Connection formulas for the asymptotics of the fifth Painlevé transcendent on the real axis. II

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Abstract

In this work, we complete the asymptotic description of general solutions of the fifth Painlevé transcendent. In particular, our results contain asymptotics of the solutions with an infinite number of poles in the neighborhood of infinity. Our results are immediately applicable to constructing the connection formulas for small- and large-time asymptotics.

1 Introduction

In this part of our work, we continue the study of asymptotics of solutions of the fifth Painlevé equation, P_5, on the real axis. For the convenience of the reader, let us recall the basic notations from [1].

We consider the following linear ordinary differential equation,

$$\frac{d}{d\lambda}Y = (\frac{t}{2}\sigma_3 + \frac{A_0}{\lambda} + \frac{A_1}{\lambda - 1})Y. \tag{1.1}$$

Here, matrices A_p, $p = 0, 1$, are as follows:

$$A_0 = \begin{pmatrix} z + \frac{\Theta_0}{2} & -u(z+\Theta_0) \\ \frac{z}{u} & -z - \frac{\Theta_0}{2} \end{pmatrix}, \ A_1 = \begin{pmatrix} -z - \frac{\Theta_0+\Theta_\infty}{2} & uy(z + \frac{\Theta_0-\Theta_1+\Theta_\infty}{2}) \\ -\frac{1}{uy}(z + \frac{\Theta_0+\Theta_1+\Theta_\infty}{2}) & z + \frac{\Theta_0+\Theta_\infty}{2} \end{pmatrix}. \tag{1.2}$$

The isomonodromy condition for Eq. (1.1) leads to the following system of nonlinear, ordinary differential equations:

$$\begin{aligned} t\frac{dy}{dt} &= ty - 2z(y-1)^2 - (y-1)(\frac{\Theta_0 - \Theta_1 + \Theta_\infty}{2}y - \frac{3\Theta_0 + \Theta_1 + \Theta_\infty}{2}), \\ t\frac{dz}{dt} &= yz(z + \frac{\Theta_0 - \Theta_1 + \Theta_\infty}{2}) - \frac{1}{y}(z+\Theta_0)(z + \frac{\Theta_0 + \Theta_1 + \Theta_\infty}{2}), \\ t\frac{d}{dt}\log u &= -2z - \Theta_0 + y(z + \frac{\Theta_0 - \Theta_1 + \Theta_\infty}{2}) + \frac{1}{y}(z + \frac{\Theta_0 + \Theta_1 + \Theta_\infty}{2}). \end{aligned} \tag{1.3}$$

It is known [2] that, the first two equations are equivalent to the fifth Painlevé transcendent, P_5:

$$\frac{d^2y}{d^2t} = (\frac{1}{2y} + \frac{1}{y-1})(\frac{dy}{dt})^2 - \frac{dy}{tdt} + \frac{(y-1)^2}{t^2}(\alpha y + \frac{\beta}{y}) + \frac{\gamma y}{t} + \frac{\delta y(y+1)}{y-1}, \tag{1.4a}$$

$$\hat{\alpha} = \frac{1}{2}(\frac{\Theta_0 - \Theta_1 + \Theta_\infty}{2})^2, \hat{\beta} = -\frac{1}{2}(\frac{\Theta_0 - \Theta_1 - \Theta_\infty}{2})^2, \hat{\gamma} = 1 - \Theta_0 - \Theta_1, \hat{\delta} = -\frac{1}{2}. \tag{1.4b}$$

The main object of the theory of isomonodromic deformations of Eq. (1.1) is the so-called manifold of monodromy data, $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$. Here, we briefly repeat the definition of this manifold (see [1] for details).

To define the monodromy data, we have to introduce the canonical solution of Eq. (1.1), $Y_2(\lambda)$. This solution is fixed by its asymptotic expansion

$$Y_2(\lambda) = (I + O(\frac{1}{\lambda}))\exp\{\frac{\lambda t}{2}\sigma_3 - \frac{\Theta_\infty}{2}\ln\lambda\sigma_3\}$$

as $\lambda \to \infty$ in sector

$$-\frac{\pi}{2} < \arg\lambda < \frac{3\pi}{2}.$$

Also, we use the notation $Y_2(\lambda)$ for the analytical continuation of this solution along the paths lying in the upper half-plane, $\Im\lambda > 0$.

The function $Y_2(\lambda)$ has the following expansions as $\lambda \to p$, $p = 0, 1$:

$$Y_2(\lambda) = \sum_{n=0}^{\infty} Y_n^p(\lambda - p)^{n + \frac{\Theta_p}{2}\sigma_3}E^p, \Theta_p \notin \mathbb{Z},$$

where Y^p_n, $E^p \in GL(2,\mathbb{C})$ and independent of λ. The monodromy matrices at the regular singular points ($\lambda = p$, $p = 0, 1$) can be defined as follows:

$$M^p = (E^p)^{-1} e^{\pi \imath \Theta_p \sigma_3} E^p. \tag{1.5}$$

The monodromy matrix at infinity is defined by considering the analytical continuation of Y_2 around the point $\lambda = \infty$, namely,

$$Y_2(\lambda e^{-2\pi \imath}) = Y_2(\lambda) M^\infty.$$

The matrix M^∞ can be parametrized via the complex parameters s_k as follows,

$$M^\infty = \begin{pmatrix} e^{\pi\imath\Theta_\infty} + s_1 s_2 e^{-\pi\imath\Theta_\infty} & s_2 e^{-\pi\imath\Theta_\infty} \\ s_1 e^{-\pi\imath\Theta_\infty} & e^{-\pi\imath\Theta_\infty} \end{pmatrix}.$$

These parameters, s_k, $k = 1, 2$, are called the Stokes multipliers. The manifold of monodromy data, $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$, can be considered as a submanifold in $\mathbb{C}^8$ defined by the following equations,

$$\det M^p = 1,\ \mathrm{tr}\, M^p = 2\cos\pi\Theta_p,\ m^1_{11} m^0_{11} + m^1_{12} m^0_{21} = e^{-\pi\imath\Theta_\infty}, \tag{1.6}$$

where $m^p_{ij} \equiv (M^p)_{ij}$, ($p = 0, 1$, $i, j = 1, 2$) are considered as eight complex coordinates. Note that $\dim_{\mathbb{C}} \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty) = 3$. The parameters m^p_{ij} and s_k (the latters can be uniquely expressed through m^p_{ij}, see (2.13) in [1]) are called the monodromy data. The parameters Θ_p and Θ_∞ are specified as the formal monodromy.

The isomonodromic condition means that the monodromy data are independent of t. Note that, there is one-to-one correspondence between the monodromy data and solutions of system (1.3). In the framework of the isomonodromy theory, we have to parametrize asymptotic expansions (as $t \to +\infty$ and $t \to 0$) of solutions of P_5 by the monodromy data. As soon as this parametrization is obtained, one can easily construct *the connection formulas* (see [1]).

The results for asymptotics as $t \to 0$ for general solutions are obtained in [1]. In this work, we complete the asymptotic description of general solutions of system (1.3) as $t \to +\infty$. To formulate our results, it is convenient to define the following domains of $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$:

$$\begin{aligned} \mathcal{R}_1 &= \{m^p_{ij} \colon m^1_{11} \neq 0, m^1_{12} \neq 0, m^0_{21} \neq 0\}, \\ \mathcal{R}_2 &= \{m^p_{ij} \colon m^1_{11} = 0, m^1_{12} \neq 0, m^0_{21} \neq 0\}, \\ \mathcal{R}_3 &= \{m^p_{ij} \colon m^1_{12} = 0, m^0_{21} \neq 0, m^1_{11} \neq 0\}, \\ \mathcal{R}_4 &= \{m^p_{ij} \colon m^0_{21} = 0, m^1_{12} \neq 0, m^1_{11} \neq 0\}, \\ \mathcal{R}_5 &= \{m^p_{ij} \colon m^1_{12} = 0, m^0_{21} = 0, m^1_{11} \neq 0\}, \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} \mathcal{R}_{3\pm} &= \{m^p_{ij} \colon m^1_{12} = 0, m^1_{11} = e^{\pm\pi\imath\Theta_1}, m^0_{21} \neq 0, m^1_{11} \neq 0\}, \\ \mathcal{R}_{4\pm} &= \{m^p_{ij} \colon m^0_{21} = 0, m^0_{11} = e^{\pm\pi\imath\Theta_0}, m^1_{12} \neq 0, m^1_{11} \neq 0\}. \end{aligned}$$

Proposition 1.1

$$\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty) = \cup_{k=1}^5 \mathcal{R}_k,\ \ \mathcal{R}_k \cap \mathcal{R}_l = \emptyset \ \textit{for}\ k \neq l,$$

and

$$\mathcal{R}_3 = \mathcal{R}_{3+} \cup \mathcal{R}_{3-}, \mathcal{R}_4 = \mathcal{R}_{4+} \cup \mathcal{R}_{4-}.$$

If $\Theta_p \in \mathbb{Z}$, *then* $\mathcal{R}_{4-p} = \mathcal{R}_{(4-p)+} = \mathcal{R}_{(4-p)-}$, *otherwise* $\mathcal{R}_{(4-p)+} \cap \mathcal{R}_{(4-p)-} = \emptyset$.

$$\mathcal{R}_5 \neq \emptyset \ \textit{iff}\ \varepsilon_1\Theta_1 + \varepsilon_0\Theta_0 + \Theta_\infty \in \mathbb{Z},\ \textit{where}\ \varepsilon_p = \pm 1.$$

The proof of Proposition 1.1 contains nothing but the analysis of the definition of the manifold of monodromy data (1.6).

Remark 1.1 Let us present, for each domain $\mathcal{R}_k$, a list of proper coordinates:

$$\mathcal{R}_1\colon\{m^1_{11}, m^1_{12}, m^0_{21}\}, \mathcal{R}_2\colon\{m^0_{11}, m^1_{12}\} \text{ or } \{m^0_{11}, m^0_{21}\},$$
$$\mathcal{R}_{3\pm}\colon\{m^1_{21}, m^0_{21}\}, \mathcal{R}_{4\pm}\colon\{m^0_{12}, m^1_{12}\}, \mathcal{R}_5\colon\{m^1_{21}, m^0_{12}\}.$$

The notation $\mathcal{R}_{3+} : \{m^1_{21}, m^0_{21}\}$, for example, means the following: to uniquely determine the position of a point on $\mathcal{R}_{3+}$, it is sufficient to know two coordinates of the point, m^1_{21} and m^0_{21}. To uniquely determine the position of the point on the whole of $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$, it is necessary to know whether the point lies on $\mathcal{R}_{3+}$ or on $\mathcal{R}_{3-}$.

In our previous work [1], we found and parametrized the asymptotics of solutions of (1.3) corresponding to the subdomain $\mathcal{R}_1^{\mathrm{reg}}$,

$$\mathcal{R}_1^{\mathrm{reg}}\colon= \{m^p_{ij} \in \mathcal{R}_1\colon -\frac{\pi}{2} < \Im \ln(1 - m^0_{11} m^1_{11} e^{\pi\imath\Theta_\infty}) < \frac{\pi}{2}\}.$$

In this work, we find and parametrize, by the monodromy data, asymptotics of the solutions of (1.3) corresponding to the case $\mathcal{R}_1\backslash\mathcal{R}_1^{\mathrm{reg}}$: also, we completely characterize $\mathcal{R}_2$.

The organization of this paper is as follows. Sec. 2 contains the results for general solutions corresponding to the domain $\mathcal{R}_1\backslash\mathcal{R}_1^{\mathrm{reg}}$. The results for special solutions from $\mathcal{R}_1\backslash\mathcal{R}_1^{\mathrm{reg}}$ and $\mathcal{R}_2$, as well as sketches of their proofs are given in Sec. 3. In Sec. 4, we present an outline of the derivation of the results of Sec. 2. Appendix contains a simpler calculation of m^0_{21} and m^1_{12}.

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2 General solutions

For $\beta \in \mathbb{C}$ and $\hat{v} \in \mathbb{C}\backslash\{0\}$ define the set $\mathcal{T}_{\beta,\hat{v}} \equiv \{t_k\}_{k\in\mathbb{Z}}$, where $t_k \in \mathbb{R}^+$ and satisfies the the following equation

$$\frac{t_k}{4} + (\frac{1}{4\imath} - \frac{\beta}{2\imath})\ln t_k - \frac{1}{2\imath}\ln(-\frac{\hat{v}}{\sqrt{2}}e^{\frac{\pi\imath}{4}}) = \frac{\pi}{4}k.$$

Define the domain

$$\Omega_{\varepsilon,\beta,\hat{v}} = \{t \in \mathbb{R}^+\backslash\mathcal{T}_{\beta,\hat{v}}\colon \mathrm{dist}(t, T_{\beta,\hat{v}}) > \varepsilon\}.$$

Note that, if $\Re\beta \neq \frac{1}{2}$ or $|\hat{v}| \neq 1$, then $\mathcal{T}_{\beta,\hat{v}} = \emptyset$ and hence $\Omega_{\varepsilon,\beta,\hat{v}} = \mathbb{R}^+$.

Theorem 2.1 *Let $\beta \in \mathbb{C}$, $\frac{1}{4} < \Re\beta < \frac{3}{4}$, $\hat{u} \in \mathbb{C}\backslash\{0\}$, and $\hat{v} \in \mathbb{C}\backslash\{0\}$. There exists the unique solution of system (1.3) with asymptotic expansion*

$$y = \frac{\cos^2\tilde{x}}{\sin^2\tilde{x}} + O(t^{-1+2|\Re\beta-\frac{1}{2}|}), z = \frac{\cos\tilde{x}\sin\tilde{x}}{4(\cos\tilde{x} - \sin\tilde{x})^2}t + O(t^{2|\Re\beta-\frac{1}{2}|}), u = \hat{u}e^{\frac{t}{2}+\frac{\pi\imath}{2}}\frac{\sin\tilde{x}}{\cos\tilde{x}}(1 + o(1)), \tag{2.1}$$

as $t \to \infty$, $t \in \Omega_{\varepsilon,\beta,\hat{v}}$, where

$$\tilde{x} \equiv \frac{t}{4} + (\frac{1}{4\imath} - \frac{\beta}{2\imath})\ln t - \frac{1}{2\imath}\ln(-\frac{\hat{v}}{\sqrt{2}}e^{\frac{\pi\imath}{4}}) \tag{2.2}$$

and the branch of the second logarithm on the r.-h.s. is arbitrary, but the same in all formulas.

Theorem 2.2 *The solution of system (1.3), described in Theorem 2.1, defines the isomonodromy deformation of Eq. (1.1) with the following monodromy data,*

$$
\begin{aligned}
m_{11}^0 &= -\frac{\sqrt{2\pi}}{\hat{v}\Gamma(\beta)}\exp(\beta\ln 2 - \frac{\pi\imath\Theta_\infty}{2} + \frac{\pi\imath\beta}{2}),\ m_{21}^0 = \frac{1}{\hat{u}}\exp(\pi\imath\beta - \frac{\pi\imath}{2} + 2\Theta_\infty\ln 2 - \pi\imath\Theta_\infty),\\
m_{11}^1 &= \frac{\sqrt{2\pi}\hat{v}}{\Gamma(1-\beta)}\exp(\frac{\pi\imath}{2} - \beta\ln 2 - \frac{\pi\imath\Theta_\infty}{2} + \frac{\pi\imath\beta}{2}),\ m_{12}^1 = \hat{u}\exp(\pi\imath\beta + \frac{\pi\imath}{2} - 2\Theta_\infty\ln 2),\\
s_2 &= -\hat{u}\exp(-2\Theta_\infty\ln 2 - \pi\imath\beta + \pi\imath\Theta_\infty)\{\frac{\sqrt{2\pi}\hat{v}}{\Gamma(1-\beta)}\exp(\frac{\pi\imath\Theta_\infty}{2} - \beta\ln 2 + \frac{\pi\imath\beta}{2}) +\\
&+ \frac{\imath\sqrt{2\pi}}{\hat{v}\Gamma(\beta)}\exp(\beta\ln 2 - \frac{\pi\imath\Theta_\infty}{2} + \frac{\pi\imath\beta}{2}) + 2\imath\cos\pi\Theta_0\},\\
s_1 &= \frac{1}{\hat{u}}\exp(2\Theta_\infty\ln 2 - \pi\imath\beta)\{\frac{\sqrt{2\pi}\hat{v}}{\Gamma(1-\beta)}\exp(-\frac{\pi\imath\Theta_\infty}{2} - \beta\ln 2 + \frac{\pi\imath\beta}{2}) +\\
&+ \frac{\imath\sqrt{2\pi}}{\hat{v}\Gamma(\beta)}\exp(\beta\ln 2 + \frac{\pi\imath\Theta_\infty}{2} + \frac{\pi\imath\beta}{2}) + 2\imath\cos\pi\Theta_1\}.
\end{aligned}
$$

Remark 2.1 The remaining monodromy data can be computed via formulas (1.6).

Remark 2.2 In the case when $|\Re\beta| \neq \frac{1}{2}$, it is natural to rewrite (2.1) in the following way:
(1) if $\frac{1}{4} < \Re\beta < \frac{1}{2}$, then

$$
y = -1 + 2\sqrt{2}e^{\frac{\pi\imath}{4}}\hat{v}t^{\beta-\frac{1}{2}}e^{\frac{-\imath t}{2}} + O(t^{-1+2\Re\beta}),\ \ z = -\frac{t}{8} - \frac{\hat{v}}{4\sqrt{2}}e^{-\frac{\pi\imath}{4}}t^{\beta+\frac{1}{2}}e^{\frac{-\imath t}{2}} + O(t^{2\Re\beta}), \tag{2.3a}
$$

$$
u = \hat{u}e^{\frac{t}{2}}(1+o(1)),
$$

(2) if $\frac{1}{2} < \Re\beta < \frac{3}{4}$, then

$$
y = -1 + \frac{4\sqrt{2}}{\hat{v}}e^{\frac{-\pi\imath}{4}}t^{-\beta+\frac{1}{2}}e^{\frac{\imath t}{2}} + O(t^{1-2\Re\beta}) = -1 - \frac{2\sqrt{2}e^{\frac{\pi\imath}{4}}}{\tilde{v}}\tilde{\beta}e^{\frac{\imath t}{2}}t^{-\frac{1}{2}-\tilde{\beta}} + O(t^{-1-2\Re\tilde{\beta}}),
$$

$$
z = -\frac{t}{8} - \frac{\imath\sqrt{2}}{4\hat{v}}e^{-\frac{\pi\imath}{4}}t^{-\beta+\frac{3}{2}}e^{\frac{\imath t}{2}} + O(t^{2-2\Re\beta}) = -\frac{t}{8} - \frac{\tilde{\beta}}{4\sqrt{2}\tilde{v}}e^{-\frac{\pi\imath}{4}}e^{\frac{\imath t}{2}}t^{\frac{1}{2}-\tilde{\beta}} + O(t^{-2\Re\tilde{\beta}}), \tag{2.3b}
$$

$$
u = -\hat{u}e^{\frac{t}{2}}(1+o(1)) = \tilde{u}e^{\frac{t}{2}},
$$

where

$$
\tilde{\beta} = \beta - 1,\ \tilde{v} = \frac{v(\beta-1)}{2i},\ \tilde{u} = -\hat{u}. \tag{2.4}
$$

Thus, Theorem 2.1 completes the proof of the statement made in Remark 3.1 [1], where it was stated that formulas (2.3a,b) are valid for $-\frac{1}{2} < \beta < \frac{1}{2}$.

Remark 2.3 Complete asymptotic expansions for the functions $y(t)$, $z(t)$, and $\zeta(t)$ (see Eq. (2.7) below) can be constructed in a manner analogous to Appendix A of [1].

Theorem 2.3 *To any point $\mathcal{M} \in \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ with coordinates satisfying the restrictions*

$$
m_{21}^0 \neq 0,\ m_{12}^1 \neq 0,\ m_{11}^1 \neq 0,\ m_{11}^0 \neq 0,\ \frac{\pi}{2} < \Im\ln(1 - m_{11}^0 m_{11}^1 e^{\pi\imath\Theta_\infty}) < \frac{3\pi}{2} \tag{2.5}
$$

there corresponds the unique solution of system (1.3) with asymptotics given by formulas (2.1), *where the parameters β, $\hat{u}$, and $\hat{v}$ are given as follows,*

$$
\beta = \frac{1}{2\pi\imath}\ln(1 - m_{11}^0 m_{11}^1 e^{\pi\imath\Theta_\infty}),\ \hat{u} = \frac{2^{2\Theta_\infty}e^{\pi\imath(\beta-\Theta_\infty)}}{\imath m_{21}^0},\ \hat{v} = \frac{m_{11}^1}{\imath\sqrt{2\pi}}\Gamma(1-\beta)2^{\beta}e^{\frac{\pi\imath}{2}(\Theta_\infty-\beta)}. \tag{2.6}
$$

Remark 2.3 One can see that Theorems 3.1, 3.2, 3.3, and 3.4 of reference [1] describe the region $-\frac{1}{4} < \Re\beta < \frac{1}{4}$, while Theorems 2.1, 2.2, and 2.3 describe the region $\frac{1}{4} < \Re\beta < \frac{3}{4}$. In Remark 2.2, using transformation (2.4), we characterize the region $-\frac{1}{2} < \Re\beta < -\frac{1}{4}$. As the obtained asymptotics match and the corresponding monodromy data coincide, we conclude that, formulas (2.3a) and (2.3b) are valid for $\Re\beta = \frac{1}{4}$ and $\Re\tilde{\beta} = -\frac{1}{4}$, respectively. This result, together with Theorem 2.3, shows that (2.1) gives us asymptotics of solutions of system (1.3) corresponding to the $\mathcal{R}_1\backslash\mathcal{R}_1^{\mathrm{reg}}$ with $m_{11}^0 \neq 0$. The solutions with $m_{11}^0 = 0$ are considered in the next section, Corollary 3.1.

The important role in applications of P_5 is played by the so-called τ-function and, closely related with it (see [3] and [1]), the function

$$\zeta(t) = \frac{1}{4}((\Theta_0+\Theta_\infty)^2-\Theta_1^2)-zt-(z-\frac{1}{y}(z+\frac{\Theta_0+\Theta_1+\Theta_\infty}{2}))(z+\Theta_0-y(z+\frac{\Theta_0-\Theta_1+\Theta_\infty}{2})).$$

Corollary 2.1 *The function $\zeta(t)$ corresponding to the solution of system (1.3) defined by Theorem 2.1 has the following asymptotic expansion as $t \to +\infty$, $t \in \Omega_{\varepsilon,\beta,\hat{v}}$,*

$$\zeta(t) = \frac{t^2}{16} + (\Theta_\infty + 2\Theta_0 + 2\imath\beta - \imath + \frac{\sin\tilde{x}+\cos\tilde{x}}{\sin\tilde{x}-\cos\tilde{x}})\frac{t}{4} + o(t), \tag{2.7}$$

where $\tilde{x}$ is given by (2.2).

3 Special solutions

Corollary 3.1 *Let $\hat{u}, \hat{v} \in \mathbb{C}\backslash\{0\}$. Then, there exists the unique solution of system (1.3) with asymptotic expansion as $t \to +\infty$*

$$\begin{aligned} y &= -1 + 2\sqrt{\frac{2}{t}}e^{\frac{\pi\imath}{4}}\hat{v}e^{\frac{-\imath t}{2}} + O(t^{-1}),\ u = \hat{u}e^{\frac{t}{2}}(1+o(1)), \\ z &= -\frac{t}{8} - \frac{1}{4}\sqrt{\frac{t}{2}}e^{\frac{-\pi\imath}{4}}\hat{v}e^{\frac{-\imath t}{2}} - \frac{\imath\hat{v}^2}{4}e^{-\imath t} - \frac{2\Theta_0+\Theta_\infty}{4} + O(t^{-\frac{1}{2}}). \end{aligned} \tag{3.1}$$

The correspondig function $\zeta(t)$ has the following asymptotic expansion,

$$\zeta(t) = \frac{t^2}{16} + (\Theta_\infty+2\Theta_0)\frac{t}{4} + \sqrt{\frac{t}{2}}\frac{e^{\frac{\pi\imath}{4}}}{2}\hat{v}e^{-\imath t/2} + \frac{\hat{v}^2}{2}e^{-\imath t} + \frac{3}{4}\Theta_0^2 + \frac{1}{4}\Theta_1^2 + \frac{1}{4}\Theta_\infty^2 + \frac{1}{2}\Theta_0\Theta_\infty - \frac{1}{4} + O(t^{-\frac{1}{2}}).$$

Solution (3.1) defines the isomonodromy deformation of Eq. (1.1) with the following monodromy data,

$$\begin{aligned} m_{11}^0 &= 0, m_{21}^0 = \frac{2^{2\Theta_\infty}}{\imath\hat{u}}e^{-\pi\imath\Theta_\infty}, m_{11}^1 = \imath\sqrt{2\pi}\hat{v}e^{-\frac{\pi\imath}{2}\Theta_\infty}, m_{12}^1 = \imath\hat{u}2^{-2\Theta_\infty}, \\ s_1 &= \frac{2^{2\Theta_\infty}}{\hat{u}}[\sqrt{2\pi}\hat{v}e^{-\frac{\pi\imath}{2}\Theta_\infty} + 2\imath\cos\pi\Theta_1],\ s_2 = -\hat{u}2^{-2\Theta_\infty}e^{\pi\imath\Theta_\infty}[\sqrt{2\pi}\hat{v}e^{\frac{\pi\imath}{2}\Theta_\infty} + 2\imath\cos\pi\Theta_0]. \end{aligned}$$

Inversely, to any point $\mathcal{M} \in \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ with coordinates satisfying the restrictions

$$m_{11}^0 = 0, m_{21}^0 \neq 0, m_{11}^1 \neq 0, m_{12}^1 \neq 0$$

there corresponds the unique solution of system (1.3) with asymptotics given by formulas (3.1), *where the parameters $\hat{u}$ and $\hat{v}$ are as follows,*

$$\hat{u} = \frac{2^{2\Theta_\infty}}{\imath m_{21}^0}e^{-\pi\imath\Theta_\infty},\ \hat{v} = \frac{m_{11}^1}{\imath\sqrt{2\pi}}e^{\frac{\pi\imath}{2}\Theta_\infty}.$$

Corollary 3.2 *Let $\hat{u}, \hat{w} \in \mathbb{C}\backslash\{0\}$. Then, there exists a solution of system (1.3) with asymptotic expansion as $t \to +\infty$*

$$
\begin{aligned}
y &= -1 + 2\sqrt{\frac{2}{t}} e^{\frac{\pi\imath}{4}} \hat{w} e^{\frac{\imath t}{2}} + O(t^{-1}), \; u = \hat{u} e^{\frac{t}{2}}(1+o(1)), \\
z &= -\frac{t}{8} + \frac{1}{4}\sqrt{\frac{t}{2}} e^{\frac{-\pi\imath}{4}} \hat{w} e^{\frac{\imath t}{2}} + \frac{\imath \hat{w}^2}{4} e^{\imath t} - \frac{2\Theta_0 + \Theta_\infty}{4} + O(t^{-\frac{1}{2}}).
\end{aligned} \tag{3.2}
$$

The corresponding asymptotic expansion of function $\zeta(t)$ is

$$
\zeta(t) = \frac{t^2}{16} + (\Theta_\infty + 2\Theta_0)\frac{t}{4} + \sqrt{\frac{t}{2}} \frac{e^{\frac{\pi\imath}{4}}}{2} \hat{w} e^{\imath t/2} - \frac{\hat{w}^2}{2} e^{\imath t} + \frac{3}{4}\Theta_0^2 + \frac{1}{4}\Theta_1^2 + \frac{1}{4}\Theta_\infty^2 + \frac{1}{2}\Theta_0\Theta_\infty - \frac{1}{4} + O(t^{-\frac{1}{2}}).
$$

Solution (3.2) defines the isomonodromy deformation of Eq. (1.1) with the following monodromy data,

$$
\begin{aligned}
m_{11}^0 &= \sqrt{2\pi} \hat{w} e^{-\frac{\pi\imath}{2}\Theta_\infty}, m_{11}^1 = 0, m_{21}^0 = \frac{2^{2\Theta_\infty}}{\imath \hat{u}} e^{-\pi\imath\Theta_\infty}, m_{12}^1 = \imath \hat{u} 2^{-2\Theta_\infty}, \\
s_1 &= \frac{2^{2\Theta_\infty}}{\imath \hat{u}}[\sqrt{2\pi} \hat{w} e^{\frac{\pi\imath}{2}\Theta_\infty} - 2\cos\pi\Theta_1], \; s_2 = \imath \hat{u} 2^{-2\Theta_\infty} e^{\pi\imath\Theta_\infty}[\sqrt{2\pi}\hat{w} e^{-\frac{\pi\imath}{2}\Theta_\infty} - 2\cos\pi\Theta_0].
\end{aligned}
$$

Inversely, to any point $\mathcal{M} \in \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ with coordinates satisfying the restrictions

$$
m_{11}^1 = 0, \; m_{21}^0 \neq 0, \; m_{11}^0 \neq 0, \; m_{12}^1 \neq 0,
$$

there corresponds the unique solution of system (1.3) with asymptotics given by formulas (3.2), *where the parameters $\hat{u}$ and $\hat{w}$ are as follows,*

$$
\hat{u} = \frac{2^{2\Theta_\infty}}{\imath m_{21}^0} e^{-\pi\imath\Theta_\infty}, \; \hat{w} = \frac{m_{11}^0}{\sqrt{2\pi}} e^{\frac{\pi\imath}{2}\Theta_\infty}.
$$

Proof of Corollaries 3.1 and 3.2. These corollaries are obtained from the results formulated in Sec. 3 of [1]. Let us denote $\hat{w} = -\beta_0/\hat{v}$ (see notations in Sec. 3 of [1]). Then, to get the results formulated in Corollary 3.2 (Corollary 3.1), one takes the limit passage as $\hat{v} \to 0$ ($\hat{w} \to 0$) in formulas (3.1)-(3.5) [1]. To justify the possibility of taking such a limit, we note that, all the error estimates of Sec. 3 are polynomialss with respect to $\hat{v}$ and $\hat{w}$.

Corollary 3.3 *Let $\hat{u} \in \mathbb{C}\backslash\{0\}$. Then, there exists the unique solution of system (1.3) with asymptotic expansion as $t \to +\infty$*

$$
y = -1 + \frac{4(\Theta_0 + \Theta_1 - 1)}{t} + O(t^{-2}), \; z = -\frac{t}{8} - \frac{1}{4}(2\Theta_0 + \Theta_\infty) + O(t^{-1}), \; u = \hat{u} e^{\frac{t}{2}}(1+o(1)). \tag{3.3}
$$

The corresponding asymptotic expansion of function $\zeta(t)$ is

$$
\zeta(t) = \frac{t^2}{16} + (\Theta_\infty + 2\Theta_0)\frac{t}{4} + \frac{3}{4}\Theta_0^2 + \frac{1}{4}\Theta_1^2 + \frac{1}{4}\Theta_\infty^2 + \frac{1}{2}\Theta_0\Theta_\infty - \frac{1}{4} + O(t^{-1}).
$$

Solution (3.3) defines the isomonodromy deformation of Eq. (1.1) with the following monodromy data,

$$
\begin{gathered}
m_{11}^0 = 0, \; m_{21}^0 = \tfrac{2^{2\Theta_\infty}}{\imath \hat{u}} e^{-\pi\imath\Theta_\infty}, m_{11}^1 = 0, m_{12}^1 = \imath \hat{u} 2^{-2\Theta_\infty}, \\
s_2 = -2\imath \hat{u} e^{\pi\imath\Theta_\infty} 2^{-2\Theta_\infty} \cos\pi\Theta_0, \; s_1 = \tfrac{2\imath}{\hat{u}} 2^{2\Theta_\infty} \cos\pi\Theta_1.
\end{gathered}
$$

Inversely, to any point $\mathcal{M} \in \mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$ with coordinates satisfying the restrictions

$$m_{11}^0 = m_{11}^1 = 0,\; m_{21}^0 \neq 0,\; m_{12}^1 \neq 0$$

there corresponds the unique solution of system (1.3) with asymptotics given by formulas (3.3), *where* $\hat{u} = \frac{2^{2\Theta_\infty}}{\imath m_{21}^0} e^{-\pi\imath\Theta_\infty}$.

Proof. The proof is analogous to the proof of Corollaries 3.1 and 3.2. To get the results formulated in Corollary 3.3, one takes the double limit passage as $\hat{v} \to 0$ and $\hat{w} \to 0$ in formulas (3.1)-(3.5) [1]; then, we use results of Appendix A [1].

Remark 3.1 According to Remark 2.3, Corollary 3.1 completes the asymptotic description of the solutions corresponding to the domain $\mathcal{R}_1$ of $\mathcal{M}_5(\Theta_0, \Theta_1, \Theta_\infty)$. Corollaries 3.2 and 3.3 give us the complete asymptotic description of solutions corresponding to $\mathcal{R}_2$ (see (1.7)).

4 Derivation of the results of Sec. 2

A general scheme for the justification of the asymptotics in the framework of the Isomonodromy Deformation Method has been given in Introduction of [1]. Here we present only the formal steps related to this scheme.

It is convenient to remove the function $u = u(t)$ from Eq. (1.1) via the following transformation,

$$Y_2(\lambda) = u^{\frac{1}{2}\sigma_3} Y(\lambda). \tag{4.1}$$

Then, the function $Y = Y(\lambda)$ solves the following equation,

$$\partial_\lambda Y = t\mathcal{A}(\lambda, t)Y; \tag{4.2}$$

here,

$$\mathcal{A}(\lambda, t) = \frac{1}{2}\sigma_3 + \frac{\hat{A}_0}{\lambda} + \frac{\hat{A}_1}{\lambda - 1}.$$

Matrices $\hat{A}_p$, $p = 0, 1$, are as follows:

$$\hat{A}_0 = \begin{pmatrix} \hat{z} + \frac{\Theta_0}{2t} & -\hat{z} - \frac{\Theta_0}{t} \\ \hat{z} & -\hat{z} - \frac{\Theta_0}{2t} \end{pmatrix}, \hat{A}_1 = \begin{pmatrix} -\hat{z} - \frac{\Theta_0 + \Theta_\infty}{2t} & y(\hat{z} + \frac{\Theta_0 - \Theta_1 + \Theta_\infty}{2t}) \\ -\frac{1}{y}(\hat{z} + \frac{\Theta_0 + \Theta_1 + \Theta_\infty}{2t}) & \hat{z} + \frac{\Theta_0 + \Theta_\infty}{2t} \end{pmatrix}, \hat{z} \equiv \frac{z}{t}.$$

The calculation of the monodromy data is carried out using the asymptotic matching procedure. This procedure is quite analogous to one presented in the first part of this work [1], but here, we use different asymptotic conditions on the functions $y(t)$ and $z(t)$:

$$\hat{z} \equiv z/t = O(1),\; y = O(1),\; \frac{d}{dt}\hat{z} = O(1),\; \frac{d}{dt}y = O(1) \text{ as } t \to +\infty, \text{ for } t \in \Omega_{\epsilon,\beta,\hat{v}}. \tag{4.3}$$

Note that, in [1], we considered the following conditions as $t \to +\infty$: $y \to -1$, $\hat{z} \to -\frac{1}{8}$ and $\frac{d}{dt}y, \frac{d}{dt}\hat{z} = o(1)$.

As in [1], we use the WKB-method to construct asymptotic expansion for $Y(\lambda)$. Since in the neigborhoods of the regular singularities of Eq. (4.2), $\lambda = 0, 1$, and in the neighborhood of the turning point, $\lambda = \frac{1}{2}$, the WKB-asymptotics fail, we use proper model equations to construct the asymptotics of $Y(\lambda)$ at these points. Below, we present the necessary details for our asymptotic constructions.

4.1 WKB-formula

The leading term of the WKB-asymptotics for the fundamental solutions of Eq. (4.2), $\Psi_q(\lambda)$ in the corresponding canonical domain Γ_q (see [1] for definition), is given by the following equation,

$$\Psi_q(\lambda) = T \exp(t\sigma_3 \int_{\lambda_q^*}^{\lambda} l(\lambda)\, d\lambda - \int_{\lambda_q^*}^{\lambda} \operatorname{diag} T^{-1} \partial_\lambda T \, d\lambda), \tag{4.4}$$

where the path of integration $[\lambda_q^*, \lambda] \in \Gamma_q$. Here, the matrix T is defined as follows:

$$T^{-1} \mathcal{A} T = l(\lambda)\sigma_3, \;\; l(\lambda)^2 = -\det \mathcal{A}.$$

According to (2.1), in addition to (4.3), we have that

$$t l^2(\frac{1}{2}) = 2\imath\beta_0 \underset{t\to+\infty}{=} O(1). \tag{4.5}$$

To specify the concrete solutions of Eq. (4.2), we have to define a canonical domain in which these solutions have WKB-asymptotics (4.4). These canonical domains are defined by the Stokes graphs (see details in [1]). Conditions (4.3) guarantee that, topologically, the Stokes graphs will be the same as in [1]. In particular, $\lambda = \frac{1}{2}$ remains the double turning point.

The difference is that, here, we need to calculate the integral

$$\int_{\lambda_q^*}^{\lambda} \operatorname{diag} T^{-1} \partial_\lambda T \, d\lambda.$$

This integral makes a contribution to the WKB-formula (4.4):

$$\int_{\lambda_q^*}^{\lambda} \operatorname{diag} T^{-1} \partial_\lambda T \, d\lambda \underset{t\to+\infty}{=} \hat{z}^2 (y - \tfrac{1}{y})(1 + o(1))\sigma_3 \int_{\lambda_q^*}^{\lambda} \frac{d\lambda}{\sqrt{\lambda(\lambda-1)}} \frac{1}{\lambda - 1/2} \frac{1}{\lambda(\lambda-1) - 2\hat{z} + (\lambda - 1/2)\sqrt{\lambda(\lambda-1)}} =$$

$$\underset{t\to+\infty}{=} (F_1(\lambda) - F_1(\lambda^*))(1 + o(1))\sigma_3,$$

where

$$F_1(\lambda) = \beta_1 \ln \frac{\imath\lambda - \sqrt{\lambda(\lambda-1)}}{\imath\lambda + \sqrt{\lambda(\lambda-1)}} - \frac{1}{2} \ln \frac{\lambda(\gamma + \sqrt{\gamma^2 - 1}) + \sqrt{\lambda(\lambda-1)}}{\lambda(\gamma - \sqrt{\gamma^2 - 1}) + \sqrt{\lambda(\lambda-1)}}, \beta_1 = \frac{2\imath\hat{z}}{\sqrt{y}}(y-1), \gamma = -1 - \frac{1}{8\hat{z}}. \tag{4.6}$$

The square root in the previous expressions is defined in the following way, $\sqrt{\lambda(\lambda-1)} \to \lambda$ as $\lambda \to +\imath\infty$.

We can rewrite the WKB-formula (4.4) as follows,

$$\Psi_q(\lambda) \underset{t\to+\infty}{=} T \exp\{\sigma_3(F(\lambda) - F(\lambda_q^*) + o(1))\}, \quad \lambda \in \Gamma_q.$$

Here,

$$F(\lambda) = F_0(\lambda) - F_1(\lambda), \tag{4.7}$$

where

$$F_0(\lambda) = t \int l(\lambda)\, d\lambda$$

was calculated in [1],

$$F_0(\lambda) = \frac{t}{2}\sqrt{\lambda(\lambda-1)} - \frac{\Theta_\infty}{2} \ln(2\lambda - 1 + 2\sqrt{\lambda(\lambda-1)}) - \beta_0 \ln \frac{2\imath\sqrt{\lambda(\lambda-1)} - 1}{2\lambda - 1}.$$

Below, we present the asymptotics of $F_0(\lambda)$, $F_1(\lambda)$, and T near the singular and turning points.

As $\lambda \to +\imath\infty$, we have

$$\begin{aligned}F_0(\lambda) &= \frac{\lambda t}{2} - \frac{\Theta_\infty}{2}\ln\lambda - \frac{t}{4} - \Theta_\infty \ln 2 - \frac{\pi\imath\beta_0}{2} + O(\frac{t}{\lambda}) + O(\frac{\ln\lambda}{\lambda}),\\ F_1(\lambda) &= \frac{\pi\imath\beta_1}{2} - \frac{1}{2}\ln\frac{\sqrt{\gamma^2-1}+\gamma+1}{-\sqrt{\gamma^2-1}+\gamma+1} + O(\lambda^{-1}),\ T = \imath\sigma_3(1+O(\lambda^{-1})).\end{aligned}$$

As $\lambda \to +0$ ($\lambda \in \mathbb{R}$) on the upper side of the cut $[0,1]$, we have

$$F_0(\lambda) = \tfrac{\imath t}{2}\sqrt{\lambda}(1+O(\lambda)) - \tfrac{\pi\imath\Theta_\infty}{2} + O(\lambda),\ \ F_1(\lambda) = \pi\imath\beta_1 + O(\lambda),$$
$$T \sim \sigma_3(-\tfrac{1}{4\imath\sqrt{\lambda}}) + \tfrac{\hat{A}_0}{\lambda} - \hat{A}_1 + \tfrac{1}{2}\sigma_3 + O(\lambda) + O(\sqrt{\lambda})\sigma_3 + O(\tfrac{1}{t\sqrt{\lambda}})\sigma_3.$$

As $\lambda \to 1$ ($\lambda \in \mathbb{R}$), $1 > \lambda$ on the upper side of the cut $[0,1]$, we have

$$F_0(\lambda) = \frac{\imath t}{2}\sqrt{1-\lambda}(1+O(1-\lambda)) - \pi\imath\beta_0 + O(1-\lambda),\ F_1(\lambda) = -\frac{1}{2}\ln\frac{\sqrt{\gamma^2-1}+\gamma}{-\sqrt{\gamma^2-1}+\gamma} + O(1-\lambda),$$
$$T \sim \sigma_3(\frac{1}{4\imath\sqrt{1-\lambda}}) - \frac{\hat{A}_1}{1-\lambda} + \hat{A}_0 + \frac{1}{2}\sigma_3 + O(\sqrt{1-\lambda}) + O(\frac{1}{t\sqrt{1-\lambda}})\sigma_3.$$

As $\lambda \to 1/2$, we have

$$\begin{aligned}F_0(\lambda) &= \frac{\imath t}{4}(1+O(\Lambda^2)) - \frac{\imath\mu^2}{2} - \frac{\pi\imath\Theta_\infty}{4} - \beta_0\ln(-\Lambda^{-1}) + O(\Lambda\ln\Lambda),\\ F_1(\lambda) &= \beta_1\ln(\Lambda)(1+O(\Lambda^2)) - \frac{1}{2}\ln\frac{\sqrt{\gamma^2-1}+\gamma+\imath}{-\sqrt{\gamma^2-1}+\gamma+\imath} + O(\Lambda),\end{aligned}$$
$$\begin{aligned}T \sim\ &\sigma_3(-\imath\Lambda)(1+2\Lambda^2+O(\Lambda^3))(1+\tfrac{1}{2t\Lambda^2}(2\imath\beta_0 - 2\imath\Theta_\infty\Lambda) + o(t^{-1})) +\\ &+\tfrac{1}{2}\sigma_3 + 2(\hat{A}_0 - \hat{A}_1) - 4\Lambda(\hat{A}_0+\hat{A}_1) + 8\Lambda^2(\hat{A}_0 - \hat{A}_1) + O(\Lambda^3),\end{aligned}$$

where the symbol $\sim$, as in [1], means that the quantities are equal up to a scalar factor and

$$\Lambda = \lambda - \frac{1}{2}. \tag{4.8}$$

4.2 Solutions near the points 0 and 1

Let us introduce the model function

$$\Phi(x,\Theta) = x\begin{pmatrix} J_{-1+\Theta}(x) & J_{1-\Theta}(x)\\ J_{1+\Theta}(x) & J_{-1-\Theta}(x)\end{pmatrix},\quad \det\Phi(x,\Theta) = -\frac{4\Theta\sin\pi\Theta}{\pi},$$

where $J_{(\cdot)}(x)$ is the Bessel function and $\Theta \notin \mathbb{Z}$. It has the following asymptotics [4], [5]:

$$\Phi(x,\Theta) \underset{\substack{x\to 0\\ |\arg x|<\pi}}{=} 2\,\mathrm{diag}\{\frac{2^{-\Theta}}{\Gamma(\Theta)}, \frac{2^{\Theta}}{\Gamma(-\Theta)}\}(1+O(x^2))x^{\Theta\sigma_3},$$

$$\Phi^{-1}(x,\Theta) \underset{\substack{x\to\infty\\ |\arg x|<\pi}}{\sim} \begin{pmatrix} \cos(x+\frac{\pi}{4}+\frac{\pi\Theta}{2}) & \cos(x+\frac{\pi}{4}+\frac{\pi\Theta}{2})\\ \cos(x+\frac{\pi}{4}-\frac{\pi\Theta}{2}) & \cos(x+\frac{\pi}{4}-\frac{\pi\Theta}{2})\end{pmatrix}(1+O(x^{-2})) -$$
$$-\frac{1}{2x}\begin{pmatrix} ((1+\Theta)^2-\frac{1}{4})\sin(x+\frac{\pi}{4}+\frac{\pi\Theta}{2}) & ((1-\Theta)^2-\frac{1}{4})\sin(x+\frac{\pi}{4}+\frac{\pi\Theta}{2})\\ ((1+\Theta)^2-\frac{1}{4})\sin(x+\frac{\pi}{4}-\frac{\pi\Theta}{2}) & ((1-\Theta)^2-\frac{1}{4})\sin(x+\frac{\pi}{4}-\frac{\pi\Theta}{2})\end{pmatrix}(1+O(x^{-2})),$$

and satisfies the model ordinary differential equation,

$$\frac{d}{dx}\Phi(x,\Theta) = (-\frac{x}{2\Theta}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \frac{\Theta}{x}\sigma_3)\Phi(x,\Theta).$$

Proposition 4.1 *There exists a solution $Y(0,\lambda)$ of Eq.(4.2) with the following asymptotic expansion*

$$Y(0,\lambda) \underset{t\to+\infty}{\sim} (\frac{\Theta_0}{2}\sigma_3 + t\hat{A}_0)\sigma_3\Phi(\frac{\sqrt{\lambda}t}{2},\Theta_0)(1+O(t^{-1})), \lambda \le O(t^{-2+\varepsilon}), \lambda\in[0,1]^+.$$

Here and below

$$0 < \varepsilon < \frac{1}{6}.$$

Corollary 4.1 *In the region $\lambda t^2 = O(t^{-\varepsilon})$, the function $Y(0,\lambda)$ has canonical asymptotic expansion,*

$$Y(0,\lambda) \underset{t\to+\infty}{=} Y_0^0(1+o(1))\lambda^{\frac{\Theta_0}{2}\sigma_3}.$$

In the region $\lambda t^2 = O(t^{+\varepsilon})$ we have

$$Y^{-1}(0,\lambda) \sim \tfrac{1}{2\hat{z}}\begin{pmatrix} \cos(x+\frac{\pi}{4}+\frac{\pi\Theta_0}{2}) & \cos(x+\frac{\pi}{4}+\frac{\pi\Theta_0}{2}) \\ \cos(x+\frac{\pi}{4}-\frac{\pi\Theta_0}{2}) & \cos(x+\frac{\pi}{4}-\frac{\pi\Theta_0}{2}) \end{pmatrix})(1+O(\tfrac{1}{t}))-$$
$$-\tfrac{4}{\sqrt{|\lambda|}}\begin{pmatrix} \sin(x+\frac{\pi}{4}+\frac{\pi\Theta_0}{2}) & -\sin(x+\frac{\pi}{4}+\frac{\pi\Theta_0}{2}) \\ \sin(x+\frac{\pi}{4}-\frac{\pi\Theta_0}{2}) & -\sin(x+\frac{\pi}{4}-\frac{\pi\Theta_0}{2}) \end{pmatrix}(1+O(\tfrac{1}{t}))+O(\lambda^{-\frac{1}{2}}t^{-1}),\ x=\tfrac{\sqrt{\lambda}t}{2}.$$

Proposition 4.2 *There exists a solution $Y(1,\lambda)$ of Eq.(4.2) with the following asymptotic expansion,*

$$Y(1,\lambda) \underset{t\to+\infty}{\sim} (\frac{\Theta_1}{2}\sigma_3 + t\hat{A}_1)(\imath\sqrt{y})^{\sigma_3}\Phi(\frac{t\sqrt{1-\lambda}}{2},\Theta_1)(1+O(t^{-1})),\ 1-\lambda \le O(t^{-2+\varepsilon}), \lambda\in[0,1]^+.$$

Note that in the region $(1-\lambda)t^2 = O(t^{-\varepsilon})$ the function $Y(1,\lambda)$ has canonical asymptotic expansion

$$Y(1,\lambda) \underset{t\to+\infty}{=} Y_0^1(1+o(1))(\lambda-1)^{\frac{\Theta_0}{2}\sigma_3}.$$

Remark 4.1 The proof of Propositions 4.1 and 4.2 is obtained in the following manner. We rewrote Eq. (4.2) as an ordinary differential equation of the second order. Then, we rescale the variable λ in a suitable way and apply the results of Olver (see [6]).

4.3 Solution near the turning point

Let us denote

$$T_{\frac{1}{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\sqrt{y} \\ \frac{1}{\sqrt{y}} & 1 \end{pmatrix},$$

and

$$c_+ = -\sqrt{y}(\frac{1}{2}+4\hat{z}+2\hat{z}\sqrt{y}(1+\frac{1}{y})),\ c_- = -\frac{t}{\sqrt{y}}(\frac{1}{2}+4\hat{z}-2\hat{z}\sqrt{y}(1+\frac{1}{y})).$$

Using conditions (4.3) and (4.5), one proves that $c_\pm \sim O(1)$. Also, we need the following notations,

$$c_2^- = \frac{2}{\sqrt{y}},\ c_1 = -\frac{4\hat{z}}{\sqrt{y}}(y-1), \kappa = -\frac{c_1}{c_+} \pm \frac{\sqrt{c_1^2 + c_2^- c_+}}{c_+},$$
$$v = \frac{e^{-\frac{\pi\imath}{4}}}{\sqrt{2}} c_+,\ w = \frac{e^{-\frac{\pi\imath}{4}}}{\sqrt{2}}(c_- - \kappa), \beta \equiv 1 + \nu = -vw.$$

Note that,

$$c_1^2 + c_+ c_2^- = -1 + o(1),\ \beta = \beta_0 - \beta_1 + \frac{1}{2}. \tag{4.9}$$

Proposition 4.3 *There exists a solution $Y(\frac{1}{2}, \lambda)$ of Eq.(4.2) with the following asymptotic expansion,*

$$Y(\frac{1}{2}, \lambda) \underset{t\to+\infty}{=} T_{\frac{1}{2}} t^{\frac{\sigma_3}{4}} \begin{pmatrix} 1 & 0 \\ \kappa\mu & 1 \end{pmatrix} D(\sqrt{2} e^{\frac{\pi\imath}{4}}\mu, v, w)(1 + O(t^{-\frac{1}{2}+3\varepsilon})),$$

where $\mu \equiv \Lambda t^{\frac{1}{2}} \leq O(t^\varepsilon)$ (see (4.8)).

Remark 4.3 The function $D(x; v, w)$ is defined in [1] (see also [7]). For our purposes, it is sufficient to know the following asymptotics as $|x| \to \infty$, $\arg x = \frac{\pi\imath}{4} + \frac{\pi\imath}{2} m$,

$$D(x; v, w) = (I + \begin{pmatrix} 0 & -v \\ w & 0 \end{pmatrix} x^{-1} + O(x^{-2})) \exp[(\frac{x^2}{4} - (1+\nu)\ln x)\sigma_3] R_m,$$

where the matrices R_m are given in [1]. The function $Y(\frac{1}{2}, \lambda)$ has the following asymptotic expansion as $t \to +\infty$ and $\mu = O(t^\varepsilon)$, $\arg\mu = \frac{\pi\imath}{2} m$, $m = 0, 1, 2$,

$$Y(\tfrac{1}{2}, \lambda) = T_{\frac{1}{2}} \Lambda^{-\frac{1}{2}\sigma_3} \begin{pmatrix} 1 & -\frac{e^{-\frac{\pi\imath}{4}}}{\sqrt{2}} v \\ \kappa & -\frac{e^{-\frac{\pi\imath}{4}}}{\sqrt{2}} v\kappa + 1 \end{pmatrix} (1 + O(t^{-\frac{\varepsilon}{2}}))\times$$
$$\times \exp[(\tfrac{\imath\mu^2}{2} - (1+\nu)\ln(\sqrt{2}e^{\frac{\pi\imath}{4}}\mu)\sigma_3] R_m (1 + O(t^{-\frac{1}{2}+3\varepsilon})).$$

Let us note that calculations of this subsection, as soon as they are local, can be justified using the scheme realized in [8].

4.4 Matching

We have defined the functions $Y(\lambda)$, $\Psi_q(\lambda)$, $Y(0,\lambda)$, $Y(1,\lambda)$, and $Y(\frac{1}{2},\lambda)$. All of them are solutions of Eq.(4.2). Hence, the following matrices,

$$C_+ = \Psi_+^{-1}(\lambda) Y(\lambda),\ L_p = Y^{-1}(p, \lambda)\Psi_p(\lambda),\ N_q = Y^{-1}(\frac{1}{2}, \lambda)\Psi_q(\lambda),$$

where $p = 0, 1$, $q = 0, 1, +$, are independent of λ. Our final goal is to calculate the monodromy matrices which were defined in terms of the matrices E^p. The latters can be obtained as follows:

$$E^p \underset{t\to\infty}{\sim} L_p N_p^{-1} N_+ C_+ u^{-\frac{1}{2}\sigma_3}(1 + o(1)). \tag{4.10}$$

Proposition 4.4 *The matrices C_+, L_p, and N_q have the following asymptotic expansions as $t \to +\infty$:*

$$C_+ = -\imath\sigma_3 \exp[(\frac{t}{4} + F(\lambda_+^*) + \Theta_\infty \ln 2 + \frac{\pi\imath\beta_0}{2} + \frac{\pi\imath\beta_1}{2} - \frac{1}{2}\ln\frac{\sqrt{\gamma^2-1}+\gamma+1}{-\sqrt{\gamma^2-1}+\gamma+1})\sigma_3], \tag{4.11}$$

$$L_p = (1+O(t^{-\varepsilon}))\begin{pmatrix} e^{-\frac{\pi\imath}{2}\Theta_p} & -e^{\frac{\pi\imath}{2}\Theta_p} \\ e^{\frac{\pi\imath}{2}\Theta_p} & -e^{-\frac{\pi\imath}{2}\Theta_p} \end{pmatrix}(\sigma_3)^p(\imath\sqrt{y})^{p\sigma_3}\exp[(-\frac{\pi\imath}{4}+\tilde{F}(\lambda\to p)-F(\lambda_p^*))\sigma_3], \tag{4.12}$$

$$N_q^{-1} = e^{F(\lambda_q^*)\sigma_3}\begin{pmatrix} 0 & n \\ 1 & 0 \end{pmatrix} R_m,$$

$$n = 2\imath\hat{z}\sqrt{y}(1+y)\exp\{-\frac{\ln t}{2}+\beta\ln t+\beta\ln 2+\frac{\pi\imath\beta}{2}-\frac{\imath t}{2}+\frac{\pi\imath\Theta_\infty}{2}+2\beta_0\pi\imath-\ln\frac{\sqrt{\gamma^2-1}+\gamma+\imath}{-\sqrt{\gamma^2-1}+\gamma+\imath}\}.$$

The notation $\tilde{F}(\lambda\to\lambda_0)$ means asymptotic expansion as $\lambda\to\lambda_0$ of the function $F(\lambda)$ (4.7) without terms depending on λ.

Proof. Since the matrices C_+, N_q, and L_p are independent of λ, they can be calculated by taking the following limit passages:

$$\begin{aligned} C_+ &= \lim_{\lambda\to+\imath\infty}\Psi_+^{-1}(\lambda)Y(\lambda), \quad \lambda\in\Gamma_+, \\ N_q &= \lim_{\lambda-1/2\sim t^{-1/2+\varepsilon}} Y^{-1}(\frac{1}{2},\lambda)\Psi_q(\lambda), t\to\infty, \lambda\in\Gamma_q,\ q=0,1,+, \\ L_p &= \lim_{\lambda-p\sim t^{-2+\varepsilon}} Y^{-1}(p,\lambda)\Psi_p(\lambda), t\to\infty, \lambda\in\Gamma_p,\ p=0,1. \end{aligned}$$

Remark 4.2 For the calculations of E^p (see (4.10), we need $N_0^{-1}N_+$ and $N_1^{-1}N_+$, rather than N_q;

$$N_0^{-1}N_+ = \begin{pmatrix} 1 & \imath\frac{\sqrt{2\pi}n}{\Gamma(\beta)v}e^{-2\pi\imath\beta} \\ 0 & 1 \end{pmatrix}, \quad N_1^{-1}N_+ = \begin{pmatrix} 1 & 0 \\ \frac{\sqrt{2\pi}v}{\Gamma(1-\beta)n}e^{\pi\imath\beta} & 1 \end{pmatrix}.$$

Finally, the monodromy matrices are calculated by using their definition (1.5).

Appendix

Here we explain how one can obtain two elements m_{21}^0 and m_{12}^1 of the corresponding monodromy matrices without matching at the turning point.

We define a class $\mathcal{Y}$ of solutions of Eq. (1.1), $Y_{(0)}(\lambda)$, by their asymptotic expansion

$$Y_{(0)}(\lambda) \underset{|\lambda|\to\infty}{=} (I+\hat{Y}_\infty\frac{1}{\lambda}+O\left(\frac{1}{\lambda^2}\right))\exp(\frac{\lambda t}{2}\sigma_3-\frac{\Theta_\infty}{2}\ln\lambda\sigma_3), \tag{A.1}$$

as $\lambda\to\infty$ in sector

$$\frac{\pi}{2} < \arg\lambda < \frac{3\pi}{2}. \tag{A.2}$$

All such defined functions differ by a right multiplicative factor,

$$Y_{(0)}(\lambda)\to Y_{(0)}(\lambda)\begin{pmatrix} 1 & s_{(0)} \\ 0 & 1 \end{pmatrix}. \tag{A.3}$$

Transformation (A.3) doesn't change the left off-diagonal element of the monodromy matrix of $Y_{(0)}(\lambda)$, $\{M^0(Y_{(0)})\}_{21}$. Thus, it is the same for all $Y_{(0)}(\lambda)\in\mathcal{Y}$; moreover,

$$\{M^0(Y_{(0)})\}_{21} = m_{21}^0, \tag{A.4}$$

since the canonical solution $Y_2(\lambda) \in \mathcal{Y}$. Due to (A.4), as soon as we want to calculate m^0_{21}, we don't need to select the canonical solution $Y_2(\lambda)$ in the class $\mathcal{Y}$. Therefore, for this calculation one can avoid the turning point of Eq. (4.2) and, hence, use only the WKB-asymptotics for its solutions.

Below, we use the WKB-analysis of Eq. (4.2) carried out in the previous section to get an explicit expression for m^0_{21}. The sector (A.2) is contained in the Stokes domain Γ_0. Thus,

$$M^0(Y_{(0)}) = E^{-1}_{(0)} e^{\pi\imath\Theta_0\sigma_3} E_{(0)}, \tag{A.5}$$

where the matrix $E_{(0)}$ is defined via the asymptotic expansion

$$Y_{(0)} \underset{\lambda\to 0}{\sim} G_0 \lambda^{\frac{\Theta_0}{2}\sigma_3} E_{(0)}$$

of the analytical continuation of $Y_{(0)}(\lambda)$ along the paths lying in Γ_0. In the domain Γ_0 we can factorize $E_{(0)}$ as follows:

$$E_{(0)} = L_0 C_0,\ L_0 = Y^{-1}(0,\lambda)\Psi_0(\lambda),\ C_0 = \Psi_0^{-1}(\lambda) Y_{(0)}(\lambda), \tag{A.6}$$

where $\Psi_0(\lambda)$ is a solution of Eq. (4.2) having the WKB-asymptotics (4.4) in Γ_0 and $Y(0,\lambda)$ is the canonical solution at $\lambda = 0$ (see Sec. 4). Now it is clear that modulo transformations (A.3) the matrix C_0 is given by Eq. (4.11). Matrix L_0 see in (4.12). Hence, using the first equation in (A.6) we obtain the following result

$$E_{(0)} \underset{t\to\infty}{=} \sigma_3 (u e^{\frac{\pi\imath}{2}})^{\frac{1}{2}\sigma_3} \begin{pmatrix} e^{-\frac{\pi\imath}{2}\Theta_0} & -e^{\frac{\pi\imath}{2}\Theta_0} \\ e^{\frac{\pi\imath}{2}\Theta_0} & -e^{-\frac{\pi\imath}{2}\Theta_0} \end{pmatrix} \begin{pmatrix} \alpha_0 & -\Delta_0 \\ 0 & \alpha_0^{-1} \end{pmatrix} u^{-\frac{1}{2}\sigma_3}\sigma_3$$

where $\Delta_0 = \Delta_0(t)$ is a complex-valued funtion and α_0 in the notations of Sec. 4 reads

$$\alpha_0 = \exp(-\frac{\pi\imath}{4} + \tilde{F}(\lambda\to 0) - \tilde{F}(\lambda\to\infty))(1 + o(1)) \tag{A.7}$$

Substituting (A.7) into (A.5) one arrives at

$$M^0 = \begin{pmatrix} -\Delta_0\alpha_0 & -u(\Delta_0^2 + \alpha_0^{-2} + 2\Delta_0\alpha_0^{-1}\cos\pi\Theta_0) \\ u^{-1}\alpha_0^2 & \Delta_0\alpha_0 + 2\cos\pi\Theta_0 \end{pmatrix}. \tag{A.8}$$

Let us briefly outline the analogous calculation for the monodromy matrix related with the other regular singularity of Eq. (1.1), $\lambda = 1$. For this purpose, we introduce a class $\mathcal{Y}_1$ of solutions of Eq. (1.1), $Y_{(1)}(\lambda)$, which have the asymptotic expansion (A.1) but in the other sector

$$-\frac{\pi}{2} < \arg\lambda < \frac{\pi}{2}.$$

The functions in $\mathcal{Y}_1$ are related by transformation like (A.3) but with the low-triangular factor in the right-hand side. Instead of Eq. (A.4), we get

$$M^1(Y_{(1)})_{12} = m^1_{12}, \tag{A.9}$$

where m^1_{12} is the corresponding matrix element of M_1, of the monodromy matrix of the canonical solution Y_2. For calculation of $M^1(Y_{(1)})$ one uses Eq. (A.5), but with $E_{(1)}$ and Θ_1 in place of $E_{(0)}$ and Θ_0. The results of these calculations are as follows:

$$E_{(1)} = D \begin{pmatrix} e^{-\frac{\pi\imath}{2}\Theta_1} & e^{\frac{\pi\imath}{2}\Theta_1} \\ e^{\frac{\pi\imath}{2}\Theta_1} & e^{-\frac{\pi\imath}{2}\Theta_1} \end{pmatrix} \begin{pmatrix} \alpha_1 & 0 \\ \Delta_1 & \alpha_1^{-1} \end{pmatrix} u^{-\frac{1}{2}\sigma_3}\sigma_3,$$

$$M^1 = \begin{pmatrix} -\Delta_1\alpha_1^{-1} & u\alpha_1^{-2} \\ -u^{-1}(\Delta_1^2+\alpha_1^2+2\Delta_1\alpha_1\cos\pi\Theta_1) & \Delta_1\alpha_1^{-1}+2\cos\pi\Theta_1 \end{pmatrix}, \tag{A.10}$$

where D is a diagonal matrix function of t, Δ_1 is a scalar function of t, and α_1 is given by the equation:

$$\alpha_1 = \exp(-\frac{\pi\imath}{4} + \tilde{F}(\lambda\to 1) - \tilde{F}(\lambda\to\infty))(1+o(1)). \tag{A.11}$$

Finally, substituting the results for the symbol $\tilde{F}(\lambda\to\lambda_0)$ obtained in the previous section into Eqs. (A.7), (A.8), (A.10), and (A.11) we obtain the following

Proposition 4.5 *The matrix elements m^0_{21} and m^1_{12} are given by*

$$m^0_{21} = \tfrac{1}{u}\tfrac{\sqrt{\gamma^2-1}-\gamma-1}{\sqrt{\gamma^2-1}+\gamma+1}\exp\{\tfrac{t}{2}+\pi\imath(\beta-\Theta_\infty)+2\Theta_\infty\ln 2\}(1+o(1)),$$
$$m^1_{12} = uy\tfrac{\sqrt{\gamma^2+1}-\gamma}{\sqrt{\gamma^2+1}+\gamma}\tfrac{\gamma+1+\sqrt{\gamma^2+1}}{\gamma+1-\sqrt{\gamma^2+1}}\exp\{-\tfrac{t}{2}+\pi\imath\beta-2\Theta_\infty\ln 2\}(1+o(1)),$$

where γ and β are given in (4.6) and in (4.9), correspondingly.

Remark A.1 Using the last equation in (1.6), one finds the following relation between the functions Δ_0 and Δ_1

$$\Delta_0\Delta_1 = \frac{e^{-\pi\imath\Theta_\infty}}{\frac{\alpha_0}{\alpha_1}(1+\frac{\alpha_0}{\alpha_1})}.$$

To get Δ_0 (Δ_1), we need more complicated calculations involving the turning point, which are presented in Sec. 4.

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