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Symmetries and Modular Intersections of von-Neumann-Algebras

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Abstract

Let $\mathcal{N}, \mathcal{M}$ be von-Neumann-algebras acting on a Hilbert space $\mathcal{H}$ and let $\Omega \in \mathcal{H}$ be a common cyclic and separating vector. We say that $(\mathcal{N}, \mathcal{M}, \Omega)$ have the *modular intersection property* with respect to Ω, if

1. $((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{N}, \Omega)$, $((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{M}, \Omega)$ - half-sided modular inclusions,
2. $J_{\mathcal{N}}(s - \lim_{t \to \infty} \Delta_{\mathcal{N}}^{it} \Delta_{\mathcal{M}}^{-it}) J_{\mathcal{N}} = s - \lim_{t \to \infty} \Delta_{\mathcal{M}}^{it} \Delta_{\mathcal{N}}^{-it}$.

(If 1. holds the strong limit exists, see below.)

We show that under these conditions the modular groups of $\mathcal{N}$ and $\mathcal{M}$ generate a 2-dim. Lie group.

This observation is the basis for getting group representations of $Sl(2, R)/Z_2$ generated by modular groups.

1 Introduction

In local nets of observables coming from Wightman fields the famous Bisognano/Wichmann result states that the modular groups of the wedge algebras w.r.t. the vacuum vector act as Lorentz boosts, see [Bi/Wich]. Since

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the Lorentz boosts generate the whole Poincaré group, in these theories the modular groups of certain algebras generate a representation of a finite dimensional Lie group.

In [Wi 2] we formulated the converse question. Starting with a finite set of algebras together with a common cyclic and separating vector, under what conditions do the modular groups generate a representation of a finite dimensional Lie group ?

In this article we continue this program. Our main result is to provide a mechanism for constructing representations of relevant symmetry groups which are not restricted to have positive energy, or put differently, which do not fulfill the spectrum condition. This is important to construct for example a vacuum representation of the d+1-dim. Lorentz group, $d \geq 2$. In the later the generator of the rotations for example do not have positive spectrum in contrast to the situation in chiral theories, see [Wi 2]. Due to this result we are able to construct a local net of observables out of a finite set of algebras lying in a proper modular position relative to each other. We will show this in a second paper in the case of 1+2 dim. theories, where we need only 4 algebras, see [Wi3].

2 Modular Intersections of von-Neumann Algebras and Symmetries

Let us first recall the definition of half-sided modular inclusions. Assume $\tilde{\mathcal{M}} \subset \mathcal{M}$ to be von-Neumann-Algebras acting on a Hilbert space $\mathcal{H}$, and let $\Omega \in \mathcal{H}$ be a common cyclic and separating vector. Denote $\Delta_{\mathcal{M}}, \Delta_{\tilde{\mathcal{M}}}$ the modular operators associated with $(\mathcal{M}, \Omega)$ and $(\tilde{\mathcal{M}}, \Omega)$, respectively. Let $\Delta_{\mathcal{M}}^{-it} \tilde{\mathcal{M}} \Delta_{\mathcal{M}}^{it} \subset \tilde{\mathcal{M}}$ for all $t \geq 0$. We called such an inclusion $(\tilde{\mathcal{M}} \subset \mathcal{M}, \Omega)$ $-$*half-sided modular* , see [Wi1]. If one changes from $t \geq 0$ to $t \leq 0$, we call it *+half-sided modular* , abbreviated by $\mp$hsm.

For such a situation we proved in [Wi1] the following

Theorem 1 *Let $(\tilde{\mathcal{M}} \subset \mathcal{M}, \Omega)$ be a $\mp$ -half-sided modular inclusion,*

$$\Delta_{\mathcal{M}}^{it} \tilde{\mathcal{M}} \Delta_{\mathcal{M}}^{-it} \subset \tilde{\mathcal{M}} \quad \textit{for all } \mp t \geq 0.$$

Then

$$a) \ \frac{1}{2\pi}(ln(\Delta_{\tilde{\mathcal{M}}}) - ln(\Delta_{\mathcal{M}})) \geq 0$$

is essentially selfadjoint. Denote $U(a), a \in R$, the unitary group on $\mathcal{H}$ with generator $\frac{1}{2\pi}(ln(\Delta_{\tilde{\mathcal{M}}}) - ln(\Delta_{\mathcal{M}}))^{-}$. Then

$$
\begin{array}{ll}
b) & \Delta_{\mathcal{M}}^{it} U(a) \Delta_{\mathcal{M}}^{-it} = \Delta_{\tilde{\mathcal{M}}}^{it} U(a) \Delta_{\tilde{\mathcal{M}}}^{-it} = U(e^{\mp 2\pi t} a) \ \textit{for all}\ t, a \in R \\
c) & J_{\mathcal{M}} U(a) J_{\mathcal{M}} = J_{\tilde{\mathcal{M}}} U(a) J_{\tilde{\mathcal{M}}} = U(-a) \ \textit{for all}\ a \in R \\
d) & \Delta_{\tilde{\mathcal{M}}}^{it} \mathcal{M} \Delta_{\tilde{\mathcal{M}}}^{-it} \subset \mathcal{M} \quad \textit{for all} \quad \mp t \geq 0 \\
e) & \tilde{\mathcal{M}} = U(\pm 1) \mathcal{M} U(\mp 1).
\end{array}
$$

Proof :see [Wi 1]

Hence, using inclusions one gets a spectrum condition, reflecting this 'inclusion relation', namely property a) in the theorem above. In order to get the Lorentz group representation of higher dimensional theories we have to get more general representations, since $\ln(\Delta_{\Delta_{\tilde{\mathcal{M}}}}) - \ln(\Delta_{\mathcal{M}})$ will be interpreted as a generator of rotations. To make contact with examples in physics we further have to face the problem that the modular group of intersections need not to have a nice geometrical or physical interpretation. How to resolve both problems can be found in an example partially worked out by H.-J. Borchers, [Bo, Lemma 2.6], see also [Wi 3]. The main observation goes as follows.

Let $\mathcal{N}, \mathcal{M}$ be von-Neumann-algebras acting on a Hilbert space $\mathcal{H}$, $\Omega \in \mathcal{H}$ a common cyclic and separating vector. Assume Ω to be cyclic also for $\mathcal{N} \cap \mathcal{M}$ and

$I)$ $((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{N}, \Omega)$, $((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{M}, \Omega)$- half-sided modular inclusions.

We can apply Theorem 1 to both inclusions . Let us denote the associated two unitary groups by U_1, resp. U_2, i.e.

$$\Delta_{\mathcal{N}}^{it} = U_1(-1) \Delta_{\mathcal{N} \cap \mathcal{M}}^{it} U_1(1) = U_1(-1 + e^{-2\pi t}) \Delta_{\mathcal{N} \cap \mathcal{M}}^{it} \tag{1}$$

and similar for U_2, $\mathcal{M}$ instead of U_1, $\mathcal{N}$. It is easy to see that

$$U := U_1(-1) U_2(1) = s - \lim_{t \to \infty} \Delta_{\mathcal{N}}^{it} \Delta_{\mathcal{M}}^{-it}$$

exists. Now assume

$$II) \qquad J_{\mathcal{N}} U J_{\mathcal{N}} = U^*.$$

Then we get as in Theorem 1:

Theorem 2 *Let $\mathcal{N}$ and $\mathcal{M}$ fulfill I) and II) above. Then*

$$a)\ \frac{1}{2\pi}(ln(\Delta_{\mathcal{N}}) - ln(\Delta_{\mathcal{M}}))$$

is essentially selfadjoint. Denote $V(a), a \in R$, the unitary group on $\mathcal{H}$ with generator $\frac{1}{2\pi}(ln(\Delta_{\mathcal{N}}) - ln(\Delta_{\mathcal{M}}))^{-}$. Then

$$\begin{array}{ll} b) & \Delta_{\mathcal{M}}^{it} V(a) \Delta_{\mathcal{M}}^{-it} = \Delta_{\mathcal{N}}^{it} V(a) \Delta_{\mathcal{N}}^{-it} = V(e^{-2\pi t} a) \text{ for all } t, a \in R \\ c) & J_{\mathcal{M}} V(a) J_{\mathcal{M}} = J_{\mathcal{N}} V(a) J_{\mathcal{N}} = V(-a) \text{ for all } a \in R \\ d) & \mathcal{N} = V(1) \mathcal{M} V(-1). \end{array}$$

Proof : First from the very definition and Theorem 1 we get

$$\mathcal{N} = \mathrm{Ad}\ U(\mathcal{M})$$

and $U\Omega = \Omega$. This implies $J_{\mathcal{N}} U = U J_{\mathcal{M}}$. Assumption II) yields

$$J_{\mathcal{N}} J_{\mathcal{M}} = J_{\mathcal{N}} U^* J_{\mathcal{N}} U = U^2 = U_1(-1) U_2(1) U_1(-1) U_2(1),$$

and $J_{\mathcal{M}} U J_{\mathcal{M}} = U^*$, which shows the symmetric role of $\mathcal{N}$ and $\mathcal{M}$ in the assumptions. From Theorem 1 we further have

$$J_{\mathcal{N}} J_{\mathcal{M}} = J_{\mathcal{N}} J_{\mathcal{N} \cap \mathcal{M}} J_{\mathcal{N} \cap \mathcal{M}} J_{\mathcal{M}} = U_1(-2) U_2(2).$$

Putting together both equations we arrive at

$$U_1(-1) U_2(1) = U_2(1) U_1(-1).$$

Now we can scale this relation by applying $\mathrm{Ad}\Delta_{\mathcal{N}\cap\mathcal{M}}^{it}$ on both sides, see Theorem 1 b), and taking also the adjoint relation we get

$$U_1(a) U_2(-a) = U_2(-a) U_1(a) \quad \forall a \in \mathbf{R}.$$

We show that this implies $[U_1(a), U_2(b)] = 0$ for all $a, b \in \mathbf{R}$:

First notice that

$$[U_1(na), U_2(ma)] = 0 \quad \forall a \in \mathbf{R}\ n, m \in \mathbf{N},$$

as one sees by successively commuting $U_1(a)$ through $U_2(a)$. Multiplying this equation from the left and right with $U_1^*(na) = U_1(-na)$ and taking adjoints immediately shows

$$[U_1(na), U_2(ma)] = 0 \forall n, m \in \mathbf{Z},$$

i.e. with $b = ma, q = \frac{n}{m}$,

$$[U_1(qb), U_2(b)] = 0 \quad \forall\, q \in \mathbf{Q},\ b \in \mathbf{R}.$$

Continuity then implies that the two groups commute.

Therefore we can define the unitary group

$$V(-a) := U_1(a)U_2(-a), \quad a \in \mathbf{R}.$$

Relation (1) then implies

$$\begin{aligned} \mathrm{Ad}\ \Delta_{\mathcal{N}}^{it}(V(-a)) &= \mathrm{Ad}\ U_1(-1+e^{-2\pi t}) \circ \mathrm{Ad}\ \Delta_{\mathcal{N}\cap\mathcal{M}}^{it}(U_1(a)U_2(-a)) \\ &= \mathrm{Ad}\ U_1(-1+e^{2\pi t})(U_1(e^{-2\pi t}a)U_2(-e^{-2\pi t}a)) \\ &= \mathrm{Ad}\ U_1(e^{-2\pi t}a)(U_2(-e^{-2\pi t}a)) = V(-e^{-2\pi t}a). \end{aligned}$$

Now

$$\begin{aligned} \Delta_{\mathcal{M}}^{it} = U^*\Delta_{\mathcal{N}}^{it}U &= V(-1)\Delta_{\mathcal{N}}^{it}V(1) \\ &= V(-1+e^{-2\pi t})\Delta_{\mathcal{N}}^{it}. \end{aligned} \tag{2}$$

This relation immediately proves a), b) and d) of the theorem. Assumption II) now gives

$$J_{\mathcal{N}}V(1)J_{\mathcal{N}} = V(-1)$$

which by scaling with $\mathrm{Ad}\Delta_{\mathcal{N}}^{it}$ leads to c). □

This theorem suggests to give special names to such pair of algebras:

Definition 3 *Let $\mathcal{N}, \mathcal{M}$ be von-Neumann-algebras acting on a Hilbert space $\mathcal{H}$, and let $\Omega \in \mathcal{H}$ be a common cyclic and separating vector which is also cyclic for $\mathcal{N} \cap \mathcal{M}$. We say that $(\mathcal{N}, \mathcal{M}, \Omega)$ have the* $\pm$ modular intersection property *(abbreviated by "$\pm$ mis"), if*

I) $((\mathcal{N}\cap\mathcal{M}) \subset \mathcal{N}, \Omega)$, $((\mathcal{N}\cap\mathcal{M}) \subset \mathcal{M}, \Omega)$ *are $\pm$ half-sided modular inclusions.*

II) $$J_{\mathcal{N}}(s - \lim_{t\to\mp\infty} \Delta_{\mathcal{N}}^{it}\Delta_{\mathcal{M}}^{-it})J_{\mathcal{N}} = s - \lim_{t\to\mp\infty} \Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{N}}^{-it}.$$

It is easy to see that Theorem 2 is also true for pairs with +-mis, where one has to change the last equation in b) by

$$b') \;\; \Delta_{\mathcal{M}}^{it} V(a) \Delta_{\mathcal{M}}^{-it} = \Delta_{\mathcal{N}}^{it} V(a) \Delta_{\mathcal{N}}^{-it} = V(e^{2\pi t} a) \text{ for all } t, a \in R.$$

Let us look in more detail at this structure.

Lemma 4 *Let $(\mathcal{N}, \mathcal{M}, \Omega)$ be $\pm$-mis, and let $U(a), a \in \mathbf{R}$, be the associated unitary group. Then $(Ad\ U(a)(\mathcal{N}), Ad\ U(b)(\mathcal{N}), \Omega)$ is $\pm$-mis for $a \neq b$.*

Proof : Let us prove this Lemma for the "$-$"case . The assumption 1 states

$$\begin{array}{ll} i) & \text{Ad } \Delta_{\mathcal{N}}^{it}(\mathcal{N} \cap U(-1)\mathcal{N}U(1)) \subset (\mathcal{N} \cap U(-1)\mathcal{N}U(1)) \;\; \text{for } t \leq 0, \\ ii) & \text{Ad } \Delta_{\mathcal{M}}^{it}(\mathcal{N} \cap U(-1)\mathcal{N}U(1)) \subset (\mathcal{N} \cap U(-1)\mathcal{N}U(1)) \;\; \text{for } t \leq 0. \end{array}$$

Applying Ad $\Delta_{\mathcal{N}}^{it_0}, t_0 \in \mathbf{R}$, to the first inclusion and , Ad $\Delta_{\mathcal{M}}^{it_0}, t_0 \in \mathbf{R}$, to the second inclusion , we get the inclusions for all $t \leq t_0$. We use Theorem 2 to rewrite $\Delta_{\mathcal{M}}^{it} = U(e^{-2\pi t} - 1)\Delta_{\mathcal{N}}^{it}$ and apply the commutation rules b) to get

$$\begin{array}{rlll} i') & \mathcal{N} \cap \text{Ad } U(-a)(\mathcal{N}) & \subset & \mathcal{N} \cap \text{Ad } U(-b)(\mathcal{N}) \;\; \text{for } a \geq b \geq 0 \\ ii') & \text{Ad } U(c)(\mathcal{N}) \cap \text{Ad } U(-1)(\mathcal{N}) & \subset & \text{Ad } U(d)(\mathcal{N}) \cap \text{Ad } U(-1)(\mathcal{N}) \;\; \text{for } c \geq d \geq -1. \end{array}$$

Applying Ad $U(a), a \in \mathbf{R}$ to these inclusions gives

$$\begin{array}{llll} i') & \text{Ad } U(a)(\mathcal{N}) \cap \text{Ad } U(b)(\mathcal{N}) & \subset & \text{Ad } U(a)(\mathcal{N}) \cap \text{Ad } U(c)(\mathcal{N}) \text{ for } a \geq c \geq b \\ ii') & \text{Ad } U(a)(\mathcal{N}) \cap \text{Ad } U(b)(\mathcal{N}) & \subset & \text{Ad } U(c)(\mathcal{N}) \cap \text{Ad } U(b)(\mathcal{N}) \text{ for } a \geq c \geq b. \end{array}$$

But these inclusions immediately imply

$$\begin{array}{ll} i'') & (\text{Ad } (U(a)(\mathcal{N}) \cap \text{Ad } U(b)(\mathcal{N}) \subset \text{Ad } U(a)(\mathcal{N}), \Omega) \quad \text{- hsm} \\ ii'') & (\text{Ad } (U(a)(\mathcal{N}) \cap \text{Ad } U(b)(\mathcal{N}) \subset \text{Ad } U(b)(\mathcal{N}), \Omega) \quad \text{- hsm} \end{array}$$

for $a > b$. Condition II is obviously fulfilled. □

We get as an important

Corollary 5 *$(\mathcal{N}, \mathcal{M}, \Omega)$ is $\pm$ mis iff $(\mathcal{N}', \mathcal{M}', \Omega)$ is $\mp$ mis.*

Proof : Assume $(\mathcal{N},\mathcal{M},\Omega)$ is $-$ mis. We will show $(\mathcal{N}',\mathcal{M}',\Omega)$ is $+$ mis. Because condition II is obvious for the later pair, we have to show

$$\begin{aligned} i) &\quad ((\mathcal{N}'\cap\mathcal{M}')\subset\mathcal{N}',\Omega) \text{ is + hsm,}\\ ii) &\quad ((\mathcal{N}'\cap\mathcal{M}')\subset\mathcal{M}',\Omega) \text{ is + hsm.}\end{aligned}$$

Applying Ad $J_{\mathcal{N}}$ this amounts to proving

$$\begin{aligned} i') &\quad ((\mathcal{N}\cap \text{Ad } J_{\mathcal{N}}J_{\mathcal{M}}(\mathcal{M}))\subset\mathcal{N},\Omega) \text{ is - hsm,}\\ ii) &\quad ((\mathcal{N}\cap \text{Ad } J_{\mathcal{N}}J_{\mathcal{M}}(\mathcal{M}))\subset \text{Ad } J_{\mathcal{N}}J_{\mathcal{M}}(\mathcal{M}),\Omega) \text{ is - hsm.}\end{aligned}$$

Now, by Theorem 2, we have $J_{\mathcal{N}}J_{\mathcal{M}}=U(2)$ and Ad $J_{\mathcal{N}}J_{\mathcal{M}}(\mathcal{M})=$ Ad $U(1)(\mathcal{N})$. The statement is therefore a simple corollary to Lemma 4. □

Using the results on half-sided modular inclusions obtained in [Wi1], one has

$$((\mathcal{N}'\cap\mathcal{M}')\subset\mathcal{N}',\Omega) + \text{hsm} \Longleftrightarrow (\mathcal{N}\subset(\mathcal{N}\cup\mathcal{M}),\Omega) \text{ - hsm,}$$

and analogously for $\mathcal{M}$ instead of $\mathcal{N}$. Therefore we equivalently could start from 'modular unions' instead of 'modular intersections'.

Similar to [Wi2], we can look at couples of such pairs:

Theorem 6 *Let $\mathcal{N},\mathcal{M},\mathcal{L}$ be von-Neumann algebras acting on a Hilbert space $\mathcal{H}$, and let $\Omega\in\mathcal{H}$ be a common cyclic and separating vector. Assume*

1. $(\mathcal{N},\mathcal{M},\Omega)$ *is - mis ,*
2. $(\mathcal{L},\mathcal{M},\Omega)$ *is + mis ,*
3. $(\mathcal{L},\mathcal{N}',\Omega)$ *is - mis .*

where the prime indicates the commutant. Then

$$\Delta_{\mathcal{M}}^{it},\Delta_{\mathcal{N}}^{ir},\Delta_{\mathcal{L}}^{is}, t,r,s\in\mathbf{R}$$

generate a representation of the Möbius group $Sl(2,R)/Z_2$.

Proof : The first part of the proof is an easy transcription of the proof of [Wi 2, Lemma 3]. Applying Theorem 2 to the pair $(\mathcal{N}', \mathcal{L}, \Omega)$ we get

$$\begin{aligned} \Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{N}}^{-i\tilde{t}} &= \Delta_{\mathcal{L}}^{it}U_{\mathcal{N}'-\mathcal{L}}(1)\Delta_{\mathcal{L}}^{i\tilde{t}}U_{\mathcal{N}'-\mathcal{L}}(-1) \\ &= \Delta_{\mathcal{L}}^{i(t+\tilde{t})}U_{\mathcal{N}'-\mathcal{L}}(e^{2\pi\tilde{t}}-1) \\ &= U_{\mathcal{N}'-\mathcal{L}}(e^{-2\pi t}-e^{-2\pi(t+\tilde{t})})\Delta_{\mathcal{L}}^{i(t+\tilde{t})} \end{aligned}$$

for $t, \tilde{t} \in R$, i.e. $\Delta_{\mathcal{L}}^{it}$ and $\Delta_{\mathcal{N}}^{i\tilde{t}}$ span a two dimensional unitary group, where $U_{\mathcal{N}'-\mathcal{L}}(a) = \exp(ia(\frac{1}{2\pi}(-\ln(\Delta_{\mathcal{N}}) - \ln(\Delta_{\mathcal{L}}))), a \in R$. Using the other intersection pairs, we get similarly ,

$$\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{i\tilde{t}} = \Delta_{\mathcal{L}}^{i(t+\tilde{t})}U_{\mathcal{L}-\mathcal{M}}(1-e^{-2\pi\tilde{t}})$$

and

$$\Delta_{\mathcal{N}}^{it}\Delta_{\mathcal{M}}^{i\tilde{t}} = \Delta_{\mathcal{N}}^{i(t+\tilde{t})}U_{\mathcal{N}-\mathcal{M}}(1-e^{2\pi\tilde{t}})$$

for $t, \tilde{t} \in R$ with

$$U_{\mathcal{L}-\mathcal{M}}(a) = \exp(ia(\frac{1}{2\pi}(\ln(\Delta_{\mathcal{L}}) - \ln(\Delta_{\mathcal{M}}))),$$

and

$$U_{\mathcal{N}-\mathcal{M}}(a) = \exp(ia(\frac{1}{2\pi}(\ln(\Delta_{\mathcal{N}}) - \ln(\Delta_{\mathcal{M}}))),$$

$a \in R$. This implies

$$\begin{aligned} U_{\mathcal{N}-\mathcal{M}}(e^{-2\pi t}-1) &= \Delta_{\mathcal{N}}^{it}\Delta_{\mathcal{M}}^{-it} \\ &= \Delta_{\mathcal{N}}^{it}\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{L}}^{-2it}\Delta_{\mathcal{L}}^{it}\Delta_{\mathcal{M}}^{-it} \\ &= U_{\mathcal{N}'-\mathcal{L}}(1-e^{-2\pi t})\Delta_{\mathcal{L}}^{-2it}U_{\mathcal{L}-\mathcal{M}}(1-e^{2\pi t}). \end{aligned}$$

These relations prove that $\Delta_{\mathcal{M}}^{it}, \Delta_{\mathcal{N}}^{ir}$, and $\Delta_{\mathcal{L}}^{is}$ with $t, r, s \in \mathbf{R}$ generate a 3-dimensional unitary group G. Using the commutation relations we can calculate the commutator relations of the generators of the associated Lie algebra

$$[\ln\Delta_{\mathcal{L}}, \frac{1}{2\pi}(-\ln\Delta_{\mathcal{N}} - \ln\Delta_{\mathcal{L}})] = i\frac{1}{2\pi}(-\ln\Delta_{\mathcal{N}} - \ln\Delta_{\mathcal{L}})$$

$$[\ln\Delta_{\mathcal{L}}, \frac{1}{2\pi}(\ln\Delta_{\mathcal{M}} - \ln\Delta_{\mathcal{L}})] = -i\frac{1}{2\pi}(\ln\Delta_{\mathcal{M}} - \ln\Delta_{\mathcal{L}})$$

$$[\ln \Delta_{\mathcal{N}}, \frac{1}{2\pi}(-\ln \Delta_{\mathcal{N}} + \ln \Delta_{\mathcal{M}})] = i\frac{1}{2\pi}(-\ln \Delta_{\mathcal{N}} + \ln \Delta_{\mathcal{M}}).$$

Hence the generators

$$\begin{aligned}
P &:= \frac{1}{2\pi}(\ln(\Delta_{\mathcal{N}'}) - \ln(\Delta_{\mathcal{L}})) \\
K &:= \frac{1}{2\pi}(\ln(\Delta_{\mathcal{M}}) - \ln(\Delta_{\mathcal{L}})) \\
D &:= \frac{1}{2\pi}\ln(\Delta_{\mathcal{L}})
\end{aligned}$$

yield a representation of $sl(2,R)$ and therefore G is identified to be a representation of the universal covering $\widetilde{Sl(2,R)}$.

Let us denote this representation by $\mathcal{V}$. In order to get a representation of the 2+1 dim. Lorentz group $SO^{\uparrow}(2,1)$, i.e. the Möbius group $Sl(2,R)/Z_2$, we have to look at the representation of the rotation. Let us first show

Lemma 7

$$U_{\mathcal{N}-\mathcal{M}}(-1)J_{\mathcal{N}}J_{\mathcal{L}} = U_{\mathcal{L}-\mathcal{M}}(1)U_{\mathcal{N}-\mathcal{M}}(-1)U_{\mathcal{N}'-\mathcal{L}}(1)$$

Proof : First notice that by using $U_{\mathcal{N}-\mathcal{M}}(1) = \Delta_{\mathcal{M}}^{-it_0}\Delta_{\mathcal{N}}^{it_0}$ with $t_0 = \frac{ln\ 2}{2\pi}$

and similar formulas for $U_{\mathcal{N}'-\mathcal{L}}(1)$ etc., which follow easily from Theorem 2 d) and b), we get

$$\begin{aligned}
U_{\mathcal{L}-\mathcal{M}}(1)U_{\mathcal{N}-\mathcal{M}}(-1)U_{\mathcal{N}'-\mathcal{L}}(1) &= U_{\mathcal{L}-\mathcal{M}}(1)\Delta_{\mathcal{N}}^{-it_0}\Delta_{\mathcal{M}}^{it_0}U_{\mathcal{N}'-\mathcal{L}}(1) \\
&= U_{\mathcal{L}-\mathcal{M}}(1)\Delta_{\mathcal{L}}^{it_0}\Delta_{\mathcal{L}}^{-it_0}\Delta_{\mathcal{N}}^{-it_0}\Delta_{\mathcal{M}}^{it_0}\Delta_{\mathcal{L}}^{-it_0}\Delta_{\mathcal{L}}^{it_0}U_{\mathcal{N}'-\mathcal{L}}(1) \\
&= U_{\mathcal{L}-\mathcal{M}}(1)\Delta_{\mathcal{L}}^{it_0}U_{\mathcal{N}'-\mathcal{L}}(1)U_{\mathcal{L}-\mathcal{M}}(1)\Delta_{\mathcal{L}}^{it_0}U_{\mathcal{N}'-\mathcal{L}}(1).
\end{aligned}$$

Now we show the above statement by calculating both elements in a faithful representation of $\widetilde{Sl(2,\mathbf{R})}$. We use the faithful representation of $\widetilde{Sl(2,R)}$ on functions f of a single real variable x according to

$$\begin{aligned}
(\pi_\alpha(U_D(t))(f))(x) &= t^{-\alpha}f(t^{-1}x)) && (3)\\
(\pi_\alpha(U_P(t))(f))(x) &= f(x+t)) && (4)\\
(\pi_\alpha(U_K(t))(f))(x) &= |tx+1|^{-2\alpha}e^{2i\pi\text{sign }(t)\theta(-tx-1)\alpha}f(\frac{x}{tx+1})) && (5)
\end{aligned}$$

where U_D, U_P, U_K are the 1-parameter subgroups in $Sl(\tilde{2},R)$ generated by the Lie algebra elements represented by D, P and K, $\alpha \in R$ is a generic parameter and θ is the Heavyside function. This faithful representation can be found for example in [Schr-Sw]. A short calculation shows

$$\pi_\alpha(U_{\mathcal{L}-\mathcal{M}}(1)U_{\mathcal{N}-\mathcal{M}}(-1)U_{\mathcal{N}'-\mathcal{L}}(1))(f) = (x \mapsto |x|^{-2\alpha}e^{2i\pi\theta(-x)\alpha}f(-\frac{1}{x}))$$

Similarly one computes

$$\pi_\alpha(U_{\mathcal{N}-\mathcal{M}}(-1)J_{\mathcal{N}}J_{\mathcal{L}}) = (x \mapsto |x|^{-2\alpha}e^{2i\pi\theta(-x)\alpha}f(-\frac{1}{x}))$$

from which the statement follows. $\square$

The explicit computation in a faitful representation of $\widetilde{Sl(2,R)}/Z_2$ even shows that both elements represent the rotation by $\frac{\pi}{2}$,

$$\mathcal{V}(rot(\frac{\pi}{2})) = U_{\mathcal{N}-\mathcal{M}}(-1)J_{\mathcal{N}}J_{\mathcal{L}}.^1$$

From Theorem 2 we immediately have

$$\text{Ad } U_{\mathcal{L}-\mathcal{M}}(1)U_{\mathcal{N}-\mathcal{M}}(-1)U_{\mathcal{N}'-\mathcal{L}}(1)(\mathcal{L}) = \mathcal{L}'$$

and we arrive at the conclusion

$$\text{Ad } U_{\mathcal{N}-\mathcal{M}}(-1)J_{\mathcal{N}}J_{\mathcal{L}}(\mathcal{L}) = \mathcal{L}. \tag{6}$$

Now $U_{\mathcal{N}-\mathcal{M}}(-1)J_{\mathcal{N}}\Omega = \Omega$ and modular theory tells us

$$[U_{\mathcal{N}-\mathcal{M}}(-1)J_{\mathcal{N}}, J_{\mathcal{L}}] = 0.$$

With

$$U_{\mathcal{N}-\mathcal{M}}(-1)J_{\mathcal{N}} = \Delta_{\mathcal{N}}^{it_0}U_{\mathcal{N}-\mathcal{M}}(-2)J_{\mathcal{N}}\Delta_{\mathcal{N}}^{-it_0} = \Delta_{\mathcal{N}}^{it_0}J_{\mathcal{M}}\Delta_{\mathcal{N}}^{-it_0}$$

we get therefore

$$(\mathcal{V}(\text{rot } (\frac{\pi}{2}))^2 = 1,$$

i.e. the modular representation reduces to a unitary representation of $Sl(2,R)/Z_2$. $\square$

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[1]There is a misprint in [Wi 2], namely, the calculated element in [Wi 2,Lemma 4] is of course the rotation in $\widetilde{Sl(2,R)}/Z_2$ by $\frac{\pi}{2}$.

The last result shows that the Lorentz group $SO^{\uparrow}(2,1) = Sl(2,R)/Z_2$ occurs quite naturally in the context of modular intersections. That they occur also naturally in 2+1-dim. and higher-dim. Quantum Field Theories was the main observation of H.-J. Borchers in [Bo]. We will show in [Wi 3], how these techniques enable one to construct out of a set of four algebras lying in a proper modular relative position to each other a whole quantum field theory in 2+1 dimensions.

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