_P^* \to$$

have closed range, too. The topological dual of the latter complex (up to a shift) is isomorphic in $Mod(M_PA_P)$ to the standard $\mathbf{n}_P$-cohomology complex of $mg(V_K)$ tensored with $\Lambda^d\mathbf{n}_P$, $d := \dim(\mathbf{n}_P)$. It follows that

$$H^*(\mathbf{n}_P, MG(\widetilde{V_K}))^* = H^{d-*}(\mathbf{n}_P, mg(V_K)) \otimes \Lambda^d\mathbf{n}_P \ .$$

We obtain using (12) that

$$\begin{aligned} mg(H^*(\mathbf{n}_P, V_K)) &= MG(H^*(\widetilde{\mathbf{n}_P, V_K}))^* \\ &= MG(H_{d-*}(\widetilde{\mathbf{n}_P, V_K}) \otimes \Lambda^d\mathbf{n}_P)^* \\ &= (H^{d-*}(\mathbf{n}_P, MG(\widetilde{V_K})) \otimes \Lambda^d\mathbf{n}_P)^* \\ &= H^*(\mathbf{n}, mg(V_K)) \ . \end{aligned}$$

This implies (13) and finishes the proof of Theorem 1.3.

3 Γ-cohomology

Let $\Gamma \subset G$ be a discrete, torsion-free, and cocompact subgroup. We consider $C^\infty(G)_K$ as a $(\Gamma, (\mathbf{g}, K))$-bimodule. If I is an injective $(\Gamma, (\mathbf{g}, K))$-bimodule and $V \in Mod(\mathbf{g}, K)$, then $\mathrm{Hom}_{(\mathbf{g},K)}(V, I)$ is an injective Γ-module. Let Γ–inv be the functor of taking Γ-invariants. If I is an injective bimodule, then $\Gamma\text{–inv}(I)$ is an injective $(\mathbf{g}, K)$-module. Since Γ acts freely on G we have

$$R\Gamma\text{–inv}(C^\infty(G)_K) = C^\infty(\Gamma\backslash G)_K \ ,$$

where $R\Gamma$–inv is the right-derived functor of Γ–inv (compare [4], Lemma 2.4). We hence have natural isomorphisms

$$\begin{aligned} & R\Gamma\text{–inv} \circ R\mathrm{Hom}_{(\mathbf{g},K)}(.,C^\infty(G)_K) \\ &\quad = R\mathrm{Hom}_{(\mathbf{g},K)}(.,R\Gamma\text{–inv}(C^\infty(G)_K)) \qquad (23) \\ &\quad = R\mathrm{Hom}_{(\mathbf{g},K)}(.,C^\infty(\Gamma\backslash G)_K) \ . \end{aligned}$$

If $V_K \in \mathcal{HC}(\mathbf{g}, K)$, then by Theorem 1.1 we have

$$R\Gamma\text{–inv}(MG(V_K)) = R\mathrm{Hom}_{(\mathbf{g},K)}(\widetilde{V_K}, C^\infty(\Gamma\backslash G)_K) \ .$$

We decompose

$$\begin{aligned} L^2(\Gamma\backslash G) &= \bigoplus_{\pi\in\hat{G}_u}^{Hilbert} N_\Gamma(\pi)V_\pi \qquad (24) \\ &= \bigoplus_{\mathrm{Ann}(V_K)\subset\mathrm{Ann}(V_{\pi,K})} N_\Gamma(\pi)V_\pi \ \oplus R \ , \qquad (25) \end{aligned}$$

where R is the Hilbert sum of all summands of (24) not occuring in the first summand of (25). The sum in (25) is finite. Now we take the smooth and K-finite vectors and obtain

$$C^\infty(\Gamma\backslash G)_K = \bigoplus_{\mathrm{Ann}(V_K)\subset\mathrm{Ann}(V_{\pi,K})} N_\Gamma(\pi)V_{\pi,K} \ \oplus R_{\infty,K}.$$

As in the proof of Lemma 6.3 [4] we show that $R\mathrm{Hom}_{(\mathbf{g},K)}(\widetilde{V_K}, R_{\infty,K}) = 0$, and we conclude

$$R\Gamma\text{–inv}(MG(V_K)) = \bigoplus_{\mathrm{Ann}(V_K)\subset\mathrm{Ann}(V_{\pi,K})} N_\Gamma(\pi)R\mathrm{Hom}_{(\mathbf{g},K)}(\widetilde{V_K}, V_{\pi,K}) \ .$$

Since

$$\dim\, \mathrm{Ext}^*_{(\mathbf{g},K)}(W_K, U_K) < \infty, \quad U_K, W_K \in \mathcal{HC}(\mathbf{g}, K)$$

we see that $H^*(\Gamma, MG(V_K))$ is finite-dimensional. This proves (14) and (15).

There is a projective resolution of $\widetilde{V_K}$

$$\mathcal{P} : \ldots \to P_1 \to P_0 \to \widetilde{V_K} \to 0$$

in $Mod(\mathbf{g},K)$ with $P_i = \mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{k})} E_i$, where E_i are finite-dimensional representations of K (see [2]). Thus $\mathcal{Q} := \mathrm{Hom}_{(\mathbf{g},K)}(\mathcal{P}, C^\infty(G)_K)$ is a complex of Fréchet spaces (spaces of smooth sections of vector bundles over G/K) with continuous differentials (invariant differential operators). The complex $\mathcal{Q}$ is a Γ-acyclic resolution (by Theorem 1.1) of $MG(V_K)$. Let ${}^\Gamma\mathcal{Q}$ be the complex of Γ-invariants. The spectral theory of locally invariant differential operators on bundles over the compact locally symmetric space $\Gamma\backslash G/K$ shows that the differentials of ${}^\Gamma\mathcal{Q}$ have closed range. Now we can prove Poincaré duality as in [4], Sec. 5.2. This implies (16) of Theorem 1.4.

4 Properties of the smooth globalization

Let $W_{K_P} \in \mathcal{HC}(\mathbf{m}_P \oplus \mathbf{a}_P, K_P)$ and W_∞ be its smooth globalization. We consider $U := C^\infty(G) \otimes W_\infty$ as a (Γ, P)-bimodule, where P acts on $C^\infty(G)$ as well as on W_∞. Here $\otimes$ is the topological (say injective) tensor product. Note that $C^\infty(G)$ is nuclear. U is acyclic with respect to the functor Γ–inv. In fact the proof of [4], Lemma 2.4, carries over to the present situation.

Note that U is a smooth P-module but not a $(\mathbf{p}, K_P)$-module. Nevertheless we want to consider the right-derived functor of $(\mathbf{p}, K_P)$–inv on modules like U. By abuse of notation we view $R(\mathbf{p}, K_P)$–inv(U) as the concrete relative Lie-algebra cohomology complex

$$0 \to U^{K_P} \to (\Lambda^1(\mathbf{p}/\mathbf{k}_P)^* \otimes U)^{K_P} \to (\Lambda^2(\mathbf{p}/\mathbf{k}_P)^* \otimes U)^{K_P} \to \dots \ . \tag{26}$$

U is $(\mathbf{p}, K_P)$–inv-acyclic. This can be seen by recognising in (26) the de Rham complexes along the contractible P-orbits in G/K_P (compare Deitmar, [6], Lemma 4.1).

Note that

$$\mathrm{Ind}_{\mathbf{p},K_P}^{\mathbf{g},K}(W_{K_P})_\infty = (\mathbf{p}, K_P)\text{–inv}(U)$$

and hence $\mathrm{Ind}_{\mathbf{p},K_P}^{\mathbf{g},K}(W_{K_P})_\infty = R(\mathbf{p}, K_P)$–inv$(U)$. The complex $R(\mathbf{p}, K_P)$–inv(U) is a complex of Γ–inv-acyclic Γ-modules. Hence

$$\begin{aligned} R\Gamma\text{–inv}(\mathrm{Ind}_{\mathbf{p},K_P}^{\mathbf{g},K}(W_{K_P})_\infty) &= R\Gamma\text{–inv} \circ R(\mathbf{p}, K_P)\text{–inv}(U) \\ &= R(\mathbf{p}, K_P)\text{–inv}(R\Gamma\text{–inv}(U)) \\ &= R(\mathbf{p}, K_P)\text{–inv}(C^\infty(\Gamma\backslash G) \otimes W_\infty) \ , \end{aligned} \tag{27}$$

where $\otimes$ is again the topological injective tensor product. We insert the continuous decomposition

$$C^\infty(\Gamma\backslash G) = \bigoplus_{\gamma^{-1}\mathrm{Ann}(W_{K_P}) \subset \mathrm{Ann}(V_{\pi,K})} N_\Gamma(\pi) V_{\pi,\infty} \oplus R_\infty \ .$$

The action of $D \in \mathcal{Z}(\mathbf{g})$ on $H^*(\Gamma, \mathrm{Ind}_{\mathbf{p},K_P}^{\mathbf{g},K}(W_{K_P})_\infty)$ is induced by the action of D on $C^\infty(G)$.

Applying again the technique of the proof of Lemma 6.3 in [4] we show that the summand R_∞ does not contribute to the Γ-cohomology, and we have

$$R\Gamma\text{–inv}(\mathrm{Ind}_{\mathbf{p},K_P}^{\mathbf{g},K}(W_{K_P})_\infty) = \bigoplus_{\gamma^{-1}\mathrm{Ann}(W_{K_P})\subset\mathrm{Ann}(V_{\pi,K})} N_\Gamma(\pi) R(\mathbf{p}, K_P)\text{–inv}(V_{\pi,\infty} \otimes W_\infty) \ .$$

Using the extension

$$0 \to \mathbf{n}_P \to \mathbf{p} \to \mathbf{m}_P \oplus \mathbf{a}_P \to 0$$

we can write

$$R(\mathbf{p}, K_P)\text{–inv}(V_{\pi,\infty} \otimes W_\infty) = R(\mathbf{m}_P \oplus \mathbf{a}_P, K_P)\text{–inv}(R\mathbf{n}_P\text{–inv}(V_{\pi,\infty} \otimes W_\infty))$$

viewed as an explicit double complex. The columns are standard $\mathbf{n}_P$-cohomology complexes. Let $E_2^{p,q} = H^p(\mathbf{m}_P \oplus \mathbf{a}_P, K_P, H^q(\mathbf{n}_P, V_{\pi,\infty} \otimes W_\infty))$ be the second stage of the spectral sequence converging to $H^{p+q}(\mathbf{p}, K_P, V_{\pi,\infty} \otimes W_\infty)$. Theorem 1.7 follows.

We now prove Theorem 1.6. It is another theorem of Casselman (see [14], 3.8.3) that any module $V_K \in \mathcal{HC}(\mathbf{g}, K)$ is a submodule as well as a quotient of a principal series representation associated to the minimal parabolic subgroup P_0 of G . Using the long exact Γ-cohomology sequences it is easy to reduce the proof of Theorem 1.6 to the case of principal series representations associated to P_0. Thus 1.6 is a consequence of 1.7. In fact in this case W_{K_P} is finite-dimensional and the spectral sequence reduces to

$$E_2^{p,q} = \bigoplus N_\Gamma(\pi) H^p(\mathbf{m}_P \oplus \mathbf{a}_P, K_P, H^q(\mathbf{n}_P, V_{\pi,K}) \otimes W_{K_P})$$

$(P = P_0)$ and thus consists of finite-dimensional vector spaces. The considerations above also imply that the differentials of (27) have closed range. Now Poincaré duality can be proved as in [4], Sec. 5.2. This finishes the proof of Theorem 1.6.

5 Cohomology of S-arithmetic groups

In this section we will change the notation of the groups slightly. By G we denote a reductive, connected algebraic group defined over $\mathbf{Q}$. Moreover, let S be a finite set of places of $\mathbf{Q}$ including the infinite place ω. Let $S_f := S \setminus \{\omega\}$ the set of finite places of S.

The goal of this section is to prove the finiteness of the cohomology of an S-arithmetic group $\Gamma \subset G(\mathbf{Q})$ with coefficients in a class of representations which we will now describe.

For $p \in S_f$ let C_p be the connected center of $G(\mathbf{Q}_p)$, $\chi_p : \mathbf{C}_p \to \mathbf{C}^*$ be a continuous character and $\mathrm{Alg}_{\chi_p}(G(\mathbf{Q}_p))$ be the category of smooth representations of $G(\mathbf{Q}_p)$ with central character χ_p. Fix an admissible $V_p \in \mathrm{Alg}_{\chi_p}(G(\mathbf{Q}_p))$ for each $p \in S_f$. Moreover, let $V_K \in \mathcal{HC}(\mathbf{g}, K)$ and $V_{-\omega}$ be its maximal globalization (in the notation of the preceding sections this was $MG(V_K)$). Then there is a natural representation of

$$G(\mathbf{R}) \times \prod_{p \in S_f} G(\mathbf{Q}_p)$$

on

$$W := \mathrm{Hom}_{\mathbf{C}}\left(\bigotimes_{p \in S_f} V_p, V_{-\omega}\right)$$

given by $(g\phi)(\otimes v_p) = g_\omega \phi(\otimes g_p^{-1} v_p)$, where $g = (g_\omega, (g_p | p \in S_f))$ and $v_p \in V_p$. By the usual diagonal embedding

$$\Gamma \hookrightarrow G(\mathbf{R}) \times \prod_{p \in S_f} G(\mathbf{Q}_p)$$

W becomes a Γ-module.

Theorem 5.1 *Suppose that* $\mathrm{rank}_{\mathbf{Q}}(G) = 0$. *Then* $H^*(\Gamma, W)$ *is finite-dimensional.*

The proof of the theorem also gives a way for computing the cohomology groups in terms of automorphic forms. In the case that V_K is finite-dimensional and V_p is trivial for all $p \in S_f$, this result has been obtained by Borel-Serre [1] (without the condition on $\mathrm{rank}_{\mathbf{Q}}(G)$).

The method also works in the case $\mathrm{rank}_{\mathbf{Q}}(G) \geq 0$. In this case one has to replace the maximal globalization $V_{-\omega}$ by the smooth globalization $V_{-\infty}$. Then $H^*(\Gamma, W)$ is finite-dimensional without any condition on $\mathrm{rank}_{\mathbf{Q}}(G)$. The proof of the latter result depends on unpublished recent results of Kashiwara-Schmid which were announced to us by Schmid and the results of Franke [7]. In the case that $\mathrm{rank}_{\mathbf{R}}(G) = 1$ and G is semisimple instead of these big machines on can alternatively use the results of [5].

In the present paper we will prove the theorem as stated. It heavily depends on the recent results of Kashiwara-Schmid [10], Thm. 1.1, and Schneider-Stuhler [13].

Proof.

Let X_p be the Bruhat-Tits building of $G(\mathbf{Q}_p)$. Schneider-Stuhler [13] construct a system of compact open subgroups $U_F^{(e)} \subset G(\mathbf{Q}_p)$, F a facet of X_p, $e \in \mathbf{N}_0$, satisfying

1. $gU_F^{(e)}g^{-1} = U_{gF}^{(e)}$, $\forall g \in G(\mathbf{Q}_p)$, F a facet of X_p.

2. $\{U_F^{(e)}\}_{e\in\mathbf{N}_0}$ is a fundamental system of neighbourhoods of $1 \in G(\mathbf{Q}_p)$.

3. If $F \subset \bar{F}'$, then $U_F^{(e)} \subset U_{F'}^{(e)}$.

An orientation of a $q \geq 1$-dimensional facet $F \subset X_p$ is a generator $c \in H_q(X_p^q, X_p^q \setminus F) \cong \mathbf{Z}$, where X_p^q is the q-skeleton of X_p. By $X_p^{(q)}$ we denote the set of all oriented q-facets (F, c) if $q \geq 1$, and $X_p^{(0)}$ denotes the set of vertices of X_p. If $F, F' \subset X_p$ are facets, $\dim(F) = q+1$, $\dim(F') = q$, then there is a natural map

$$\partial_{F'}^F : H_{q+1}(X_p^{q+1}, X_p^{q+1} \setminus F) \to H_q(X_p^q, X_p^q \setminus F')$$

which is an isomorphism if $F' \subset \bar{F}$ and zero otherwise.

For all $p \in S_f$ we choose $e_p \in \mathbf{N}_0$ sufficiently large such that V_p is generated by $V_p^{U_F^{(e)}}$ for some facet $F \subset X_p$. This is possible by property 2. Let $e := \max_{p\in S_f} e_p$.

Let J_f be the set of all tuples $l_f = (l_p | p \in S_f)$ with $l_p \in \{0, \ldots, \dim(X_p)\}$. For $l_f \in J_f$ set $Z^{l_f} := \prod_{p\in S_f} X_p^{(l_p)}$. If $x \in Z^{l_f}$, then the stabilizer $\Gamma_x \subset \Gamma$ of x is an arithmetic group acting cocompactly ($\mathrm{rank}_{\mathbf{Q}}(G) = 0$!) on the Riemannian symmetric space $X_\omega := G(\mathbf{R})/K(\mathbf{R})$. Let $d(x)$ be the diameter of $\Gamma_x \backslash X_\omega$. Since $\Gamma \backslash \cup_{l_f \in J_f} Z^{l_f}$ is finite and $d(gx) = d(x)$, we have

$$d := \max_{x \in \cup_{l_f\in J_f} Z^{l_f}} d(x) < \infty \ .$$

Let $U \subset X_\omega$ be the ball of radius $d+1$ centered at the origin. Let J be the set of all tuples $l = (k, l_\omega, l_f)$, where $k, l_\omega \in \mathbf{N}_0$, $l_f \in J_f$. For $l \in J$ and $x \in Z^{l_f}$ let Z_x^l be the set of tuples $A = (g_0 U, \ldots, g_{l_\omega} U)$ with $|A| := g_0 U \cap \ldots \cap g_{l_\omega} U \neq \emptyset$, where $g_i \in \Gamma_x$. Set $Z^l = \cup_{x\in Z^{l_f}} Z_x^l$ and denote a generic point of Z^l by (A, x), $x \in Z^{l_f}$, $A \in Z_x^l$.

Let

$$\mathcal{P}^\cdot : \ldots \to P_2 \to P_1 \to P_0 \to \tilde{V}_K \to 0 \tag{28}$$

be a projective resolution of $\tilde{V}_K$ in the category $Mod(\mathbf{g}, K)$ with P_i of the special form $\mathcal{U}(\mathbf{g}) \otimes_{\mathcal{U}(\mathbf{k})} L_i$, where L_i is a finite-dimensional representation of $K(\mathbf{R})$. Let $\mathcal{V}^k$ be the sheaf of smooth sections of the homogeneous bundle $G(\mathbf{R}) \times_{K(\mathbf{R})} L_k$. Sheafification of $\mathrm{Hom}_{(\mathbf{g},K)}(\mathcal{P}^\cdot, C^\infty(G)_K)$ leads to a complex of sheaves

$$0 \to \mathcal{V}^0 \stackrel{d_{SK}}{\to} \mathcal{V}^1 \stackrel{d_{SK}}{\to} \mathcal{V}^2 \stackrel{d_{SK}}{\to} \ldots \ , \tag{29}$$

where d_{SK} are differential operators.

For any $l \in J$, $l = (k, l_\omega, l_f)$ we define C^l to be the vector space of all

$$\phi \in \prod_{(A,x)\in Z^l} \mathrm{Hom}\left(\bigotimes_{p\in S_f} V_p^{U^{(e)}_{F_p}}, \mathcal{V}^k(|A|)\right)$$

which are odd with respect to the change of the orientations of the facets (F_p, c_p) ($x = ((F_p, c_p)|p \in S_f)$) and with respect to permutations of the components of A.

The differentials d_{SK} induce differentials

$$d_{SK} : C^l \to C^{l'} \ ,$$

where $l = (k, l_\omega, l_f)$ and $l' = (k+1, l_\omega, l_f)$. For any $x \in Z^{l_f}$ the cover $\{gU\}_{g\in\Gamma_x}$ leads to Čech complexes for the sheaves $\mathcal{V}^k$. The Čech differential induces a differential

$$d_\omega : C^l \to C^{l'} \ ,$$

where $l = (k, l_\omega, l_f)$ and $l' = (k, l_\omega + 1, l_f)$. For any $p \in S_f$ we define a differential

$$d_p : C^l \to C^{l'} \ ,$$

where $l = (k, l_\omega, l_f)$ and $l' = (k, l_\omega, l'_f)$, $l_f = (l_q|q \in S_f)$, $l'_f = (l'_q|q \in S_f)$, $l_q = l'_q$, $\forall q \in S_f \setminus \{p\}$ and $l'_p = l_p + 1$ as follows: Let $\phi \in C^l$ and $(A, x') \in Z^{l'}$, $x' = ((F'_q, c'_q)|q \in S_f)$, $\otimes v_q \in \bigotimes_{q\in S_f} V_q^{U^{(e)}_{F'_q}}$. Then we set

$$(d_p\phi)_{(A,x')}(\otimes v_q) := \sum_{(F_p,c_p)\in X_p^{(l_p)}, \partial^{F'_p}_{F_p} c'_p = c_p} \phi_{(A,x)}(\otimes v_q) \ ,$$

where $x = ((F_p, c_p)|p \in S_f)$ with $(F_q, c_q) = (F'_q, c'_q)$ and $\forall q \in S_f \setminus \{p\}$. The differential d_p is well defined by property 3. of the system $\{U^{(e)}_{F_q}\}$ and satisfies $d_p^2 = 0$ (see [13]).

The system of spaces C^l, $l \in J$, and differentials d_{SK}, d_q, $q \in S$, forms a multicomplex.

Lemma 5.2 *The total complex associated to $((C^l|l \in J), (d_{SK}, (d_q|q \in S)))$*

$$0 \to W \to D^0 \to D^1 \to D^2 \to \dots \tag{30}$$

is a Γ-acyclic resolution of W.

Proof. The action of Γ on C^l is defined by

$$(g\phi)_{(A,x)}(\otimes v_q) := g^*(\phi_{(g^{-1}A, g^{-1}x)}(\otimes g^{-1} v_q), \quad \phi \in C^l \ .$$

Since $g^{-1}A \in Z^l_{g^{-1}x}$, $g^{-1}v_q \in V_q^{U^{(e)}_{g^{-1}F_q}}$ (by property 1. of the system $\{U^{(e)}_{F_q}\}$),

$$\phi_{(g^{-1}A, g^{-1}x)}(\otimes g^{-1}v_q) \in \mathcal{V}^k(g^{-1}|A|) \ ,$$

and the action $C^l \ni \phi \mapsto g\phi \in C^l$ is well defined.

Let $\breve{Z}^{l_f}$ be the finite set of representatives of $\Gamma\backslash Z^{l_f}$. For any $x \in \breve{Z}^{l_f}$ let $\breve{Z}^l_x$ be the finite set of representatives of $\Gamma_x\backslash X^l_x$. Then the Γ-module C^l has the description

$$C^l = \bigoplus_{x \in \breve{Z}^{l_f}} \mathrm{Ind}^{\Gamma}_{\{1\}} \bigoplus_{(A,x) \in \breve{Z}^l_x} \mathrm{Hom}\left(\bigotimes_{q \in S_f} V_q^{U^{(e)}_{F_q}}, \mathcal{V}^k(|A|)\right)$$

$(x = ((F_p, c_p)|p \in S_f))$, and this module is Γ-acyclic by Shapiro's Lemma.

In order to finish the proof of the lemma we must show exactness of (30). Let $l_i := (k, i, l_f)$. Since $\mathcal{V}^k$ are soft sheaves, the Čech complex

$$0 \to C^{l_0} \stackrel{d_\omega}{\to} C^{l_1} \stackrel{d_\omega}{\to} C^{l_2} \stackrel{d_\omega}{\to} \ldots \tag{31}$$

is exact in degree ≥ 1 and its zeroth cohomology is

$$C^{k,l_f} := \prod_{x \in Z^{l_f}} \mathrm{Hom}\left(\bigotimes_{q \in S_f} V_q^{U^{(e)}_{F_q}}, \mathcal{V}^k(X_\omega)\right) \ ,$$

where $x = ((F_p, c_p)|p \in S_f)$.

Note that $\bigotimes_{q \in S_f} V_q^{U^{(e)}_{F_q}}$ is finite-dimensional. We can apply Theorem 1 in order to conclude that for each $l_f \in J_f$ the complex

$$0 \to C^{0,l_f} \stackrel{d_{SK}}{\to} C^{1,l_f} \stackrel{d_{SK}}{\to} C^{2,l_f} \stackrel{d_{SK}}{\to} \ldots$$

is acyclic in degree ≥ 1 and its zeroth cohomology is

$$C^{l_f} := \prod_{x \in Z^{l_f}} \mathrm{Hom}\left(\bigotimes_{q \in S_f} V_q^{U^{(e)}_{F_q}}, V_{-\omega}\right) \ ,$$

where $x = ((F_p, c_p)|p \in S_f)$.

Standard homological algebra shows that the cohomology of the total complex associated to the multicomplex $((C^{l_f}|l_f \in J_f), (d_p|p \in S_f))$ is equal to the cohomology of the complex (30).

For $q \in S_f$ let $S^q_f := \{p \in S_f | q \geq p\}$, $J^q_f := \{l^q_f = (l^q_p|p \in S^q_f), l^q_p \in \{0, \ldots, \dim(X_p)\}\}$. Assume that we have already shown that the cohomology of the total complex associated to the multicomplex $((C^{l_f^q}|l_f^q \in J_f^q),(d_p|p \in S_f^q))$ is equal to the cohomology of the complex (30). Let $r \in S_f$ be the largest prime smaller than q. Then we define $((C^{l_f^r}|l_f^r \in J_f^r),(d_p|p \in S_f^r))$ as the zeroth cohomology of

$$0 \to C^{(l_f^r,0)} \xrightarrow{d_q} C^{(l_f^r,1)} \xrightarrow{d_q} C^{(l_f^r,2)} \xrightarrow{d_q} \dots \; . \tag{32}$$

If we could prove that (32) is acyclic in degree ≥ 1, then we would know that the cohomology of the total complex associated to the multicomplex $((C^{l_f^r}|l_f^r \in J_f^r),(d_p|p \in S_f^r))$ is equal to the cohomology of the complex (30). We reduce this induction step to the result of Schneider-Stuhler.

Let E_p^i be the space of all functions $\omega : X_p^{(i)} \to V_p$ satisfying

- ω has finite support,
- $\omega((F,c)) \in V_p^{U_F^{(e)}}$,
- if $i \geq 1$, then $\omega((F,c)) = -\omega((F,-c))$.

Define the differential $\partial_p : E_p^{i+1} \to E_p^i$ by

$$(\partial_p\omega)((F',c')) = \sum_{(F,c)\in X_p^{(i+1)},\partial_{F'}^F(c)=c'} \omega((F,c)) \; .$$

Then Schneider-Stuhler (Theorem 1. and Corollary 2.,[13]) have shown that

$$\mathcal{E}_p^{\cdot} : \dots \xrightarrow{\partial_p} E_p^2 \xrightarrow{\partial_p} E_p^1 \xrightarrow{\partial_p} E_p^0 \to 0$$

is acyclic in degree $\neq 0$ and its zeroth cohomology is isomorphic to V_p. This isomorphism is induced by $\epsilon : E_p^0 \to V_p$ given by $\epsilon(\omega) := \sum_{F\in X_p^{(0)}} \omega(F)$.

We claim that the complex (32) is isomorphic to

$$\prod_{y=((F_p,c_p)|p\in S_f^r)\in\prod_{p\in S_f^r} X_p^{(l_p^r)}} \mathrm{Hom}\left(\bigotimes_{p\in S_f\setminus S_f^q} V_p \otimes \mathcal{E}_q^{\cdot} \otimes \bigotimes_{p\in S_f^r} V_p^{U_{F_p}^{(e)}}, V_{-\omega} \right) \; .$$

In fact this is true if q is the largest prime of S_f.

Now we argue by induction. By the exactness of the tensor product and the Hom-functor for complex vector spaces the complex (32) is exact at all degrees ≥ 1 and its zeroth cohomology is just

$$\prod_{y=((F_p,c_p)|p\in S_f^r)\in\prod_{p\in S_f^r} X_p^{(l_p^r)}} \mathrm{Hom}\left(\bigotimes_{p\in S_f\setminus S_f^q} V_p \otimes V_q \otimes \bigotimes_{p\in S_f^r} V_p^{U_{F_p}^{(e)}}, V_{-\omega}\right) .$$

Eventually we obtain that (30) is exact. □

We now finish the proof of the Theorem 5.1. Take the complex of Γ-invariant vectors of (30)

$$0 \to {}^\Gamma D^0 \to {}^\Gamma D^1 \to {}^\Gamma D^2 \to \dots . \tag{33}$$

Since there are equivariant partitions of unity the partial complex

$$0 \to {}^\Gamma C^{l_0} \overset{d_\omega}{\to} {}^\Gamma C^{l_1} \overset{d_\omega}{\to} {}^\Gamma C^{l_2} \overset{d_\omega}{\to} \dots \tag{34}$$

is exact in degree ≥ 1 and its zeroth cohomology is

$${}^\Gamma C^{(k,l_f)} := \prod_{x\in \check{Z}^{l_f}} \mathrm{Hom}\left(\bigotimes_{q\in S_f} V_q^{U_{F_q}^{(e)}}, \mathcal{V}^k(Y_x)\right) ,$$

where $x = ((F_p, c_p)|p \in S_f)$ and $Y_x := \Gamma\backslash X_\omega$. Here $\mathcal{V}^k(Y_x)$ denotes the space of smooth sections of the locally homogeneous vector bundle $\Gamma_x\backslash G(\mathbf{R}) \times_{K(\mathbf{R})} L_k$. Thus the cohomology of (33) is equal to the cohomology of the total complex associated to the multicomplex $((C^{(k,l_f)}|k \in \mathbf{N}_0, l_f \in J_f), (d_{SK}, (d_p|p \in S_f)))$.

We form the double complex $((C^{(k,h)}|k, h \in \mathbf{N}_0), (d_{SK}, d))$ by taking the total complex with respect to the index l_f. There is a spectral sequence converging to the cohomology of (33) with second term

$$E_2^{h,k} = H_d^h H_{d_{SK}}^k({}^\Gamma C^{(.,.)}) .$$

By the proof of Theorem 1.4 we have

$$H_{d_{SK}}^k({}^\Gamma C^{(.,h)}) = \bigoplus_{l_f\in J_f, \sum_{p\in S_f} l_p=h}\ \bigoplus_{x\in\check{Z}^{l_f}} \mathrm{Hom}\left(\bigotimes_{p\in S_f} V_p^{U_{F_p}^{(e)}}, H^k(\Gamma_x, V_{-\omega})\right) ,$$

$(l_f = (l_p, p \in S_f)$. Thus $E_2^{h,k}$ is finite-dimensional for all $h, k \in \mathbf{N}_0$, and this implies the assertion of Theorem 5.1. □

References

[1] A. Borel and J. P. Serre. Cohomologie d'immeubles et de groupes S-arithmétiques. *Topology*, 15, 211–232 (1976).

[2] A. Borel and N. Wallach. *Continuous Cohomology, Discrete Subgroups, and Representations of Reductive Groups*. Princeton University Press, 1980.

[3] T. Bratton. A comparison theorem for Lie algebra homology groups. Preprint, 1995.

[4] U. Bunke and M. Olbrich. Gamma-cohomology and the Selberg zeta function. *J. reine angew. Math.*, 467, 199–219 (1995).

[5] U. Bunke and M. Olbrich. Cohomological properties of the smooth globalization of a Harish-Chandra module. Preprint, SFB288, No.177, 1995.

[6] A. Deitmar. On some zeta functions attached to compact Shimura manifolds. Preprint, 1995.

[7] J. Franke. Harmonic analysis in weighted L_2-spaces. preprint, 1995.

[8] H. Hecht and W. Schmid. Characters, asymptotics, and $\mathbf{n}$-homology of Harish-Chandra modules. *Acta Math.*, 151, 49–151 (1983).

[9] H. Hecht and J. L. Taylor. A comparison theorem for $\mathbf{n}$-homology. *Compositio Math.*, 86, 189–207 (1993).

[10] M. Kashiwara and W. Schmid. Quasi-equivariant D-modules, equivariant derived category, and representations of reductive Lie groups. In *Lie Theory and Geometry. In Honor of Bertram Kostant*, pages 457–488. Progr. in Math. 123, Birkhäuser Boston, 1994.

[11] A. W. Knapp. *Lie Groups, Lie Algebras, and Cohomology.* Princeton University Press, 1988.

[12] W. Schmid. Boundary value problems for group invariant differential equations. *Soc. Math. de France, Asterisque, hors série*, pages 311–321, 1985.

[13] P. Schneider and U. Stuhler. Representation theory and sheaves on the Bruhat-Tits building. Preprint, 1995.

[14] N. R. Wallach. *Real Reductive Groups.* Academic Press, 1988.

[15] N. R. Wallach. *Real Reductive Groups II.* Academic Press, 1992.