Modular Intersections of von Neumann Algebras in Quantum Field Theory

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(revised version)

Abstract

We show that modular intersections of von Neumann algebras occur naturally in quantum field theory. An example are local observable algebras associated with wedge regions, which have a lightray in common, see also [Bo 2, Wi 3].
Conversely, starting from a set of four algebras lying in a specified modular position relative to each other we construct a net of local observables of a 2+1 dimensional quantum field theory.

1 Preliminaries

In this article we investigate the intimate relationship between modular theory and quantum field theory. As a fundamental feature of quantum theories we have the Reeh-Schlieder property of the vacuum. Intuitively speaking, this property reflects the vacuum fluctuation of local quantum field theories. Mathematically this feature enables one to apply the modular theory of operator algebras as developed by M. Tomita and M. Takesaki [Ta] to quantum

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field theory. It has been realized by J. Bisognano and E. Wichmann [Bi-Wich] that this rather general structure gets a very beautiful interpretation in terms of space time symmetries of the underlying quantum field theory. We will subsequently review some of these results in the second part of this work.

In '92 H.-J. Borchers [Bo1] gave an abstract version of the Bisognano/Wichmann result which led to the introduction of a special type of inclusion of algebras, the so-called modular inclusions [Wi1, Ar-Zs]. It turned out that inclusions of this type carry a strikingly rich symmetry structure, hidden in their modular theory [Wi2]. They led to a new characterization of chiral quantum field theories by one special pair of algebras lying in an appropriate relative position to each other w.r.t. their modular data [Wi4, Gu-Lo-Wi].

In order to generalize these results to higher dimensional theories we introduced the notion of modular intersections of algebras in [Wi3]. Within this setting one recovers almost all features of modular inclusions except for the spectrum condition, which reflected the inclusion order in the former case.

In Section 3 we will show how one can use these structures to construct a (2+1)-dimensional quantum field theory starting from a set of four algebras lying in an appropriate relative position to each other w.r.t. their modular data.

As mentioned already our results are mainly based on two structures, namely modular inclusions and intersections of algebras. Therefore we first recall the basic notions of half-sided modular inclusions and modular intersections and review the important results needed in the following. We often will have several of von Neumann algebras $\mathcal{M}, \mathcal{N}, \ldots$ together with a common cyclic and separating vector Ω. As a notational convention we will index the associated modular objects by the related algebra, e.g. $\Delta_{\mathcal{M}}$ resp. $J_{\mathcal{M}}$ will denote the modular operator resp. conjugation associated with $(\mathcal{M}, \Omega)$ etc..

Definition 1 *a) Let $\mathcal{N} \subset \mathcal{M}$ be von Neumann algebras acting on a Hilbert space $\mathcal{H}$. Let Ω be a common cyclic and separating vector. Denote by $\sigma^t_{\mathcal{M}}$ the modular group associated with $(\mathcal{M}, \Omega)$. If $\sigma^t_{\mathcal{M}}(\mathcal{N}) \subset \mathcal{N}$ for all $t \geq 0$ (or $t \leq 0$), we call $(\mathcal{N} \subset \mathcal{M}, \Omega)$ a* $\pm$-half-sided modular inclusion *, respectively, abbreviated by $\pm$-hsm inclusion.*

b) Let $\mathcal{N}, \mathcal{M}$ be von Neumann algebras acting on a Hilbert space $\mathcal{H}$ and let $\Omega \in \mathcal{H}$ be a common cyclic and separating vector which is also cyclic for $\mathcal{N} \cap \mathcal{M}$. If

$$I) \qquad ((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{N}, \Omega), \quad ((\mathcal{N} \cap \mathcal{M}) \subset \mathcal{M}, \Omega) \text{ are } \pm\text{-hsm inclusions,}$$

$$II) \qquad J_{\mathcal{N}}(\mathrm{s} - \lim_{t \to \mp\infty} \Delta_{\mathcal{N}}^{it} \Delta_{\mathcal{M}}^{-it}) J_{\mathcal{N}} = \mathrm{s} - \lim_{t \to \mp\infty} \Delta_{\mathcal{M}}^{it} \Delta_{\mathcal{N}}^{-it}$$

then we say that such a triple $(\mathcal{N}, \mathcal{M}, \Omega)$ has the $(\pm)$ modular intersection property*, abbreviated by $\pm$ mi property.*

Note that condition I implies that the strong limit in II exists. The main theorem for such pairs of algebras is [1]

Theorem 2 (Wi 1, Ar-Zs, Wi 3) *Let $\mathcal{N}$ and $\mathcal{M}$ fulfill a) or b) of the above definition with $\pm$ -sign. Then*

a) $\qquad \frac{1}{2\pi}(ln(\Delta_{\mathcal{N}}) - ln(\Delta_{\mathcal{M}}))$

is essentially selfadjoint, (in the case of modular inclusions even positive). Denote by $U(a), a \in \mathbf{R}$, the unitary group on $\mathcal{H}$ with generator $\frac{1}{2\pi}(ln(\Delta_{\mathcal{N}}) - ln(\Delta_{\mathcal{M}}))^{-}$. Then

b) $\quad \Delta_{\mathcal{M}}^{it} U(a) \Delta_{\mathcal{M}}^{-it} = \Delta_{\mathcal{N}}^{it} U(a) \Delta_{\mathcal{N}}^{-it} = U(e^{\pm 2\pi t} a)$ *for all* $t, a \in \mathbf{R}$

c) $\quad J_{\mathcal{M}} U(a) J_{\mathcal{M}} = J_{\mathcal{N}} U(a) J_{\mathcal{N}} = U(-a)$ *for all* $a \in \mathbf{R}$

d) $\quad \mathcal{N} = U(1) \mathcal{M} U(-1)$, *especially* $J_{\mathcal{N}} = U(1) J_{\mathcal{M}} U(-1)$.

From c) and d) we immediately conclude $U(2) = J_{\mathcal{N}} J_{\mathcal{M}}$ and with b) $\Delta_{\mathcal{M}}^{it} \Delta_{\mathcal{N}}^{-it} = U(1 - e^{\pm 2\pi t})$.

Let us introduce a convenient notation for the various unitary groups arising in Theorem 2. In the case of a $\pm$-hsm inclusion $(\mathcal{N} \subset \mathcal{M}, \Omega)$, we denote the associated unitary group by $U_{\mathcal{N} \subset_{\pm} \mathcal{M}}$, in the case of modular intersections by $U_{\mathcal{N} \cap \pm \mathcal{M}}$. Using this notation statement d) of Theorem 2 reads as $\mathcal{N} = \mathrm{Ad}\ U_{\mathcal{N} \subset_{\pm} \mathcal{M}}(1)(\mathcal{M})$ etc..
For the construction of a quantum field theory from a finite set of algebras we will use space-time symmetry representations generated by modular operators. In 2+1 dimensions the homogeneous Lorentz group $SO^{\uparrow}(1,2)$ is isomorphic to $SL(2,R)/Z_2$ and we will apply

[1] There was an error in the original proof of statement a) for the case of hsm-inclusions by [Wi 1], as was noticed by H. Araki and L. Zsido. They provided also a way of filling the gap [Ar-Zs].

Theorem 3 (Wi 3, Theorem 6) *Let $\mathcal{N}, \mathcal{M}, \mathcal{L}$ be von Neumann algebras acting on a Hilbert space $\mathcal{H}$ and let $\Omega \in \mathcal{H}$ be a common cyclic and separating vector. Assume the following modular intersection properties*

$$\begin{aligned} 1.\quad (\mathcal{N}, \mathcal{M}, \Omega) &\quad has\ -mi\ , \\ 2.\quad (\mathcal{L}, \mathcal{M}, \Omega) &\quad has\ +mi\ , \\ 3.\quad (\mathcal{L}, \mathcal{N}', \Omega) &\quad has\ -mi\ , \end{aligned}$$

where the prime indicates the commutant. Then for $t, r, s \in \mathbf{R}$ the one-parameter groups

$$\Delta^{it}_{\mathcal{M}}, \Delta^{ir}_{\mathcal{N}}, \Delta^{is}_{\mathcal{L}}$$

generate a representation of the group $SL(2,R)/Z_2 \cong SO^{\uparrow}(1,2)$.

In the proof of Proposition 11 below we will give a detailed description of that representation. We remark that Theorem 3 holds equally well if we replace the mi-properties by hsm-inclusions [Wi 2, Theorem 5]. We need a further relation which was obtained in the proof of this theorem,

Lemma 4 (Wi 3, Lemma 7)

$$U_{\mathcal{M}\cap-\mathcal{N}}(1) J_{\mathcal{N}} J_{\mathcal{L}} = U_{\mathcal{L}\cap+\mathcal{M}}(1) U_{\mathcal{M}\cap-\mathcal{N}}(1) U_{\mathcal{N}'\cap-\mathcal{L}}(1)$$

$$Ad\, U_{\mathcal{M}\cap-\mathcal{N}}(1) J_{\mathcal{N}}(\mathcal{L}) = \mathcal{L}$$

and by modular theory

$$[U_{\mathcal{M}\cap-\mathcal{N}}(1) J_{\mathcal{N}}, J_{\mathcal{L}}] = 0.$$

The same statement holds if we replace $\mathcal{N}$ by $\mathcal{L}$, $\mathcal{M}$ by $\mathcal{N}'$ and $\mathcal{L}$ by $\mathcal{M}'$, respectively $\mathcal{N}$ by $\mathcal{M}'$, $\mathcal{M}$ by $\mathcal{L}'$ and $\mathcal{L}$ by $\mathcal{N}$.

After these preparations let us show that these abstract structures occur naturally in the context of 2+1-dimensional quantum field theories.

2 Modular Intersections of von Neumann Algebras in 2+1 dimensional Quantum Field Theory

Our framework will be the description of quantum field theories in terms of nets of local algebras, see [Haa]. Instead of working with unbounded field operators one uses algebras of bounded operators which might be viewed as bounded functions of the observables. Let us recall some basic assumptions on the set of algebras, which replace the familiar assumptions on the observable fields.

Let $\mathcal{A}(O) \subset \mathcal{B}(\mathcal{H})$ be a net of von Neumann Algebras indexed by the closed double cones $O \subset \mathbf{R}^{1,2}$ which satisfies the following properties:

$$(i) \quad \mathcal{A}(O_1) \subset \mathcal{A}(O_2) \ \text{ if } O_1 \subset O_2 \quad (\text{ isotony })$$

$$(ii) \quad \mathcal{A}(O_1) \subset \mathcal{A}(O_2)' \ \text{ if } O_1 \subset O_2' \quad (\text{ locality })$$

where $\mathcal{A}(O)'$ denotes the commutant of $\mathcal{A}(O)$ in $\mathcal{B}(\mathcal{H})$ and O' is the causal complement of $O \subset \mathbf{R}^{1,2}$.

$$(iii) \quad \exists\, \mathcal{U} : SO^{\uparrow}(2,1) \triangleright \mathbf{R}^{1,2} \to \mathcal{U}(\mathcal{H}) \quad (\text{ Poincaré covariance })$$

is a unitary representation of the Poincaré group with positive energy, and

$$(iv) \quad \exists \text{ a unique } \mathcal{U}\text{-invariant vector } \Omega \in \mathcal{H} \quad (\text{vacuum}).$$

The algebra $\mathcal{A}(O)$ is called the algebra of observables localized in $O \subset \mathbf{R}^{1,2}$. Such a net describes a (2+1)-dimensional quantum field theory in the algebraic approach.

The famous Reeh-Schlieder property of the vacuum Ω states that Ω is cyclic and separating for a local algebra $\mathcal{A}(O)$, whenever the causal complement of the localization region O contains an open subset. Hence, we can apply the modular theory of Tomita-Takesaki to $(\mathcal{A}(O), \Omega)$. For some special regions, the so-called wedges, these abstract algebraic structures have nice physical and geometrical interpretations. A wedge $W[l_1, l_2]$ is determined by two linearly independent lightlike vectors $l_1, l_2 \in \mathbf{R}^{1,2}$ belonging to the forward lightcone. Using the convenient notation introduced in [Bo2] it is defined by

$$W[l_1, l_2] := \{\alpha l_1 + \beta l_2 + l^{\perp} \in \mathbf{R}^{1,2} | \alpha > 0, \beta < 0, (l^{\perp}, l_i) = 0, i = 1, 2\}$$

where $(\cdot,\cdot)$ denotes the Minkowski-product.

If the local net is generated by Wightman fields J. Bisognano and E. Wichmann [Bi-Wich] showed that the Tomita-Takesaki modular groups associated with algebras of observables localized in such wedges act as Lorentz boosts. More specifically one gets the one-parameter group of the Lorentz boosts $\Lambda_{l_1,l_2}(t), t \in \mathbf{R}$ leaving the set $W[l_1, l_2]$ invariant. For example, if $l_1 = (1,1,0)$ and $l_2 = (1,-1,0)$ then

$$\Lambda_{l_1,l_2}(t) = \begin{pmatrix} \cosh\ 2\pi t & \sinh\ 2\pi t & 0 \\ \sinh\ 2\pi t & \cosh\ 2\pi t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The precise relation is given by

$$\Delta^{it}_{\mathcal{A}(W[l_1,l_2])} = \mathcal{U}(\Lambda_{l_1,l_2}(-2\pi t)),$$

where $\Delta_{\mathcal{A}(W[l_1,l_2])}$ denotes the modular operator associated with $(\mathcal{A}(W[l_1,l_2]),\Omega)$.

In [Bo2] H.-J. Borchers looked at various relations between the Lorentz boosts in the directions of different wedges. In his investigation he made the following observation.

Lemma 5 *For all l_1, l_2 and l_3 as above*

$$a) \qquad W[l_1,l_2] \cap W[l_1,l_3] \quad \textit{contains an open neighborhood,}$$

$$b) \quad \Lambda_{l_1,l_2}(t)(W[l_1,l_2] \cap W[l_1,l_3]) \subset (W[l_1,l_2] \cap W[l_1,l_3]) \quad \textit{for } t \geq 0$$

and similar if we replace Λ_{l_1,l_2} by Λ_{l_1,l_3}.

In the case where the Bisognano-Wichmann result holds, also the modular conjugations of the wedges act geometrically, namely as reflections. More precisely, the defining lightlike vectors are reflected whereas the orthogonal spacelike line is kept fixed. Denote this reflection by $\hat{J}_{l_1,l_2}$. Then an exercise shows

Lemma 6

$$\hat{\mathcal{U}} := \lim_{t\to\infty} \Lambda_{l_1,l_2}(t)\Lambda_{l_1,l_3}(-t)$$

exists and

$$\hat{J}_{l_1,l_2}\hat{\mathcal{U}}\hat{J}_{l_1,l_2} = \hat{\mathcal{U}}^{-1} \equiv \lim_{t\to\infty} \Lambda_{l_1,l_3}(t)\Lambda_{l_1,l_2}(-t).$$

Proof : These statements are Lorentz-invariant. Let us take $l_1 = (1,1,0), l_2 = (1,-1,0), l_3 = (2,0,2)$. With this choice the computations in [Bo 2, (2.5) and Corollary 4.3] show

$$\hat{\mathcal{U}} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} & -1 \\ \frac{1}{2} & \frac{1}{2} & -1 \\ -1 & 1 & 1 \end{pmatrix}.$$

Since

$$\hat{J}_{l_1,l_2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

the second statement reduces to a simple matrix calculation. □

Putting these observations together we obtain the following important result

Proposition 7 *Let $\mathcal{A}(O), O \subset \mathbf{R}^{1,2}$, be the local observable algebras of a 2+1 dim. quantum field theory for which the Bisognano/Wichmann property holds. Then*

$$(\mathcal{A}(W[l_1,l_2]), \mathcal{A}(W[l_1,l_3]), \Omega) \quad \textit{has the } -\textit{modular intersection property.}$$

Proof : See [Bo 2, Lemma 6]. The assumptions state that the modular groups and conjugations act geometrically. Therefore it is enough to look at the geometrical picture. The statement is then a reformulation of Lemma 5 and Lemma 6. □

If instead of l_1 the above wedges have the second lightlike vector in common, then we similarly have the +modular intersection property. As an immediate corollary of Theorem 3 we get

Theorem 8 *Let $\mathcal{A}(O), O \subset \mathbf{R}^{1,2}$, be a local net fulfilling the Bisognano/Wichmann property for wedges and put*

$$\mathcal{N} := \mathcal{A}(W[l_1,l_2]),\ \mathcal{M} := \mathcal{A}(W[l_1,l_3]),\ \mathcal{L} := \mathcal{A}(W[l_2,l_3]).$$

Then we have the following intersection properties

$$\begin{array}{lll} 1. & (\mathcal{N}, \mathcal{M}, \Omega) & has\ -mi\ , \\ 2. & (\mathcal{L}, \mathcal{M}, \Omega) & has\ +mi\ , \\ 3. & (\mathcal{L}, \mathcal{N}', \Omega) & has\ -mi\ , \end{array}$$

where the prime indicates the commutant. Especially

$$\Delta_{\mathcal{M}}^{it}, \Delta_{\mathcal{N}}^{ir}, \Delta_{\mathcal{L}}^{is}, t, r, s \in \mathbf{R}$$

generate a representation of the 2+1 dim. Lorentz group $SO^{\uparrow}(1,2) \cong SL(2,R)/Z_2$.

Proof : Due to the Bisognano/Wichmann property we have $\mathcal{A}(W[l_1, l_2])' = \mathcal{A}(W[l_2, l_1])$. The theorem now follows from Theorem 3 and Proposition 7 above . □

To see how space-time translations arise by modular theory we look at translated wedges. Denote by

$$W[l_1, l_2, a] := \{x \in \mathbf{R}^{1,2} | x - a \in W[l_1, l_2]\}$$

the wedge translated by $a \in \mathbf{R}^{1,2}$. Then we have

Lemma 9 $(\mathcal{A}(W[l_1, l_2, \alpha l_1]) \subset \mathcal{A}(W[l_1, l_2]), \Omega)$ *is a* $-hsm$ *inclusion for* $\alpha > 0$.

Proof : (see [Bo2]) Denote by $\mathcal{U}(\alpha l_1), \alpha \in \mathbf{R}$, the translations in l_1-direction. Due to the spectrum condition the generator is positive. Moreover

$$\mathrm{Ad}\, \mathcal{U}(\alpha l_1)(\mathcal{A}(W[l_1, l_2]) = \mathcal{A}(W[l_1, l_2, \alpha l_1]) \subset \mathcal{A}(W[l_1, l_2])$$

for $\alpha > 0$ and $\mathcal{U}(\alpha l_1)\Omega = \Omega$. Therefore we can apply H.-J. Borchers' result, [Bo 1] to get the commutation relation

$$\mathrm{Ad}\, \Delta_{l_1,l_2}^{it}(\mathcal{U}(\alpha l_1)) = \mathcal{U}(e^{-2\pi t} \alpha l_1),\ \forall t \in \mathbf{R},$$

where Δ_{l_1,l_2} denotes the modular operator associated with $(\mathcal{A}(W[l_1, l_2]), \Omega)$. This implies

$$\mathrm{Ad}\, \Delta_{l_1,l_2}^{it}(\mathcal{A}(W[l_1, l_2, \alpha l_1])) = \mathcal{A}(W[l_1, l_2, e^{-2\pi t} \alpha l_1])$$

and therefore the statement. $\square$

Notice that Lemma 9 holds independently of the Bisognano/Wichmann property.
Proposition 7 and Lemma 9 show that modular inclusions and modular intersections of von Neumann algebras occur naturally in 2+1 dimensional quantum field theories and similar in 3+1 -dimensional theories.

3 On the Reconstruction of a $2+1$- dimensional Quantum Field Theory

In the following we conversely start from a finite set of algebras lying in a specific relative position w.r.t. their modular data. From these data we then construct a unitary representation of the 2+1 dimensional Poincaré group fulfilling the spectrum condition.

To get the Abelian translation group we need the following result.

Lemma 10 (Wi 2, Theorem 2) *Assume*

1. $(\mathcal{M}_1 \subset \mathcal{M}, \Omega)$ *is a $-$hsm inclusion,*
2. $(\mathcal{M}_2 \subset \mathcal{M}, \Omega)$ *is a $+$hsm inclusion,*
3. $[J_{\mathcal{M}_1} J_{\mathcal{M}}, J_{\mathcal{M}} J_{\mathcal{M}_2}] = 0$

Then the two unitary groups associated with the hsm inclusions 1 and 2 commute. Furthermore these 'translations' fulfill the spectrum condition.

Applying Ad $\Delta^{it}_{\mathcal{M}}$ to the equation in assumption 3 and taking products we see that only $[U_{\mathcal{M}_1 \subset_- \mathcal{M}}(a), U_{\mathcal{M}_2 \subset_+ \mathcal{M}}(b)] = 0$ for some $a, b \in \mathbf{R} \setminus \{0\}$ is needed, where $U_{\mathcal{M}_i \subset_\pm \mathcal{M}}$ denote the unitary groups associated with the hsm inclusions $(\mathcal{M}_i \subset \mathcal{M}, \Omega)$. With these preparations we can show

Proposition 11 *Let $\mathcal{N}, \mathcal{M}, \mathcal{L}$ and $\hat{\mathcal{N}}$ be von Neumann algebras acting on a Hilbert space $\mathcal{H}$ with a common cyclic and separating vector Ω. Assume*

I. 1. $(\mathcal{N}, \mathcal{M}, \Omega)$ *has the $-$mi property,*

2. $(\mathcal{L}, \mathcal{M}, \Omega)$ *has the* $+mi$ *property,*
3. $(\mathcal{L}, \mathcal{N}', \Omega)$ *has the* $-mi$ *property,*

II. 1. $(\hat{\mathcal{N}} \subset \mathcal{N}, \Omega)$ *is a* $-hsm$ *inclusion,*
2. $Ad\ J_{\mathcal{M}}(J_{\hat{\mathcal{N}}} J_{\mathcal{N}}) = J_{\mathcal{N}} J_{\hat{\mathcal{N}}}$,
3. $[Ad\ J_{\mathcal{L}}(J_{\hat{\mathcal{N}}} J_{\mathcal{N}}), J_{\hat{\mathcal{N}}} J_{\mathcal{N}}] = 0$.

Then we get a unitary representation of the 2+1 -dimensional Lorentz group $SO^{\uparrow}(1,2)$ *and of the translation group* $\mathbf{R}^{1,2}$ *by modular operators.*

(For a detailed description of these representations see the proof below.)

Before coming to the proof let us comment on the various conditions. First, according to Theorem 3 the conditions I give us a unitary representation of the 2+1 -dimensional homogeneous Lorentz group.
Comparison with the example of 2+1 -dimensional quantum field theory suggests that the hsm inclusion in condition II could be thought of as giving us a representation of translations along some lightray, see Lemma 9. Then the product of the two modular conjugations in assumption II. 2 would be a finite translation of this kind, see the remarks after Theorem 2. Due to the result of Bisognano/Wichmann [Bi-Wi] the modular conjugations of the wedge algebras act as reflections. At least they map translations to translations. Conditions II.2 and II.3 should be interpreted as a reminiscence of these expected properties.
Proof : Due to the assumptions I.1.- 3. Theorem 3 provides us with a unitary representation of $SL(2,R)/Z_2 \cong SO^{\uparrow}(1,2)$. We fix the representation of the Lorentz group by choosing a Lorentz frame: Let $l_1 = (1,1,0), l_2 = (1,-1,0), l_3 = (2,0,2)$. These are 3 linearly independent lightlike vectors in $\mathbf{R}^{1,2}$, pointing to the future. Denote by

$$\Lambda_{l_i,l_j}(t) \in SO^{\uparrow}(1,2)$$

the Lorentz boosts associated with the wedges $W[l_i, l_j]$ as described above. By definition, these boosts leave the sets $W[l_i, l_j]$ invariant and they generate the whole Lorentz group. We define

$$\mathcal{U}_{Lor}(\Lambda_{l_1,l_2}(t)) := \Delta_{\mathcal{N}}^{-i\frac{t}{2\pi}} \quad , \quad \mathcal{U}_{Lor}(\Lambda_{l_1,l_3}(t)) := \Delta_{\mathcal{M}}^{-i\frac{t}{2\pi}}$$
$$\mathcal{U}_{Lor}(\Lambda_{l_2,l_3}(t)) \ := \ \Delta_{\mathcal{L}}^{-i\frac{t}{2\pi}}.$$

In this way we fix the unitary representation of $SO^{\uparrow}(1,2)$ on $\mathcal{H}$ [Wi 3] so that we get a unitary representation

$$\mathcal{U}_{Lor} : SO^{\uparrow}(1,2) \to \mathcal{U}(\mathcal{H})$$

In the framework of Bisognano and Wichmann, the algebras $\mathcal{N}, \mathcal{M}$, and $\mathcal{L}$ could be thought of as being associated with the localization regions $W[l_1, l_2], W[l_1, l_3]$ and $W[l_2, l_3]$. We will use this picture as a guiding line through the various arguments during the proof.

Before coming to the translations let us note the following. Since the $-$mi property for $(\mathcal{N}, \mathcal{M}, \Omega)$ is equivalent to the $+$mi property for $(\mathcal{N}', \mathcal{M}', \Omega)$ [Wi 3], assumption I is symmetric in $(\mathcal{N}, \mathcal{M}, \mathcal{L})$, more precisely with $(\mathcal{N}, \mathcal{M}, \mathcal{L})$ also $(\mathcal{L}, \mathcal{N}', \mathcal{M}')$ and $(\mathcal{M}', \mathcal{L}', \mathcal{N})$ fulfill assumption I. We will exhibit this symmetry by introducing

$$\begin{aligned}
\Gamma_{\mathcal{N}} &:= U_{\mathcal{L}\cap^{+}\mathcal{M}}(1) J_{\mathcal{M}} = U_{\mathcal{M}\cap^{+}\mathcal{L}}(1) J_{\mathcal{L}} \\
\Gamma_{\mathcal{M}} &:= U_{\mathcal{N}'\cap^{-}\mathcal{L}}(1) J_{\mathcal{L}} = U_{\mathcal{L}\cap^{-}\mathcal{N}'}(1) J_{\mathcal{N}} \\
\Gamma_{\mathcal{L}} &:= U_{\mathcal{N}\cap^{-}\mathcal{M}}(1) J_{\mathcal{M}} = U_{\mathcal{M}\cap^{-}\mathcal{N}}(1) J_{\mathcal{N}}.
\end{aligned}$$

Then $\Gamma_{\mathcal{N}}$ is involutive, since

$$\Gamma_{\mathcal{N}}^2 = U_{\mathcal{L}\cap^{+}\mathcal{M}}(1) J_{\mathcal{M}} U_{\mathcal{L}\cap^{+}\mathcal{M}}(1) J_{\mathcal{M}} = U_{\mathcal{L}\cap^{+}\mathcal{M}}(1) U_{\mathcal{L}\cap^{+}\mathcal{M}}(-1) = 1,$$

where we have used Theorem 2c). This obviously also holds for $\Gamma_{\mathcal{M}}$ and $\Gamma_{\mathcal{L}}$. Moreover, Lemma 4 states

$$\mathrm{Ad}\ \Gamma_{\mathcal{N}}(\mathcal{N}) = \mathcal{N}, \quad \mathrm{Ad}\ \Gamma_{\mathcal{N}}(\Delta_{\mathcal{N}}^{it}) = \Delta_{\mathcal{N}}^{-it}, \quad t \in \mathbf{R},$$

and looking at the definitions and Theorem 2d) we see

$$\mathrm{Ad}\ \Gamma_{\mathcal{N}}(\mathcal{L}) = \mathcal{M}', \quad \mathrm{Ad}\ \Gamma_{\mathcal{N}}(\mathcal{M}) = \mathcal{L}'.$$

Modular theory tells us that in this case

$$\mathrm{Ad}\ \Gamma_{\mathcal{N}}(\Delta_{\mathcal{M}}^{it}) = \Delta_{\mathcal{L}}^{it}, \quad \mathrm{Ad}\ \Gamma_{\mathcal{N}}(\Delta_{\mathcal{L}}^{it}) = \Delta_{\mathcal{M}}^{it}, \quad t \in \mathbf{R} \tag{1}$$

holds. These facts motivated our index notation. Similar results follow for $\Gamma_{\mathcal{M}}, \Gamma_{\mathcal{L}}$.

After these preparations let us come to the translations. Due to Lemma 9 we might interprete the unitary group associated with the hsm inclusion

$(\hat{\mathcal{N}} \subset \mathcal{N}, \Omega)$ of condition II as translations in the l_1-direction. We therefore introduce the notion

$$\mathcal{U}_{trans,l_1}(a) := U_{\hat{\mathcal{N}} \subset -\mathcal{N}}(a).$$

Note that $\mathcal{U}_{trans,l_1}$ fulfills the spectrum condition, see Theorem 2.

To get the translations in l_2 and l_3 directions we will use the Lorentz boost representation. An easy group theoretical computation shows

$$\Lambda_{l_1,l_2}(-\ln 2)\Lambda_{l_2,l_3}(-\ln 2)l_1 = l_3$$

and with $t_o = \frac{1}{2\pi}\ln 2$, we get as a candidate

$$\begin{aligned}
\mathcal{U}_{trans,l_3}(a) &:= \mathrm{Ad}\ \mathcal{U}_{Lor}(\Lambda_{l_1,l_2}(-\ln 2)\Lambda_{l_2,l_3}(-\ln 2))(\mathcal{U}_{trans,l_1}(a)) \\
&= \mathrm{Ad}\ \Delta_{\mathcal{N}'}^{-it_0}\Delta_{\mathcal{L}}^{it_0}(\mathcal{U}_{trans,l_1}(a)) \\
&= \mathrm{Ad}\ U_{\mathcal{L}\cap-\mathcal{N}'}(1)(\mathcal{U}_{trans,l_1}(a))
\end{aligned}$$

where in the last line we used again the remarks following Theorem 2.

Similarly we make the Ansatz

$$\mathcal{U}_{trans,l_2}(a) := \mathrm{Ad}\ U_{\mathcal{L}\cap+\mathcal{M}}(1)(\mathcal{U}_{trans,l_1}(a)).$$

We will show that these one-parameter unitary groups $\mathcal{U}_{trans,l_i}$ indeed commute, such that they provide us with a unitary representation $\mathcal{U}_{trans}$ of $\mathbf{R}^{1,2}$. First we need

Lemma 12

$$Ad\,\mathcal{U}_{Lor}(\Lambda_{l_i,l_j}(2\pi t))(\mathcal{U}_{trans,l_i}(a)) = \mathcal{U}_{trans,l_i}(e^{2\pi t}a) \quad \forall a,t\ \in \mathbf{R}, i,j = 1,2,3. \tag{2}$$

Proof of Lemma 12: Let us begin with the translations in l_1-direction. Due to Theorem 2 we have

$$\mathcal{U}_{trans,l_1}(2) = J_{\hat{\mathcal{N}}}J_{\mathcal{N}} = U_{\hat{\mathcal{N}}\subset-\mathcal{N}}(2),$$

and assumption II.2 states that Ad $J_{\mathcal{M}}$ acts as a reflection on this discrete translation. We get :

$$\begin{aligned}
\mathrm{Ad}\ U_{\mathcal{N}\cap-\mathcal{M}}(-2)(\mathcal{U}_{trans,l_1}(2)) &= \mathrm{Ad}\ U_{\mathcal{N}\cap-\mathcal{M}}(-2)(U_{\hat{\mathcal{N}}\subset-\mathcal{N}}(2)) \\
&= \mathrm{Ad}\ J_{\mathcal{M}}J_{\mathcal{N}}(J_{\hat{\mathcal{N}}}J_{\mathcal{N}}) \\
&= \mathrm{Ad}\ J_{\mathcal{M}}(J_{\mathcal{N}}J_{\hat{\mathcal{N}}}) = J_{\hat{\mathcal{N}}}J_{\mathcal{N}} = \mathcal{U}_{trans,l_1}(2)
\end{aligned}$$

where we used Theorem 2 c) in the second line and II.2 in the third line. Scaling this equation by applying Ad $\Delta_{\mathcal{N}}^{it}$ to it and using the transformation rule Theorem 2 b) gives

$$U_{\mathcal{N}\cap-\mathcal{M}}(-2e^{-2\pi t})\mathcal{U}_{trans,l_1}(2e^{-2\pi t})U_{\mathcal{N}\cap-\mathcal{M}}(2e^{-2\pi t}) =$$

$$\text{Ad } \Delta_{\mathcal{N}}^{it}(U_{\mathcal{M}\cap-\mathcal{N}}(2)\mathcal{U}_{trans,l_1}(2)U_{\mathcal{M}\cap-\mathcal{N}}(-2)) =$$

$$\text{Ad } \Delta_{\mathcal{N}}^{it}(\mathcal{U}_{trans,l_1}(2)) = \mathcal{U}_{trans,l_1}(2e^{-2\pi t}) \quad t \in \mathbf{R}.$$

Taking products and adjoints of this equation for arbitrary t easily leads to

$$\text{Ad } U_{\mathcal{N}\cap-\mathcal{M}}(-a)(\mathcal{U}_{trans,l_1}(b)) = \mathcal{U}_{trans,l_1}(b) \quad a, b \in \mathbf{R} \tag{3}$$

(see the proof of Th. 2 in [Wi 3] for example.) But this implies

$$\begin{aligned}
\text{Ad } \mathcal{U}_{Lor}(\Lambda_{l_1,l_3}(-2\pi t))(\mathcal{U}_{trans,l_1}(a)) &= \text{Ad } \Delta_{\mathcal{M}}^{it}(\mathcal{U}_{trans,l_1}(a)) \\
&= \text{Ad } \Delta_{\mathcal{M}}^{it}\Delta_{\mathcal{N}}^{-it}(\mathcal{U}_{trans,l_1}(e^{-2\pi t}a)) \\
&= \text{Ad } U_{\mathcal{N}\cap-\mathcal{M}}(1-e^{-2\pi t})(\mathcal{U}_{trans,l_1}(e^{-2\pi t}a)) \\
&= \mathcal{U}_{trans,l_1}(e^{-2\pi t}a), \quad t, a \in \mathbf{R}
\end{aligned}$$

where in the second line we have again used Theorem 2 b) . The similar result for $\mathcal{U}_{Lor}(\Lambda_{l_1,l_2}(-2\pi t)) = \Delta_{\mathcal{N}}^{it}$ follows directly from the definition of $\mathcal{U}_{trans,l_1}$ and Theorem 2 b), so that we have proven Lemma 12, (2), for $i = 1$.
To obtain the remaining translations we will explore the symmetry in $(\mathcal{N}, \mathcal{M}, \mathcal{L})$. For example by the definition of $\mathcal{U}_{trans,l_2}$ we have

$$\text{Ad } \Gamma_{\mathcal{N}}(\mathcal{U}_{trans,l_1}(a)) = \text{Ad } U_{\mathcal{L}\cap+\mathcal{M}}(1)(\mathcal{U}_{trans,l_1}(-a)) = \mathcal{U}_{trans,l_2}(-a), \quad a \in \mathbf{R}. \tag{4}$$

Furthermore as shown above, see (1), we know

$$\text{Ad } \Gamma_{\mathcal{N}}(\Delta_{\mathcal{M}}^{it}) = \Delta_{\mathcal{L}}^{it}, \quad \text{Ad } \Gamma_{\mathcal{N}}(\Delta_{\mathcal{L}}^{it}) = \Delta_{\mathcal{M}}^{it}, \quad t \in \mathbf{R}.$$

Using this the symmetry interchanges $(l_1 \mapsto l_2, l_2 \mapsto l_1)$ and from the proven statement for $i = 1$ we get Lemma 12 (2) in case of i=2.

Next we have

$$\text{Ad } \Gamma_{\mathcal{M}}(\mathcal{U}_{trans,l_1}(a)) = \text{Ad } U_{\mathcal{L}\cap+\mathcal{N}'}(1)(\mathcal{U}_{trans,l_1}(-a)) = \mathcal{U}_{trans,l_3}(-a), \quad a \in \mathbf{R}$$

and with the analogue of (1) for $\Gamma_{\mathcal{M}}$ we can finish the proof of Lemma 12. $\square$

Note that (3) shows

$$\mathrm{Ad}\ \Gamma_{\mathcal{L}}(\mathcal{U}_{trans,l_1}(a)) = \mathcal{U}_{trans,l_1}(-a), \quad a \in \mathbf{R}$$

and similarly one gets for $a \in \mathbf{R}$

$$\begin{aligned} \mathrm{Ad}\ \Gamma_{\mathcal{N}}(\mathcal{U}_{trans,l_3}(a)) &= \mathcal{U}_{trans,l_3}(-a), \\ \mathrm{Ad}\ \Gamma_{\mathcal{M}}(\mathcal{U}_{trans,l_2}(a)) &= \mathcal{U}_{trans,l_2}(-a). \end{aligned} \tag{5}$$

We will use another simple observation. Due to Theorem 2c) we have for $a \in \mathbf{R}$

$$J_{\mathcal{N}}\mathcal{U}_{\mathrm{trans},l_1}(a)J_{\mathcal{N}} = J_{\mathcal{N}}U_{\hat{\mathcal{N}}\subset -\mathcal{N}}(a)J_{\mathcal{N}} = \mathcal{U}_{\mathrm{trans},l_1}(-a),$$

and

$$\begin{aligned} \mathrm{Ad}\ J_{\mathcal{N}}(\mathcal{U}_{\mathrm{trans},l_2}(a)) &= \mathrm{Ad}\ J_{\mathcal{N}}\Gamma_{\mathcal{N}}(\mathcal{U}_{\mathrm{trans},l_1}(-a)) \\ &= \mathrm{Ad}\ \Gamma_{\mathcal{N}}J_{\mathcal{N}}(\mathcal{U}_{\mathrm{trans},l_1}(-a)) = \mathcal{U}_{\mathrm{trans},l_2}(-a) \end{aligned}$$

where we used equation (4) and Lemma 4. Applying Ad $\Gamma_{\mathcal{M}}$ resp. Ad$\Gamma_{\mathcal{L}}$ to it we get similar results for $J_{\mathcal{M}}$ resp. $J_{\mathcal{L}}$.

Proof of Proposition 11: Hence, we end up with three unitary one-parameter groups fulfilling the standard commutation relations with the $SO^{\uparrow}(1,2)$- representation. To show that they fit together to give a unitary representation of the translations $\mathbf{R}^{1,2}$ we are left to prove that these one-parameter groups pairwise commute. Due to unitary equivalence it is enough to check this for $\mathcal{U}_{trans,l_2}(a), \mathcal{U}_{trans,l_3}(b), a, b \in \mathbf{R}$. Exploiting the fact that $(\Gamma_{\mathcal{N}}\hat{\mathcal{N}}\Gamma_{\mathcal{N}} \subset \mathcal{N}, \Omega)$ is a + hsm inclusion, see (1), we can apply Lemma 10 according to which it is enough to prove this for one pair of nonzero values. We compute for $a = b = 4$ with $t_0 := \frac{1}{2\pi}\ln 2$

$$[\mathcal{U}_{trans,l_3}(4), \mathcal{U}_{trans,l_2}(4)] = \mathrm{Ad}\ U_{\mathcal{L}\cap -\mathcal{N}'}(1)([\mathcal{U}_{trans,l_1}(4), \mathcal{U}_{trans,l_2}(4)])$$

$$= \mathrm{Ad}\ U_{\mathcal{L}\cap -\mathcal{N}'}(1)([\mathcal{U}_{trans,l_1}(4), \mathrm{Ad}\ U_{\mathcal{L}\cap +\mathcal{M}}(1)(\mathcal{U}_{trans,l_1}(4))]$$

$$= \mathrm{Ad}\ U_{\mathcal{L}\cap -\mathcal{N}'}(1)\Delta_{\mathcal{M}}^{-it_0}([\mathcal{U}_{trans,l_1}(2), \mathrm{Ad}\ U_{\mathcal{L}\cap +\mathcal{M}}(2)(\mathcal{U}_{trans,l_1}(2))])$$

where we used the definitions and (3) in the first two lines, the scaling property of $U_{\mathcal{L}\cap^+\mathcal{M}}$, see Theorem 2 b) in the last one. Assumptions II.2 and II.3 imply

$$
\begin{aligned}
[\mathcal{U}_{trans,l_1}(2), \mathrm{Ad}\ U_{\mathcal{L}\cap^+\mathcal{M}}(2)(\mathcal{U}_{trans,l_1}(2))] &= [U_{\hat{\mathcal{N}}\subset_-\mathcal{N}}(2), \mathrm{Ad}\ U_{\mathcal{L}\cap^+\mathcal{M}}(2)(U_{\hat{\mathcal{N}}\subset_-\mathcal{N}}(2))] \\
&= [J_{\hat{\mathcal{N}}}J_{\mathcal{N}},\ \mathrm{Ad}\ J_{\mathcal{L}}J_{\mathcal{M}}(J_{\hat{\mathcal{N}}}J_{\mathcal{N}})] \\
&= [J_{\hat{\mathcal{N}}}J_{\mathcal{N}}, \mathrm{Ad}\ J_{\mathcal{L}}(J_{\mathcal{N}}J_{\hat{\mathcal{N}}})] = 0,
\end{aligned}
$$

i.e.

$$
\begin{aligned}
\mathcal{U}_{trans} : \mathbf{R}^{1,2} &\to \mathcal{U}(\mathcal{H}) \\
(\alpha l_1 + \beta l_2 + \gamma l_3) &\mapsto \mathcal{U}_{trans,l_1}(\alpha)\mathcal{U}_{trans,l_2}(\beta)\mathcal{U}_{trans,l_3}(\gamma)
\end{aligned}
\tag{6}
$$

defines a unitary representation. This finishes the proof of Proposition 11. □

Before coming to the Poincaré group let us remark

Lemma 13 $\mathcal{U}_{trans} : \mathbf{R}^{1,2} \to \mathcal{U}(\mathcal{H})$ *is faithful.*

Proof : First we show

$$
\mathcal{U}_{trans}|_{\mathbf{R}l_1+\mathbf{R}l_2} \quad \text{is faithful.} \tag{7}
$$

For this assume

$$
\mathcal{U}_{trans,l_1}(a) = \mathcal{U}_{trans,l_2}(b), \quad a, b \in \mathbf{R}.
$$

Taking products and adjoints gives

$$
\mathcal{U}_{trans,l_1}(na) = \mathcal{U}_{trans,l_2}(nb), \quad \forall n \in \mathbf{Z}.
$$

and scaling this relation by applying Ad $\Delta_{\mathcal{N}}^{it_m}$ with $t_m = \frac{1}{2\pi}\ln m$ to it implies

$$
\mathcal{U}_{trans,l_1}(\frac{n}{m}a) = \mathcal{U}_{trans,l_2}(mnb), \quad \forall n, m \in \mathbf{Z},
$$

especially

$$
\mathcal{U}_{trans,l_1}(\frac{2}{3}a) = \mathcal{U}_{trans,l_2}(6b) = \mathcal{U}_{trans,l_1}(\frac{3}{2}a).
$$

Due to the nontriviality of $\mathcal{U}_{trans,l_1} = U_{\mathcal{N}\subset_-\mathcal{M}}$ and scaling invariance, see Theorem 2 b), we get $a = 0 = b$, i.e statement (7).

Using $\Gamma_\mathcal{N}, \Gamma_\mathcal{M}$, and $\Gamma_\mathcal{L}$ we easily see that $\mathcal{U}_{trans}$ is faithful on any $\mathbf{R}l_i + \mathbf{R}l_j, i,j = 1,2,3$.

Now assume

$$\mathcal{U}_{trans,l_1}(a)\mathcal{U}_{trans,l_2}(b)\mathcal{U}_{trans,l_3}(c) = 1$$

with some $a,b,c \in \mathbf{R}$. Applying Ad $\Gamma_\mathcal{N}$ to it combined with the properties shown in (4) and (5) gives

$$\mathcal{U}_{trans,l_2}(-a)\mathcal{U}_{trans,l_1}(-b)\mathcal{U}_{trans,l_3}(-c) = 1$$

and multiplying with the first equation implies

$$\mathcal{U}_{trans,l_1}(a-b)\mathcal{U}_{trans,l_2}(b-a) = 1$$

i.e. $a = b$. Using Ad $\Gamma_\mathcal{L}$ shows similarly $b = c$ so that the assumption states

$$\mathcal{U}_{trans,l_1}(a) = \mathcal{U}_{trans,l_2}(-a)\mathcal{U}_{trans,l_3}(-a).$$

But this implies

$$J_\mathcal{L}\mathcal{U}_{trans,l_1}(a)J_\mathcal{L} = \mathcal{U}_{trans,l_2}(a)\mathcal{U}_{trans,l_3}(a) = \mathcal{U}_{trans,l_1}(-a). \tag{8}$$

Now we observe

$$\begin{aligned} J_\mathcal{L}\mathcal{U}_{trans,l_1}(a)J_\mathcal{L} &= J_\mathcal{L}J_\mathcal{M}\mathcal{U}_{trans,l_1}(-a)J_\mathcal{M}J_\mathcal{L} \\ &= U_{\mathcal{L}\cap_+\mathcal{M}}(2)\mathcal{U}_{trans,l_1}(-a)U_{\mathcal{L}\cap_+\mathcal{M}}(-2) \\ &= \Delta_\mathcal{M}^{it_0}U_{\mathcal{L}\cap_+\mathcal{M}}(1)\mathcal{U}_{trans,l_1}(-2a)U_{\mathcal{L}\cap_+\mathcal{M}}(-1)\Delta_\mathcal{M}^{-it_0} \\ &= \Delta_\mathcal{M}^{it_0}\mathcal{U}_{trans,l_2}(-2a)\Delta_\mathcal{M}^{-it_0} \end{aligned} \tag{9}$$

and combined with equation (8) this gives

$$\text{Ad } \Delta_\mathcal{M}^{it_0}(\mathcal{U}_{trans,l_2}(-2a)) = J_\mathcal{L}\mathcal{U}_{trans,l_1}(a)J_\mathcal{L} = \mathcal{U}_{trans,l_1}(-a),$$

$$\mathcal{U}_{trans,l_2}(-2a) = \text{Ad } \Delta_\mathcal{M}^{-it_0}(\mathcal{U}_{trans,l_1}(-a)) = \mathcal{U}_{trans,l_1}(-2a)$$

which implies $a = 0$ and finishes the proof. $\square$

Note that similarly as in (9) we have

$$\begin{aligned} J_{\mathcal{L}} \mathcal{U}_{trans,l_1}(a) J_{\mathcal{L}} &= J_{\mathcal{L}} J_{\mathcal{N}} \mathcal{U}_{trans,l_1}(-a) J_{\mathcal{N}} J_{\mathcal{L}} \\ &= \Delta_{\mathcal{N}}^{it_0} \mathcal{U}_{trans,l_3}(-2a) \Delta_{\mathcal{N}}^{-it_0}. \end{aligned} \tag{10}$$

Moreover, we have

$$J_{\mathcal{N}} \mathcal{U}_{trans,l_3}(a) J_{\mathcal{N}} = \Delta_{\mathcal{M}}^{-it_0} \mathcal{U}_{trans,l_2}(-2a) \Delta_{\mathcal{M}}^{it_0} = \Delta_{\mathcal{L}}^{-it_0} \mathcal{U}_{trans,l_1}(-2a) \Delta_{\mathcal{L}}^{it_0},$$

$$J_{\mathcal{M}} \mathcal{U}_{trans,l_2}(a) J_{\mathcal{M}} = \Delta_{\mathcal{N}}^{-it_0} \mathcal{U}_{trans,l_3}(-2a) \Delta_{\mathcal{N}}^{it_0} = \Delta_{\mathcal{L}}^{it_0} \mathcal{U}_{trans,l_1}(-2a) \Delta_{\mathcal{L}}^{-it_0}. \tag{11}$$

The next step in constructing a (2+1)-dimensional quantum field theory out of a set of 4 algebras is to show that the Lorentz group representation of Proposition 11 and the representation of the translations properly combine to give a representation of the Poincaré group of $\mathbf{R}^{1,2}$. For this we need as a further assumption property III below.

Theorem 14 *Let $\mathcal{N}, \mathcal{M}, \mathcal{L}$ and $\hat{\mathcal{N}}$ be von Neumann algebras acting on a Hilbert space $\mathcal{H}$ with a common cyclic and separating vector Ω so that the assumptions I. and II. of Proposition 11 are fulfilled. Denote $\mathcal{U}_{trans}$ the representation of the translations $\mathbf{R}^{1,2}$ and assume further*

III. *To all $a \in \mathbf{R}^{1,2}$ there exists $b \in \mathbf{R}^{1,2}$ with*

$$Ad\ J_{\mathcal{N}}(\mathcal{U}_{trans}(a)) = \mathcal{U}_{trans}(b)$$

i.e. $Ad\ J_{\mathcal{N}}$ maps translations to translations. Then the modular groups

$$\Delta_{\mathcal{N}}^{it}, \Delta_{\mathcal{M}}^{ir}, \Delta_{\mathcal{L}}^{is} \text{ and } \Delta_{\hat{\mathcal{N}}}^{iv} \quad t, r, s, v \in \mathbf{R}$$

generate a unitary representation of the 2+1 dimensional Poincaré group with spectrum condition.

We will use the shortcut notation

$$\mathrm{Ad}\ J_{\mathcal{N}}(\mathcal{U}_{trans}(\mathbf{R}^{1,2})) = \mathcal{U}_{trans}(\mathbf{R}^{1,2})$$

for the assumption made in III.

Proof : The idea of the proof goes as follows. First we will show that assumption III implies for all $a \in \mathbf{R}$

$$J_{\mathcal{N}} \mathcal{U}_{trans,l_3}(a) J_{\mathcal{N}} = \mathcal{U}_{trans,l_1}(-2a)\mathcal{U}_{trans,l_2}(-2a)\mathcal{U}_{trans,l_3}(a).$$

Due to equation (11) together with Proposition 12 we thereby get the correct action (spin 1 action of $SO^{\uparrow}(1,2)$ on $\mathbf{R}^3$) of

$$\mathrm{Ad}\, \Delta_{\mathcal{M}}^{-it_0} = \mathrm{Ad}\, \mathcal{U}_{Lor}(\Lambda_{l_1,l_3}(\ln 2))$$

on the translations, where $t_0 = \frac{1}{2\pi} \ln 2$. Using the symmetry in the algebras we get this result for the other boosts and taking products of theses elements yields the correct action for a dense set of $SO^{\uparrow}(1,2)$. Strong continuity of the representation will finish the proof.

We use the notations introduced in the proof of Proposition 11. First one observes

$$J_{\mathcal{N}} \mathcal{U}_{trans,l_3}(a) J_{\mathcal{N}} = \mathcal{U}_{trans,l_1}(\lambda_1 a)\mathcal{U}_{trans,l_2}(\lambda_2 a)\mathcal{U}_{trans,l_3}(\lambda_3 a), \quad \forall a \in \mathbf{R} \quad (12)$$

with

$$J_{\mathcal{N}} \mathcal{U}_{trans,l_3}(1) J_{\mathcal{N}} = \mathcal{U}_{trans,l_1}(\lambda_1)\mathcal{U}_{trans,l_2}(\lambda_2)\mathcal{U}_{trans,l_3}(\lambda_3).$$

(Note that λ_i are uniquely defined due to Lemma 13.) For this let $a_n, b_n, c_n, n \in \mathbf{N}$ be given by

$$J_{\mathcal{N}} \mathcal{U}_{trans,l_3}(\frac{1}{n}) J_{\mathcal{N}} = \mathcal{U}_{trans,l_1}(a_n)\mathcal{U}_{trans,l_2}(b_n)\mathcal{U}_{trans,l_3}(c_n).$$

Then taking products of this relation yields

$$\mathcal{U}_{trans,l_1}(na_n)\mathcal{U}_{trans,l_2}(nb_n)\mathcal{U}_{trans,l_3}(nc_n) = (J_{\mathcal{N}} \mathcal{U}_{trans,l_3}(\frac{1}{n}) J_{\mathcal{N}})^n$$

$$= J_{\mathcal{N}} \mathcal{U}_{trans,l_3}(1) J_{\mathcal{N}} = \mathcal{U}_{trans,l_1}(\lambda_1)\mathcal{U}_{trans,l_2}(\lambda_2)\mathcal{U}_{trans,l_3}(\lambda_3)$$

which shows $a_n = \frac{\lambda_1}{n}, b_n = \frac{\lambda_2}{n}, c_n = \frac{\lambda_3}{n}$. Taking products and adjoints gives

$$J_{\mathcal{N}} \mathcal{U}_{trans,l_3}(\frac{m}{n}) J_{\mathcal{N}} = \mathcal{U}_{trans,l_1}(\frac{m}{n}\lambda_1)\mathcal{U}_{trans,l_2}(\frac{m}{n}\lambda_2)\mathcal{U}_{trans,l_3}(\frac{m}{n}\lambda_3) \quad n, m \in \mathbf{Z}$$

and strong continuity of both sides proves (12).

Now apply Ad $J_{\mathcal{N}}$ to the equation (12). This gives

$$\mathcal{U}_{trans,l_3}(a) = \mathcal{U}_{trans,l_1}(-\lambda_1 a)\mathcal{U}_{trans,l_2}(-\lambda_2 a)J_{\mathcal{N}}\mathcal{U}_{trans,l_3}(\lambda_3 a)J_{\mathcal{N}},$$

and multiplying both equations implies

$$J_{\mathcal{N}}\mathcal{U}_{trans,l_3}((\lambda_3 - 1)a)J_{\mathcal{N}} = \mathcal{U}_{trans,l_3}(-(\lambda_3 - 1)a).$$

As in (8) we conclude $\lambda_3 = 1$.

Using the symmetry $\Gamma_{\mathcal{N}}$ gives

$$\begin{aligned} J_{\mathcal{N}}\mathcal{U}_{trans,l_3}(a)J_{\mathcal{N}} &= \text{Ad } \Gamma_{\mathcal{N}}(J_{\mathcal{N}}\mathcal{U}_{trans,l_3}(-a)J_{\mathcal{N}}) \\ &= \text{Ad } \Gamma_{\mathcal{N}}(\mathcal{U}_{trans,l_1}(-\lambda_1 a)\mathcal{U}_{trans,l_2}(-\lambda_2 a)\mathcal{U}_{trans,l_3}(-a)) \\ &= \mathcal{U}_{trans,l_2}(\lambda_1 a)\mathcal{U}_{trans,l_1}(\lambda_2 a)\mathcal{U}_{trans,l_3}(a) \qquad (13) \end{aligned}$$

which compared with (12) and due to Lemma 13 shows $\lambda_1 = \lambda_2 =: \lambda$.

For determining λ we first use again the symmetry in the algebras, namely

$$\begin{aligned} \text{Ad } \Gamma_{\mathcal{M}}(J_{\mathcal{N}}\mathcal{U}_{trans,l_3}(a)J_{\mathcal{N}}) &= \text{Ad } U_{\mathcal{L}\cap-\mathcal{N}'}(1)(\mathcal{U}_{trans,l_3}(a)) \\ &= \text{Ad } J_{\mathcal{L}}U_{\mathcal{L}\cap-\mathcal{N}'}(-1)(\mathcal{U}_{trans,l_3}(-a)) \\ &= J_{\mathcal{L}}\mathcal{U}_{trans,l_1}(-a)J_{\mathcal{L}} \end{aligned}$$

which gives as in (13), see (10)

$$\Delta_{\mathcal{N}}^{it_0}\mathcal{U}_{trans,l_3}(-2a)\Delta_{\mathcal{N}}^{-it_0} =$$

$$J_{\mathcal{L}}\mathcal{U}_{trans,l_1}(a)J_{\mathcal{L}} = \mathcal{U}_{trans,l_3}(\lambda a)\mathcal{U}_{trans,l_2}(\lambda a)\mathcal{U}_{trans,l_1}(a)$$

where we used also (4) and (5).

Therefore we can apply Ad $\Delta_{\mathcal{N}}^{it_0}$ to equation (12),

$$\begin{aligned} &\Delta_{\mathcal{N}}^{it_0}J_{\mathcal{N}}\mathcal{U}_{trans,l_3}(\lambda a)J_{\mathcal{N}}\mathcal{U}_{trans,l_3}(a)\Delta_{\mathcal{N}}^{-it_0} \\ &= \Delta_{\mathcal{N}}^{it_0}\mathcal{U}_{trans,l_1}(\lambda a)\mathcal{U}_{trans,l_2}(\lambda a)\mathcal{U}_{trans,l_3}(2a)\Delta_{\mathcal{N}}^{-it_0} \\ &= \mathcal{U}_{trans,l_1}(\frac{\lambda}{2}a)\mathcal{U}_{trans,l_2}(2\lambda a)\Delta_{\mathcal{N}}^{it_0}\mathcal{U}_{trans,l_3}(2a)\Delta_{\mathcal{N}}^{-it_0} \\ &= \mathcal{U}_{trans,l_1}((\frac{\lambda}{2} - 1)a)\mathcal{U}_{trans,l_2}(\lambda a)\mathcal{U}_{trans,l_3}(-\lambda a) \end{aligned}$$

Now this expression has to be Ad $J_{\mathcal{N}}$ -invariant. On the other hand we compute

$$J_{\mathcal{N}}\mathcal{U}_{trans,l_1}((\frac{\lambda}{2}-1)a)\mathcal{U}_{trans,l_2}(\lambda a)\mathcal{U}_{trans,l_3}(-\lambda a)J_{\mathcal{N}}$$

$$= \mathcal{U}_{trans,l_1}(-(\frac{\lambda}{2}-1)a)\mathcal{U}_{trans,l_2}(-\lambda a)J_{\mathcal{N}}\mathcal{U}_{trans,l_3}(-\lambda a)J_{\mathcal{N}}$$

$$= \mathcal{U}_{trans,l_1}(-(\frac{\lambda}{2}-1+\lambda^2)a)\mathcal{U}_{trans,l_2}((-\lambda-\lambda^2)a)\mathcal{U}_{trans,l_3}(-\lambda a)$$

which by comparison leads to

$$\begin{aligned} -(\frac{\lambda}{2}-1+\lambda^2) &= \frac{\lambda}{2}-1 \\ -\lambda-\lambda^2 &= \lambda \end{aligned}$$

whose unique solution is $\lambda = -2$. Therefore we finally conclude

$$J_{\mathcal{N}}\mathcal{U}_{trans,l_3}(a)J_{\mathcal{N}} = \mathcal{U}_{trans,l_1}(-2a)\mathcal{U}_{trans,l_2}(-2a)\mathcal{U}_{trans,l_3}(a).$$

This implies, using (10)

$$\text{Ad } \mathcal{U}_{Lor}(\Lambda_{l_1,l_3}(\ln 2))(\mathcal{U}_{trans,l_2}(-2a)) = \Delta_{\mathcal{M}}^{-it_0}\mathcal{U}_{trans,l_2}(-2a)\Delta_{\mathcal{M}}^{it_0}$$

$$= J_{\mathcal{N}}\mathcal{U}_{trans,l_3}(a)J_{\mathcal{N}} = \mathcal{U}_{trans}((\Lambda_{l_1,l_3}(\ln 2))(al_2))$$

showing according to Lemma 12

$$\text{Ad } \mathcal{U}_{Lor}(\Lambda_{l_1,l_3}(n\ln 2))(\mathcal{U}_{trans}(\vec{a})) = \mathcal{U}_{trans}(\Lambda_{l_1,l_3}(n\ln 2)\vec{a}) \tag{14}$$

for all $\vec{a} \in \mathbf{R}^{1,2}, n \in \mathbf{Z}$. Similarly we get

$$\text{Ad } \mathcal{U}_{Lor}(\Lambda_{l_1,l_2}(n\ln 2))(\mathcal{U}_{trans}(\vec{a})) = \mathcal{U}_{trans}(\Lambda_{l_1,l_2}(n\ln 2)\vec{a})$$

and

$$\text{Ad } \mathcal{U}_{Lor}(\Lambda_{l_2,l_3}(n\ln 2))(\mathcal{U}_{trans}(\vec{a})) = \mathcal{U}_{trans}(\Lambda_{l_2,l_3}(n\ln 2)\vec{a})$$

for all $\vec{a} \in \mathbf{R}^{1,2}, n \in \mathbf{Z}$.

Next we will show that products of such elements form a dense subset of $SO^{\uparrow}(1,2)$. By continuity we then get the correct action of the representation $\mathcal{U}_{Lor}$ on the translations, and therefore a unitary representation of the semidirect product $SO^{\uparrow}(1,2) \triangleright \mathbf{R}^{1,2} \equiv$ (the Poincaré group in (2+1) dimensions).

Let

$$U_{\mathcal{LM}}(a) := \Lambda_{l_2,l_3}(-\ln(a+1))\Lambda_{l_1,l_3}(+\ln(a+1)) \quad a \geq 0, \;\; U_{\mathcal{LM}}(-a) := (U_{\mathcal{LM}}(a))^{-1}.$$

(Notice that $\mathcal{U}_{Lor}(U_{\mathcal{LM}}(n)) = U_{\mathcal{L}\cap+\mathcal{M}}(n), n \in \mathbf{Z}$.)

This defines a one-parameter subgroup of $SO^{\uparrow}(1,2)$ and some group relations are

$$\text{Ad}\; \Lambda_{l_1,l_3}(t)(U_{\mathcal{LM}}(a)) = U_{\mathcal{LM}}(e^{-t}a), \quad \forall t, a \in \mathbf{R}.$$

The relations (14) show that the representors of

$$U_{\mathcal{LM}}(\pm 1) \;\; \text{and} \; \Lambda_{l_1,l_3}(n \ln 2), \quad n \in \mathbf{Z}$$

act in the right way on the translations. Taking products this is true for

$$\text{Ad}\; \Lambda_{l_1,l_3}(n \ln 2)(U_{\mathcal{LM}}(m)) = U_{\mathcal{LM}}(2^{-n}m), \quad \forall n, m \in \mathbf{Z}$$

and by strong continuity we get it for all $U_{\mathcal{LM}}(a), a \in \mathbf{R}$. Similarly we define

$$U_{\mathcal{NM}}(a) := \Lambda_{l_1,l_3}(+\ln(a+1))\Lambda_{l_1,l_2}(-\ln(a+1)) \quad a \geq 0, \;\; U_{\mathcal{NM}}(-a) := (U_{\mathcal{NM}}(a))^{-1}$$

and

$$U_{\mathcal{NL}}(a) := \Lambda_{l_1,l_2}(-\ln(a+1))\Lambda_{l_1,l_3}(+\ln(a+1)) \quad a \geq 0, \;\; U_{\mathcal{NL}}(-a) := (U_{\mathcal{NL}}(a))^{-1}$$

and as above we get that the representors of $U_{\mathcal{NM}}(a), U_{\mathcal{NL}}(a), a \in \mathbf{R}$ act in the right way on the translations. But these three 1-dimensional subgroups generate the whole Lorentz group $SO^{\uparrow}(1,2)$ and therefore we finally get a representation of the Poincaré group of dimension (2+1) by modular theory.

The spectrum of the representation of the translations is Lorentz invariant. Since the translations in the l_1-direction are represented by $U_{\hat{\mathcal{N}}\subset_{-}\mathcal{N}}$ the generator is positive according to Theorem 2 a). From these remarks we conclude that we arrived at a positive energy representation of the Poincaré group. □

As in the case of chiral quantum field theories [Wi4] we can use Theorem 14 in order to construct a 2+1 -dim. quantum field theory starting from a

set of 4 algebras $\mathcal{N}, \mathcal{M}, \mathcal{L}, \hat{\mathcal{N}}$. Let us assume conditions I.-III. for this set. We define a local net of observables in the following way:
Let $l_1 = (1,1,0), l_2 = (1,-1,0)$ and $l_3 = (2,0,2) \in \mathbf{R}^{1,2}$ be as in the proof of Proposition 11. We define the local algebras of observables associated with wedges by

$$\mathcal{A}(W[l_1,l_2]) := \mathcal{N} \quad , \quad \mathcal{A}(W[l_1,l_3]) := \mathcal{M}$$
$$\mathcal{A}(W[l_2,l_3]) \;:=\; \mathcal{L}. \tag{15}$$

For arbitrary linearly independent light rays $l_i, l_j \in \mathbf{R}^{1,2}$ pointing to the future we choose a $g_{l_i,l_j} \in SO^{\uparrow}(1,2)$ with

$$l_i = g_{l_i,l_j} l_1, \quad l_j = g_{l_i,l_j} l_2.$$

Such a group element always exists in $SO^{\uparrow}(1,2)$ and is unique up to a multiplication by a boost of type $\Lambda_{l_1,l_2}(t), t \in \mathbf{R}$. (These are the only elements in $SO^{\uparrow}(1,2)$ which leave fixed the wedge region $W[l_1,l_2]$.) [2]

We will use this group element in order to define the local observable algebra associated with $W[l_i,l_j]$. Let $\mathcal{U}$ denote the unitary representation of the Poincaré group according to Theorem 14, i.e. let

$$\mathcal{U}(\Lambda_{l_1,l_2}(t)) := \Delta_{\mathcal{N}}^{-i\frac{t}{2\pi}} \text{ etc. with } t \in \mathbf{R}.$$

Then we define the observable algebra associated with $W[l_i,l_j]$ by

$$\mathcal{A}(W[l_i,l_j]) := \mathrm{Ad}\, \mathcal{U}(g_{l_i,l_j})(\mathcal{N}).$$

For this to be a good definition we need $\mathrm{Ad}\, \mathcal{U}(\Lambda_{l_1,l_2}(t))(\mathcal{N}) = \mathcal{N}$, which trivially holds by modular theory. A look at the proof of Proposition 11 shows that this definition is consistent with the choice in (15). Similarly we define for translated wedges

$$\mathcal{A}(W[l_i,l_j,\vec{a}]) := \mathrm{Ad}\, \mathcal{U}(\vec{a})\mathcal{U}(\Lambda_{l_i,l_j})(\mathcal{N}), \quad \vec{a} \in \mathbf{R}^{1,2}.$$

In this way we associate with any wedge region in $\mathbf{R}^{1,2}$ a unique von Neumann algebra. Obviously this map respects isotony, i.e.

$$W[l_i,l_j,\vec{a}] \subset W[l_i,l_j,\hat{\vec{a}}] \Rightarrow \mathcal{A}(W[l_i,l_j,\vec{a}]) \subset \mathcal{A}(W[l_i,l_j,\hat{\vec{a}}]) \quad (\text{ isotony })$$

[2] Note that the reflection across the plane orthogonal to the basis line of $W[l_1,l_2]$ is not in the group $SO^{\uparrow}(1,2)$. It has determinant = -1.

which follows from the properties of modular inclusions.

We have

$$W[l_i, l_j, \vec{a}] \text{ and } W[\hat{l}_i, \hat{l}_j, \hat{\vec{a}}] \text{ spacelike separated } \Rightarrow$$
$$l_i = \hat{l}_j, l_j = \hat{l}_i, \text{ and } \vec{a} - \hat{\vec{a}} \text{ spacelike.}$$

The definition tells us now

$$\mathcal{A}(W[l_2, l_1]) = \mathrm{Ad}\, \mathcal{U}_{\mathcal{N}'\cap-\mathcal{L}}(1)\mathcal{U}_{\mathcal{L}\cap+\mathcal{M}}(1)\mathcal{U}_{\mathcal{M}\cap-\mathcal{N}}(1)(\mathcal{N}) = \mathcal{N}',$$

see Lemma 4, and using the Lorentz invariance of the construction we immediately get

Lemma 15 *If $W[l_i, l_j, \vec{a}]$ and $W[\hat{l}_i, \hat{l}_j, \hat{\vec{a}}]$ are spacelike separated then*

$$\mathcal{A}(W[l_i, l_j, \vec{a}]) \subset \mathcal{A}(W[\hat{l}_i, \hat{l}_j, \hat{\vec{a}}])'.$$

Now we can define local observable algebras belonging to double cones by taking intersections of appropriate observable algebras associated with wedges. Due to Lorentz covariance, isotony and locality for the wedge algebras, the double cone algebras will inherit these structures. For details one may look up [Bi-Wich], where the sketched construction, starting from a proper set of wedge algebras, is carried out. We should mention here that this construction does not necessarily imply the Reeh-Schlieder property for observables localized in double cones.

In summary we arrive at

Theorem 16 *Let $\mathcal{A}(\mathcal{O}), \mathcal{O} \subset \mathbf{R}^{1,2}$, be a local net fulfilling the Bisognano/Wichmann property for wedges, l_1, l_2, l_3 be linearly independent lightlike vectors, pointing to the future. Let*

$$\mathcal{N} := \mathcal{A}(W[l_1, l_2]),\ \mathcal{M} := \mathcal{A}(W[l_1, l_3]),\ \mathcal{L} := \mathcal{A}(W[l_2, l_3])$$
$$\text{and } \hat{\mathcal{N}} := \mathcal{A}(W[l_1, l_2, l_1]).$$

Then this set of algebras together with the vacuum vector Ω fulfil the assumptions of Theorem 14.

Conversely, given a set of 4 algebras $\mathcal{N}, \mathcal{M}, \mathcal{L}$ and $\hat{\mathcal{N}}$ acting on a Hilbert space $\mathcal{H}$ together with a common cyclic and separating vector $\Omega \in \mathcal{H}$, which *fulfill the assumptions in Theorem 14. Then these data determine a local (Bisognano-Wichmann) net $\mathcal{A}(\mathcal{O}) \subset \mathcal{B}(\mathcal{H}), \mathcal{O} \subset \mathbf{R}^{1,2}$, such that the algebras one started with become wedge algebras of the constructed net as in the first part.*

The generalization of these results to the 1+3 dimensional quantum field theory is under work.

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