Group cohomology and the singularities of the Selberg zeta function associated to a Kleinian group

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1 Introduction

Let $G := SO(1,n)$ denote the isometry group of the n-dimensional hyperbolic space $X := H^n$ equipped with the Riemannian metric of constant sectional curvature -1. We consider a convex-cocompact, torsion-free subgroup $\Gamma \subset G$. The quotient $Y := \Gamma\backslash X$ is a complete hyperbolic

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manifold and we assume that $\mathrm{vol}(Y) = \infty$. The Selberg zeta function $Z_S(s)$, $s \in \mathbf{C}$, associated to this geometric situation encodes the lengths of the closed geodesics of Y and the eigenvalues of their Poincaré maps. In order to fix our conventions we will give a definition of $Z_S(s)$ in group-theoretic terms below. The goal of this paper is a description of the singularities of the Selberg zeta function in group-cohomological terms.

Let $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ be a Cartan decomposition of the Lie algebra of G, where $\mathbf{k}$ is the Lie-algebra of a maximal compact subgroup $K \subset G$, $K \cong SO(n)$. Fix a one-dimensional subspace $\mathbf{a} \subset \mathbf{p}$ and let $M \subset K$, $M \cong SO(n-1)$, denote the centralizer of $\mathbf{a}$. The Riemannian metric of X induces a metric on $\mathbf{a}$. We fix an isometry $\mathbf{a} \cong \mathbf{R}$. Let $\mathbf{a}_+$ denote the half-space corresponding to the ray $[0, \infty)$ and set $A := \exp(\mathbf{a})$, $A_+ := \exp(\mathbf{a}_+)$. By $\mathbf{n} \subset \mathbf{g}$ we denote the positive root-space of $(\mathbf{g}, \mathbf{a})$. We define $N := \exp(\mathbf{n})$, $N \cong \mathbf{R}^{n-1}$, and obtain the Iwasawa decomposition $G = KAN$. Any element $g \in \Gamma$ is conjugated in G to an unique element of the form $m_g a_g \in MA_+$. By $l_g := \log(a_g)$ we denote the length of the closed geodesic of Y corresponding to g.

Definition 1.1 *The exponent $\delta_\Gamma \in \mathbf{R}$ of Γ is defined to be the smallest number such that the Poincaré series*

$$\sum_{g \in \Gamma} \mathrm{e}^{-(s + \frac{n-1}{2}) l_g} \tag{1}$$

converges for all $s > \delta_\Gamma$.

Let $C\Gamma$ denote the set of conjugacy classes $[g] \neq 1$ of Γ. If $[g] \in C\Gamma$, then $n_\Gamma(g) \in \mathbf{N}$ is the multiplicity of $[g]$, i.e., the largest number $k \in \mathbf{N}$ such that $[g] = [h^k]$ for some $[h] \in C\Gamma$. If $A : V \to V$ is a linear homomorphism of a complex vector space V, then by $S^k A : S^k V \to S^k V$ we denote its k'th symmetric power.

Definition 1.2 *The Selberg zeta function $Z_S(s)$, $s \in \mathbf{C}$, associated to Γ is defined by the infinite product*

$$Z_S(s) := \prod_{[g] \in C\Gamma, n_\Gamma(g) = 1} \prod_{k=0}^{\infty} \det \left(1 - \mathrm{e}^{-(\frac{n-1}{2} + s) l_g} S^k (\mathrm{Ad}(m_g a_g)^{-1}_{|\mathbf{n}})\right)$$

Fact 1.3 *The infinite product converges for $\mathrm{Re}(s) > \delta_\Gamma$. The Selberg zeta function has a meromorphic continuation to all of $\mathbf{C}$ [15].*

In the case of cocompact Γ the Selberg zeta function has proved to be a useful tool connecting the geometric data given by the closed geodesics of Y and the spectrum of the Laplace-Beltrami operator Δ_Y. More precisely the spectral data can be seen in the singularities of the Selberg zeta function [2].

In our present case the spectrum of Δ_Y consists of finitely many isolated eigenvalues and an absolute continuous spectrum $[(\frac{n-1}{2})^2, \infty)$ of infinite multiplicity [16]. The distributional kernel of the inverse $(\Delta_Y - (\frac{n-1}{2})^2 + \lambda^2)^{-1}$ which is initially defined for $\mathrm{Re}(\lambda) >> 0$ has a meromorphic continuation to all of $\mathbf{C}$. The poles of this continuation with positive real part correspond to the eigenvalues of Δ_Y. The poles with non-positive real part are called resonances. Let us consider the resonances as a sort of eigenvalues associated to the continuous spectrum. Then the relation of the spectral data with the singularities of $Z_S(s)$ is the same as in the cocompact case. If n is even this has been worked out in detail in [15].

For cocompact Γ we have given (proving a conjecture of S. Patterson) a description of the singularities of $Z_S(s)$ in terms of group cohomology of Γ with coefficients in the space of hyperfunction vectors of the spherical principal series representation of G. One advantage of this picture is that it gives a uniform description of the singularities of $Z_S(s)$ without distinction between topological singularities (at negative half integers) and spectral singularities (the remaining ones).

In the present paper we extend the group cohomological picture to the convex-cocompact case. Proving (essentially) the corresponding conjecture of S. Patterson [14] we relate the order of $Z_S(s)$ at $s = \lambda$ (a pole has negative and a zero positive order) with the dimensions of the cohomology groups of Γ with coefficients in the representation $\mathcal{O}_{(\lambda,k)}C^{-\omega}(\Lambda)$ which we will now describe.

The geodesic boundary $\partial X \cong S^{n-1}$ can be identified with G/P, where $P = MAN$. The group G appears here as the group of conformal automorphisms of the standard sphere S^{n-1}. There is a Γ-invariant partition $\partial X = \Omega \cup \Lambda$, where Λ is the limit set, and Ω is a domain of discontinuity for Γ with compact quotient $B := \Gamma\backslash\Omega$.

Fact 1.4 *Patterson [12] ($n = 2$) and Sullivan [19] have shown that*

$$\delta_\Gamma = \dim_H(\Lambda) - \frac{n-1}{2} \in (-\frac{n-1}{2}, \frac{n-1}{2}) \ ,$$

where $\dim_H(\Lambda)$ *is the Hausdorff dimension of* Λ.

For any $\lambda \in \mathbf{C}$ let V_λ be the representation of P on $\mathbf{C}$ given by $man \mapsto \mathrm{e}^{(\frac{n-1}{2}-\lambda)\log(a)}$. Let $V(\lambda) := G \times_P V_\lambda$ be the associated homogeneous vector bundle. Note that $V(-\frac{n-1}{2}) \cong \Lambda^{n-1}T^*_{\mathbf{C}}\partial X$ is the complexified bundle of volume forms. Moreover, $V(\lambda) \cong (\Lambda^{n-1}T^*_{\mathbf{C}}\partial X)^{-\frac{2\lambda-(n-1)}{2(n-1)}}$, and if we choose a nowhere vanishing volume form vol, then $vol^{-\frac{2\lambda-(n-1)}{2(n-1)}}$ is a section trivializing $V(\lambda)$. Sections of $V(\lambda)$ can be viewed as functions which transform under G according to a conformal weight related with λ.

The union $\cup_{\lambda\in\mathbf{C}}V(\lambda) \to \partial X$ has the structure of a holomorphic family of vector bundles. Choosing a nowhere vanishing volume form $vol \in C^\omega(\partial X, V(-\frac{n-1}{2}))$ we define isomorphisms $vol^{-\frac{\mu}{n-1}} : V(\lambda) \cong V(\lambda+\mu)$. Using these isomorphisms we can identify the space of sections of $V(\lambda)$ of a given regularity with the corresponding space of sections of one fixed bundle, say, $V(0)$. This allows us to speak of holomorphic families of sections or homomorphisms.

By $\pi^\lambda(g) : C^{-\omega}(\partial X, V(\lambda)) \to C^{-\omega}(\partial X, V(\lambda))$, $g \in G$, we denote the representation of G on the space of hyperfunction sections of $V(\lambda)$. As a topological vector space $C^{-\omega}(\partial X, V(\lambda))$ is the space of continuous linear functionals on $C^{\omega}(\partial X, V(-\lambda))$. If $g \in G$ is fixed, then $\pi^\lambda(g)$ depends holomorphically on λ.

Since Λ is Γ-invariant the space of hyperfunctions $C^{-\omega}(\Lambda, V(\lambda)) \subset C^{-\omega}(\partial X, V(\lambda))$ with support in Λ is a representation of Γ. Let $\mathcal{O}_\lambda C^{-\omega}(\Lambda)$ denote the space of germs at λ of holomorphic families of sections $f_\mu \in C^{-\omega}(\partial X, V(\mu))$ with $\mathrm{supp}(f_\mu) \in \Lambda$. The representation of Γ on that space is given by $(\pi(g)f)_\mu := \pi^\mu(g) f_\mu$, $g \in \Gamma$.

If L^k denotes the multiplication operator $L^k : f_\mu \mapsto (\mu-\lambda)^k f_\mu$, $k \in \mathbf{N}$, then we have a short exact sequence

$$0 \to \mathcal{O}_\lambda C^{-\omega}(\Lambda) \stackrel{L^k}{\to} \mathcal{O}_\lambda C^{-\omega}(\Lambda) \to \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda) \to 0$$

of Γ-modules defining $\mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda)$. There are canonical projections

$$p_k^{k+1} : \mathcal{O}_{(\lambda,k+1)} C^{-\omega}(\Lambda) \to \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda) \ .$$

Now we can formulate the main theorem of the present paper. It essentially settles a conjecture of Patterson [14].

Theorem 1.5 *For any $\lambda \in \mathbf{C}$ there is $k(\lambda) \in \mathbf{N}$ such that for all $k \geq k(\lambda)$*

1. *the projection p_k^{k+1} induces an isomorphism*

$$(p_k^{k+1})_* : H^*(\Gamma, \mathcal{O}_{(\lambda,k+1)} C^{-\omega}(\Lambda)) \cong H^*(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda))$$

   of cohomology groups,

2. *the cohomology groups $H^*(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda))$ are finite-dimensional,*
3. *the Euler characteristic vanishes: $\chi(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda)) = 0$,*
4. *the order of the Selberg zeta function at λ is given by*

$$\mathrm{ord}_{s=\lambda} Z_S(s) = -\chi_1(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda)) \ , \qquad (2)$$

   where

$$\chi_1(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda)) := \sum_{p=1}^{n} p(-1)^p \dim\, H^p(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda)) \ .$$

It will be shown in Proposition 4.17 that one can take $k(\lambda) := \mathrm{ord}_{\mu=\lambda} ext_\mu + \epsilon$, where $\epsilon = 0$ iff $\lambda \notin -\mathbf{N}_0 - \frac{n-1}{2}$ and $\epsilon = 1$ else, and where ext is the extension map defined in Section 2. In the original conjecture $\mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda)$ was replaced by $C^{-\infty}(\Lambda, V(\lambda))$, the space of distribution sections of $V(\lambda)$ with support in Λ. With this replacement the conjecture is likely to be true for most λ. More precisely, if $k(\lambda) \leq 1$, then for $k \geq k(\lambda)$

$$\chi_1(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda)) = \chi_1(\Gamma, C^{-\omega}(\Lambda, V(\lambda))) \ .$$

We believe that in this formula hyperfunctions can be replaced by distributions.

The theorem above deals with the so-called spherical and hyperbolic case. More general one could consider a group Γ acting convex-cocompactly on a rank-one symmetric space X of non-compact type and a representation $\sigma \in \hat{M}$, where $M \subset G$ is the group fixing a point in the unit-sphere bundle of X. Then the associated Selberg zeta function $Z_S(s,\sigma)$ has a meromorphic continuation and its singularities should be related to the cohomology of Γ with coefficients in the space of hyperfunction vectors with support in the limit set of the principal series representation of G associated to σ. For cocompact Γ the general case was solved in [1], but it is still open for convex-cocompact groups.

The theory developed in the present paper can be extended to the general case. This generalization will be the topic of a longer paper which is still in progress. Hereby the problem which is still unsolved is to understand the behaviour of the scattering matrix near the points where the normalized Knapp-Stein intertwining operator has a pole with a residue which has an infinite-dimensional cokernel (these notions are introduced in the next section).

The strategy of the proof of Theorem 1.5 is as follows. We first develop a method which eventually allows us to compute the cohomology groups $H^*(\Gamma, \mathcal{O}_{(\lambda,k)}C^{-\omega}(\Lambda))$. The result will be formulated in terms of the spectrum of Δ_Y and of resonances. The first three assertions of the theorem are consequences of these computations.

Using the embedding $SO(1,n) \hookrightarrow SO(1,n+m)$ (a similar idea has been employed by Mandouvalos), we reduce the fourth assertion of the theorem to the case that n is even and $\delta_\Gamma < 0$. In the case of even n the description of the singularities of $Z_S(s)$ in terms of the spectrum of Δ_Y and resonances was given in [15]. This description simplifies considerably if $\delta_\Gamma < 0$. Equation (2) is verified relating both sides by explicit computation.

In [15] the order of $Z_S(s)$ at $s=0$ was not given explicitly. As a corollary of our computations we obtain:

$$\mathrm{ord}_{s=0} Z_S(s) = \dim {}^\Gamma C^{-\omega}(\Lambda, V(0)) = \dim \ker(S_0 + \mathrm{id}) \ ,$$

where S_0 is the normalized scattering matrix (introduced in the next section) at zero.

In order to keep the paper short and to stress the main ideas we omit the formal proofs of a number of analytic results. They are stated as **Facts** and can be found in the given literature or be proved in a straight-forward way.

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2 Restriction, extension, and the scattering matrix

Since the sheaf of hyperfunction sections of $V(\lambda)$ is flabby there is an exact sequence of Γ-modules

$$0 \to C^{-\omega}(\Lambda, V(\lambda)) \to C^{-\omega}(\partial X, V(\lambda)) \stackrel{res_\Omega}{\to} C^{-\omega}(\Omega, V(\lambda)) \to 0 \ .$$

Let $V_B(\lambda) := \Gamma \backslash V(\lambda)_{|\Omega}$. If we identify ${}^\Gamma C^{-\omega}(\Omega, V(\lambda)) \cong C^{-\omega}(B, V_B(\lambda))$, then res_Ω induces a map $res_\lambda : {}^\Gamma C^{-\omega}(\partial X, V(\lambda)) \to C^{-\omega}(B, V_B(\lambda))$.

The main topic of the present section is a construction of a meromorphic family of right-inverse maps $ext_\lambda : C^{-\omega}(B, V_B(\lambda)) \to {}^\Gamma C^{-\omega}(\partial X, V(\lambda))$. The poles of ext_λ will correspond exactly to those points $\lambda \in \mathbf{C}$ where res_λ is not isomorphic.

The strategy of the construction of ext_λ is the following. We first construct ext_λ for $\mathrm{Re}(\lambda) > \delta_\Gamma$. Then we introduce the scattering matrix $\hat{S}_\lambda : C^{-\omega}(B, V_B(\lambda)) \to C^{-\omega}(B, V_B(-\lambda))$ by

$$\hat{S}_\lambda := res_{-\lambda} \circ \hat{J}_\lambda \circ ext_\lambda \ . \tag{3}$$

Here $\hat{J}_\lambda : C^{-\omega}(\partial X, V(\lambda)) \to C^{-\omega}(\partial X, V(-\lambda))$ is the Knapp-Stein intertwining operator [5] introduced below. Assuming for a moment that $\delta_\Gamma < 0$ we obtain a meromorphic continuation of $\hat{S}_\lambda$ using results of Patterson [10]. Then we construct the meromorphic continuation of ext_λ by

$$ext_\lambda := J_{-\lambda} \circ ext_{-\lambda} \circ S_\lambda, \quad \mathrm{Re}(\lambda) < 0 \ . \tag{4}$$

If $\delta_\Gamma \ge 0$, then we employ the embedding $SO(1,n) \hookrightarrow SO(1,n+m)$, m sufficiently large, in order to reduce to the case $\delta_\Gamma < 0$.

A holomorphic family of vectors in a locally convex vector space $\mathcal{F}$ defined on $U \subset \mathbf{C}$ is just a continuous function from U to $\mathcal{F}$ which is weakly holomorphic. In order to speak of holomorphic families of homomorphisms from $\mathcal{F}$ to $\mathcal{E}$, where $\mathcal{E}$ is another locally convex vector space we equip $\mathrm{Hom}(\mathcal{F}, \mathcal{E})$ with the topology of uniform convergence on bounded sets. We will frequently use the following

Fact 2.1 *All topological vector spaces which appear below are Montel. The composition of holomorphic families of homomorphisms, the adjoint of a holomorphic family of homomorphisms, and the limit of a sequence of holomorphic families of homomorphism is a holomorphic family of homomorphisms.*

Using the identification $C^\infty(B, V_B(-\lambda)) \cong {}^\Gamma C^\infty(\Omega, V(-\lambda))$ we define the push down $\pi_{*,-\lambda}$ by

$$\pi_{*,-\lambda}(f)(b) = \sum_{g \in \Gamma} (\pi^{-\lambda}(g) f)(b), \quad b \in \Omega, \quad f \in C^\infty(\partial X, V(-\lambda)) \tag{5}$$

if the sum converges. The following fact can be checked in a straight-forward way though the formal details are somewhat lengthy. The basic estimate is implied by the convergence of the Poincaré series (1) for $\mathrm{Re}(s) > \delta_\Gamma$.

Fact 2.2 *If* $\mathrm{Re}(\lambda) > \delta_\Gamma$*, then the sum (5) converges and defines a holomorphic family of continuous maps*
$$\pi_{*,-\lambda} : C^\sharp(\partial X, V(-\lambda)) \to C^\sharp(B, V_B(-\lambda)), \quad \sharp = \infty, \omega \ .$$

Note that we also need an extension of distributions. Therefore we consider the push-down of smooth functions, too.

Definition 2.3 *For* $\mathrm{Re}(\lambda) > \delta_\Gamma$ *and* $* = \omega, \infty$ *we define the extension map* $ext_\lambda : C^{-*}(B, V_B(\lambda)) \to {}^\Gamma C^{-*}(\partial X, V(\lambda))$ *by*
$$ext_\lambda := {}^t\pi_{*,-\lambda} \ .$$

Fact 2.4 ext_λ *is a holomorphic family of continuous maps satisfying*
$$res_\lambda \circ ext_\lambda = id \ .$$
If ${}^\Gamma C^{-\omega}(\Lambda, V(\lambda)) = 0$ *and if* ext_λ *is defined, then we have* $ext_\lambda \circ res_\lambda = \mathrm{id}$.

In order to apply the last assertion of the fact we need vanishing results for ${}^\Gamma C^{-\omega}(\Lambda, V(\lambda))$.

Lemma 2.5 *If* $\mathrm{Re}(\lambda) > 0$ *and* $\mathrm{Im}(\lambda) \neq 0$*, then* ${}^\Gamma C^{-\omega}(\Lambda, V(\lambda)) = 0$.

Proof. Let $g = \kappa(g)a(g)n(g) \in KAN$ be the Iwasawa decomposition of $g \in G$.

Definition 2.6 *The Poisson transform*
$$P_\lambda : C^{-\omega}(\partial X, V(\lambda)) \to C^\infty(X)$$
is defined by
$$(P_\lambda\phi)(gK) := \int_K a(g^{-1}k)^{-\frac{n-1}{2}-\lambda}\phi(k)dk \ ,$$
where $a(h)^\mu := \mathrm{e}^{\mu\log(a(h))}$*,* $h \in G$*,* $\mu \in \mathbf{C}$*. Here* $\phi \in C^{-\omega}(\partial X, V(\lambda))$ *is viewed as a "function" on* G *with values in* $V_\lambda \cong \mathbf{C}$ *and the integral is a formal notation meaning that the analytic functional* ϕ *has to be applied to the analytic integral kernel.*

The Poisson transform P_λ is G-equivariant and we have $(\Delta_X - (\frac{n-1}{2})^2 + \lambda^2)P_\lambda(\phi) = 0$. For $\mathrm{Im}(\lambda) \neq 0$ the principal series representation of G on $C^{-\omega}(\partial X, V(\lambda))$ is irreducible. Since the Poisson transform does not vanish it is injective.

Let $V \subset \partial X$ and $U \subset X$ be such that $\mathrm{clo}(U) \cap V = \emptyset$, where we take the closure of U in $X \cup \partial X$. Then for $\mathrm{Re}(\lambda) > 0$ the integral kernel of the Poisson transform $(gK, kM) \mapsto p_\lambda(gK, kM) := a(g^{-1}k)^{-\frac{n-1}{2}-\lambda}$ is a real analytic function from V to $L^2(U)$.

If $\phi \in {}^\Gamma C^{-\omega}(\Lambda, V(\lambda))$, then $P_\lambda\phi$ is Γ-invariant, and it descends to an eigenfunction of Δ_Y in $L^2(Y)$.

Since Y is complete the operator Δ_Y is selfadjoint with domain $\{f \in L^2(Y) | \Delta_Y f \in L^2(Y)\}$. In particular Δ_Y can not have non-trivial eigenvectors in $L^2(Y)$ to eigenvalues with non-trivial imaginary part. Since $\mathrm{Im}(\lambda^2) \neq 0$ we conclude that $P_\lambda\phi = 0$ and hence $\phi = 0$ by the injectivity of the Poisson transform. $\square$

We now recall the definition of the Knapp-Stein intertwining operator $\hat{J}_\lambda$ and some of its properties. Consider $f \in C^\infty(\partial X, V(\lambda))$ as a right P-invariant function on G with values in V_λ. For $\mathrm{Re}(\lambda) < 0$ the intertwining operator is defined by the convergent integral

$$(\hat{J}_\lambda f)(g) := \int_{\bar{N}} f(gw\bar{n}) d\bar{n} \ ,$$

where $\bar{N} = \exp(\bar{\mathbf{n}})$ and $\bar{\mathbf{n}}$ is the negative root space of $(\mathbf{g}, \mathbf{a})$, and $w \in K$ is a representative of the non-trivial element of the Weyl-group $W \cong \mathbf{Z}_2$ of $(\mathbf{g}, \mathbf{a})$. For $\mathrm{Re}(\lambda) \geq 0$ the operator $\hat{J}_\lambda$ is defined by meromorphic continuation. It is known that $\hat{J}_\lambda$ has poles of at most first order [21].

Fact 2.7 *The intertwining operators restrict or extend to meromorphic families of operators*

$$\hat{J}_\lambda : C^*(\partial X, V(\lambda)) \to C^*(\partial X, V(-\lambda)), \quad * = \pm\omega, \pm\infty \ .$$

The intertwining operators satisfy the functional equation

$$\hat{J}_\lambda \circ \hat{J}_{-\lambda} = \frac{\mathrm{id}}{P(\lambda)} \ , \tag{6}$$

where the meromorphic function $P(\lambda)$ is the Plancherel density.

Let $W \subset \partial X$ be a closed set with non-empty interior and consider the space

$$\mathcal{F}_\lambda := \{f \in C^{-\infty}(\partial X, V(\lambda)) | f_{|W} \in C^\omega(\partial X, V(\lambda))\} \ .$$

We equip $\mathcal{F}_\lambda$ with the weakest topology such that the maps $\mathcal{F}_\lambda \hookrightarrow C^{-\infty}(\partial X, V(\lambda))$ and $\mathcal{F}_\lambda \to C^\omega(W, V(\lambda))$ are continuous. Using a volume form we can identify the spaces $\mathcal{F}_\lambda$ for different λ. Let $F \subset \mathrm{int}(W)$ be closed. Using the fact that the integral kernel of $\hat{J}_\lambda$ is analytic outside of the diagonal one can show

Fact 2.8 *The composition $res_F \circ \hat{J}_\lambda : \mathcal{F}_\lambda \to C^\omega(F, V(-\lambda))$ is well defined and depends meromorphically on λ.*

A similar property holds in the smooth situation. Let $W \subset \partial X$ be a closed subset and let

$$\mathcal{G}_\lambda := \{f \in C^{-\infty}(\partial X, V(\lambda)) | f_{|\partial X \setminus W} \in C^\infty(\partial X \setminus W, V(\lambda))\} \ .$$

We equip $\mathcal{G}_\lambda$ with the weakest topology such that the maps $\mathcal{G}_\lambda \hookrightarrow C^{-\infty}(\partial X, V(\lambda))$ and $\mathcal{G}_\lambda \to C^\infty(\partial X \setminus W, V(\lambda))$ are continuous. Let $U \subset \bar{U} \subset \partial X \setminus W$ be open.

Fact 2.9 *The composition*

$$res_U \circ \hat{J}_\lambda : \mathcal{G}_\lambda \to C^\infty(U, V(-\lambda))$$

is well-defined and depends meromorphically on λ.

Fact 2.10 *The scattering matrices defined by*

$$\hat{S}_\lambda := res_{-\lambda} \circ \hat{J}_\lambda \circ ext_\lambda : C^*(B, V_B(\lambda)) \to C^*(B, V_B(-\lambda)), \quad * = \pm\omega, \pm\infty$$

form meromorphic families of operators for $\mathrm{Re}(\lambda) > \delta_\Gamma$.

In order to define $\hat{S}_\lambda$ for $* = \omega, \infty$ we employ the fact that ext_λ factors over $\mathcal{F}_\lambda$ and $\mathcal{G}_\lambda$, respectively.

To prove the following fact one employs the corresponding property of $\hat{J}_\lambda$.

Fact 2.11 *The adjoint*

$${}^t\hat{S}_\lambda : C^\infty(B, V_B(\lambda)) \to C^\infty(B, V_B(-\lambda))$$

of

$$\hat{S}_\lambda : C^{-\infty}(B, V_B(\lambda)) \to C^{-\infty}(B, V_B(-\lambda))$$

coincides with the restriction of $\hat{S}_\lambda$ *to* $C^\infty(B, V_B(\lambda))$.

The following fact was obtained by Patterson [10].

Fact 2.12 *If* $\delta_\Gamma < 0$, *then there exists a meromorphic continuation of*

$$\hat{S}_\lambda : C^\infty(B, V_B(\lambda)) \to C^\infty(B, V_B(-\lambda))$$

to all of $\mathbf{C}$. *The singularities of* S_λ *for* $\mathrm{Re}(\lambda) < 0$ *are at most finite-dimensional.*

We extend this continuation to distributions by taking the adjoint.

Fact 2.13 *If* $\delta_\Gamma < 0$, *then formula (4) defines a meromorphic continuation of*

$$ext_\lambda : C^{-\infty}(B, V_B(\lambda)) \to {}^\Gamma C^{-\infty}(\partial X, V(\lambda))$$

to all of $\mathbf{C}$. *The singularities of* ext_λ *are at most finite-dimensional.*

We now show how to drop the assumption $\delta_\Gamma < 0$ using the embedding trick.

Proposition 2.14
$$ext_\lambda : C^{-\infty}(B, V_B(\lambda)) \to {}^\Gamma C^{-\infty}(\partial X, V(\lambda))$$
has a meromorphic continuation to all of $\mathbf{C}$ *with at most finite-dimensional singularities.*

Proof. Set $G^n := SO(1,n)$. Then we have a sequence $\ldots \subset G^n \subset G^{n+1} \subset \ldots$ of embeddings inducing embeddings of the corresponding Iwasawa constituents $K^n \subset K^{n+1}$, $N^n \subsetneq N^{n+1}$, $M^n \subset M^{n+1}$ such that $A = A^n \cong A^{n+1}$. Moreover there are totally geodesic embeddings $X^n \subset X^{n+1}$ inducing embeddings of their boundaries $\partial X^n \subset \partial X^{n+1}$. If one identifies $\partial X^{n+1} \cong S^n$, then $\partial X^n \cong S^{n-1}$ can be viewed as an equator.

The subgroup $\Gamma \subset G^n$ remains to be convex-cocompact when it is viewed as a subgroup of G^{n+1}. We obtain embeddings $\Omega^n \subset \Omega^{n+1}$ inducing $B^n \subset B^{n+1}$ while the limit set Λ^n is identified with Λ^{n+1}.

The exponent of Γ now depends on n and is denoted by δ_Γ^n. We have the relation $\delta_\Gamma^{n+1} = \delta_\Gamma^n - \frac{1}{2}$ (see Fact 1.4). Thus $\delta_\Gamma^{n+m} \to -\infty$ as $m \to 0$, and hence taking m large enough one has $\delta_\Gamma^{n+m} < 0$.

Let ext_λ^n denote the extension map associated to $\Gamma \subset G^n$. The aim of the following discussion is to show how the meromorphic continuation ext_λ^{n+1} leads to the continuation of ext_λ^n.

Let $P^n := M^n A^n N^n$. The representation V_λ of P^{n+1} restricts to the representation $V_{\lambda-\frac{1}{2}}$ of P^n. This induces an isomorphism of bundles
$$V_{B^{n+1}}(\lambda)_{|B^n} = V_{B^n}(\lambda - \frac{1}{2}) \ .$$
The adjoint of the restriction of smooth (analytic) sections to B^n is a push forward of distributions (hyperfunctions)
$$i_* : C^{-\sharp}(B^n, V_{B^n}(\lambda)) \to C^{-\sharp}(B^{n+1}, V_{B^{n+1}}(\lambda - \frac{1}{2})), \quad \sharp = \omega, \infty \ .$$
For $\phi \in C^{-\infty}(B^n, V_{B^n}(\lambda))$ the push forward $i_*(\phi)$ has support in $B^n \subset B^{n+1}$. It follows that also $\mathrm{supp}(ext^{n+1}_{\lambda-\frac{1}{2}} \circ i_*(\phi)) \subset \partial X^n$. We try to define a pull back $ext_\lambda^n(\phi) := i^* \circ ext^{n+1}_{\lambda-\frac{1}{2}} \circ i_*(\phi)$ as follows. For $f \in C^\infty(\partial X^n, V(-\lambda))$ let $\tilde f \in C^\infty(\partial X^{n+1}, V(-\lambda + \frac{1}{2}))$ be an arbitrary extension.

Lemma 2.15 ext_λ^n *is well defined by*
$$\langle ext^n(\phi), f\rangle := \langle ext^{n+1}_{\lambda-\frac{1}{2}} \circ i_*(\phi), \tilde f\rangle \ .$$

Proof. We must show that this definition does not depend on the choice of the extension $\tilde f$ of f. It is sufficient to show that if $h \in C^\infty(\partial X^{n+1}, V(-\lambda + \frac{1}{2}))$ vanishes on ∂X^n, then $\langle ext^{n+1}_{\lambda-\frac{1}{2}} \circ i_*(\phi), h\rangle = 0$. Embed ϕ, h into holomorphic families ϕ_μ, h_μ, $\phi_\mu \in C^{-\infty}(B^n, V_{B^n}(\mu))$, $h_\mu \in$

$C^\infty(\partial X^{n+1}, V(-\mu+\frac{1}{2}))$ such that $(h_\mu)_{|\partial X^n} = 0$. Then for $\mathrm{Re}(\mu)$ large enough $\pi^{n+1}_{*,-\mu+\frac{1}{2}}(h_{-\mu})_{|B^n} =$ 0 and thus

$$\langle ext^{n+1}_{\mu-\frac{1}{2}} \circ i_*(\phi_\mu), h_{-\mu}\rangle = \langle i_*(\phi_\mu), \pi^{n+1}_{*,-\mu+\frac{1}{2}} h_{-\mu}\rangle = 0 \ .$$

By meromorphic continuation this identity holds for all μ, in particular at $\mu = \lambda$. □

Now the properties of ext^n_λ can be deduced from those of ext^{n+1}_λ. In particular, ext^n_λ is a meromorphic family of maps and has at most finite-dimensional singularities. □

Fact 2.16 *Formula (3) defines a meromorphic continuation of the scattering matrix*

$$\hat{S}_\lambda : C^{-\infty}(B, V_B(\lambda)) \to C^{-\infty}(B, V_B(-\lambda)) \ .$$

Lemma 2.17 *The restriction (extension) of the scattering matrix to analytic (hyperfunction) sections is a meromorphic family of continuous operators*

$$\hat{S}_\lambda : C^{\pm\omega}(B, V_B(\sigma_\lambda)) \to C^{\pm\omega}(B, V_B(-\lambda)) \ .$$

Proof. Note that ext_λ defines a meromorphic family

$$ext_\lambda : C^\omega(B, V_B(\lambda)) \to \mathcal{F}_\lambda \ .$$

Now we apply Fact 2.8 in order to conclude that

$$\hat{S}_\lambda := res_{-\lambda} \circ \hat{J}_\lambda \circ ext_\lambda : C^\omega(B, V_B(\lambda)) \to C^\omega(B, V_B(-\lambda))$$

is meromorphic. The corresponding statement for the hyperfunction sections follows by duality and Fact 2.1. □

Let $\psi_\lambda \in C^\omega(\partial X, V(\lambda))$ be the unique K-invariant section normalized such that $\psi_\lambda(1) = 1$ (when viewed as a function on G with values in $V(\lambda) \cong \mathbf{C}$).

Definition 2.18 *The meromorphic function $c(\lambda)$ is defined by*

$$\hat{J}_\lambda \psi_\lambda = c(-\lambda)\psi_{-\lambda} \ .$$

Definition 2.19 *We define the normalized intertwining operator and scattering matrix by*

$$J_\lambda := c(-\lambda)^{-1}\hat{J}_\lambda, \quad S_\lambda := c(-\lambda)^{-1}\hat{S}_\lambda \ .$$

The following fact is a consequence of of Equation (6) and

$$\frac{1}{P(\lambda)} = c(\lambda)c(-\lambda)\ . \tag{7}$$

Fact 2.20 *The scattering matrix satisfies the functional equation*

$$S_\lambda \circ S_{-\lambda} = 1\ .$$

Proposition 2.21 *The extension map*

$$ext_\lambda : C^{-\omega}(B, V_B(\lambda)) \to {}^\Gamma C^{-\omega}(\partial X, V(-\lambda))$$

has a meromorphic continuation to all of $\mathbf{C}$.

Proof. If $\delta_\Gamma < 0$, then we employ formula (4) together with Lemma 2.17. The case $\delta_\Gamma \geq 0$ is reduced to the case $\delta_\Gamma < 0$ as in the proof of Proposition 2.14. □

Proposition 2.22 *If* $\mathrm{Re}(\lambda) > 0$, *then the order of a singularity of* ext_λ *is at most* 1.

Proof. Let $f_\mu \in C^\infty(B, V_B(\mu))$, $\mu \in \mathbf{C}$, be a holomorphic family such that $ext_\mu(f_\mu)$ has a pole of order $k \geq 1$ at $\mu = \lambda$, $\mathrm{Re}(\lambda) > 0$.

We assume that $k \geq 2$ and argue by contradiction. Let $0 \neq \phi \in {}^\Gamma C^{-\infty}(\partial X, V(\lambda))$ be the leading singular part of $ext_\mu(f_\mu)$ at $\mu = \lambda$. Because of $res_\mu \circ ext_\mu = \mathrm{id}$ we have $res_\lambda \phi = 0$ and hence $\phi \in {}^\Gamma C^{-\infty}(\Lambda, V(\lambda))$. Then $(\lambda^2 - \mu^2)^{k-1} ext_\mu(f_\mu)$ and hence $(\lambda^2 - \mu^2)^{k-1} P_\mu \circ ext_\mu(f_\mu)$ have a first-order pole at $\mu = \lambda$ with residue $(-2\lambda)^{k-1} P_\lambda \phi$.

Since $res_\Omega \circ ext_\mu(f_\mu)$ is smooth we have (see also Fact 3.4)

$$\begin{aligned} P_\mu \circ ext_\mu(f_\mu)(kaK) &= O(a^{\mu - \frac{n-1}{2}}), \quad kM \in \Omega \\ P_\lambda \phi(kaK) &= O(a^{-\lambda - \frac{n-1}{2}}), \quad kM \in \Omega\ , \end{aligned}$$

and both estimates hold uniformly for kM in compact subsets of Ω, μ near λ, and large $a \in A_+$. These estimates justify the following computation using partial integration:

$$\begin{aligned} \infty &= \lim_{\mu \to \lambda,\ \mathrm{Re}(\mu) < \mathrm{Re}(\lambda)} \langle (\lambda^2 - \mu^2)^{k-1} P_\mu \circ ext_\mu(f_\mu), P_\lambda \phi \rangle_{L^2(Y)} \\ &= \lim_{\mu \to \lambda,\ \mathrm{Re}(\mu) < \mathrm{Re}(\lambda)} \langle (\Delta_Y - (\frac{n-1}{2})^2 + \lambda^2)^{k-1} P_\mu \circ ext_\mu(f_\mu), P_\lambda \phi \rangle_{L^2(Y)} \\ &= 0\ . \end{aligned}$$

This is a contradiction and thus $k = 1$. $\square$

For $\lambda \in \imath\mathbf{R}$ there is a conjugate-linear pairing $V_\lambda \otimes V_\lambda \to V_{-\frac{n-1}{2}}$ and hence a natural L^2-scalar product $C^\infty(B, V_B(\lambda)) \otimes C^\infty(B, V_B(\lambda)) \to \mathbf{C}$. Let $L^2(B, V_B(\lambda))$ be the associated Hilbert space. Using the Fact 2.11 we see that the adjoint S_λ^* with respect to this Hilbert space structure is just $S_{-\lambda}$.

Proposition 2.23 *If* $\mathrm{Re}(\lambda) = 0$, *then* S_λ *is regular and unitary.*

Proof. Let $f \in C^\infty(B, V_B(\lambda))$. Then by Fact 2.20

$$\|S_\lambda f\|^2_{L^2(B,V_B(\lambda))} = \langle S_{-\lambda} \circ S_\lambda f, f\rangle_{L^2(B,V_B(\lambda))} = \|f\|^2_{L^2(B,V_B(\lambda))} \ .$$

This implies that S_λ is regular and unitary. $\square$

3 Green's formula and applications

Fact 3.1 *There exists a cut-off function* $\chi \in C^\infty(X)$ *such that*

1. $\mathrm{supp}(\chi) \subset \bigcup_{g\in L} gF$ *for some finite* $L \subset \Gamma$, *where* $F \subset X$ *is a fundamental domain for* Γ,
2. $\sum_{g\in\Gamma} g^*\chi = 1$,
3. $|\nabla^2\chi(ka)| < Ca^{-1}$, $\forall k \in K$, $a \in A_+$,
4. $\lim_{a\to\infty} \chi(ka) =: \chi_\infty(k)$ *is smooth.*

Let $\phi \in {}^\Gamma C^{-\omega}(\Lambda, V(\lambda))$. Then by Fact 2.8 $res_{-\lambda} \circ \hat{J}_\lambda(\phi) \in C^\omega(B, V_B(-\lambda))$ is well defined even if $\hat{J}_\mu$ has a pole at $\mu = \lambda$. In the latter case the residue of $\hat{J}_\mu$ at $\mu = \lambda$ is a differential operator D_λ and $res_{-\lambda} \circ D_\lambda(\phi) = 0$.

Proposition 3.2 *If* $\mathrm{Re}(\lambda) \geq 0$, $\phi \in {}^\Gamma C^{-\omega}(\Lambda, V(\lambda))$, *and* $f \in {}^\Gamma C^{-\omega}(\partial X, V(\lambda))$, *then*

$$\langle res_{-\lambda} \circ \hat{J}_\lambda(\phi), res_\lambda(f)\rangle = 0 \ .$$

Proof. By Lemma 2.5 we can assume that $\lambda \in \mathbf{R} \cup \imath\mathbf{R}$.

Fact 3.3 *By a density argument it is sufficient to show the proposition for $f \in {}^\Gamma C^{-\omega}(\partial X, V(\lambda))$ with $f_{|\Omega}$ smooth.*

Thus we assume that $f_{|\Omega}$ is smooth.

Let $A = \Delta_X - (\frac{n-1}{2})^2 + \lambda^2$, $P = P_\lambda$ be the Poisson transform and χ be the cut-off function given by Fact 3.1. Let $B_R \subset X = G/K$ be the metric ball of radius R centered at the origin given by the class $[K] \subset G/K$. The following computation is essentially Green's formula:

$$\begin{aligned} 0 &= \langle \chi A P\phi, Pf\rangle_{L^2(B_R)} - \langle \chi P\phi, APf\rangle_{L^2(B_R)} \\ &= \langle A\chi P\phi, Pf\rangle_{L^2(B_R)} - \langle \chi P\phi, APf\rangle_{L^2(B_R)} - \langle [A,\chi]P\phi, Pf\rangle_{L^2(B_R)} \\ &= -\langle \nabla_n \chi P\phi, Pf\rangle_{L^2(\partial B_R)} + \langle \chi P\phi, \nabla_n Pf\rangle_{L^2(\partial B_R)} - \langle [A,\chi]P\phi, Pf\rangle_{L^2(B_R)}\ , \end{aligned} \tag{8}$$

where n is the exterior unit normal vector field at ∂B_R. For the following discussion we distinguish between the three cases:

- $\lambda \in \mathbf{R}, \lambda > 0$,
- $\mathrm{Re}(\lambda) = 0$, $\lambda \neq 0$, and
- $\lambda = 0$.

We first consider the case $\lambda > 0$. Recall the asymptotics of the Poisson transform. Note that $\hat{J}_\lambda\phi$ is analytic on Ω by Fact 2.8 and f is smooth on Ω by assumption.

Fact 3.4 *For $kM \in \Omega$ we have*

$$\begin{aligned} P(f)(ka) &= a^{\lambda - \frac{n-1}{2}} c(\lambda) f(k) + O(a^{\lambda - \frac{n-1}{2} - \epsilon}) \\ P(\phi)(ka) &= a^{-\lambda - \frac{n-1}{2}} (\hat{J}_\lambda\phi)(k) + O(a^{-\lambda - \frac{n-1}{2} - \epsilon}) \end{aligned}$$

where $\epsilon > 0$.

The estimate can be differentiated with respect to a and holds locally uniformly on Ω. By property (3) of χ we see that $\langle [A,\chi]P\phi, Pf\rangle$ is integrable and by property (2) and the Γ-invariance of f and ϕ we have $\langle [A,\chi]P\phi, Pf\rangle_{L^2(X)} = 0$. Taking the limit $R \to \infty$ in (8) we obtain

$$\begin{aligned} 0 &= (\lambda + \frac{n-1}{2}) \int_{\partial X} \chi_\infty(k) \langle (\hat{J}_\lambda\phi)(k), c(\lambda) f(k)\rangle \\ &\quad + (\lambda - \frac{n-1}{2}) \int_{\partial X} \chi_\infty(k) \langle (\hat{J}_\lambda\phi)(k), c(\lambda) f(k)\rangle \\ &= 2\lambda c(\lambda) \int_{\partial X} \chi_\infty(k) \langle (\hat{J}_\lambda\phi)(k), f(k)\rangle \\ &= 2\lambda c(\lambda) \langle res_{-\lambda} \circ \hat{J}_\lambda(\phi), res_\lambda(f)\rangle\ . \end{aligned}$$

This is the assertion of the proposition for $\mathrm{Re}(\lambda) > 0$ since $c(\lambda) \neq 0$.

Now we discuss the case $\mathrm{Re}(\lambda) = 0$ and $\lambda \neq 0$.

Fact 3.5 *In this case we have the following asymptotic expansion*

$$P(f)(ka) = a^{\lambda - \frac{n-1}{2}} c(\lambda) f(k) + a^{-\lambda - \frac{n-1}{2}} \hat{J}_\lambda f(k) + O(a^{-\frac{n-1}{2} - \epsilon}), \quad kM \in \Omega \ .$$

Instead of taking the limit $R \to \infty$ in (8) we apply $\lim_{r\to\infty} \frac{1}{r} \int_0^r dR$. Again we have

$$\lim_{r\to\infty} \frac{1}{r} \int_0^r \langle [A,\chi] P\phi, Pf \rangle_{L^2(B_R)} dR = 0 \ .$$

Moreover, the asymptotic term $a^{-\lambda - \frac{n-1}{2}} \hat{J}_\lambda f(k)$ does not contribute to the limit because of

$$-2\lambda \lim_{r\to\infty} \frac{1}{r} \int_0^r R^{-2\lambda - 2\frac{n-1}{2}} \langle \chi(.R) \hat{J}_\lambda \phi, \hat{J}_\lambda f \rangle_{L^2(\partial X)} dR = 0 \ .$$

The contribution of the term $a^{\lambda - \frac{n-1}{2}} c(\lambda) f(k)$ leads to

$$0 = 2\lambda c(\lambda) \langle res_{-\lambda} \circ \hat{J}_\lambda(\phi), res_\lambda(f) \rangle$$

as in the case $\lambda > 0$. The proposition again follows since $c(\lambda) \neq 0$.

Now we consider the last case $\lambda = 0$. Note that $c(\lambda)$ has a first order pole at $\lambda = 0$ with residue $c_1 \neq 0$. We do the computation (8) using the Poisson transform $P = P_\mu$ at μ in a neighbourhood of 0. Of course in general $P_\mu(f), P_\mu(\phi)$ are not Γ-invariant except for $\mu = 0$. Nevertheless,

$$\lim_{R\to\infty} \langle [A,\chi] P_{1/R}\phi, P_{1/R} f \rangle_{L^2(B_R)} = 0$$

by the theorem of Lebesgue about dominated convergence. Moreover we have

$$\begin{aligned}
-\lim_{R\to\infty} & (\langle \nabla_n \chi P_{1/R}\phi, Pf \rangle_{L^2(\partial B_R)} - \langle \chi P_{1/R}\phi, \nabla_n P_{1/R} f \rangle_{L^2(\partial B_R)}) \\
&= \lim_{R\to\infty} ((1/R + \frac{n-1}{2}) \int_{\partial X} \chi_\infty(k) \langle (\hat{J}_{1/R}\phi)(k), c(1/R) f(k) \rangle dk \\
&\quad + (1/R - \frac{n-1}{2}) \int_{\partial X} \chi_\infty(k) \langle (\hat{J}_{1/R}\phi)(k), c(1/R) f(k) \rangle dk) \\
&= 2 \int_{\partial X} \chi_\infty(k) \langle (\hat{J}_0\phi)(k), \lim_{R\to\infty} \frac{c(1/R)}{R} f(k) \rangle dk \\
&= 2c_1 \langle res_0 \circ \hat{J}_0(\phi), res_0(f) \rangle \ .
\end{aligned}$$

It follows from (8) that the latter pairing vanishes. This finishes the proof of the proposition in the last case $\lambda = 0$. □

The following fact is a consequence of results of van den Ban-Schlichtkrull [20].

Fact 3.6 *Let* $\lambda \notin \frac{1}{2}\mathbf{Z}$ *or* $\mathrm{Re}(\lambda) \geq 0$, *and let* $U \subset \partial X$ *be open. Assume that* $\phi \in C^{-\omega}(\partial X, V(\lambda))$ *satisfies* $\phi_{|U} = 0$. *Then* $(\hat{J}_\lambda \phi)_{|U} = 0$ *implies* $\phi = 0$ *(note that* $(\hat{J}_\lambda \phi)_{|U}$ *is defined even if* $\hat{J}_\lambda$ *has a pole).*

Corollary 3.7 *Let* $\lambda \notin \frac{1}{2}\mathbf{Z}$ *or* $\mathrm{Re}(\lambda) \geq 0$. *Then*

$$res_\Omega \circ \hat{J}_\lambda : C^{-\omega}(\Lambda, V(\lambda)) \to C^\omega(\Omega, V(-\lambda))$$

is injective.

Lemma 3.8 *Assume that* $\mathrm{Re}(\lambda) \geq 0$. *If*

$$ext_\lambda : C^\infty(B, V_B(\lambda)) \to {}^\Gamma C^{-\omega}(\partial X, V(\lambda))$$

is regular, then ${}^\Gamma C^{-\omega}(\Lambda, V(\sigma_\lambda)) = 0$.

Proof. Let $\phi \in {}^\Gamma C^{-\omega}(\Lambda, V(\lambda))$. Then for all $f \in C^\infty(B, V_B(\lambda))$ we have by Proposition 3.2

$$0 = \langle res_{-\lambda} \circ \hat{J}_\lambda(\phi), res_\lambda \circ ext_\lambda(f) \rangle = \langle res_{-\lambda} \circ \hat{J}_\lambda(\phi), f \rangle \ .$$

Thus $res_{-\lambda} \circ \hat{J}_\lambda(\phi) = 0$ and by Corollary 3.7 we conclude that $\phi = 0$. □

Lemma 3.9 ext_λ *has a pole of at most first order at* $\lambda = 0$. ext_0 *is singular iff* $S_0 - \mathrm{id} \neq 0$. *Moreover,* $S_0 - \mathrm{id}$ *is finite-dimensional.*

Proof. Since S_0 is regular and unitary by Lemma 2.23 and $c(\lambda)$ has a first order pole at $\lambda = 0$ the unnormalized scattering matrix $\hat{S}_\lambda$ has a first order pole at $\lambda = 0$, too. The principal series representation $C^\infty(\partial X, V(0))$ of G is irreducible ([4], Cor. 14.30). Thus $J_0 = \mathrm{id}$ by the choice of normalization.

We expand $\hat{J}_\lambda = \mathrm{res}_{\lambda=0} c(-\lambda)\, \lambda^{-1} + J^0_\lambda$ and $ext_\lambda = \sum_{k<0} \lambda^k ext^k + ext^0_\lambda$, where J^0_λ and ext^0_λ depend holomorphically on λ near $\lambda = 0$. Since for $k < 0$ we have $\mathrm{supp}(ext^k(f)) \subset \Lambda$ for all f Corollary 3.7 implies that $res_\Omega \circ J^0_0 \circ ext^k = 0$ iff $ext^k = 0$. Since $\hat{S}_\lambda$ has a first order pole it follows $ext^k = 0$ for $k \leq -2$. We now consider the residue of $\hat{S}_\lambda$.

$$\begin{aligned} \mathrm{res}_{\lambda=0} c(-\lambda) S_0 &= \mathrm{res}_{\lambda=0} \hat{S}_\lambda \\ &= \mathrm{res}_{\lambda=0} res_{-\lambda} \circ \hat{J}_\lambda \circ ext_\lambda \\ &= \mathrm{res}_{\lambda=0} c(-\lambda) \mathrm{id} + res_0 \circ J^0_0 \circ ext^{-1} \ . \end{aligned}$$

We conclude that

$$\mathrm{res}_{\lambda=0} c(-\lambda)(S_0 - \mathrm{id}) = res_0 \circ J^0_0 \circ ext^{-1} \ .$$

The right-hand side of this equation is finite-dimensional. By Corollary 3.7 we conclude that $ext^{-1} \neq 0$ iff $S_0 \neq \mathrm{id}$. □

Lemma 3.10 *If* $\mathrm{Re}(\lambda) = 0$, $\lambda \neq 0$, *then*

$$ext_\lambda : C^{-\omega}(B, V_B(\sigma_\lambda)) \to {}^\Gamma C^{-\omega}(\partial X, V(\sigma_\lambda))$$

is regular.

Proof. Since $\mathrm{Re}(\lambda) = 0$ by Lemma 2.23 the scattering matrix S_λ is unitary, in particular regular. We also have $S_\lambda = res_{-\lambda} \circ J_\lambda \circ ext_\lambda$. Corollary 3.7 and the fact that the singular part of ext_λ maps to hyperfunctions supported on the limit set Λ show that a pole of ext_λ would necessarily imply a singularity of S_λ. Thus ext_λ is regular for $\mathrm{Re}(\lambda) = 0$, $\lambda \neq 0$. □

Lemma 3.11 *Let* $\mathrm{Re}(\lambda) \geq 0$. *Then the residue of*

$$ext_\mu : C^{-\omega}(B, V_B(\mu)) \to {}^\Gamma C^{-\omega}(\partial X, V(\mu))$$

at $\mu = \lambda$ *spans* ${}^\Gamma C^{-\omega}(\Lambda, V(\lambda))$.

Proof. By the Lemmas 2.5, 3.9, and 3.10 we can assume that $\lambda \in \mathbf{R}$. Moreover by the Lemmas 2.22, 3.9 and 3.10 the family ext_λ has a poles of at most first order for $\mathrm{Re}(\lambda) \geq 0$. The identity $res_\lambda \circ ext_\lambda = \mathrm{id}$ implies on the one hand that the dimension d of the range of the residue of

$$ext_\lambda : C^{-\omega}(B, V_B(\lambda)) \to {}^\Gamma C^{-\omega}(\partial X, V(\lambda))$$

is smaller than $\dim {}^\Gamma C^{-\omega}(\Lambda, V(\lambda))$ (the dimension of the kernel of res_λ). On the other hand this identity shows that d is larger than the codimension of the range of res_λ. Since by Proposition 3.2 and Corollary 3.7 the codimension of the range of res_λ is larger than $\dim {}^\Gamma C^{-\omega}(\Lambda, V(\lambda))$ it follows

$$d = \dim {}^\Gamma C^{-\omega}(\Lambda, V(\lambda)) .$$

This proves the lemma. □

4 Cohomology

Let $U \subset \partial X$ and $T \subset \mathbf{C}$ be open. Using a real-analytic volume form we trivialize the holomorphic family of bundles $\bigcup_{\lambda \in T} V(\lambda)_{|U}$. By $\mathcal{O}C^{-\omega}(U)(T)$ we denote the space of holomorphic families of hyperfunction sections $\lambda \mapsto f_\lambda \in C^{-\omega}(U, V(\lambda))$. If U is non-compact, then the vector space $C^{-\omega}(U, V(\lambda))$ does not carry a good topology. Hence to be precise using the above trivialization we define $\mathcal{O}C^{-\omega}(U)(T)$ to be the space of hyperfunctions on $T \times U$ which contain $\lambda \in \mathbf{C}$ as holomorphic parameter in the sense of [6], Ch.5.

Note that $C^{-\omega}(\partial X, V(\lambda))$ is a Frechét space. The following fact follows from the results of [3], Ch.5.

Fact 4.1 *$\mathcal{O}C^{-\omega}(\partial X)(T)$ is canonically identified with the space of holomorphic families of sections $T \ni \mu \mapsto f_\mu \in C^{-\omega}(\partial X, V(\lambda))$, i.e., f_μ is a holomorphic function with values in a locally convex vector space.*

If $T \subset \mathbf{C}$ is Stein, then the sheaf on ∂X defined by

$$\partial X \supset U \mapsto \mathcal{O}C^{-\omega}(U)(T) \tag{9}$$

is flabby. This implies the exactness of the sequence

$$0 \to \mathcal{O}C^{-\omega}(\Lambda)(T) \to \mathcal{O}C^{-\omega}(\partial X)(T) \stackrel{res_{T\times\Omega}}{\to} \mathcal{O}C^{-\omega}(\Omega)(T) \to 0 \ ,$$

where $res_{T\times\Omega}$ is the restriction from $T \times \partial X$ to $T \times \Omega$ and $\mathcal{O}C^{-\omega}(\Lambda)(T)$ is just defined as the kernel of $res_{T\times\Omega}$. Any $\lambda \in \mathbf{C}$ admits a fundamental sequence of Stein neighbourhoods $\{T_\alpha\}$. We define the space of germs of holomorphic families of hyperfunctions at $\lambda \in \mathbf{C}$ by

$$\mathcal{O}_\lambda C^{-\omega}(\partial X) := \lim_{\stackrel{\to}{\alpha}} \mathcal{O}C^{-\omega}(\partial X)(T_\alpha)$$

and similar for ∂X replaced by Ω or Λ. Then the sequence of germs

$$0 \to \mathcal{O}_\lambda C^{-\omega}(\Lambda) \to \mathcal{O}_\lambda C^{-\omega}(\partial X) \stackrel{res_\Omega}{\to} \mathcal{O}_\lambda C^{-\omega}(\Omega) \to 0 \tag{10}$$

is an exact sequence of Γ-modules.

Lemma 4.2 *For all $\lambda \in \mathbf{a}^*_{\mathbf{C}}$ the Γ-module $\mathcal{O}_\lambda C^{-\omega}(\Omega)$ is acyclic, i.e.,*

$$H^p(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Omega)) = 0, \quad \forall p \geq 1 \ .$$

Proof. The assertion follows from the flabbyness of the sheaf (9) and the exactness of the direct limit functor (compare [1], Lemma 2.6). $\square$

Proposition 4.3 *If $\mathrm{Re}(\lambda) \geq 0$, then $\mathcal{O}_\lambda C^{-\omega}(\partial X)$ is Γ-acyclic.*

Proof. Let $\mathcal{O}_\lambda C^\infty(X)$ be the space of germs at λ of holomorphic families $\mu \mapsto \phi_\mu \in C^\infty(X)$ of smooth functions on X. Let A denote the holomorphic family of operators $\mu \mapsto \Delta_X - (\frac{n-1}{2})^2 + \mu^2$ acting on $\mathcal{O}_\lambda C^\infty(X)$ in the natural way. Using Fact 4.1 we see that the family of Poisson transforms $\mu \to P_\mu$ identifies $\mathcal{O}_\lambda C^{-\omega}(\partial X)$ with the kernel of A. In fact, the inverse map is given by the (normalized) boundary value map β. If $\phi \in \ker(A)$, then $\beta(\phi)_\mu := \beta_\mu \phi_\mu$, where β_μ the the boundary value map $\beta_\mu : \ker(\Delta_X - (\frac{n-1}{2})^2 + \mu^2) \to C^{-\omega}(\partial X, V(\mu))$.

The main technical result needed in order to prove the proposition is the following lemma.

Lemma 4.4 *If* $\mathrm{Re}(\lambda) \geq 0$, *then the operator* $A : \mathcal{O}_\lambda C^\infty(X) \to \mathcal{O}_\lambda C^\infty(X)$ *is surjective.*

Proof. As an intermediate step we prove

Lemma 4.5 *Let* D *be an elliptic operator with real-analytic coefficients on a real-analytic non-compact manifold* X. *Then for any* $\lambda \in \mathbf{C}$ *there is a neighbourhood* $U \subset \mathbf{C}$ *of* λ *such that* $A : C^\infty(U \times X) \to C^\infty(U \times X)$ *given by* $(Af)_\mu = (D + \mu^2) f_\mu$ *is surjective.*

Proof. We consider the adjoint operator ${}^tA : C_c^{-\infty}(U \times X) \to C_c^{-\infty}(U \times X)$. We must show that ${}^tA$ is injective and has closed range.

First we prove injectivity. Let $f \in C_c^{-\infty}(U \times X)$ satisfy ${}^tAf = 0$. The hyperplanes $\{\mu = const\}$ are non-characteristic for ${}^tA$. By a theorem of Sato $f(\mu, x)$, viewed as a hyperfunction in $C^{-\omega}(U \times X)$, contains the variable $x \in X$ as a real-analytic parameter. Since f has compact support, by [3], Thm. 1.5 we conclude $f = 0$. This shows injectivity of ${}^tA$.

We now prove that the range of ${}^tA$ is closed. Let $f_i \in C_c^{-\infty}(U \times X)$ be a sequence such that ${}^tAf_i =: h_i$ converges to $h \in C_c^{-\infty}(U \times X)$. Then there is a compact subset $T \times M \subset U \times X$ such that $\mathrm{supp}(h_i) \subset T \times M$ for all i. The unique continuation result [3], Thm. 1.5 implies that $\mathrm{supp}(f_i) \subset T \times M$ for all i, too.

It is not difficult to show the following

Fact 4.6 *If* U *is a sufficiently small neighbourhood of* $\lambda \in \mathbf{C}$, *then there is a fundamental solution* E_μ *of* $D + \mu^2$ *on* M *depending holomorphically on* $\mu \in U$.

Hence $f_i = Eh_i$ converges to $f \in C^{-\infty}(T \times M)$, and $\mathrm{supp}(f) \subset T \times M$ is compact. It follows $f \in C_c^{-\infty}(U \times X)$ and ${}^tAf = h$. This proves closedness of the range of ${}^tA$. □

We now finish the proof of Lemma 4.4. Let $[f] \in \mathcal{O}_\lambda C^\infty(X)$ be represented by $f \in C^\infty(U \times X)$, U being a neighbourhood of λ, such that f satisfies $\bar\partial f = 0$, where $\bar\partial$ acts on the first variable. By Lemma 4.5 we can find $h \in C^\infty(U \times X)$ with $Ah = f$ after possibly making U smaller. We have $0 = \bar\partial Ah = A\bar\partial h$. Thus $\beta(\bar\partial h)$ is defined.

Since $\lambda \mapsto V(\lambda)$ is a trivial holomorphic family of bundles we can solve the $\bar\partial$-problem and find a smooth family $U' \ni \mu \mapsto k_\mu \in C^{-\omega}(\partial X, V(\mu))$, where $U' \subset U$ is a neighbourhood of λ, such that $\bar\partial k = \beta(\bar\partial h)$. Then $\phi \in C^\infty(U' \times X)$ given by $\phi_\mu := h_\mu - P_\mu(k_\mu)$ is holomorphic for $\mu \in U'$ and satisfies $A\phi = f$. Thus $A[\phi] = [f]$. This proves the required surjectivity of A and thus Lemma 4.4. □

Lemma 4.7 *The module $\mathcal{O}_\lambda C^\infty(X)$ is Γ-acyclic.*

Proof. We employ the proof of Lemma 2.4 [1], which is not sensitive to the additional variable $\mu \in \mathbf{C}$. □

We continue with the proof of Proposition 4.3. The following sequence is a Γ-acyclic resolution of $\mathcal{O}_\lambda C^{-\omega}(\partial X)$:

$$0 \to \mathcal{O}_\lambda C^{-\omega}(\partial X) \stackrel{P}{\to} \mathcal{O}_\lambda C^\infty(X) \stackrel{A}{\to} \mathcal{O}_\lambda C^\infty(X) \to 0 .$$

The spaces $H^*(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\partial X))$ are isomorphic to the cohomology spaces of the sequence of Γ-invariant vectors:

$$0 \to \mathcal{O}_\lambda C^\infty(Y) \stackrel{A_Y}{\to} \mathcal{O}_\lambda C^\infty(Y) \to 0 .$$

It follows $H^p(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\partial X)) = 0$, $\forall p \ge 2$.

We claim that $H^1(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\partial X)) = 0$, or equivalently, that A_Y is surjective.

The proof of that claim is similar to the proof of Lemma 4.4. Let $[f] \in \mathcal{O}_\lambda C^\infty(Y)$ be represented by $f \in C^\infty(U \times Y)$ satisfying $\bar\partial f = 0$. By Lemma 4.5 we can find $h \in C^\infty(U \times Y)$ with $A_Y h = f$ after possibly making U smaller. We have $0 = \bar\partial A_Y h = A_Y \bar\partial h$. Then $\mu \mapsto res_\mu \circ \beta_\mu(\bar\partial h)_\mu \in C^{-\omega}(B, V_B(\mu))$ is a smooth family of analytic functionals. The family of spaces $C^{-\omega}(B, V_B(\lambda))$ is again holomorphically trivial. Solving the $\bar\partial$-problem we find a smooth family k satisfying $(\bar\partial k)_\mu = res_\mu \circ \beta_\mu(\bar\partial h)_\mu$ for $\mu \in U$ after possibly making U smaller. Modifying k further by a holomorphic family we can assume that $k_\lambda = 0$. Since $\mathrm{Re}(\lambda) \ge 0$ the order of a possible pole of ext_λ is at most one by Lemmas 2.22 and 3.10.

We obtain that the family $\phi \in C^\infty(U \times Y)$ given by $\mu \mapsto \phi_\mu := h_\mu - P_\mu \circ ext_\mu(k_\mu)$ is holomorphic for $\mu \in U$ and satisfies $A_Y \phi = f$. Thus $A_Y[\phi] = [f]$. This proves the claim, and the proposition follows. □

Proposition 4.8 *If* $\mathrm{Re}(\lambda) \geq 0$, *then*

$$\begin{aligned} H^0(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) &= 0\ , \\ H^1(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) &= {}^\Gamma C^{-\omega}(\Lambda, V(\lambda))\ , \\ H^p(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) &= 0, \quad \forall p \geq 2\ . \end{aligned}$$

In particular, if $H^1(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) \neq 0$, *then* λ *is real.*

*Proof.*By Lemma 4.2 and Proposition 4.3 the sequence (10) is a Γ-acyclic resolution of $\mathcal{O}_\lambda C^{-\omega}(\Lambda)$. The cohomology groups $H^*(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda))$ are the cohomology groups of the subcomplex of Γ-invariant vectors:

$$0 \to {}^\Gamma\mathcal{O}_\lambda C^{-\omega}(\partial X) \stackrel{res}{\to} \mathcal{O}_\lambda C^{-\omega}(B) \to 0\ ,$$

where $\mathcal{O}_\lambda C^{-\omega}(B)$ is the space of germs at λ of holomorphic families of hyperfunction sections $\mu \mapsto \phi_\mu \in C^{-\omega}(B, V_B(\lambda))$, and res is defined by $res(f)_\mu := res_\mu f_\mu$ for $f \in {}^\Gamma\mathcal{O}_\lambda C^{-\omega}(\partial X)$. Since res_μ is injective for generic μ we have

$$\begin{aligned} H^0(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) &= 0\ , \\ H^1(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) &= \mathrm{coker}(res)\ , \\ H^p(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) &= 0, \quad \forall p \geq 2\ . \end{aligned}$$

If ext_λ is regular, then res is surjective. In fact if $\phi \in \mathcal{O}_\lambda C^{-\omega}(B)$, then we define $(ext\phi)_\mu := ext_\mu \phi_\mu \in {}^\Gamma\mathcal{O}_\lambda C^{-\omega}(\partial X)$ and have $res \circ ext(\phi) = \phi$. If ext is singular, then it has a first order pole and λ is real (Lemmas 3.9 and 2.22). We consider the sequence

$$0 \to {}^\Gamma\mathcal{O}_\lambda C^{-\omega}(\partial X) \stackrel{res}{\to} \mathcal{O}_\lambda C^{-\omega}(B) \stackrel{ev}{\to} {}^\Gamma C^{-\omega}(\Lambda, V(\lambda)) \to 0\ , \tag{11}$$

where ev is defined by $ev(\phi) := \lim_{\mu\to\lambda}(\mu-\lambda)ext_\mu(\phi_\mu)$. It is exact at the first term and at the last by Lemma 3.11. Since ext is defined on $\ker(ev)$ and inverts res, this sequence is exact at the middle term, too. Thus we have identified $\mathrm{coker}(res)$ with ${}^\Gamma C^{-\omega}(\Lambda, V(\lambda))$. $\square$

Note that we have

$$\begin{aligned} {}^\Gamma C^{-\omega}(\Lambda, V(\lambda)) &\cong \ker_{L^2}(\Delta_Y - (\frac{n-1}{2})^2 + \lambda^2), \quad \mathrm{Re}(\lambda) > 0 \\ {}^\Gamma C^{-\omega}(\Lambda, V(0)) &\cong \mathrm{im}(S_0 - \mathrm{id})\ , \end{aligned}$$

where the last equation follows from Lemma 3.11 and Lemma 3.10.

We now discuss points $\lambda \in \mathbf{C}$ with $\mathrm{Re}(\lambda) < 0$. We distinguish between two cases.

Fact 4.9 *If* $\lambda \in -\mathbf{N}_0 - \frac{n-1}{2}$, *then the range of the Knapp-Stein intertwining operator* $\hat{J}_\lambda$ *is an irreducible finite-dimensional representation* F_λ *of* G *(case (i)). If* $\lambda \notin -\mathbf{N}_0 - \frac{n-1}{2}$ *and* $\mathrm{Re}(\lambda) < 0$, *then* $\hat{J}_\lambda$ *is an isomorphism (case (ii)).*

In case (i) we define $b : \mathcal{O}_\lambda C^{-\omega}(\Lambda) \to F^0_\lambda \subset F_\lambda$ by $b(\phi) := \hat{J}_\lambda \phi_\lambda$, where the Γ-module F^0_λ is just the range of b. Then we define $\mathcal{O}^0_\lambda C^{-\omega}(\Lambda)$ to be the kernel of b such that the following sequence is exact:

$$0 \to \mathcal{O}^0_\lambda C^{-\omega}(\Lambda) \to \mathcal{O}_\lambda C^{-\omega}(\Lambda) \xrightarrow{b} F^0_\lambda \to 0 \ . \tag{12}$$

Note that the family $\mu \mapsto \tilde{J}_\mu := (\mu - \lambda)^{-1} \hat{J}_\mu$ induces a well-defined operator

$$\tilde{J} : \mathcal{O}^0_\lambda C^{-\omega}(\Lambda) \to \mathcal{O}_{-\lambda} C^{-\omega}(\Lambda) \ .$$

In case (ii) we set $\mathcal{O}^0_\lambda C^{-\omega}(\Lambda) := \mathcal{O}_\lambda C^{-\omega}(\Lambda)$, $\tilde{J}_\mu := \hat{J}_\mu$, and $F^0_\lambda = 0$.

Fact 4.10 *If $\lambda \in -\mathbf{N}_0$, then $\hat{J}_{-\lambda}$ has a first-order pole (case (a)). If $\lambda \notin -\mathbf{N}_0$, then $\hat{J}_{-\lambda}$ is regular (case (b)).*

In case (a) we set $\tilde{J}_{-\mu} := (\mu - \lambda)\hat{J}_{-\mu}$, while in case (b) we set $\tilde{J}_{-\mu} := \hat{J}_{-\mu}$.

Fact 4.11 *If λ is of case (i), then the range of $\tilde{J}_{-\lambda}$ is exactly the kernel of $\hat{J}_\lambda$.*

The functional equation of the intertwining operators (6) and the relation (7) imply the following fact.

Fact 4.12 *If $\mathrm{Re}(\lambda) < 0$, then there exists a holomorphic function $q(\mu)$ near $\mu = \lambda$ with $q(\lambda) \neq 0$, such that $\tilde{J}_\mu \circ \tilde{J}_{-\mu} = q(\mu)\mathrm{id}$.*

Lemma 4.13 *If $\mathrm{Re}(\lambda) < 0$, then the following sequence is exact:*

$$0 \to \mathcal{O}^0_\lambda C^{-\omega}(\Lambda) \xrightarrow{\tilde{J}} \mathcal{O}_{-\lambda} C^{-\omega}(\partial X) \xrightarrow{res \circ \tilde{J}} \mathcal{O}_\lambda C^{-\omega}(\Omega) \to F^1_\lambda \to 0 \ , \tag{13}$$

where F^1_λ is a finite-dimensional Γ-module fitting into the exact sequence

$$0 \to F^0_\lambda \to F_\lambda \to F^1_\lambda \to 0 \ . \tag{14}$$

Proof. Exactness at the first place is clear since $\tilde{J}_\mu$ is an isomorphism for $\mu \neq \lambda$ in a small neighbourhood of λ.

We show exactness at the second place. Let $f \in \mathcal{O}_{-\lambda} C^{-\omega}(\partial X)$ satisfy $res \circ \tilde{J}(f) = 0$. Then we consider $\tilde{J}(f) =: g \in \mathcal{O}_\lambda C^{-\omega}(\partial X)$. Since the composition $\tilde{J}_\mu \circ \tilde{J}_{-\mu}$ is regular at $\mu = \lambda$ we have in case (i) that $g_\lambda \in \ker(\hat{J}_\lambda)$. We also have $res(g) = 0$ and thus $g \in \mathcal{O}^0_\lambda C^{-\omega}(\Lambda)$. It follows that $f_{-\mu} = \tilde{J}_\mu(q(\mu)^{-1} g_\mu)$.

Now we compute the cokernel of $res \circ \tilde{J}$. Let $f \in \mathcal{O}_\lambda C^{-\omega}(\Omega)$. Since res is surjective we can find $h \in \mathcal{O}_\lambda C^{-\omega}(\partial X)$ with $res(h) = f$. By Fact 4.11 the obstruction for h being in the range of $\tilde{J}$ is exactly $\hat{J}_\lambda h_\lambda \in F_\lambda$. Since h is determined uniquely up to elements of $\mathcal{O}_\lambda C^{-\omega}(\Lambda)$, the obstruction space for f being in the range of $res \circ \tilde{J}$ is just F_λ / F^0_λ. $\square$

If $ext_{-\lambda}$ is singular, then we define $\mathcal{O}^0_{-\lambda} C^{-\omega}(B) = \ker(ev) = \mathrm{im}(res) \subset \mathcal{O}_{-\lambda} C^{-\omega}(B)$. If $ext_{-\lambda}$ is regular, then we set $\mathcal{O}^0_{-\lambda} C^{-\omega}(B) := \mathcal{O}_{-\lambda} C^{-\omega}(B)$. Let $\tilde{S}^0 := res \circ \tilde{J} \circ ext$ be the scattering matrix viewed as an operator

$$\tilde{S}^0 : \mathcal{O}^0_{-\lambda} C^{-\omega}(B) \to \mathcal{O}_\lambda C^{-\omega}(B) \ .$$

Lemma 4.14 *If* $\mathrm{Re}(\lambda) < 0$*, then the cohomology groups* $H^*(\Gamma, \mathcal{O}^0_\lambda C^{-\omega}(\Lambda))$ *are finite-dimensional. Moreover we have*

$$\chi(\Gamma, \mathcal{O}^0_\lambda C^{-\omega}(\Lambda)) = \chi(\Gamma, F^1_\lambda) - \dim \mathrm{coker} \tilde{S}^0 \ .$$

Proof. We split (13) into two short exact sequences

$$\begin{array}{l} 0 \to \mathcal{O}^0_\lambda C^{-\omega}(\Lambda) \stackrel{\tilde{J}}{\to} \mathcal{O}_{-\lambda} C^{-\omega}(\partial X) \stackrel{\alpha}{\to} R \to 0 \\ 0 \to R \to \mathcal{O}_\lambda C^{-\omega}(\Omega) \stackrel{\beta}{\to} F^1_\lambda \to 0 \ . \end{array}$$

Then using Lemma 4.2 and Proposition 4.3 we can read off

$$\begin{aligned} H^0(\Gamma, \mathcal{O}^0_\lambda C^{-\omega}(\Lambda)) &= 0 \\ H^1(\Gamma, \mathcal{O}^0_\lambda C^{-\omega}(\Lambda)) &= \mathrm{coker}\left(\alpha_* : H^0(\Gamma, \mathcal{O}_{-\lambda} C^{-\omega}(\partial X)) \to H^0(\Gamma, R)\right) \\ H^p(\Gamma, \mathcal{O}^0_\lambda C^{-\omega}(\Lambda)) &= H^{p-1}(\Gamma, R), \quad p \ge 2 \\ H^1(\Gamma, R) &= \mathrm{coker}\left(\beta_* : H^0(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Omega)) \to H^0(\Gamma, F^1_\lambda)\right) \qquad (15) \\ H^p(\Gamma, R) &= H^{p-1}(\Gamma, F^1_\lambda), \quad p \ge 2 \ . \qquad (16) \end{aligned}$$

We conclude

$$\begin{aligned} H^p(\Gamma, \mathcal{O}^0_\lambda C^{-\omega}(\Lambda)) &= H^{p-2}(\Gamma, F^1_\lambda), \quad p \ge 3 \qquad (17) \\ H^2(\Gamma, \mathcal{O}^0_\lambda C^{-\omega}(\Lambda)) &= \mathrm{coker}\left(\beta_* : H^0(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Omega)) \to H^0(\Gamma, F^1_\lambda)\right) \ . \qquad (18) \end{aligned}$$

The finite-dimensionality statement is non-obvious only for the first cohomology group. Let $f \in H^0(\Gamma, R)$ be given by a section $f \in \mathcal{O}_\lambda C^{-\omega}(B)$. Assume that $ext(f)_\lambda$ is regular. If $\hat{J}_\lambda \circ ext(f)_\lambda = 0$, then we can form $g := res \circ \tilde{J} \circ ext(f)$. We have $\tilde{S}^0(g)_\mu = res \circ \tilde{J} \circ ext(g)_\mu = q(\mu) f_\mu$. In particular, the dimension of $\mathrm{coker}(\alpha_*)$ is smaller than the finite codimension of the range of $\tilde{S}^0$.

We now compute the Euler characteristic and obtain

$$\chi(\Gamma, \mathcal{O}^0_\lambda C^{-\omega}(\Lambda)) = \chi(\Gamma, F^1_\lambda) - h^0(\Gamma, F^1_\lambda) + \dim \mathrm{coker}(\beta_*) - \dim \mathrm{coker}(\alpha_*) \ .$$

The discussion above yields that $\dim \operatorname{coker}(\alpha_*) + h^0(\Gamma, F^1_\lambda) - \dim \operatorname{coker}(\beta_*) = \dim \operatorname{coker}(\tilde{S}^0)$. □

Proposition 4.15 *If* $\mathrm{Re}(\lambda) < 0$*, then the cohomology groups* $H^*(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda))$ *are finite-dimensional. Moreover we have*

$$\chi(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) = \chi(\Gamma, F_\lambda) - \dim \operatorname{coker} \tilde{S}^0 \ .$$

Proof. It follows from the long exact cohomology sequence associated to (12) that the cohomology $H^*(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda))$ is finite-dimensional. The same sequence also implies

$$\chi(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) = \chi(\Gamma, \mathcal{O}^0_\lambda C^{-\omega}(\Lambda)) + \chi(\Gamma, F^0_\lambda) \ .$$

By (14) we have

$$\chi(\Gamma, F_\lambda) = \chi(\Gamma, F^0_\lambda) + \chi(\Gamma, F^1_\lambda) \ .$$

Combining these two equations with Lemma 4.14 we get the desired result. □

For any $k \in \mathbf{N}$ we consider the quotient

$$0 \to \mathcal{O}_\lambda C^{-\omega}(\Lambda) \stackrel{L^k}{\to} \mathcal{O}_\lambda C^{-\omega}(\Lambda) \to \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda) \to 0 \ , \tag{19}$$

where L is the multiplication operator by the function $\mu \mapsto (\mu - \lambda)$. Note that $\mathcal{O}_{(\lambda,1)} C^{-\omega}(\Lambda) = C^{-\omega}(\Lambda, V(\lambda))$ as a Γ-module.

Definition 4.16 *For any* $\lambda \in \mathbf{C}$ *define*

$$k(\lambda) := \mathrm{ord}_{\mu=\lambda} ext_\mu + \epsilon \ ,$$

where $\epsilon = 0$ *iff* $\lambda \notin -\mathbf{N}_0 - \frac{n-1}{2}$ *and* $\epsilon = 1$ *else.*

If $\mathrm{Re}(\lambda) \geq 0$, then by the Lemmas 2.22, 3.9 and 3.10 $k(\lambda) \leq 1$.

The following proposition contains the first three assertions of Theorem 1.5.

Proposition 4.17 *If* $\lambda \in \mathbf{C}$ *and* $k \geq k(\lambda)$*, then for* $l \geq 1$ *the projection*

$$p^{k+l}_k : \mathcal{O}_{(\lambda,k+l)} C^{-\omega}(\Lambda) \to \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda)$$

induces an isomorphism in Γ*-cohomology. Moreover, we have*

$$\begin{aligned} \dim H^p(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda)) &< \infty, \quad \forall p \in \mathbf{N}_0 \ , \\ \chi(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda)) &= 0 \ , \\ \chi_1(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda)) &= \chi(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) \ , \end{aligned}$$

where

$$\chi_1(\Gamma, \mathcal{O}_{(\lambda,k)}C^{-\omega}(\Lambda)) := \sum_{p=0}^{\infty} p(-1)^p \dim H^p(\Gamma, \mathcal{O}_{(\lambda,k)}C^{-\omega}(\Lambda))$$

is the first derived Euler characteristic.

Proof. It follows from the long exact cohomology sequence associated to (19) that it is sufficient to show that

$$L_*^m : H^*(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) \to H^*(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda))$$

vanishes for $m \geq k$.

We first show that

$$L_p : H^p(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\partial X)) \to H^p(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\partial X)), \quad p \geq 1 \ ,$$

vanishes. For $\mathrm{Re}(\lambda) \geq 0$ this is trivial since $H^p(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\partial X)) = 0$. If $\mathrm{Re}(\lambda) < 0$, then the assertion follows from the long exact cohomology sequence associated to the short exact sequence of Γ- and $\mathbf{C}[L]$-modules

$$0 \to \mathcal{O}_{-\lambda}C^{-\omega}(\partial X) \stackrel{\tilde{J}}{\to} \mathcal{O}_\lambda C^{-\omega}(\partial X) \to F \to 0$$

(see Fact 4.11), where we view F as a trivial $\mathbf{C}[L]$-module.

Consider now (10) as an exact sequence of Γ- and $\mathbf{C}[L]$-modules. The associated long exact cohomology sequence of $\mathbf{C}[L]$-modules gives

$$0 \to {}^\Gamma\mathcal{O}_\lambda C^{-\omega}(\partial X) \stackrel{res}{\to} \mathcal{O}_\lambda C^{-\omega}(B) \stackrel{\delta}{\to} H^1(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) \stackrel{w}{\to} H^1(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\partial X)) \to 0 \qquad (20)$$

$$H^p(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) = H^p(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\partial X)), \quad p \geq 2 \ . \qquad (21)$$

Equation (21) implies the vanishing of L_p, $p \geq 2$. The map L_0 is zero since the corresponding cohomology groups are trivial. Thus it remains to show that

$$L_1^m : H^1(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) \to H^1(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda))$$

vanishes for $m \geq k$.

We employ the exact sequence (20). Assume that $\lambda \not\in -\mathbf{N}_0 - \frac{n-1}{2}$. If in addition $\mathrm{Re}(\lambda) < 0$, then $\tilde{J} : \mathcal{O}_{-\lambda}C^{-\omega}(\partial X) \to \mathcal{O}_\lambda C^{-\omega}(\partial X)$ is an isomorphism. Hence $H^1(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\partial X)) = 0$ by Proposition 4.3. If $[f] \in H^1(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda)) \stackrel{\delta}{=} \mathrm{coker}(res)$ is represented by $f \in \mathcal{O}_\lambda C^{-\omega}(B)$, then $ext \circ L^m f =: g \in {}^\Gamma\mathcal{O}_\lambda C^{-\omega}(\partial X)$ exists for $m \geq \mathrm{ord}_{\mu=\lambda} ext_\mu$. It follows $L_1^m[f] = [L^m f] = [res(g)] = 0$.

Finally we assume that $\lambda \in -\mathbf{N}_0 - \frac{n-1}{2}$. Consider $\phi \in H^1(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda))$. Then $0 = L_1 \circ w\phi = w \circ L_1\phi$. Thus $L_1\phi = \delta(f)$ for some $f \in \mathcal{O}_\lambda C^{-\omega}(B)$. Note that $m \geq \mathrm{ord}_{\mu=\lambda} ext_\mu + 1$. Putting $g := ext \circ L_1^{m-1} f$ we obtain $L_1^m\phi = \delta \circ res(g) = 0$. This finishes the proof of the proposition. $\square$

The final result of the present section is an immediate consequence of Propositions 4.17, 4.15, 4.8, and the long exact cohomology sequence associated to (19).

Proposition 4.18 *If $\lambda \in \mathbf{C}$ and $k \geq k(\lambda)$, then*

$$\chi_1(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda)) = \begin{cases} -\dim \ker_{L^2}(\Delta_Y - (\frac{n-1}{2})^2 + \lambda^2) & \mathrm{Re}(\lambda) \geq 0, \lambda \neq 0 \\ -\dim \ker(S_0 + \mathrm{id}) & \lambda = 0 \\ -\dim \mathrm{coker}(\tilde{S}^0) + \chi(\Gamma, F_\lambda) & \mathrm{Re}(\lambda) < 0 \end{cases} .$$

5 The embedding trick

We adopt the notations G^n, ∂X^n introduced in the proof of Proposition 2.14. The limit set of $\Gamma \subset G^n \subset G^{n+m}$, $m \geq 0$, considered as a subset of ∂X^{n+m} is denoted by Λ^{n+m}.

Proposition 5.1 *For $k \geq 1$ we have*

$$H^p(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda^{n+1})) = \bigoplus_{j=0}^{\infty} H^p(\Gamma, \mathcal{O}_{(\lambda+j+\frac{1}{2},k)} C^{-\omega}(\Lambda^n)) \quad (22)$$

$$\chi_1(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda^{n+1})) = \sum_{j=0}^{\infty} \chi_1(\Gamma, \mathcal{O}_{(\lambda+j+\frac{1}{2},k)} C^{-\omega}(\Lambda^n)) . \quad (23)$$

The sums are finite.

Proof. Since $H^p(\Gamma, \mathcal{O}_{(\lambda+j+\frac{1}{2},k)} C^{-\omega}(\Lambda^n)) = 0$ for all $p \geq 0$ and for large j the sums are finite. Equation (23) is a consequence of (22). Thus it remains to prove (22).

Let ∂X^{n+1} be identified with the standard sphere S^n, and $\partial X^n \cong S^{n-1}$ with an equator. Let $p \in \partial X^n$ be the north pole and $\partial X^{n+1} \setminus \{p\} \cong \mathbf{R}^n$ be the stereographic projection. Then $\partial X^n \setminus \{p\}$ is identified with $\mathbf{R}^{n-1}$. Let (x, y), $x \in \mathbf{R}^{n-1}$, $y \in \mathbf{R}$, be coordinates of $\mathbf{R}^n$, and let N be a vector field on $S^n \setminus \{p\}$ defined near S^{n-1} which under stereographic projection corresponds to ∂_y.

We trivialize $V(\lambda)^{n+1}_{|\partial X^{n+1} \setminus \{p\}}$ using the standard volume form of $\mathbf{R}^n$. Then we can differentiate elements of $C^{-\omega}(\Lambda^{n+1}, V(\lambda)^{n+1})$ with support in $S^n \setminus \{p\}$ with respect to N. We denote this differential operator by the same symbol N.

We claim that the differential operator N continues across $\{p\}$ and defines an Γ-equivariant embedding
$$N : \mathcal{O}_\lambda C^{-\omega}(\Lambda^{n+1}) \hookrightarrow \mathcal{O}_{\lambda-1} C^{-\omega}(\Lambda^{n+1}) \ ,$$
and that we can identify the cokernel of N with the Γ-module $\mathcal{O}_{\lambda-\frac{1}{2}} C^{-\omega}(\Lambda^n)$ such that the exact sequence of Γ-modules
$$0 \to \mathcal{O}_\lambda C^{-\omega}(\Lambda^{n+1}) \stackrel{N}{\to} \mathcal{O}_{\lambda-1} C^{-\omega}(\Lambda^{n+1}) \stackrel{q}{\to} \mathcal{O}_{\lambda-\frac{1}{2}} C^{-\omega}(\Lambda^n) \to 0$$
splits.

By an explicit calculation one can check that N is Γ-equivariant. A right-inverse of q is given by the push-forward $i_* : C^{-\omega}(\partial X^n, V(\lambda - \frac{1}{2})) \to C^{-\omega}(\partial X^{n+1}, V(\lambda-1))$. The exactness of the sequence is now obvious.

We conclude that
$$H^p(\Gamma, \mathcal{O}_\lambda C^{-\omega}(\Lambda^{n+1})) \cong H^p(\Gamma, \mathcal{O}_{\lambda+1} C^{-\omega}(\Lambda^{n+1})) \oplus H^p(\Gamma, \mathcal{O}_{\lambda+\frac{1}{2}} C^{-\omega}(\Lambda^n)) \ .$$
Equation (22) now follows by induction and from Proposition 4.17. □

The following fact was noted in [10]

Fact 5.2 *If $Z_{S,n+m}(s)$ denotes the Selberg zeta function associated to $\Gamma \subset G^{n+m}$, then we have*
$$Z_{S,n+1}(s) = \prod_{j=0}^{\infty} Z_{S,n}(s + j + \frac{1}{2}) \ .$$
In particular,
$$\mathrm{ord}_{s=\lambda} Z_{S,n+1}(s) = \sum_{j=0}^{\infty} \mathrm{ord}_{s=\lambda+j+\frac{1}{2}} Z_{S,n}(s) \ .$$

For $k \geq 1$ we have
$$\begin{aligned} \chi_1(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda^n)) &= \chi_1(\Gamma, \mathcal{O}_{(\lambda-\frac{1}{2},k)} C^{-\omega}(\Lambda^{n+1})) - \chi_1(\Gamma, \mathcal{O}_{(\lambda+\frac{1}{2},k)} C^{-\omega}(\Lambda^{n+1})) \\ \mathrm{ord}_{s=\lambda} Z_{S,n}(s) &= \mathrm{ord}_{s=\lambda-\frac{1}{2}} Z_{S,n+1}(s) - \mathrm{ord}_{s=\lambda+\frac{1}{2}} Z_{S,n+1}(s) \end{aligned}$$
Combining these two equations we obtain

Corollary 5.3 *Let $\Gamma \subset G^n$ be convex-cocompact. If Equation (2) holds true for Γ viewed as a subgroup of G^{n+1}, so it does for Γ viewed as a subgroup of G^n. In particular, it is sufficient to prove (2) when n is even and $\delta_\Gamma < 0$.*

6 The singularities of the Selberg zeta function

Let n be even and assume that $\delta_\Gamma < 0$. Consider $\lambda \in \mathbf{C}$ with $\mathrm{Re}(\lambda) > 0$.

First we give an expression for dim coker$(\tilde{S} : \mathcal{O}_{-\lambda}C^{-\omega}(B) \to \mathcal{O}_\lambda C^{-\omega}(B))$ in terms of the trace of the residue of the logarithmic derivative of $\tilde{S}$. Because of the assumption $\delta_\Gamma < 0$ the scattering matrix $\tilde{S}_\lambda := res_{-\lambda} \circ \tilde{J}_\lambda \circ ext_\lambda$ is regular at λ, and thus $\tilde{S}_\lambda = \tilde{S}^0_\lambda$.

We start with reviewing results of Patterson-Perry [15]. We fix an analytic Riemannian metric on B in the canonical conformal class. This metric induces a volume form which we employ in order to identify all bundles $V_B(\lambda)$ with $B \times \mathbf{C} = V_B(\frac{n-1}{2})$. For any $\lambda \in \mathbf{C}$ the scattering matrix becomes a meromorphic family of operators $\tilde{S}(\mu)$ on $C^*(B)$, $* = \pm\omega, \pm\infty$, μ near λ.

Let $P^2 := \Delta_B + c$, where we choose $c >> 0$ such that P^2 is positive. Then for μ close to λ the scattering matrix can be factored as $\tilde{S}(\mu) = P^\mu(\mathrm{id} + K(\mu))P^\mu$, where $K(\mu)$ is a holomorphic family of operators in the $(n+1)$'th Schatten class (i.e. $K(\mu)^{n+1}$ is of trace class) acting on $\mathcal{H} := L^2(B)$. The inverse $(1 + K(\mu))^{-1}$ is a meromorphic family of operators with finite-dimensional residues.

Proposition 6.1 *We have*

$$\dim \operatorname{coker}(\tilde{S} : \mathcal{O}_\lambda C^{-\omega}(B) \to \mathcal{O}_{-\lambda}C^{-\omega}(B)) = \operatorname{Tr} \operatorname{res}_{\mu=\lambda}(1 + K(\mu))^{-1}K'(\mu) \ ,$$

where $K'(\mu)$ denotes the derivative of $K(\mu)$ with respect to μ.

Proof. Let $\mathcal{O}_\lambda\mathcal{H}$ denote the space of germs at λ of holomorphic families of vectors in $\mathcal{H}$ and let $(1 + K) : \mathcal{O}_\lambda\mathcal{H} \to \mathcal{O}_\lambda\mathcal{H}$ be given by $((1 + K)f)_\mu := (1 + K(\mu))f_\mu$, $f \in \mathcal{O}_\lambda\mathcal{H}$.

Lemma 6.2

$$\dim \operatorname{coker}(\tilde{S}) = \dim \operatorname{coker}(1 + K)$$

Proof. For any $f \in \mathcal{O}_{-\lambda}C^{-\omega}(B)$ let $f(\mu) = \sum_i f_i\, (\mu + \lambda)^i$ be the Taylor expansion of f at $-\lambda$. There is a finite-dimensional space V of linear forms on $\mathcal{O}_{-\lambda}C^{-\omega}(B)$ such that a section $f \in \mathcal{O}_{-\lambda}C^{-\omega}(B)$ is in the range of $\tilde{S}$ iff $\langle\phi, f\rangle = 0$, $\forall\phi \in V$. Moreover, V has the property that for all $\phi \in V$ there is an $r \in \mathbf{N}_0$ and $\phi_\alpha \in C^\omega(B)$ such that $\langle\phi, f\rangle = \sum_{\alpha=0}^r \langle\phi_\alpha, f_\alpha\rangle$ for all $f \in \mathcal{O}_{-\lambda}C^{-\omega}(B)$. Note that if $f \in \mathcal{O}_{-\lambda}C^{-\omega}(B)$ satisfies $\langle\phi, f\rangle = 0$ for all $\phi \in V$, then $\tilde{S}^{-1}f$ is regular.

The space $C^{-\infty}(B)$ is dense in $C^{-\omega}(B)$. Thus the restriction of V to $\mathcal{O}_{-\lambda}C^{-\infty}(B)$ has the same dimension as V. Since $\tilde{S}^{-1}$ is given by the family $q(\mu)^{-1}\tilde{S}_{-\mu}$ it also acts on distributions and

$$\dim V = \dim \operatorname{coker}(\tilde{S} : \mathcal{O}_\lambda C^{-\infty}(B) \to \mathcal{O}_{-\lambda}C^{-\infty}(B)) \ .$$

Since $P^\lambda : \mathcal{O}_\lambda C^{-\infty}(B) \to \mathcal{O}_\lambda C^{-\infty}(B)$ is a holomorphic family of invertible operators

$$\dim\ V = \dim \operatorname{coker}((1+K) : \mathcal{O}_\lambda C^{-\infty}(B) \to \mathcal{O}_\lambda C^{-\infty}(B)) \ .$$

Now $(1+K(\mu))$ and its inverse for $\mu \neq \lambda$ is defined also on $\mathcal{H}$. A similar argument as above shows that the dimension of the cokernel of $(\mathrm{id}+K) : \mathcal{O}_\lambda C^{-\infty}(B) \to \mathcal{O}_\lambda C^{-\infty}(B)$ coincides with the dimension of the cokernel of $(\mathrm{id}+K) : \mathcal{O}_\lambda \mathcal{H} \to \mathcal{O}_\lambda \mathcal{H}$. □

The following lemma implies the proposition.

Lemma 6.3

$$\mathrm{Tr}\,\mathrm{res}_{\mu=\lambda}(1+K(\mu))^{-1}K'(\mu) = \dim\ \operatorname{coker}(1+K)$$

Proof. Let $P(\mu)$ be the holomorphic family of finite-dimensional projections given by

$$P(\mu) := \frac{1}{2\pi\imath}\oint_c \frac{1}{z-1-K(\mu)}dz \ ,$$

where the path of integration is a small circle enclosing $0 \in \mathbf{C}$ counter-clock wise and μ is close to λ. There is a holomorphic family of invertible operators $U(\mu)$ ([18], Thm. XII.12) such that $U(\mu)^{-1}P(\mu)U(\mu) = P(\lambda)$. We define $T(\mu) := U(\mu)^{-1}(1+K(\mu))U(\mu)$. Then

$$T(\mu)P(\lambda) = U(\mu)^{-1}(1+K(\mu))P(\mu)U(\mu) = U(\mu)^{-1}P(\mu)(1+K(\mu))U(\mu) = P(\lambda)T(\mu) \ .$$

Let $V := P(\lambda)\mathcal{H}$ and $W := (1-P(\lambda))\mathcal{H}$. Then

$$T(\mu) = \left(\begin{array}{cc} A(\mu) & 0 \\ 0 & B(\mu) \end{array}\right) : \begin{array}{c} V \\ \oplus \\ W \end{array} \to \begin{array}{c} V \\ \oplus \\ W \end{array} \ ,$$

where $B(\mu)$ is invertible for $\mu - \lambda$ small.

We claim that

$$\mathrm{Tr}\,\mathrm{res}_{\mu=\lambda}T(\mu)^{-1}T'(\mu) = \mathrm{Tr}\,\mathrm{res}_{\mu=\lambda}(1+K(\mu))^{-1}K'(\mu) \ .$$

In fact $T'(\mu) = U(\mu)^{-1}(1+K(\mu))U'(\mu) + U(\mu)^{-1}K'(\mu)U(\mu) - U(\mu)^{-1}U'(\mu)U(\mu)^{-1}(1+K(\mu))U(\mu)$. Using the fact that all singular terms of $(1+K(\mu))^{-1}$ are finite-dimensional and the cyclicity of the trace we compute

$$\begin{aligned}
& \mathrm{Tr}\,\mathrm{res}_{\mu=\lambda}T(\mu)^{-1}T'(\mu) \\
= \ & \mathrm{Tr}\,\mathrm{res}_{\mu=\lambda}U(\mu)^{-1}(1+K(\mu))^{-1}U(\mu)[U(\mu)^{-1}(1+K(\mu))U'(\mu) + U(\mu)^{-1}K'(\mu)U(\mu) \\
& \qquad -U(\mu)^{-1}U'(\mu)U(\mu)^{-1}(1+K(\mu))U(\mu)] \\
= \ & \mathrm{Tr}\,\mathrm{res}_{\mu=\lambda}[U(\mu)^{-1}U'(\mu) + U(\mu)^{-1}(1+K(\mu))^{-1}K'(\mu)U(\mu) \\
& -U(\mu)^{-1}(1+K(\mu))^{-1}U'(\mu)U(\mu)^{-1}(1+K(\mu))U(\mu)] \\
= \ & \mathrm{Tr}\,\mathrm{res}_{\mu=\lambda}(1+K(\mu))^{-1}K'(\mu) \ .
\end{aligned}$$

This proves the claim.

Let $T : \mathcal{O}_\lambda \mathcal{H} \to \mathcal{O}_\lambda \mathcal{H}$ be given by $(Tf)(\mu) = T(\mu) f(\mu)$, $f \in \mathcal{O}_\lambda \mathcal{H}$. Then we have

$$\dim \operatorname{coker}(T) = \dim \operatorname{coker}(1+K) \ .$$

Now

$$\operatorname{Tr} \operatorname{res}_{\mu=\lambda} T(\mu)^{-1} T'(\mu) = \operatorname{Tr} \operatorname{res}_{\mu=\lambda} A(\mu)^{-1} A'(\mu)$$

for the holomorphic family of operators $A(\mu)$ on the finite-dimensional space V. Moreover $\dim \operatorname{coker}(T) = \dim \operatorname{coker}(A)$, where $A : \mathcal{O}_\lambda V \to \mathcal{O}_\lambda V$ is given by $(Af)(\mu) := A(\mu) f(\mu)$, $f \in \mathcal{O}_\lambda V$. In order to finish the proof of the proposition we must show

$$\operatorname{Tr} \operatorname{res}_{\mu=\lambda} A(\mu)^{-1} A'(\mu) = \dim \operatorname{coker}(A) \ .$$

We have

$$\begin{aligned} \operatorname{Tr} \operatorname{res}_{\mu=\lambda} A(\mu)^{-1} A'(\mu) &= \operatorname{res}_{\mu=\lambda} \operatorname{Tr} A(\mu)^{-1} A'(\mu) \\ &= \operatorname{res}_{\mu=\lambda} \frac{\det(A(\mu))'}{\det(A(\mu))} \\ &= \operatorname{ord}_{\mu=\lambda} \det(A(\mu)) \ . \end{aligned}$$

Using Gauß's algorithm $A(\mu)$ can be transformed to a holomorphic family of diagonal matrices $\tilde{A}(\mu)$ through multiplication from the left and right with holomorphic matrix functions with invertible determinants. We have $\dim \operatorname{coker}(\tilde{A}) = \dim \operatorname{coker}(A)$, where $\tilde{A} : \mathcal{O}_\lambda V \to \mathcal{O}_\lambda V$ is given by $(\tilde{A} f)(\mu) := \tilde{A}(\mu) f(\mu)$, $f \in \mathcal{O}_\lambda V$, and $\operatorname{ord}_{\mu=\lambda} \det(\tilde{A}(\mu)) = \operatorname{ord}_{\mu=\lambda} \det(A(\mu))$. But for holomorphic families of diagonal matrices the equation $\dim \operatorname{coker} \tilde{A} = \operatorname{ord}_{\mu=\lambda} \det(\tilde{A}(\mu))$ is obvious. This finishes the proof of the lemma. $\square$

We now recall the description of the divisor of the Selberg zeta function given in [15]. Note that our standing hypothesis is n even and $\delta_\Gamma < 0$. This simplifies things considerably because the point spectrum of Δ_Y is absent. Since the infinite product 1.2 converges at $s = 0$ we also know that $\operatorname{ord}_{s=0} Z_S(s) = 0$. For $\operatorname{Re}(\lambda) < 0$ let $n_\lambda := \operatorname{Tr} \operatorname{res}_{\mu=-\lambda} (1 + K(\mu))^{-1} K'(\mu)$.

Fact 6.4 *We have*

$$\operatorname{ord}_{s=\lambda} Z_S(s) = \begin{cases} 0 & \operatorname{Re}(\lambda) \geq 0 \\ n_\lambda & \operatorname{Re}(\lambda) < 0, \lambda \notin -\mathbf{N}_0 - \frac{n-1}{2} \\ n_\lambda - \chi(Y) \dim F_\lambda & \lambda \in -\mathbf{N}_0 - \frac{n-1}{2} \ . \end{cases}$$

In order to finish the proof of Equation (2) we just recall the results of Section 4. Note that $\chi(\Gamma, F_\lambda) = \chi(Y) \dim F_\lambda$ and by Proposition 6.1

$$\dim \operatorname{coker}(\tilde{S} : \mathcal{O}_{-\lambda} C^{-\omega}(B) \to \mathcal{O}_\lambda C^{-\omega}(B)) = n_\lambda \ .$$

For $k \geq k(\lambda)$ we have

$$-\chi_1(\Gamma, \mathcal{O}_{(\lambda,k)} C^{-\omega}(\Lambda)) = \begin{cases} 0 & \mathrm{Re}(\lambda) \geq 0 \\ n_\lambda & \mathrm{Re}(\lambda) < 0, \lambda \notin -\mathbf{N}_0 - \frac{n-1}{2} \\ n_\lambda - \chi(Y) \dim\, F_\lambda & \lambda \in -\mathbf{N}_0 - \frac{n-1}{2} \end{cases} .$$

Equation (2) follows.

In view of Corollary 5.3 we have completed the proof of Theorem 1.5.

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