Deformations of isotropic curves in $\mathbb{C}^3$ and minimal surfaces in $\mathbb{R}^3$

Hubert Gollek
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Abstract

We study properties of the natural parametrization of E. Vessiot and E. Study of isotropic curves in $\mathbb{C}^3$ and their Weierstraß-representation formula $\mathbf{Wei}_{f,g}$. Demanding, that the parametrization of the curve $\mathbf{Wei}_{f,g}$ is the natural one, we arrive at some special case $\mathbf{Wei}^*_g := \mathbf{Wei}_{i/g,g}$ of this formula. We show that the mapping, assigning to a meromorphic function f the isotropic curve $\mathbf{Wei}^*_g$, is equivariant with respect to a certain group-homomorphism $\mu : \mathrm{Sl}(2,\mathbb{C}) \longrightarrow \mathrm{SO}(3,\mathbb{C})$ and study some consequences of this fact for symmetries of minimal surfaces.

The *Study invariant* or *minimal curvature* $\kappa_\Phi(z)$ of an isotropic curve Φ resembles the classical notion of curvature. We use it here for the construction of *variations*, defined by $\delta_{\Phi,h}(z) = \left(-h(z)\,\kappa^2_\Phi(z) + h''(z)\right)\Phi'(z) - h''(z)\Phi''(z) + h(z)\Phi^{(3)}(z)$, and *deformations*, defined by $\Psi_{\Phi,h}(z) = \Phi(z) + \delta_{\Phi,h}(z)$, of Φ. They are new isotropic curves that depend on Φ and an arbitrary meromorphic function $h(z)$. Putting $\Phi = \mathbf{Wei}^*_f$, the assignment $(f,h) \longrightarrow \delta_{\Phi,h}(z)$ yields a new representation formula of minimal surfaces, depending on two arbitrary meromorphic functions f and h and their derivatives only, not involving integrations.

Contents

1 Introduction

Minimal surfaces $\mathcal{M}$ in the 3-dimensional Euclidean space $\mathbb{R}^3$ are in 1-1 correspondence with isotropic curves $\mathcal{C}$ in the complex Euclidean space $\mathbb{C}^3$ (see [2] or [3]). The easiest way to establish this correspondence is, to assign to a parametrized isotropic curve $\Psi(z)$ the parametrized real surface $\mathbf{x}(u,v) = \Re\,(\Psi(u+i\,v))$ – the real part of $\Psi(u+i\,v)$. On the other hand, similar to arc length of curves in the real Euclidean space, on isotropic curves of $\mathcal{C} \subset \mathbb{C}^3$ there is a natural parameter $\Psi : p \in \mathbb{C} \longrightarrow \mathcal{C} \subset \mathbb{C}^3$, defined by the condition $\left|\dfrac{\mathbf{d}^2\Psi}{\mathbf{d}p^2}\right|^2 = -1$.

Let us denote by $\langle\cdot,\cdot\rangle$ the usual scalar product of $\mathbb{R}^3$ as well as it's bilinear extension to $\mathbb{C}^3$ and by $(\cdot\ \cdot\ \cdot)$ the associated vector triple product. The norm $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$ is defined with respect to some fixed branch of the square root function on $\mathbb{C}$. The following result is proved in [1].

Proposition 1.1 *Let $\Psi{:}\,\mathcal{U} \subset \mathbb{C} \longrightarrow \mathbb{C}^3$ be a regular isotropic curve and $\mathbf{p} \in \mathbb{C}^3$ an arbitrary vector. Then we have*

$$\text{(1.1)} \qquad \langle\Psi''(z),\Psi''(z)\rangle = \sqrt[3]{-\left(\Psi'(z)\ \Psi''(z)\ \Psi'''(z)\right)^2} = -\left(\frac{(\Psi'(z)\ \Psi''(z)\ \mathbf{p})}{\langle\Psi'(z),\mathbf{p}\rangle}\right)^2.$$

We infer from (1.1) that $\langle\Psi''(z),\Psi''(z)\rangle \neq 0$ if and only if Ψ', Ψ'' and Ψ''' are linearly independent, that is, if Ψ is nowhere planar. Therefore, for nonplanar isotropic curves, a new parameter can be defined as follows:

Definition 1 *The* **natural parameter** *of a regular, nonplanar, isotropic curve $\Psi : \mathcal{U} \subset \mathbb{C} \longrightarrow \mathbb{C}^3$ is the function $p(z)$ given by*

$$\text{(1.2)} \qquad p(z) = \int_{z_0}^{z} \sqrt{-\frac{(\Psi'(\zeta)\ \Psi''(\zeta)\ \mathbf{p}(\zeta))}{\langle\Psi'(\zeta),\mathbf{p}(\zeta)\rangle}}\mathbf{d}\zeta, \quad (z_0 \in \mathbb{C} \quad \textit{initial value})$$

where $\mathbf{p}(\zeta) \in \mathbb{C}^3$ is some arbitrary but fixed complex vector function (see [1]).

As a consequence of proposition 1.1, denoting by $z(p)$ a local inverse of the function $p(z)$ and defining a new parametrization of Ψ by $\widetilde{\Psi}(p) = \Psi(z(p))$, we obtain

$$\text{(1.3)} \qquad \left\langle \frac{d^2\widetilde{\Psi}}{dp^2}, \frac{d^2\widetilde{\Psi}}{dp^2} \right\rangle = -1.$$

The *Weierstrass representation formula* (see [2] or [3]) assigns to any pair $f(z)$ and $g(z)$ of meromorphic functions an isotropic curve $\mathcal{C}$ given by the parametrization

$$\text{(1.4)} \qquad \Psi(z) = \mathbf{Wei}_{f,g}(z) = \frac{1}{2}\int_{z_0}^{z} \begin{pmatrix} f(\zeta)\,(1-g(\zeta))^2\,\mathbf{d}\zeta \\ i f(\zeta)\,(1+g(\zeta))^2\,\mathbf{d}\zeta \\ 2f(\zeta)g(\zeta)\mathbf{d}\zeta \end{pmatrix},$$

where $z_0 \in \mathbb{C}$ is arbitrary.

Similarly, the *Björling-representation formula* $\mathbf{Bj}_{\alpha,\gamma}(z)$ (see also [2] or [3]) assigns to a so-called real analytic strip $(\alpha(t), \gamma(t))$ an isotropic curve $\mathcal{C}$. By a strip we mean a pair of parametrized curves $(\alpha(t), \gamma(t))$ of $\mathbb{R}^3$ such that $\langle \alpha'(z), \gamma(z) \rangle = 0$ and $|\gamma'(z)| = 1$. The idea behind this notion is that of a curve α of unit speed on a surface $\mathcal{M}$ together with a unit normal vector field γ of $\mathcal{M}$ along α. $\mathcal{C}$ is given by a parameter representation $\Psi(z)$ as follows: Let $\alpha(z)$ and $\gamma(z)$ be local holomorphic extensions of $\alpha(t)$ and $\gamma(t)$ to the complex plane. Then

$$\Psi(z) = \mathbf{Bj}_{\alpha,\gamma}(z) = \alpha(z) - i \int_{z_0}^{z} \alpha'(\zeta) \times \gamma(\zeta) \mathbf{d}\zeta. \tag{1.5}$$

We denote by $\mathcal{IC}^*$ the set of parametrized, by $\mathcal{IC}$ the set of unparametrized isotropic curves, that can be parametrized by a unique coordinate patch, that is, $\mathcal{IC}$ is the set of all curves obtained from parametrized curves of $\mathcal{IC}^*$ by forgetting the parametrization. We consider in $\mathcal{IC}$ curves as equal, if they are are congruent with respect to the group of complex Euclidean motions. By $\mathbf{A}$ we denote the algebra of meromorphic functions on $\mathbb{C}$. The two operations, **Wei** and **Bj**, considered as mappings acting from $\mathbf{A} \times \mathbf{A}$ (or the set of analytic strips, respectively) into $\mathcal{IC}^*$ as well as into $\mathcal{IC}$, are surjective. For the operator **Bj**, this assertion is proved in [2] and for **Wei**, one infers from (1.4), that putting $\Psi'(z) = (\varphi_1(z), \varphi_2(z), \varphi_3(z))$ and defining the functions f and g as in (1.6) below yields an inverse of the operator $\mathbf{Wei} : \mathbf{A} \times \mathbf{A} \longrightarrow \mathcal{IC}^*$.

$$f(z) = \varphi_1(z) - i\,\varphi_2(z) \quad \text{and} \quad g(z) = \frac{\varphi_3(z)}{\varphi_1(z) - i\,\varphi_2(z)}. \tag{1.6}$$

The following proposition can be proved by direct evaluation of the defining equations (1.2), (1.4) and (1.5). When trying to verify the expressions (1.7) of below by means of (1.2), the reader is advised to choose $\mathbf{p}(\zeta)$ as $(1, 0, 0)$ in the case of **Wei** and as $\alpha'(\zeta)$ in the case of **Bj**.

Proposition 1.2 *Both complex curves,* $\Psi_{\mathbf{Wei}}(z) = \mathbf{Wei}_{f,g}(z)$ *and* $\Psi_{\mathbf{Bj}}(z) = \mathbf{Bj}_{\alpha,\gamma}(z)$*, are isotropic and their natural parameters are given by*

$$p'_{\mathbf{Wei}}(z) = \sqrt{-i\, f(z)\, g'(z)} \quad \textit{and} \quad p'_{\mathbf{Bj}}(z) = \sqrt{-\,(\alpha'\ \gamma\ \gamma')} \tag{1.7}$$

respectively.

2 Weierstrass-representation and the natural Parameter

We focus our attention to the Weierstraß representation formula and define a new formula $\mathbf{Wei}^*_g$, depending only on a single function g and giving an isotropic curve, parametrized by its natural parameter. Equation (1.7) teaches us, that choosing $f(z) = i/g'(z)$ yields $p'(z) = 1$ for the isotropic curve $\Psi(z) = \mathbf{Wei}_{g,i/g'}(z)$. Therefore, we define

Definition 2 *The* **natural Weierstraß representation** *is given by*

$$\mathbf{Wei}^*_g(z) = \mathbf{Wei}_{g,i/g'}(z) = \int_{z_0}^{z} \frac{1}{2g'(\zeta)} \begin{pmatrix} i\,\left(1 - g(\zeta)^2\right) \\ -\left(1 + g(\zeta)^2\right) \\ 2\,i\,g(\zeta) \end{pmatrix} \mathbf{d}\zeta. \tag{2.1}$$

We will consider $\mathbf{Wei}^*$ and $\mathbf{Wei}$ not only as mappings into $\mathcal{IC}^*$, but also as mappings into the set $\mathcal{IC}$ of unparametrized isotropic curves. Therefore, $\mathbf{Wei}^*$ is the composition of mappings

$$\mathbf{Wei}^* : \mathbf{A} \overset{f \to (i/f', f)}{\longrightarrow} \mathbf{A} \times \mathbf{A} \overset{\mathbf{Wei}}{\longrightarrow} \mathcal{IC}^* \longrightarrow \mathcal{IC}, \tag{2.2}$$

and we arrive at the question, whether this restriction is still surjective. The full Weierstraß mapping $\mathbf{Wei}_{f,g}$ was seen to be surjective, by considering the formula (1.6) as an inverse of $\mathbf{Wei}$ acting from $\mathcal{IC}$ into $\mathbf{A} \times \mathbf{A}$. The following proposition shows, that it remains surjective when restricting it to the image of the first mapping of (2.2)

Proposition 2.1 *If $\Psi{:}\mathcal{U} \longrightarrow \mathbb{C}^3$ is an isotropic curve, z its natural parameter and $\Psi'(z) = (\varphi_1(z), \varphi_2(z), \varphi_3(z))$. Then the functions $f(z)$ and $g(z)$ given by (1.6) are related by $f(z)g'(z) = i$, that is, $\Psi'(z) = \mathbf{Wei}^*_g(z)$.*

Proof. We have by (1.6) $f = \varphi_1 - i\,\varphi_2$ and $g = \dfrac{\varphi_3}{\varphi_1 - i\,\varphi_2}$. Differentiation of g yields

$$g' = \frac{\varphi_3'\varphi_1 - \varphi_3\varphi_1' - i\,(\varphi_3'\varphi_2 - \varphi_3\varphi_2')}{(\varphi_1 - i\,\varphi_2)^2}. \tag{2.3}$$

Since by assumption, z is the natural parameter on Ψ, the integrands in (1.2) must be equal to 1, for any choice of $\mathbf{p}(\zeta)$. Choosing simply $\mathbf{p} = (0,1,0)$ we get $\varphi_1\varphi_3' - \varphi_1'\varphi_3 = \varphi_2$ and for $\mathbf{p} = (0,0,1)$, $\varphi_2\varphi_3' - \varphi_2'\varphi_3 = -\varphi_1$. Therefore

$$g' = \frac{\varphi_2 + i\,\varphi_1}{(\varphi_1 - i\,\varphi_2)^2} = \frac{i}{\varphi_1 - i\,\varphi_2} = \frac{i}{f}.. \tag{2.4}$$

■

The question of injectivity of the transformation $\mathbf{Wei}^*$ will be discussed in the next section. By proposition 2.1, for any minimal surface $\mathcal{M}$ there exists a meromorphic function f, such that $\mathcal{M}$ is (at least locally) parametrized by the real part of $\mathbf{Wei}^*_f(u + i\,v)$. Geometric quantities such as spherical map and the first fundamental form for a minimal patch, given by the Weierstraß representation formula $\mathbf{x}(u,v) = \Re\left(\mathbf{Wei}_{f,g}(u + i\,v)\right)$, can be found for instance in [2]. Here we want to mention an expression for the second fundamental form $\mathbf{II}$ of $\Re\left(\mathbf{Wei}_{f,g}(u + i\,v)\right)$ and a special property of the second fundamental form of $\Re\left(\mathbf{Wei}^*_f(u + i\,v)\right)$: The coordinate lines of this parametrization are asymptotic and the matrix of $\mathbf{II}$ with respect to this coordinate system is constant. More precisely, we have the following proposition.

Proposition 2.2 *Let $f, g \in \mathbf{A}$ and $\mathbf{x}_{f,g} = \Re(\mathbf{Wei}_{f,g}(u+i\,v))$ the associated minimal surface. Decompose $f(u+iv) = a(u,v)+i\,b(u,v)$ and $g(u+iv) = c(u,v)+i\,d(u,v)$ into real and imaginary parts. Then, the matrix of the second fundamental form of $\mathbf{x}_{f,g}$ is given by*

$$\mathbf{II} = \begin{pmatrix} -bc_v - ac_u & -ac_v + bc_u \\ -ac_v + bc_u & bc_v + ac_u \end{pmatrix} \tag{2.5}$$

(where c_v, c_u denote the partial drivatives of c). If in addition to this, $f = i/g'$, that is, $\mathbf{x}_g = \Re\left(\mathbf{Wei}^*_g(u+i\,v)\right)$ is given in it's natural Weierstraß representation, then $\mathbf{II} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3 Equivariance of $\mathbf{Wei}^*_g$

Let the group $\mathrm{SO}(3,\mathbb{C})$ act on $\mathcal{IC}$ in the usual way and $\mathrm{Gl}(2,\mathbb{C})$ on $\mathbf{A}$ by fractional linear transformations, that is,

$$\text{if} \quad \mathbf{g} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}(2,\mathbb{C}), \quad f \in \mathbf{A}, \quad \text{then} \quad \mathbf{g}\cdot f = \frac{a\,f+b}{c\,f+d)}. \tag{3.1}$$

The transformation $\mathbf{Wei}^* : \mathbf{A} \longrightarrow \mathcal{IC}$ is not injective. So for instance, f, $1/f$, $(1+f)/(1-f), \ldots$ all yield the same isotropic curve. However, as a mapping into $\mathcal{IC}^*$, it is equivariant with respect to the group homomorphism $\mu : \mathrm{Gl}(2,\mathbb{C}) \longrightarrow \mathrm{SO}(3,\mathbb{C})$ defined by

$$\begin{cases} \mu : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Gl}(2,\mathbb{C}) \longrightarrow A = \dfrac{1}{2(ad-bc)} \star \\ \begin{pmatrix} a^2-b^2-c^2+d^2 & i\,\left(a^2+b^2-c^2-d^2\right) & 2\,(c\,d-a\,b) \\ -i\,\left(a^2-b^2+c^2-d^2\right) & a^2+b^2+c^2+d^2 & 2\,i\,(a\,b+c\,d) \\ 2\,(b\,d-a\,c) & -2\,i\,(a\,c+b\,d) & 2\,(b\,c+a\,d) \end{pmatrix} \end{cases} \tag{3.2}$$

Moreover, the counterimage $(\mathbf{Wei}^*)^{-1}\,(\mathrm{SO}(3,\mathbb{C})\cdot\mathcal{C})$ of the orbit of a curve $\mathcal{C} \in \mathcal{IC}$ is a $\mathrm{Gl}(2,\mathbb{C})$-orbit in $\mathbf{A}$.

Proposition 3.1 *The mapping μ is a group homomorphism of $Gl(2,\mathbb{C})$ onto $SO(3,\mathbb{C})$. The kernel of μ is the center of $Gl(2,\mathbb{C})$. $\mathbf{Wei}^*_f$ is equivariant with respect to μ, that is,*

$$\mu(\mathbf{g})\cdot\mathbf{Wei}^*_f(z) = \mathbf{Wei}^*_{\mathbf{g}\cdot f}(z), \quad (\textit{for } \mathbf{g} \in Sl(2,\mathbb{C}), f \in \mathbf{A}, z \in \mathbb{C}). \tag{3.3}$$

The proof of this equivariance property is elementary. One can do it, as well as several other computations in this article, with a computer algebra program.

In order to apply this result to (real) minimal surfaces, we have to find out, which elements $\mathbf{g} \in \mathrm{Gl}(2,\mathbb{C})$ yield real matrices in $\mathrm{SO}(3,\mathbb{C})$. We have the following

Proposition 3.2 *The subgroup $G_{quat} \subset Gl(2,\mathbb{C})$, consisting of all matrices of the form $\begin{pmatrix} z & w \\ -\varepsilon\bar{w} & \varepsilon\bar{z} \end{pmatrix}$ with $z, w \in \mathbb{C}$, $|z|^2 + |w|^2 = 1$ and $\varepsilon = \pm 1$ is mapped by μ onto the subgroup $SO(3,\mathbb{R}) \subset SO(3,\mathbb{C})$*

Consequently, if a function $f \in \mathbf{A}$ is invariant with respect to some element $\mathbf{g} \in G_{\text{quat}}$ in the sense, that $\mathbf{g} \cdot f(z) = f(z+t)$ for some fixed $t \in \mathbb{C}$, then the corresponding unparametrized minimal surface $\mathcal{M}$ will be *almost invariant* with respect to the real orthogonal transformation $\mu(\mathbf{g})$, that is, there exist a vector $\mathbf{p} \in \mathbb{R}^3$ such that $\mu(\mathbf{g}) \cdot \mathcal{M} = \mathbf{p} + \mathcal{M}$

We give some detailed rules for finding invariant functions of this kind for special elements $\mathbf{g} \in G_{\text{quat}}$. Let us introduce the following elements of $\mathrm{O}(3,\mathbb{R})$ and determine their inverse images under μ in G_{quat}.

$$\begin{aligned}
\mathbf{d}_{xy}^r &= \text{rotation around the } z\text{-axis by the angle } r \\
\mathbf{d}_{xz}^r &= \text{rotation around the } y\text{-axis by the angle } r \\
\mathbf{d}_{yz}^r &= \text{rotation around the } x\text{-axis by the angle } r
\end{aligned}$$

There is a simple formula for a right-inverse mapping of μ on $\mathrm{SO}(3,\mathbb{R})$. If an element element $\mathbf{d} \in \mathrm{SO}(3,\mathbb{R})$ is represented as a product $\mathbf{d} = \mathbf{d}_{xy}^\alpha\, \mathbf{d}_{xz}^\beta\, \mathbf{d}_{yz}^\gamma$ by it's Euler angles α, β and γ, then the G_{quat}-matrix

$$\tag{3.4} \frac{1}{\sqrt{-2iq}} \begin{pmatrix} q\left(\cos(\frac{\beta-\gamma}{2}) - i\cos(\frac{\beta+\gamma}{2})\right) & iq\left(\sin(\frac{\beta-\gamma}{2}) + i\sin(\frac{\beta+\gamma}{2})\right) \\ \sin(\frac{\beta-\gamma}{2}) - i\sin(\frac{\beta+\gamma}{2}) & -i\cos(\frac{\beta-\gamma}{2}) + \cos(\frac{\beta+\gamma}{2}) \end{pmatrix}$$

(where $q = \cos(\alpha) - i\,\sin(\alpha)$) is mapped by μ onto $\mathbf{d}$.

Proposition 3.3 *Let $\mathbf{g}_{xy}^r$, $\mathbf{g}_{xz}^r$ and $\mathbf{g}_{yz}^r$ be the following elements of G_{quat}:*

$$\mathbf{g}_{xy}^r = \frac{1-i}{\sqrt{-2\,i\cos r - 2\,\sin r}} \begin{pmatrix} \cos r & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\frac{-ir}{2}} & 0 \\ 0 & e^{\frac{ir}{2}} \end{pmatrix}$$

$$\mathbf{g}_{xz}^r = \begin{pmatrix} \cos\frac{r}{2} & -\sin\frac{r}{2} \\ \sin\frac{r}{2} & \cos\frac{r}{2} \end{pmatrix}, \quad \mathbf{g}_{yz}^r = \begin{pmatrix} \cos\frac{r}{2} & -i\,\sin\frac{r}{2} \\ -i\,\sin\frac{r}{2} & \cos(r/2) \end{pmatrix}$$

Then we have $\mu\left(\mathbf{g}_{xy}^r\right) = \mathbf{d}_{xy}^r$, $\mu\left(\mathbf{g}_{xz}^r\right) = \mathbf{d}_{xz}^r$ and $\mu\left(\mathbf{g}_{yz}^r\right) = \mathbf{d}_{yz}^r$ and for any $f \in \mathbf{A}$

$$\begin{aligned}
\mathbf{g}_{xy}^r \cdot f(z) &= \cos(r)\, f(z) - i\, f(z)\,\sin(r) = e^{-i\,r} f(z) \\
\mathbf{g}_{xz}^r \cdot f(z) &= (\cos(r/2)\, f(z) - \sin(r/2)) \big/ (\cos(r/2) + f(z)\,\sin(r/2)) \\
\mathbf{g}_{yz}^r \cdot f(z) &= (\cos(r/2)\, f(z) - i\,\sin(r/2)) \big/ (\cos(r/2) - i\, f(z)\,\sin(r/2))
\end{aligned}$$

Proposition 3.4 *Let $t \in \mathbb{C}$, $t \neq 0$ a complex, $r \in \mathbb{R}$ a real number, $\widetilde{f}(z) \in \mathbf{A}$ a periodic function with complex period t, that is, $\widetilde{f}(z+t) = \widetilde{f}(z)$ for any z. Consider* *the functions*

$$(3.5)\qquad \begin{cases} f_{xy}(z) &= e^{-i\,r\,z/t}\widetilde{f}(z), \quad f_{xz}(z) = \dfrac{e^{-i\,r\,z/t}\widetilde{f}(z)+i}{i\,e^{-i\,r\,z/t}\widetilde{f}(z)+1} \quad \text{and} \\ f_{yz}(z) &= \dfrac{e^{-i\,r\,z/t}\widetilde{f}(z)-1}{e^{-i\,r\,z/t}\widetilde{f}(z)+1} \end{cases}$$

Then we have $\mathbf{g}^r_{xy}\cdot f_{xy}(z) = f_{xy}(z+t)$, $\mathbf{g}^r_{xz}\cdot f_{xz}(z) = f_{xz}(z+t)$ *and* $\mathbf{g}^r_{yz}\cdot f_{yz}(z) = f_{yz}(z+t)$. *The parametrized minimal surfaces* $\mathbf{w}_{xy}(u,v) = \Re\left(\mathbf{Wei}^*_{f_{xy}}(u+i\,v)\right)$, $\mathbf{w}_{xz}(u,v) = \Re\left(\mathbf{Wei}^*_{f_{xz}}(u+i\,v)\right)$ *and* $\mathbf{w}_{yz}(u,v) = \Re\left(\mathbf{Wei}^*_{f_{yz}}(u+i\,v)\right)$ *are almost invariant with respect to the rotations* $\mathbf{d}^r_{xy}$, $\mathbf{d}^r_{xz}$ *and* $\mathbf{d}^r_{yz}$ *respectively in the sense explained above.*

4 Frenet Frames and Natural Equations

An other way to obtain a more or less unique representation of isotropic curves (or minimal surfaces respectively), is given by an analogue of the natural equations of ordinary curve theory for isotropic curves. We define an orthonormal moving frame and some curvature quantities as follows.

Definition 3 *Let* $\Psi(z)$ *be a isotropic curve such that* z *is the natural parameter of* Ψ *and* $\langle\Psi''(z),\Psi''(z)\rangle = -1$. *The functions* $\kappa(z) = |\Psi'''(z)|$ *and* $\tau(z) = \kappa'(z)/\kappa(z)$ *are called the* **minimal curvature** *and the* **minimal torsion** *of* $\Psi(z)$ *respectively. We define the vector fields* $\mathbf{e}_1$, $\mathbf{e}_2$ *and* $\mathbf{e}_3$ *(the* **Frenet frame** *of* $\Psi(z)$*) by* $\mathbf{e}_1(z) = i\,\Psi''(z)$, $\mathbf{e}_2(z) = \Psi'''(z)/\kappa(z)$ *and* $\mathbf{e}_3(z) = \mathbf{e}_1(z)\times\mathbf{e}_2(z)$.

We have the following analogue of the natural equations of curve theory.

Proposition 4.1 *The Frenet frame of an isotropic curve* $\Psi(z)$, *given in it's natural parametrization, is orthonormal. It satisfies the following natural equation:*

$$(4.1)\qquad \begin{cases} \mathbf{e}_1'(z) &= \qquad -i\,\kappa(z)\mathbf{e}_2(z) \\ \mathbf{e}_2'(z) &= i\,\kappa(z)\mathbf{e}_1(z) + \qquad i\,\tau(z)\mathbf{e}_3(z) \\ \mathbf{e}_3'(z) &= \qquad -i\,\tau(z)\mathbf{e}_2(z) \qquad . \end{cases}$$

Representing the isotropic curve by the natural Weierstraß representation formula, these equations can be solved in a simple way, as described by the following proposition.

Proposition 4.2 *Let* $f(z)\in\mathbf{A}$. *The minimal curvature and the minimal torsion of the isotropic curve* $\mathbf{Wei}^*_f(z)$ *are given by*

$$(4.2)\qquad \kappa(z) = \frac{\sqrt{3\,f''(z)^2 - 2\,f'(z)\,f^{(3)}(z)}}{f'(z)} \quad \text{and}$$

$$\tau(z)=\frac{-3\,f''(z)^3+4\,f'(z)\,f''(z)\,f^{(3)}(z)-f'(z)^2\,f^{(4)}(z)}{3\,f'(z)\,f''(z)^2-2\,f'(z)^2\,f^{(3)}(z)}. \tag{4.3}$$

Equation (4.2) shows a way to reduce the system (4.1) to some integrations and to solving a Riccati equation. Namely, under the substitution $y(z)=f''(z)/f'(z)$, (4.2) turns into the equation $y'(z)=\left(y(z)^2-\kappa(z)^2\right)/2$. $\Psi(z)$ can be reconstructed from $\kappa(z)$ by solving this Riccati equation $y'(z)=\left(y(z)^2-\kappa(z)^2\right)/2$ for $y(z)$, integrating twice to obtain $f(z)$ and finally applying the operator $\mathbf{Wei}^*$ to f.

Note, that the right hand side of equation (4.2) is the Schwartz derivative of the function f. It is an invariant with respect to the action of $\mathrm{Sl}(2,\mathbb{C})$.

If $\Psi(z)$ is an isotropic curve parametrized by its natural parameter, the scalar products $\left\langle\Psi^{(i)}(z),\Psi^{(j)}(z)\right\rangle$ of the higher order derivatives of Ψ can be expressed by the minimal curvature $\kappa(z)$ of Ψ and its derivatives. If we confine to the case $i,j\le 5$, the matrix of these scalar products is

$$\begin{pmatrix} 0 & 0 & 1 & 0 & \kappa^2(z) \\ 0 & -1 & 0 & -\kappa^2(z) & m_{25} \\ 1 & 0 & \kappa^2(z) & \kappa(z)\,\kappa'(z) & m_{35} \\ 0 & -\kappa^2(z) & \kappa(z)\,\kappa'(z) & -\kappa^4(z) & -2\,\kappa^3(z)\,\kappa'(z) \\ \kappa^2(z) & m_{52} & m_{53} & -2\,\kappa^3(z)\,\kappa'(z) & m_{55} \end{pmatrix}, \tag{4.4}$$

where $m_{25}=m_{52}=-3\,\kappa(z)\,\kappa'(z)$, $m_{35}=m_{53}=\kappa^4(z)+(\kappa'(z))^2+\kappa(z)\,\kappa''(z)$ and $m_{55}=\kappa^2(z)\,\left(\kappa^4(z)-7\,(\kappa'(z))^2+2\,\kappa(z)\,\kappa''(z)\right)$.

5 Variations and Deformations

We are going to develop some operations on $\mathcal{IC}^*$ that lead to two new representation formulas of isotropic curves. Suppose that $\Phi(z)$ is a nonplanar isotropic curve and z the natural parameter on $\Phi(z)$. Then, $\Phi'(z)$, $\Phi''(z)$ and $\Phi^{(3)}(z)$ are linearly independend and any other curve $\Psi(z)$ in $\mathbb{C}^3$ can be represented by

$$\Psi(z)=\Phi(z)+v_1(z)\Phi'(z)+v_2(z)\Phi''(z)+v_3(z)\Phi^{(3)}(z) \tag{5.1}$$

with certain functions $v_1,v_2,v_3\in\mathbf{A}$. We want to determine them so, that $\Psi(z)$ is again an isotropic curve

Using the scalar products of the various derivatives of $\Phi(z)$ as displayed by the matrix (4.4), a direct computation shows, that the length $|\Psi'(z)|$ can be expressed in terms of the minimal curvature $\kappa_\Phi(z)$ of Φ in the following way:

$$\left\{\begin{array}{rcl} |\Psi'(z)|^2 &=& -v_1^2+2v_2+2v_1'\,v_2-2v_1\,v_2'-(v_2')^2+2v_3'+2v_1'\,v_3' \\ &+& (2v_2\,v_3+2v_3\,v_3')\,\kappa_\Phi\,\kappa_\Phi' \\ &+& \left({v_2}^2-2v_1\,v_3-2v_2'\,v_3+2v_2\,v_3'+(v_3')^2\right)\kappa_\Phi^2-v_3^2\,\kappa_\Phi^4. \end{array}\right. \tag{5.2}$$

For given $\Phi(z)$, the condition $|\Psi' z)| = 0$ is an ordinary differential equation for v_1, v_2 and v_3, that is a Riccati-equation for v_1 and still more involved for v_2 and v_3. However, a computation, similar to that of (5.2), shows, that the length of $\Psi'(z) - \Phi'(z)$ and that of $\Psi'(z)$ differ by

$$|\Psi'(z))|^2 - |\Psi'(z) - \Phi'(z)|^2 = 2\ (v_2(z) + v_3'(z)) \tag{5.3}$$

This result shows, that putting

$$v_2(z) = -v_3'(z), \tag{5.4}$$

yields the same length for $\Psi'(z) - \Phi'(z)$ and $\Psi'(z)$. Moreover, under the condition (5.4), the right hand side of equation (5.2) simplifies significantly. It becomes

$$|\Psi' z)|^2 = -\left(-v_1 - v_3\,\kappa_\Phi^2 + v_3''\right)^2. \tag{5.5}$$

Therefore, a partial solution of the differential equation $|\Psi' z)| = 0$ is obtained by choosing $v_3(z) = h(z)$ as an arbitrary meromorphic function and defining v_2 by (5.4) and v_1 by $v_1(z) = h(z)\,\kappa_\Phi^2(z) - h''(z)$.

Let us assume that v_1, v_2, v_3 are given in this way and denote the difference $\Psi(z) - \Phi(z)$ by $\delta_{\Phi,h}(z)$, that is,

$$\delta_{\Phi,h}(z) = \left(-h(z)\,\kappa_\Phi^2(z) + h''(z)\right)\Phi'(z) - h''(z)\Phi''(z) + h(z)\Phi^{(3)}(z). \tag{5.6}$$

By $\Psi_{\Phi,h}(z)$ we denote the curve $\Psi(z)$ given in (5.1) or explicitely by

$$\Psi_{\Phi,h}(z) = \Phi(z) + \delta_{\Phi,h}(z). \tag{5.7}$$

$\Psi_{\Phi,h}(z)$ and $\delta_{\Phi,h}(z)$ are two new isotropic curves, constructed from a given isotropic curve $\Phi(z)$ and a meromorphic function $h(z)$, such that for fixed Φ the mapping $h \in \mathbf{A} \longrightarrow \delta_{\Phi,h} \in \mathcal{IC}^*$ is $\mathbb{C}$−linear. The following proposition lists its main properties.

Proposition 5.1 *For any isotropic curve $\Phi(z)$, parametrized by its natural parameter, and for any meromorphic function $h(z)$, the two curves $\delta_{\Phi,h}$ and $\Psi_{\Phi,h}$ defined above are isotropic.*

The derivatives of the natural parameters $p_\delta(z)$ and $p_\Psi(z)$ of $\delta_{\Phi,h}$ and $\Psi_{\Phi,h}(z)$ respectively are given by

$$\begin{cases} p_\delta'(z) &= \sqrt{-\kappa_\Phi(z)\,\kappa_\Phi'(z)\,h(z) - \kappa_\Phi^2(z)\,h'(z) + h^{(3)}(z)} \\ p_\Psi'(z) &= \sqrt{1 - \kappa_\Phi(z)\,\kappa_\Phi'(z)\,h(z) - \kappa_\Phi(z)^2\,h'(z) + h^{(3)}(z)} \end{cases} \tag{5.8}$$

We have in particular $\left(p_\Psi'(z)\right)^2 - \left(p_\delta'(z)\right)^2 = 1$.

The derivatives $\delta_{\Phi,h}'(z)$ and $\Psi_{\Phi,h}'(z)$ both are multiples of $\Phi'(z)$, namely, we have

$$\begin{cases} \delta_{\Phi,h}'(z) = \left(p_\delta'(z)\right)^2\ \Phi'(z), \\ \Psi_{\Phi,h}'(z) = \left(p_\Psi'(z)\right)^2\ \Phi'(z). \end{cases} \tag{5.9}$$

If $f(z)$ is a meromorphic function with $\Phi(z) = \mathbf{Wei}_f^*(z)$ then functions f_Ψ and g_Ψ with $\Psi_{\Phi,h}(z) = \mathbf{Wei}_{f_\Psi,g_\Psi}(z)$ and functions $f_\delta(z)$ and $g_\delta(z)$ with $\delta_{\Phi,h}(z) = \mathbf{Wei}_{f_\delta,g_\delta}(z)$ are given by

$$f_\Psi(z) = i\,\frac{(p'_\Psi(z))^2}{f'(z)}, \quad f_\delta(z) = i\,\frac{(p'_\delta(z))^2}{f'(z)}, \quad g_\Psi(z) = g_\delta(z) = f(z). \tag{5.10}$$

Proof. The proof of these results is gained by some elementary, but involved computations. The basic problem perhaps is to reduce the very long expressions obtained by differentiation and evaluation of the relevant scalar products to the short form we have given in these formulas. ∎

We call the isotropic curve $\delta_{\Phi,h}$ the *variation* and $\Psi_{\Phi,h} = \Phi+\delta_{\Phi,h}$ the *deformation* of Φ by h. Note that by the formulas (5.8) and (5.10), the operations, assigning to a pair $(f,h) \in \mathbf{A}\times\mathbf{A}$ the pairs $(f_\Psi, g_\Psi) \in \mathbf{A}\times\mathbf{A}$ and $(f_\delta, g_\delta) \in \mathbf{A}\times\mathbf{A}$ respectively, are surjective. Since the Weierstraß representation is surjective as a mapping from $\mathbf{A}\times\mathbf{A}$ into $\mathcal{IC}$, the same is true for the variation and the deformation of isotropic curves, considered as mappings from $\mathcal{IC}\times\mathbf{A}$ into $\mathcal{IC}$.

Since in general, z is not the natural parameter on $\delta_{\Phi,h}(z)$ and $\Psi_{\Phi,h}(z)$, their minimal curvatures can not be computed easily as the norm of their third derivatives. With the help of the higher order chain rule we can express the derivative of order 3 of the curve $\Phi(z) = \mathbf{Wei}_{f,g}(z)$ with respect to the canonical parameter $p(z)$ in terms $\Phi'(z)$, $\Phi''(z)$ and $\Phi^{(3)}(z)$ and $p'(z)$, $p''(z)$ and $p^{(3)}(z)$ Using the result of proposition 1.2 we get the following expression for the square of the minimal curvature of an isotropic curve that is given by the Weierstraß representation formula.

$$\left\{\begin{array}{rcl} \kappa_\Phi^2(z) & = & \dfrac{i}{4\,f^3(z)\,g'^3(z)}\big(5\,f'^2(z)\,g'^2(z) - 4\,f(z)\,g'^2(z)\,f''(z) \\ & & +2\,f(z)\,f'(z)\,g'(z)\,g''(z) - 7\,f^2(z)\,g''(z)^2 \\ & & +4\,f^2(z)\,g'(z)\,g^{(3)}(z)\big) \end{array}\right. \tag{5.11}$$

From this and the Weierstraß functions (5.10) we obtain the following result.

Proposition 5.2 *Let $\delta_{\Phi,h}$ and $\Psi_{\Phi,h}$ be as in proposition 5.1, $\kappa_\delta(z)$ and $\kappa_\Psi(z)$ their minimal curvatures and $\kappa_\Phi(z)$ the minimal curvature of the isotropic curve Φ. Then we have*

$$\left\{\begin{array}{rcl} \kappa_\delta^2(z) & = & \dfrac{-\big(\kappa_\Phi^2(z)\,q^2(z)\big) + 3\,\left(q'(z)\right)^2 - 2\,q(z)\,q''(z)}{q^4(z)} \\ \kappa_\Psi^2(z) & = & \left(1+q^2(z)\right)^{-3}\big(-\kappa_\Phi^2(z) - 2\,\kappa_\Phi^2(z)\,q^2(z) \\ & & -\kappa_\Phi^2(z)\,q^4(z) - 2\,\left(q'(z)\right)^2 + 3\,q^2(z)\,\left(q'(z)\right)^2 \\ & & -2\,q(z)\,q''(z) - 2\,q^3(z)\,q''(z)\big) \end{array}\right. \tag{5.12}$$

where $q(z) = p'_\delta(z)$.

6 Examples and Equivariance of Deformations

There are a lots of surfaces that can be parametrized with elementary functions by $\mathbf{Wei}^*$. The more interesting ones such as those, described in proposition 3.4 need more effort. First we give here some remarks on isotropic curves of simplest form. These are for instance curves with constant minimal curvature or constant minimal torsion. One of the most symmetric functions in $\mathbf{A}$ with respect to $\mathrm{Gl}(2,\mathbb{C})$ is $f(z) = \tan(a\,z)$. It has a 1-dimensional symmetry group in the following sense: If $r \in \mathbb{C}$ and $\mathbf{g} = \begin{pmatrix} \cos r & \sin r \\ -\sin r & \cos r \end{pmatrix}$, then $(\mathbf{g}\cdot f)(z) = f(z+r)$. The minimal curve $\Psi(z) = \mathbf{Wei}^*_f(z)$ is $\left(\sin(2\,a\,z)/2\,a^2, -z/2\,a, -i\cos(a\,z)^2/2\,a^2\right)$. It belongs to the $\mathrm{Gl}(2,\mathbb{C})$-orbit of the exponential function $\exp(a\,z/2)$. All curves corresponding to functions in the orbit of $\exp(a\,z/2)$ have $\kappa(z) = -a = \mathrm{const}$ and $\tau(z) = 0$. Among the minimal surfaces corresponding to the curves of this orbit are the helicoid and the catenoid. By the remark of section 5, finding the curves with constant minimal torsion t by means of the Weierstrass formula, leads to the solution of the Riccati-equation $y'(z) = \left(y(z)^2 - c\,\exp(t\,z)\right)/2$

Other simple functions $f \in \mathbf{A}$ are the polynomials. They give well known surfaces, some of them with a rotational symmetry, but in general with many selfintersections. For the power function $f(z) = z^n$, curvature and torsion are $\kappa(z) = -\sqrt{-1+n^2}/z$ and $\tau/z = -1/z$ resp.

If we choose $\Phi(z) = \mathbf{Wei}^*_f(z)$, then $\widetilde{\Phi}(z) = \delta_{\Phi,h}(z)$ is a lengthy expression, containing the derivatives of f and h. Its components are the following

$$
\left\{
\begin{array}{lcl}
\widetilde{\Phi}_1 & = & -\left(h'\left(-i\,f + \frac{\frac{i}{2}\left(-1+f^2\right)f''}{f'^2}\right)\right) \\
 & + & h\left(-i\,f' + \frac{i\,f\,f''}{f'} - \frac{i\left(-1+f^2\right)f''^2}{f'^3} + \frac{\frac{i}{2}\left(-1+f^2\right)f^{(3)}}{f'^2}\right) \\
 & - & \frac{\frac{i}{2}\left(-1+f^2\right)\left(-3\,h\,f''^2+f'^2\,h''+2\,h\,f'\,f^{(3)}\right)}{f'^3} \\
\widetilde{\Phi}_2 & = & -\left(h'\left(-f + \frac{\left(1+f^2\right)f''}{2\,f'^2}\right)\right) \\
 & + & h\left(-f' + \frac{f\,f''}{f'} - \frac{\left(1+f^2\right)f''^2}{f'^3} + \frac{\left(1+f^2\right)f^{(3)}}{2\,f'^2}\right) \\
 & + & \frac{\left(1+f^2\right)\left(3\,h\,f''^2-f'^2\,h''-2\,h\,f'\,f^{(3)}\right)}{2\,f'^3} \\
\widetilde{\Phi}_3 & = & \frac{i\left(-\left(f'^3\,h'\right)-h\,f'^2\,f''+f\,f'\,h'\,f''-f\,h\,f''^2+f\,f'^2\,h''+f\,h\,f'\,f^{(3)}\right)}{f'^3}
\end{array}
\right.
\tag{6.1}
$$

Let us denote the operation assigning to two meromorphic functions $f, h \in \mathbf{A}$ the isotropic curve (6.1) by $\mathbf{Var} : (f,h) \in \mathbf{A}\times\mathbf{A} \longrightarrow \widetilde{\Phi} = \mathbf{Var}_{f,h} \in \mathcal{IC}^*$. This is an useful representation formula avoiding integrations. Simple parametrizations of interesting isotropic curves and minimal surfaces can be constructed with its help. The functions $f(z) = e^{a\,z}$ and $h(z) = e^{b\,z}$ give for instance

$$
(6.2) \qquad \begin{cases}
\widetilde{\Phi}_1(z) &= \frac{i\,b}{2\,a}\left((a+b)\,e^{(-a+b)\,z} + (a-b)\,e^{(a+b)\,z}\right) \\
\widetilde{\Phi}_2(z) &= \frac{1}{2\,a\,e^{a\,z}}\big(-\left(a^2\,e^{b\,z}\right) - a\,b\,e^{b\,z} - a^2\,e^{2\,a\,z+b\,z} \\
&+ \; a\,b\,e^{2\,a\,z+b\,z} + a^2\,e^{a\,z+(-a+b)\,z} - b^2\,e^{a\,z+(-a+b)\,z} \\
&+ \; a^2\,e^{3\,a\,z+(-a+b)\,z} - b^2\,e^{3\,a\,z+(-a+b)\,z}\big) \\
\widetilde{\Phi}_3(z) &= \frac{-i\left(a^2-b^2\right)e^{b\,z}}{a}
\end{cases}
$$

More examples and plots can be found in [4]. We hope that variations and deformations can be effectively used to construct minimal surfaces with prescribed properties of shape.

A simple special case is that of $\mathbf{Var}_{f,1}$. Let an isotropic $\widetilde{\Phi}$ curve be given by (6.1) with $h(z) = 1$, that is, put $\Phi(z) = \mathbf{Wei}_f^*(z)$ and $h(z) = 1$ in (5.6) to obtain

$$(6.3) \qquad \mathbf{Var}_{f,1}(z) = \widetilde{\Phi}(z) = \Phi^{(3)}(z) - \left(\kappa_\Phi(z)\right)^2 \Phi'(z),$$

where $\kappa_\Phi(z)$ is the minimal curvature of Φ. Then the derivative $p'(z)$ of the natural parameter of $\widetilde{\Phi}$ is related to $\kappa_\Phi(z)$ and the minimal torsion $\tau_\Phi(z)$ of $\Phi(z)$ by

$$(6.4) \qquad \left(\frac{p'(z)}{\kappa_\Phi(z)}\right)^2 + \tau_\Phi(z) = 0.$$

The representation formula (6.1) has equivariance properties similar to that of $\mathbf{Wei}^*$ with respect to the group homomorphism μ. We have

Proposition 6.1 *For any fixed $h \in \mathbf{A}$ the mapping $f \in \mathbf{A} \longrightarrow \mathbf{Var}_{f,h} \in \mathcal{IC}^*$ is euivariant with respect to the group homomorphism $\mu : Sl(2,\mathbb{C}) \longrightarrow SO(3,\mathbb{C})$.*

Proof. Let $\widetilde{\Phi}(z) = \mathbf{Var}_{f,h}(z)$ and $\Phi(z) = \mathbf{Wei}_f^*(z)$. By proposition 5.1, equation (5.9), we have

$$(6.5) \qquad \widetilde{\Phi}'(z) = \left(p'(z)\right)^2 \Phi'(z),$$

where $p'(z)$ is the derivative of the natural parameter of $\widetilde{\Phi}$. By equation (5.8), $p'(z)$ is expressed in terms of $h(z)$ and the minimal curvature $\kappa_\Phi(z)$ of $\Phi(z)$. Since $\kappa_\Phi(z)$ is an invariant of $\Phi(z)$, changing f to $\mathbf{g}\cdot f$, where $\mathbf{g} \in \mathrm{Sl}(2,\mathbb{C})$, changes the right hand side of (6.5) to $\mu(\mathbf{g})\big(\left(p'(z)\right)^2 \Phi'(z)\big)$ (see proposition 3.1). Therefore, $\widetilde{\Phi}'$ is transformed into $\mu(\mathbf{g})\widetilde{\Phi}'$ and the same is true for $\widetilde{\Phi}$. ∎

The representation formula obtained by assigning to functions $(f,h) \in \mathbf{A}\times\mathbf{A}$ the deformation of $\Phi(z) = \mathbf{Wei}_f^*$ by h is equivariant with respect to μ in a similar way. Therefore, the results of section 3 are true for the variation and the deformation in a similar way. In the case of the variation, the minimal surfaces associated to the functions $f_{xy}(z)$, $f_{xz}(z)$, $f_{yz}(z)$ of (3.5) are periodic (not only almost periodic) with respect to the elementary rotations $\mathbf{d}^r_{xy}$, $\mathbf{d}^r_{xz}$, $\mathbf{d}^r_{yz}$.

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