On symmetries of constant mean curvature surfaces

J. Dorfmeister*
Department of Mathematics
University of Kansas
Lawrence, KS 66045

G. Haak†
Fachbereich Mathematik
TU Berlin
D-10623 Berlin

1 Introduction

The goal of this note is to start the investigation of conformal CMC-immersions $\Psi : \mathcal{D} \to \mathbb{R}^3$, $\mathcal{D}$ an open, simply connected subset of $\mathbb{C}$, which allow for groups of spatial symmetries

$$\mathrm{Aut}\Psi(\mathcal{D}) = \{\tilde{T}\,\text{proper Euclidean motion of}\,\mathbb{R}^3 | \tilde{T}\Psi(\mathcal{D}) = \Psi(\mathcal{D})\}.$$

More precisely (see the definition in Section 2), we consider a Riemann surface M with universal covering $\pi : \mathcal{D} \to M$, and a conformal CMC-immersion $\Phi : M \to \mathbb{R}^3$ with nonzero mean curvature, such that $\Phi \circ \pi = \Psi$. Then we consider the groups

$$\mathrm{Aut}\mathcal{D} = \{g : \mathcal{D} \to \mathcal{D}\,\text{biholomorphic}\},$$

$$\mathrm{Aut}M = \{g : M \to M\,\text{biholomorphic}\},$$

$$\mathrm{Aut}_\pi\mathcal{D} = \{g \in \mathrm{Aut}\mathcal{D} |\ \text{there exists}\ \hat{g} \in \mathrm{Aut}M : \pi \circ g = \hat{g} \circ \pi\},$$

$$\mathrm{Aut}_\Phi M = \{\hat{g} \in \mathrm{Aut}M |\ \text{there exists}\ \tilde{T} \in \mathrm{Aut}\Psi(\mathcal{D}) : \Phi \circ \hat{g} = \tilde{T} \circ \Phi\},$$

and

$$\mathrm{Aut}_\Psi\mathcal{D} = \{g \in \mathrm{Aut}\mathcal{D} |\ \text{there exists}\ \tilde{T} \in \mathrm{Aut}\Psi(\mathcal{D}) : \Psi \circ g = \tilde{T} \circ \Psi\}.$$

There are many well known examples of CMC-surfaces with large spatial symmetry groups. The classic Delaunay surfaces (see [8]) have a nondiscrete group $\mathrm{Aut}\Psi(\mathcal{D})$ containing the group of all rotations around their generating axis. Other examples are the Smyth surfaces [19], which were visualized by D. Lerner, I. Sterling, C. Gunn and U. Pinkall. These surfaces have an $m+2$-fold rotational symmetry in $\mathbb{R}^3$, where the axis of rotation passes through the single umbilic of order m. More recent is the large class of examples provided by K. Große-Brauckmann and K. Polthier (see e.g. [11, 12]) of singly, doubly and triply periodic CMC-surfaces.

Other interesting classes of surfaces are the ones with a large group $\mathrm{Aut}_\Psi\mathcal{D}$. Examples for this are the compact CMC-surfaces, whose Fuchsian or elementary group is contained in $\mathrm{Aut}_\Psi\mathcal{D}$.

Yet another class of surfaces (M, Φ), for which $\mathrm{Aut}_\Psi\mathcal{D}$ is interesting are the surfaces with branch-points. If one deletes the set $B \subset M$ of those points in M which are mapped by Φ to the branchpoints, then such a surface can be constructed as an immersion $\hat{\Phi}$ of the non-simply connected Riemann surface $M \setminus B$ into $\mathbb{R}^3$. To get an immersion Ψ of a simply connected domain $\mathcal{D}$ into $\mathbb{R}^3$, which covers $(M \setminus B, \hat{\Phi})$, one can also apply the discussion of this paper as will be shown in Section 5.5.

It is our goal to describe properties of the group $\mathrm{Aut}\Psi(\mathcal{D})$ in terms of biholomorphic automorphisms of the Riemann surface M or the simply connected cover $\mathcal{D}$, i.e., in terms of $\mathrm{Aut}_\Phi M$ or $\mathrm{Aut}_\Psi\mathcal{D}$. To this end we investigate the relation between these groups. This is done in Chapter 2. After defining in Section 2.1, what we mean by a CMC-immersions (M, Φ), we start in Section 2.3 by

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listing some well known properties of the groups $\mathrm{Aut}M$, $\mathrm{Aut}\mathcal{D}$, $\mathrm{Aut}_\pi\mathcal{D}$. These follow entirely from the underlying Riemannian structure of M and $\mathcal{D}$. After fixing the conventions for conformal CMC-immersions in Section 2.4 we derive in Section 2.6 the transformation properties of the metric and the Hopf differential under an automorphism in $\mathrm{Aut}_\Psi\mathcal{D}$. These will lead in Section 2.7 to some general restrictions on $\mathrm{Aut}_\Psi\mathcal{D}$ in the case $\mathcal{D} = \mathbb{C}$. In Section 2.8–2.10 we will introduce group homomorphisms $\bar\pi : \mathrm{Aut}_\pi\mathcal{D} \to \mathrm{Aut}M$, $\phi : \mathrm{Aut}_\Phi M \to \mathrm{Aut}\Psi(\mathcal{D})$ and $\psi : \mathrm{Aut}_\Psi\mathcal{D} \to \mathrm{Aut}\Psi(\mathcal{D})$. We will also prove, that, in case M with the metric induced by Ψ is complete, ψ is surjective (Corollary 2.10). In Section 2.11 it will be shown that we furthermore can restrict our investigation to those CMC-immersions $\Phi : M \to \mathbb{R}^3$, for which ϕ is an isomorphism of Lie groups. In Sections 2.12 and 2.13 we will investigate, for which complete CMC-surfaces the group $\mathrm{Aut}_\Psi\mathcal{D}$ is nondiscrete. To this end we give a simple condition on the image $\Psi(\mathcal{D})$ of Ψ under which the group $\mathrm{Aut}\Psi(\mathcal{D})$ is a closed Lie subgroup of $\mathrm{OAff}(\mathbb{R}^3)$. Here we denote by $\mathrm{OAff}(\mathbb{R}^3)$ the group of proper (i.e., orientation preserving) Euclidean motions of $\mathbb{R}^3$. Using these results and the work of Smyth [19], we will prove in Section 3.4, that if $\mathrm{Aut}\Psi(\mathcal{D})$ is closed, $\mathrm{Aut}\Psi(\mathcal{D})$ is nondiscrete, iff the surface $\Psi(\mathcal{D})$ is isometric to a CMC-surface of revolution, i.e., a Delaunay surface. In Section 2.14 we illustrate the discussion in Chapter 2, using the examples of Delaunay and Smyth. The groups $\mathrm{Aut}_\Phi M$, $\mathrm{Aut}_\Psi\mathcal{D}$ and $\mathrm{Aut}\Psi(\mathcal{D})$ are explicitly given for these examples.

In Chapter 3 we will recall the only known constructive approach to general CMC-immersions, the DPW construction (see [6]). This will lead to the notion of the extended frame for a conformal CMC-immersion (Section 3.2). In Section 3.1–3.3 we will derive (Theorem 3.3), how the elements of $\mathrm{Aut}_\Psi\mathcal{D}$ act on the extended frame of a conformal CMC-immersion Ψ. This will be used in Section 3.4 to show, how the elements of $\mathrm{Aut}_\Psi\mathcal{D}$ act on the whole associated family of a CMC-surface. In Section 3.5 we define the meromorphic potential ξ of a CMC surface and derive its transformation properties under $\mathrm{Aut}_\Psi\mathcal{D}$. This discussion will be used in Section 3.6 to characterize CMC-surfaces with symmetries in terms of their meromorphic potentials (Proposition 3.6).

The discussion up to Section 3.6 will be used in Section 3.7 to prove (Theorem 3.7), that there are no nontrivial, translationally symmetric surfaces in the dressing orbit of the cylinder. In the forthcoming paper [4], the authors will use the results of these sections to give an alternative derivation of the classification of CMC-tori in terms of algebro-geometric data [16, 1, 9, 14].

Chapter 3 closes with an investigation of the coefficients of the meromorphic potential and their transformation properties. We will give (Theorem 3.9) a necessary condition on the meromorphic potential for the surface (M, Φ) to be complete.

In Chapter 4 we start a closer investigation of the transformation properties we derived for the meromorphic potential. In Section 4.2 we will prove necessary and sufficient conditions on the coefficient functions of the meromorphic potential in order to give surfaces with symmetries under the DPW construction. Unfortunately, these conditions still involve the Birkhoff splitting of the extended frames. Nevertheless, we expect them to be suitable starting points for the investigation and construction of many concrete examples of surfaces with symmetries and, in particular, of compact CMC-surfaces.

We finish the paper in Chapter 5 with a closer look at the meromorphic potential of Smyth surfaces and the treatment of a CMC-immersion with a branchpoint.

The results of sections 3.1, 3.3 and 3.5 are based on discussions of one of the authors (J.D.) with Franz Pedit.

2 Automorphisms of CMC-surfaces

2.1 Before we can start the investigation of CMC-immersions (M, Φ) we have to define, what we mean by a CMC-immersion, if M is not a domain in $\mathbb{R}^2$. We will restrict our investigations to the case of nonzero mean curvature, i.e. we will exclude the special case of minimal surfaces.

Definition: Let M be a connected C^2-manifold and let $\Phi : M \to \mathbb{R}^3$ be an immersion of type C^2. Φ is called a CMC-immersion, if there exists an atlas of M, s.t. every chart (U, φ) in this atlas defines a C^2-surface $\Phi \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^3$ with nonzero constant mean curvature.

That this definition makes sense is shown by the following

Theorem: *Let M be a connected C^2-manifold for which there exists a C^2-immersion $\Phi : M \to \mathbb{R}^3$, s.t. (M, Φ) is a CMC-immersion. Then there exists an atlas of M, s.t. the mean curvature is globally constant. In particular, M can be oriented and the mean curvature depends only on the chosen orientation of M.*

Proof: Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha), \alpha \in \mathcal{I}\}$, $\mathcal{I}$ some index set, be an atlas of M, s.t. for every chart $\Phi \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \to \mathbb{R}^3$ defines a C^2-surface of constant mean curvature $H_\alpha \neq 0$. By changing, if necessary, the orientation of the surfaces $\Phi \circ \varphi_\alpha^{-1}$, we can assume that $H_\alpha > 0$ for all $\alpha \in \mathcal{I}$. Let U_α and U_β, $\alpha \neq \beta$, be s.t. $U_\alpha \cap U_\beta \neq \{\}$. Then, on $U_\alpha \cap U_\beta$, $\Phi \circ \varphi_\alpha^{-1}$ and $\Phi \circ \varphi_\beta^{-1}$ define two surfaces which differ only by a C^2-reparametrization $\varphi_\beta \circ \varphi_\alpha^{-1}$. The absolute value of the mean curvature is independent of reparametrizations. Since H_α and H_β are positive, we get $H_\alpha = H_\beta$ and $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive Jacobian. From this it follows that M is orientable. Since M is also connected, we get that all H_α coincide. Therefore, the mean curvature defined by $\mathcal{A}$ on M is globally constant.

By the arguments above, for a chosen orientation on M, the absolute value of the mean curvature H is constant on M and does not depend on the chosen atlas, as long as this atlas defines the same orientation. Changing the orientation of the atlas changes the sign of H, which finishes the proof. □

2.2 We will not work with the fairly general Definition 2.1. Instead we use Lichtenstein's theorem to turn M into a Riemann surface (see also [21, Theorem 5.13]) and Φ into a conformal immersion.

Theorem: *Let (M, Φ) be a CMC-immersion. Then there exists a conformal structure on M, s.t. M becomes a Riemann surface and Φ becomes a conformal CMC-immersion.*

Proof: Let $\mathcal{A} = \{(U_\alpha, \varphi_\alpha), \alpha \in \mathcal{I}\}$ be an oriented atlas of M. W.l.o.g. we can choose $\mathcal{A}$ s.t. $\Phi|_{U_\alpha}$ is injective for all $\alpha \in \mathcal{I}$. Then, by Lichtenstein's theorem (see e.g. [2, Section I.1.4]), for every $\alpha \in \mathcal{I}$, there exists an orientation preserving diffeomorphism $q_\alpha : V_\alpha \to \varphi(U_\alpha)$, V_α a domain in $\mathbb{R}^2$, s.t. $\Phi \circ \varphi_\alpha^{-1} \circ q_\alpha$ is a conformal parametrization of the surface defined by $\Phi \circ \varphi_\alpha^{-1}$. Then $\{(U_\alpha, \tilde{\varphi}_\alpha = q_\alpha^{-1} \circ \varphi_\alpha), \alpha \in \mathcal{I}\}$ defines an atlas on M, for which every transfer function is holomorphic, which finishes the proof. □

In this paper we will therefore restrict ourselves to conformal CMC-immersions $\Phi : M \to \mathbb{R}^3$, where M is a Riemann surface. I.e., if we write "*(M, Φ) is a CMC-immersion*", we always mean "*M is a Riemann surface and Φ is a conformal CMC-immersion*".

2.3 In this section we strip the CMC-surface M of its metric and leave only the complex structure. We will recollect some well known facts about Riemann surfaces.

Up to conformal equivalence the only Riemann surfaces, which are simply connected are the sphere $\mathbb{CP}_1 \cong \mathbb{C} \cup \{\infty\}$, the complex plane $\mathbb{C}$ and the upper half plane $\Delta = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$, which is conformally equivalent to the open unit disk. Each of these surfaces is equipped with its standard complex structure.

For convenience of language we will not distinguish between Fuchsian groups (as in the case of $\mathcal{D} = \Delta$) and elementary groups (as in the case of $\mathcal{D} = \mathbb{C}$). For a more thorough treatment of the uniformization of Riemann surfaces see e.g. [10].

Every Riemann surface M can be represented as the quotient of one of these three Riemann surfaces by a freely acting Fuchsian group Γ of biholomorphic automorphisms, i.e., we may write $M = \Gamma \setminus \mathcal{D}$, where $\mathcal{D}$ is the simply connected cover of M.

If $\pi : \mathcal{D} \to M$ is the covering map, then Γ is also the covering group of π. Then by [10, V.4.5]

$$\mathrm{Aut} M = N(\Gamma)/\Gamma, \tag{2.3.1}$$

where $N(\Gamma)$ is the normalizer of Γ in $\mathrm{Aut}\mathcal{D}$. Since $g \in \mathrm{Aut}_\pi \mathcal{D}$ is equivalent to

$$\pi \circ g \circ \gamma = \pi \circ g \tag{2.3.2}$$

for all $\gamma \in \Gamma$, we have $g \circ \gamma = \gamma_1 \circ g$ for some $\gamma_1 \in \Gamma$, and thus

$$\mathrm{Aut}_\pi \mathcal{D} = N(\Gamma). \tag{2.3.3}$$

Therefore,

$$\mathrm{Aut} M \cong \mathrm{Aut}_\pi \mathcal{D}/\Gamma, \tag{2.3.4}$$

where $\mathrm{Aut}_\pi \mathcal{D}$ is a closed subgroup of $\mathrm{Aut}\mathcal{D}$, and Γ is a normal subgroup of $\mathrm{Aut}_\pi \mathcal{D}$.

The following is also well known (see e.g. [10, IV.5,IV.6,V.4]):

Lemma:

a) *The group Γ is discrete, i.e., either finite or countable, and consists of conformal (biholomorphic) automorphisms of $\mathcal{D}$, which act fixed point free on $\mathcal{D}$.*

b) *If $\mathcal{D} = \mathbb{C}\mathrm{P}_1$, then Γ is trivial and M is the sphere.*

c) *If $\mathcal{D} = \mathbb{C}$, then Γ is abelian and M is either the plane, the cylinder or a torus.*

d) *For a Riemann surface $M = \Gamma \setminus \mathcal{D}$ the following are equivalent:*

- *The Fuchsian group is abelian.*
- *The Lie group $\mathrm{Aut} M$ of conformal automorphisms of M is nondiscrete.*

Surfaces of this kind are called exceptional surfaces.

2.4 Now we also take into account the metric of the CMC-surface.

Let (M, Φ) be a CMC-immersion. The Riemann surface M is endowed with the induced metric. As usual, M is called (geodesically) complete as a manifold with metric, if each geodesic can be extended to a curve parametrized over $\mathbb{R}$. It is not possible to reach the boundary of a complete manifold by going along a curve of finite length.

We are interested in orientation preserving isometries of the surface onto itself. They are automatically biholomorphic automorphisms, but may have fixed points.

We consider the commutative diagram

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, "\pi"'] \arrow[dr, "\Psi"] & \\ M \arrow[rr, "\Phi"'] & & \mathbb{R}^3 \end{tikzcd} \tag{2.4.1}$$

where M is a Riemann surface and Φ is a CMC-immersion of M. Moreover, π is the universal covering map of M and $\Psi = \Phi \circ \pi$. Remember, that we always assume, that Φ and therefore also Ψ is a conformal immersion.

We would like to remind the reader of the following well known result (see e.g. the Appendix of [3]):

Theorem: *Let $\Psi : \mathcal{D} \subset \mathbb{C} \to \mathbb{R}^3$ be a conformal immersion with metric*

$$ds^2 = \frac{1}{2} e^u \mathrm{d}z \mathrm{d}\bar{z} \tag{2.4.2}$$

where $u = u(z, \bar{z}) : \mathcal{D} \to \mathbb{R}$. Define the function $E : \mathcal{D} \to \mathbb{C}$ by

$$E = \langle \Psi_{zz}, N \rangle \tag{2.4.3}$$

where $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^3$ and

$$N = \frac{\Psi_z \times \Psi_{\bar{z}}}{|\Psi_z \times \Psi_{\bar{z}}|} \tag{2.4.4}$$

is the Gauß map of $(\mathcal{D}, \Psi)$. Then the second fundamental form of $(\mathcal{D}, \Psi)$ has in real coordinates $x = \frac{1}{2}(z + \bar{z})$, $y = \frac{1}{2i}(z - \bar{z})$, the form

$$II = \frac{1}{2} \begin{pmatrix} (E + \overline{E}) + He^u & i(E - \overline{E}) \\ i(E - \overline{E}) & -(E + \overline{E}) + He^u \end{pmatrix}, \tag{2.4.5}$$

where $H = 2e^{-u} \langle \Psi_{z\bar{z}}, N \rangle$ is the mean curvature. The Gauß-Codazzi equations take the form

$$u_{z\bar{z}} + \frac{1}{2} e^u H^2 - 2e^{-u} |E|^2 = 0, \tag{2.4.6}$$

$$E_{\bar{z}} = \frac{1}{2} e^u H_z. \tag{2.4.7}$$

The Gauß curvature is given in terms of u, E and H by

$$K = H^2 - 4|E|^2 e^{-2u}. \tag{2.4.8}$$

From this the following corollary follows immediately

Corollary: *Let $\mathcal{D}$ be the open unit disk or the complex plane.*

1. *Let $\Psi : \mathcal{D} \to \mathbb{R}^3$ be a CMC-immersion. Let u and E be defined as above. Then $E\mathrm{d}z^2$ is a holomorphic two-form on $\mathcal{D}$ and the function u satisfies the partial differential equation*

$$u_{z\bar{z}} + \frac{H^2}{2} e^u - 2|E|^2 e^{-u} = 0, \tag{2.4.9}$$

   where H is the mean curvature.

2. *Let $E\mathrm{d}z^2$ be a holomorphic differential on $\mathcal{D}$ and $u(z, \bar{z})$ be a real valued function on $\mathcal{D}$. Assume, that u and E satisfy Eq. (2.4.9) for some constant H. Then there exists a conformal immersion $\Psi : \mathcal{D} \to \mathbb{R}^3$ with constant mean curvature H, s.t. u and E are given by Eqs. (2.4.2) and (2.4.3), respectively. The conformal immersion Ψ is unique up to a proper Euclidean transformation.*

Proof: 1. follows from Eqs. (2.4.9) and (2.4.7).

2. For u and E the Gauss-Codazzi equations (2.4.9) and (2.4.7) are satisfied. The functions u and E determine, by (2.4.2) and (2.4.5), the first and second fundamental form uniquely. The existence and the uniqueness of Ψ, up to proper Euclidean motion, follows therefore from the fundamental theorem of surface theory (see e.g. [20, III.2.3]). □

2.5 For a CMC-immersion the holomorphic two-form $E\mathrm{d}z^2$ is called the Hopf differential. It is identically zero iff the surface $(\mathcal{D}, \Psi)$ is part of a sphere. Its zeroes are mapped to the umbilical points of the surface. The cylinder of mean curvature H can be represented by the "vacuum solution" $E \equiv \text{const} \neq 0$, $e^{2u} = \frac{4}{H^2}|E|^2 \equiv \text{const}$, of Eq. (2.4.9). As a special case of Corollary 2.4 we get the

Proposition: *If $\Psi : \mathcal{D} \to \mathbb{R}^3$ is a complete CMC-immersion with $u \equiv \text{const}$ and $E \equiv \text{const} \neq 0$, then, up to orientation, $\mathcal{D} = \mathbb{C}$ and $\Psi(\mathcal{D})$ is a cylinder of mean curvature $H = 2e^{-u}|E|$.*

Proof: Since, by assumption, $\mathcal{D}$ with the constant metric $\mathrm{d}s^2 = \frac{1}{2}e^u \mathrm{d}z\mathrm{d}\bar{z}$ is complete, we have $\mathcal{D} = \mathbb{C}$. If we define $H = 2e^{-u}|E|$ or $H = -2e^{-u}|E|$, then u and E satisfy Eq. (2.4.9). Moreover, these are the only two choices of H, for which Eq. (2.4.9) is satisfied. Therefore, Ψ has the same metric and Hopf differential as a cylinder of mean curvature $H = \pm 2e^{-u}|E|$, where the sign depends only on the choice of the orientation. The rest of the proposition then follows from the second part of Corollary 2.4. □

Since, by a scaling in $\mathbb{R}^3$ and the choice of an orientation of M, H can be fixed to any numerical value in $\mathbb{R} \setminus \{0\}$, we will from now on set

$$H = -\frac{1}{2}. \tag{2.5.1}$$

Let us fix a CMC-immersion $(\mathcal{D}, \Psi)$ with Hopf differential E and let u be defined by Eq. (2.4.2). Then u solves Eq. (2.4.9) for the given E.

It is clear, that u still satisfies Eq. (2.4.9) if we replace E by λE, $|\lambda| = 1$. In this way, by Corollary 2.4, we get for every $\lambda \in S^1$ a CMC-immersion $\Psi_\lambda : \mathcal{D} \to \mathbb{R}^3$ with constant mean curvature H. The surface $(\mathcal{D}, \Psi_\lambda)$ is uniquely determined up to a proper Euclidean motion. The family $\{\Psi_\lambda, \lambda \in S^1\}$ of CMC-immersions is called the *associated family* of $(\mathcal{D}, \Psi = \Psi_{\lambda=1})$. Note, that up to a proper Euclidean motion,

$$(\Psi_\lambda)_\mu = \Psi_{\lambda\mu}, \qquad \text{for } \lambda, \mu \in S^1. \tag{2.5.2}$$

Let Ψ_1 and Ψ_2 be two conformal immersions from $\mathcal{D} \subset \mathbb{C}$ to $\mathbb{R}^3$ with metrics $\mathrm{d}s_1^2 = \frac{1}{2}e^{u_1}\mathrm{d}z\mathrm{d}\bar{z}$ and $\mathrm{d}s_2^2 = \frac{1}{2}e^{u_2}\mathrm{d}z\mathrm{d}\bar{z}$. We call Ψ_1 and Ψ_2 *isometric* iff there is a conformal automorphism $g : \mathcal{D} \to \mathcal{D}$, s.t. $\Psi_1 \circ g$ and Ψ_2 have the same metric, i.e., iff $e^{(u_1 \circ g)}|g'|^2 = e^{u_2}$.

Lemma: *Let Ψ_1 and Ψ_2 be two CMC-immersions from $\mathcal{D} \subset \mathbb{C}$ to $\mathbb{R}^3$ with the same mean curvature. Then Ψ_1 and Ψ_2 are isometric iff there is a biholomorphic automorphism g, s.t. $\Psi_1 \circ g$ and Ψ_2 are in the same associated family.*

Proof: If g is a conformal automorphism of $\mathcal{D}$, then the metric of $\hat{\Psi} = \Psi \circ g$ is given by $\frac{1}{2}e^{\hat{u}}\mathrm{d}z\mathrm{d}\bar{z}$ with $e^{\hat{u}} = e^{(u_1 \circ g)}|g'|^2$, and the Hopf differential of $\Psi \circ g$ is $\hat{E} = (E \circ g)(g')^2$.

Let Ψ_1 and Ψ_2 be two CMC-immersions with the same mean curvature, and let g be a conformal automorphism, s.t. $\hat{\Psi} = \Psi_1 \circ g$ and Ψ_2 have the same metric, i.e. $e^{\hat{u}} = e^{(u_1 \circ g)}|g'|^2 = e^{u_2}$. Since Ψ_1 and Ψ_2 have the same mean curvature, g is orientation preserving, i.e. g is a biholomorphic automorphism of $\mathcal{D}$. In addition, since the Gauß curvature is invariant under isometries, it follows from Eq. (2.4.8) that

$$|(E_1 \circ g)(z)| \cdot |g'(z)|^2 = |E_2(z)|, \tag{2.5.3}$$

i.e. $\hat{E}(z) = (E_1 \circ g)(z)(g'(z))^2 = E_2(z)e^{i\theta(z)}$, where $\theta(z) : \mathcal{D} \to \mathbb{R}$. Since $\hat{E}$ and E_2 are holomorphic we get that θ is constant. Consequently, $\hat{\Psi}$ and Ψ_2 are in the same associated family.

Conversely, assume that there exists a biholomorphic automorphism g, s.t. $\Psi_1 \circ g$ and Ψ_2 are in the same associated family. Then the functions $e^{\hat{u}} = e^{(u_1 \circ g)}|g'|$ and e^{u_2} coincide and therefore Ψ_1 and Ψ_2 are isometric. $\square$

It is clear from the definition of a conformal CMC-immersion, that with (M, Φ), M a Riemann surface, also $(M, \Phi \circ \hat{g})$, $\hat{g} \in \mathrm{Aut}M$, is a conformal CMC-immersion. If $(\mathcal{D}, \Psi)$ is a CMC-immersion from $\mathcal{D} \subset \mathbb{C}$ to $\mathbb{R}^3$, then we call $g \in \mathrm{Aut}\mathcal{D}$ a *self-isometry* of $(\mathcal{D}, \Psi)$ if the CMC-immersions $\Psi \circ g$ and Ψ have the same metric. If (M, Φ) is a CMC-immersion with universal cover $(\mathcal{D}, \Psi)$, $\pi : \mathcal{D} \to M$, then $\hat{g} \in \mathrm{Aut}M$ will be called a self-isometry of (M, Φ) if there exists a self-isometry g of $(\mathcal{D}, \Psi)$, s.t. $\pi \circ g = \hat{g} \circ \pi$. We denote by $\mathrm{Iso}_\Psi \mathcal{D} \subset \mathrm{Aut}\mathcal{D}$ and $\mathrm{Iso}_\Phi M \subset \mathrm{Aut}M$ the groups of self-isometries of $(\mathcal{D}, \Psi)$ and (M, Φ), respectively. Note, that by definition, a self-isometry is orientation preserving. From the proof of Lemma 2.5 we get

Corollary 1: *Let $\Psi : D \to \mathbb{R}^3$ be a CMC-immersion with Hopf differential $E dz^2$ and define the function u as in Eq. (2.4.2). Let $g \in \mathrm{Aut}\mathcal{D}$. Then the following are equivalent:*

1. *The automorphism g is in $\mathrm{Iso}_\Psi \mathcal{D}$.*
2. *The function u transforms under g as*

$$e^{(u \circ g)(z,\bar{z})}|g'(z)|^2 = e^{u(z,\bar{z})}. \tag{2.5.4}$$

If $g \in \mathrm{Iso}_\Psi \mathcal{D}$, then E transforms under g as

$$|(E \circ g)(z)| \cdot |g'(z)|^2 = |E(z)|. \tag{2.5.5}$$

Since in Chapter 3 we will consider only complete CMC immersions, we also want to state the following

Corollary 2: *Let $(\mathcal{D}, \Psi)$ be a CMC-immersion. If $(\mathcal{D}, \Psi)$ is complete (w.r.t. the induced metric), then all elements $(\mathcal{D}, \Psi_\lambda)$, $\lambda \in S^1$, of its associated family are complete. If $(\mathcal{D}, \Psi)$ is not complete, then no element of its associated family is complete.*

Proof: Since all surfaces in the associated family share the same metric, they are either all complete or none of them is complete. $\square$

2.6 We define $\mathrm{OAff}(\mathbb{R}^3)$ to be the group of proper Euclidean motions in $\mathbb{R}^3$. It will be convenient at times to decompose an element of $\tilde{T} \in \mathrm{OAff}(\mathbb{R}^3)$ into a rotational and a translational part:

$$\tilde{T}v = R_{\tilde{T}}v + t_{\tilde{T}}, \qquad v \in \mathbb{R}^3. \tag{2.6.1}$$

We will also write $\tilde{T} = (R_{\tilde{T}}, t_{\tilde{T}})$.

Definition: As already mentioned in the introduction we define

$$\mathrm{Aut}_\pi \mathcal{D} = \{g \in \mathrm{Aut}\mathcal{D} |\text{ there exists } \hat{g} \in \mathrm{Aut}M : \pi \circ g = \hat{g} \circ \pi\}, \tag{2.6.2}$$

$$\mathrm{Aut}_\Phi M = \{\hat{g} \in \mathrm{Aut}M |\text{ there exists } \tilde{T} \in \mathrm{Aut}\Psi(\mathcal{D}) : \Phi \circ \hat{g} = \tilde{T} \circ \Phi\}, \tag{2.6.3}$$

$$\mathrm{Aut}_\Psi \mathcal{D} = \{g \in \mathrm{Aut}\mathcal{D} |\text{ there exists } \tilde{T} \in \mathrm{Aut}\Psi(\mathcal{D}) : \Psi \circ g = \tilde{T} \circ \Psi\}, \tag{2.6.4}$$

and

$$\mathrm{Aut}\Psi(\mathcal{D}) = \{\tilde{T} \in \mathrm{OAff}(\mathbb{R}^3) | \tilde{T}\Psi(\mathcal{D}) = \Psi(\mathcal{D})\}. \tag{2.6.5}$$

Lemma: *Let $\Psi : \mathcal{D} \to \mathbb{R}^3$ be a CMC-immersion with Hopf differential $E\mathrm{d}z^2$ and define the function u as in Eq. (2.4.2). Let $g \in \mathrm{Aut}\mathcal{D}$. Then the following are equivalent:*

1. *The automorphism g is in $\mathrm{Aut}_\Psi\mathcal{D}$.*
2. *The functions u and E transform under g as*

$$e^{(u\circ g)(z,\bar{z})}|g'(z)|^2 = e^{u(z,\bar{z})}, \tag{2.6.6}$$

$$(E\circ g)(z)(g'(z))^2 = E(z). \tag{2.6.7}$$

Proof: Let us define the immersion $\Psi_1 = \Psi \circ g$. Then $\Psi_1 : \mathcal{D} \to \mathbb{R}^3$ is also a CMC-immersion. By the definition of u we have

$$e^{u_1(z,\bar{z})} = e^{(u\circ g)(z,\bar{z})}|g'(z)|^2. \tag{2.6.8}$$

Since the Hopf differential is a holomorphic two-form we get

$$E_1(z) = (E\circ g)(z)(g'(z))^2. \tag{2.6.9}$$

We have $g \in \mathrm{Aut}_\Psi\mathcal{D}$ iff Ψ_1 and Ψ give the same surface up to a proper Euclidean motion. By the fundamental theorem of surface theory this is the case iff both surfaces have the same first and second fundamental form, which by Eqs. (2.4.2) and (2.4.5) is equivalent to $E_1 = E$ and $u_1 = u$. This, together with Eq. (2.6.8) and Eq. (2.6.9), proves the lemma. □

Corollary: *The elements of $\mathrm{Aut}_\Psi\mathcal{D}$ act as self-isometries of $(\mathcal{D}, \Psi)$, i.e. $\mathrm{Aut}_\Psi\mathcal{D} \subset \mathrm{Iso}_\Psi\mathcal{D}$.*

Proof: By Eq. (2.6.6) and the definition (2.4.2) of u, the metric $\mathrm{d}s^2$ is invariant under $g \in \mathrm{Aut}_\Psi\mathcal{D}$. □

2.7 We will draw some further conclusions from the automorphicity of the Hopf differential, Eq. (2.6.7). We recall that CMC-surfaces with Hopf differential identically zero are part of a round sphere. Such surfaces will be called *spherical.*

Proposition: *Let (M, Φ) be a CMC-surface with simply connected cover $(\mathcal{D}, \Psi)$, $\mathcal{D} = \mathbb{C}$. Then either $E \equiv 0$ or the group $\mathrm{Iso}_\Psi\mathcal{D} \subset \mathrm{OAff}(\mathbb{R}^3)$ of self-isometries of $(\mathcal{D}, \Psi)$ consists only of rigid motions of the plane, i.e., every $g \in \mathrm{Iso}_\Psi\mathcal{D}$ can be written as $g : z \mapsto az + b$, with $a, b \in \mathbb{C}$ and $|a| = 1$.*

Proof: Let us assume, that there exists an automorphism g in $\mathrm{Iso}_\Psi\mathcal{D} \subset \mathrm{Aut}\mathbb{C}$, which is of the form

$$g(z) = az + b \tag{2.7.1}$$

with a, b being complex constants and $|a| \neq 1$.

Case I: If $b \neq 0$ then we can, by a biholomorphic change of coordinates

$$z \mapsto \tilde{z} = \frac{a-1}{b}z + 1, \tag{2.7.2}$$

turn g into a scaling with rotation $g(\tilde{z}) = a\tilde{z}$. Let us also define $\tilde{E} : \mathcal{D} \to \mathbb{C}$ by $\tilde{E}\mathrm{d}\tilde{z}^2 = E\mathrm{d}z^2$, then

$$\tilde{E}(\tilde{z}) = \frac{b^2}{(a-1)^2}E(\frac{b}{a-1}(z-1)). \tag{2.7.3}$$

The Hopf differential $\tilde{E}$ transforms under $\tilde{g}$ according to (2.5.5). We therefore get for all $n \in \mathbb{Z}$:

$$|\tilde{E}(a^n\tilde{z})| \cdot |a^{2n}| = |\tilde{E}(\tilde{z})|. \tag{2.7.4}$$

For $|a| > 1$ this implies that the absolute value of $\tilde{E}$ is decreasing from the fixed point $\tilde{z} = 0$ of g in all directions in the $\tilde{z}$-plane to zero. Since $\tilde{E}$ is holomorphic in $\tilde{z}$, this gives $\tilde{E} \equiv 0$ and therefore $E \equiv 0$.

If $|a| < 1$, consider the inverse $g^{-1} \in \mathrm{Iso}_\Psi \mathcal{D}$:

$$g^{-1}(z) = \frac{1}{a} z - \frac{b}{a}. \tag{2.7.5}$$

Since $\left|\frac{1}{a}\right| > 1$ we can use the first part of the proof again.

Case II: If $b = 0$ then Eq. (2.5.5) gives directly

$$|E(a^n z)| = |a^{-2n}||E(z)|. \tag{2.7.6}$$

We can therefore argue in the same way as in the first case. $\square$

The following Lemma will also be important:

Lemma: *Let (M, Φ) be a complete, nonspherical CMC-surface with conformal covering immersion $(\mathcal{D}, \Psi)$ and $\mathcal{D} = \mathbb{C}$. Then the following holds*

1. *If $\mathrm{Iso}_\Psi \mathcal{D}$ contains the one-parameter group of rotations $\mathcal{R}$ around a fixed point and also a translation, then $\Phi(M)$ is a cylinder.*

2. *If $\mathrm{Iso}_\Psi \mathcal{D}$ contains a one-parameter group of translations $\mathcal{T}$ and also a rotation R, s.t. $R^2 \neq \mathrm{id}$, then $\Phi(M)$ is a cylinder.*

Proof: 1. Let u be defined by Eq. (2.4.2). W.l.o.g. we can choose the center of the rotation group $\mathcal{R}$ as $z = 0$ and the translation as $g : z \mapsto z + 1$. Then $\mathcal{R} = \{r_\phi | r_\phi(z) = e^{i\phi} z, \phi \in [0, 2\pi)\}$. By Eqs. (2.5.4) and (2.5.5) we have

$$e^{u(z+1)} = e^{u(z)}, \qquad |E(z+1)| = |E(z)| \tag{2.7.7}$$

$$|e^{u(e^{i\phi} z)}| = |e^{u(z)}|, \qquad |E(r_\phi(z))| = |E(e^{i\phi} z)| = |E(z)| \tag{2.7.8}$$

for all $z \in \mathbb{C}$. Therefore, e^u and $|E|$ are constant on the whole orbit of $z = 0$ under the group $\mathcal{R} \times \mathcal{T}$, where $\mathcal{T}$ is the group generated by $z \mapsto z + 1$. The translation $z \mapsto z + 1$ takes the origin to $z = 1$, which is mapped to the whole unit circle by the rotation group $\mathcal{R}$. The group $\mathcal{T}$ takes the unit circle into a connected set reaching from 0 to ∞. This set therefore contains for each $r \in \mathbb{R}$ a complex number z_r with $|z_r| = r$. Thus, the orbit of this set under $\mathcal{R}$ is the whole complex plane. This shows, that $\mathcal{R} \times \mathcal{T}$ acts transitively on $\mathbb{C}$ and e^u and $|E|$ are constant functions. Since u is real valued and E is holomorphic, we get that u and E are constant functions. The image $\Phi(M)$ is then, by Proposition 2.5, a cylinder.

2. W.l.o.g. we choose the group $\mathcal{T}$ as the group of translations along the real axes $\{z \mapsto z + r | r \in \mathbb{R}\}$ and R as a rotation around the origin $R(z) = e^{i\phi}$ with $\phi \neq m\pi$, $m \in \mathbb{Z}$. Let us denote the group generated by R as $\mathcal{R}$. The orbit of the origin $z = 0$ under the group $\mathcal{R} \times \mathcal{T}$ contains the straight line $\{z | z = re^{i\phi}, r \in \mathbb{R}\}$, which does not coincide with the real line. Therefore, the action of $\mathcal{T}$ takes this line into the whole complex plane and the group $\mathcal{R} \times \mathcal{T}$ acts transitively on $\mathbb{C}$. As in the proof of the first part of the lemma this shows, that E and u are constant and that $\Phi(M)$ is a cylinder. $\square$

From the results of this section we can draw the following conclusion for $\mathrm{Iso}_\Psi(\mathcal{D})$, if $\mathcal{D} = \mathbb{C}$:

Theorem: *Let (M, Φ) be a complete, nonspherical CMC-surface with universal covering immersion $(\mathcal{D}, \Psi)$ and $\mathcal{D} = \mathbb{C}$. Let $\mathrm{Iso}_\Psi \mathcal{D} \subset \mathrm{Aut}\mathcal{D}$ be the group of self-isometries of $(\mathcal{D}, \Psi)$.*

1. If $\mathrm{Iso}_\Psi \mathcal{D}$ contains the one-parameter group $\mathcal{R}$ of rotations around a fixed point $z_0 \in \mathbb{C}$, then

- *either* $\mathrm{Iso}_\Psi \mathcal{D} = \mathcal{R}$
- *or* $\Phi(M)$ *is a cylinder.*

2. *If* $\mathrm{Iso}_\Psi \mathcal{D}$ *contains a one-parameter group* $\mathcal{T}$ *of translations, then*

- *either* $\Phi(M)$ *is a cylinder,*
- *or* $\mathrm{Iso}_\Psi \mathcal{D} = \mathcal{T} \times Q$,
- *or* $\mathrm{Iso}_\Psi \mathcal{D} = \mathcal{T} \times Q \times R$,

where $\times$ *denotes the product of sets,* Q *is a, possibly trivial, discrete group of translations, not contained in* $\mathcal{T}$, *and* R *is the group generated by the* $180°$*-rotation around a fixed point,* $z \to 2z_0 - z$, $z_0 \in \mathbb{C}$.

Proof: 1. W.l.o.g. we can assume that $\mathcal{R}$ is the set of rotations around the origin in $\mathbb{C}$. If $\mathrm{Iso}_\Psi \mathcal{D} \neq \mathcal{R}$, then there exists $g \in \mathrm{Iso}_\Psi \mathcal{D}$ s.t. $g(z) = az+b$ with $b \neq 0$ and, by Proposition 2.7, $|a| = 1$. If $\tilde{g}(z) = az$, then $\tilde{g} \in \mathcal{R}$ and $(g \circ \tilde{g}^{-1})(z) = z + b$, i.e., $\mathrm{Iso}_\Psi \mathcal{D}$ contains a pure translation. Lemma 2.7 then shows that $\Phi(M)$ is a cylinder.

2. Assume, that $\mathrm{Iso}_\Psi \mathcal{D}$ contains a one-parameter group $\mathcal{T}$ of translations in $\mathbb{C}$, i.e.,

$$\mathcal{T} = \{g_r | g_r(z) = z + rv,\ r \in \mathbb{R}, v \in \mathbb{C} \setminus \{0\}\}, \tag{2.7.9}$$

and an automorphism $g(z) = az + b$ with $a \neq 1$. Then $|a| = 1$ by Proposition 2.7. If $b = 0$, then g is a rotation around the origin. If $b \neq 0$ we can, as in Proposition 2.7, apply the coordinate transformation $\tilde{z} = \frac{a-1}{b} z + 1$, s.t. g becomes a rotation around the origin, $g(\tilde{z}) = a\tilde{z}$. In the new coordinates $\mathcal{T}$ is still a one-parameter group of pure translations. Its direction in the $\tilde{z}$-plane is given by the vector $\tilde{v} = \frac{a-1}{b} v$. Therefore, we can restrict our investigation (after an additional rotation around the origin) to the case that $\mathcal{T}$ is the group of translations along the real axis and g is a rotation around the origin in $\mathbb{C}$, $g(z) = az$. Lemma 2.7 shows, that then either $a = -1$ or $\Phi(M)$ is a cylinder. Therefore, if $\Phi(M)$ is not a cylinder, $Iso_\Psi \mathcal{D} = T \times R$ or $\mathrm{Iso}_\Psi \mathcal{D} = T$, where T is a group of translations and $\mathcal{T} \subset T$. By the same argument as in Lemma 2.7, for a noncylindrical surface, the orbit of $z = 0$ under $\mathrm{Iso}_\Psi \mathcal{D}$ cannot be the whole complex plane. This shows, that $T = \mathcal{T} \times Q$, where Q is a, possibly trivial, discrete group of translations. □

Corollary: *Let* (M, Φ), $(\mathbb{C}, \Psi)$ *and* $\mathrm{Iso}_\Psi \mathcal{D}$ *be defined as in Theorem 2.7. Let* $E\mathrm{d}z^2$ *be the Hopf differential of* $(\mathbb{C}, \Psi)$.

1. *If* $\mathrm{Iso}_\Psi \mathbb{C}$ *contains the group of all rotations around a fixed point, then, up to a biholomorphic change of coordinates, we have* $E = d(z - z_0)^m$, *where* $d \in \mathbb{C} \setminus 0$ *and* $m = 0, 1, 2, \ldots$ *is an integer.*

2. *If* $\mathrm{Iso}_\Psi \mathbb{C}$ *contains a one-parameter group* $\mathcal{T}$ *of translations, then, up to a biholomorphic change of coordinates, we have* $E \equiv 1$.

Proof: 1. W.l.o.g. we can choose $z_0 = 0$. Let $\mathcal{R} = \{g_\varphi \in \mathrm{Aut}\mathbb{C} | g_\varphi(z) = e^{i\varphi} z,\ \varphi \in [0, 2\pi)\}$ be the one-parameter group of rotations around the origin. By Corollary 2.5.1, $|E|$ is invariant under all automorphisms in $\mathcal{R} \subset \mathrm{Iso}_\Psi \mathcal{D}$. Therefore, for each $\varphi \in \mathbb{R}$, we get $E(g_\varphi(z)) = e^{i\theta} E(z)$, where $\theta = \theta(\varphi) \in \mathbb{R}$ depends linearly on φ. It follows, since E is holomorphic, that $\theta = m\phi$, m a nonnegative integer, and, up to a biholomorphic change of coordinates, $E = dz^m$, $d \in \mathbb{C}$. Since by assumption $E \not\equiv 0$, we have $d \neq 0$.

2. By Corollary 2.5, $|E|$ and therefore also the set of zeroes of E, is invariant under the group $\mathcal{T}$. Therefore, since the set of zeroes of a holomorphic function is discrete, E cannot have any zeroes. It can therefore, by a biholomorphic change of coordinates $\mathrm{d}w^2 = E(z)\mathrm{d}z^2$, be transformed into $E \equiv 1$. □

Remark: The immersions considered in the theorem and in the corollary will be investigated in more detail in Section 2.15.

2.8 In the next sections we will investigate some properties of the groups defined in Definition 2.6. We begin with the following

Lemma:

a) Let $g \in \mathrm{Aut}_\pi \mathcal{D}$ and $\hat{g} \in \mathrm{Aut}M$ be as in (2.6.2), then $\hat{g}$ is uniquely defined.

b) Let $\hat{g} \in \mathrm{Aut}_\Phi M$ and $\tilde{T} \in \mathrm{Aut}\Psi(\mathcal{D})$ be as in (2.6.3), then $\tilde{T}$ is uniquely defined.

c) Let $g \in \mathrm{Aut}_\Psi \mathcal{D}$ and $\tilde{T} \in \mathrm{Aut}\Psi(\mathcal{D})$ be as in (2.6.4), then $\tilde{T}$ is uniquely defined.

Proof: a) Assume $\hat{g}$ and $\hat{g}'$ both satisfy (2.6.2), then $\hat{g}(\pi(z)) = \hat{g}'(\pi(z))$ for all $z \in \mathcal{D}$. This implies $\hat{g} = \hat{g}'$, since π is surjective.

b) A proper Euclidean motion in $\mathbb{R}^3$ is determined uniquely by its restriction to an affine two dimensional subspace. If we choose a point $z \in M$, then for each point p of the affine tangent plane $\Phi(z) + \mathrm{d}\Phi(T_z M)$, $p = \Phi(z) + \mathrm{d}\Phi(v)$, we have

$$\tilde{T}(p) = \Phi(\hat{g}(z)) + (\hat{g}_*\mathrm{d}\Phi)(v). \tag{2.8.1}$$

Therefore $\tilde{T}$ is uniquely determined.

c) Similarly. □

Remark: It actually follows from the proof, that $\tilde{T}$ is already determined by the restriction of $\hat{g}$ to an arbitrary open subset of M.

2.9 Using Lemma 2.8 we define the following maps:

Definition:

$$\begin{aligned} \bar{\pi} : \mathrm{Aut}_\pi \mathcal{D} &\longrightarrow \mathrm{Aut}M, \\ \bar{\pi} : g &\longmapsto \hat{g}, \end{aligned} \tag{2.9.1}$$

where g and $\hat{g}$ are as in (2.6.2),

$$\begin{aligned} \phi : \mathrm{Aut}_\Phi M &\longrightarrow \mathrm{OAff}(\mathbb{R}^3), \\ \phi : \hat{g} &\longmapsto \tilde{T}, \end{aligned} \tag{2.9.2}$$

where $\hat{g}$ and $\tilde{T}$ are as in (2.6.3), and

$$\begin{aligned} \psi : \mathrm{Aut}_\Psi \mathcal{D} &\longrightarrow \mathrm{OAff}(\mathbb{R}^3), \\ \psi : g &\longmapsto \tilde{T}, \end{aligned} \tag{2.9.3}$$

where g and $\tilde{T}$ are as in (2.6.4).

By Lemma 2.8, ϕ and ψ are group homomorphisms. Also note, that the images of ϕ and ψ are contained in $\mathrm{Aut}\Psi(\mathcal{D})$.

Theorem:

a) The groups $\mathrm{Aut}_\Psi\mathcal{D}$ *and* $\mathrm{Aut}_\Phi M$ *are closed Lie subgroups of* $\mathrm{Aut}\mathcal{D}$ *and* $\mathrm{Aut}M$*, respectively.*

b) The maps $\bar{\pi}$*,* ϕ *and* ψ *are analytic homomorphisms of Lie groups.*

Proof: a) Let $g_n \in \mathrm{Aut}_\Psi\mathcal{D}$ be a sequence which converges to $g \in \mathrm{Aut}\mathcal{D}$. Then g_n converges uniformly on each compact subset of $\mathcal{D}$. In particular, $\Psi \circ g_n = \tilde{T}_n \circ \Psi$ converges uniformly to $\Psi \circ g$ on each sufficiently small closed ball around any point $z \in \mathcal{D}$. Therefore, also the differentials converge, whence $(\tilde{T}_n)_* \mathrm{d}_z\Psi = R_{\tilde{T}_n}\mathrm{d}_z\Psi$ converges, where we have written $\tilde{T}_n = (R_{\tilde{T}_n}, t_{\tilde{T}_n})$ as in (2.6.1). This implies, that $R_{\tilde{T}_n}$ converges to a rotation R in $\mathbb{R}^3$. Since also $\tilde{T}_n \circ \Psi = t_{\tilde{T}_n} + R_{\tilde{T}_n} \circ \Psi$ converges, $t_{\tilde{T}_n} \to t$ for some $t \in \mathbb{R}^3$. Altogether, this shows $\tilde{T}_n \to \tilde{T} = (R, t)$. But now $\Psi \circ g_n \to \Psi \circ g = \tilde{T} \circ \Psi$. This shows, that $\tilde{T} \in \mathrm{Aut}\Psi(\mathcal{D})$ and $g \in \mathrm{Aut}_\Psi\mathcal{D}$. The argument for $\mathrm{Aut}_\Phi M$ is similar.

b) We know, that $\mathrm{Aut}M$ and $\mathrm{Aut}\mathcal{D}$ are Lie groups and, by the argument above, we know that $\mathrm{Aut}_\Psi\mathcal{D}$ and $\mathrm{Aut}_\Phi M$ are closed subgroups of $\mathrm{Aut}\mathcal{D}$ and $\mathrm{Aut}M$, respectively. Therefore, with the induced topology $\mathrm{Aut}_\Psi\mathcal{D}$ and $\mathrm{Aut}_\Phi M$ are Lie groups. We show, that in this topology the maps ϕ and ψ are continuous, from which analyticity follows [13, Th. II.2.6].

Assume $g_n \to g$ in $\mathrm{Aut}_\Psi\mathcal{D}$. Then g_n converges to g uniformly on each compact subset of $\mathcal{D}$. In particular, $\Psi \circ g_n = \tilde{T}_n \circ \Psi$ converges uniformly to $\Psi \circ g = \tilde{T} \circ \Psi$ on each sufficiently small closed ball around any point $z \in \mathcal{D}$. By the proof of Lemma 2.8, $\tilde{T}$ and $\tilde{T}_n$ are uniquely determined by the restriction of g and g_n to an arbitrary open subset of $\mathcal{D}$. Therefore, $\tilde{T}_n$ converges to $\tilde{T}$. This shows, that ψ is continuous. For ϕ we proceed analogously. For $\bar{\pi}$ the claim is trivial. □

2.10 We will need the following

Theorem: *We retain the notation of Section 2.6. If* M *with the metric induced by* Φ *is complete, then for every Euclidean motion* $\tilde{T} \in \mathrm{Aut}\Psi(\mathcal{D})$ *there exists a* $g \in \mathrm{Aut}_\Psi\mathcal{D}$*, s.t.* $\Psi \circ g = \tilde{T} \circ \Psi$*, i.e.,* ψ *maps* $\mathrm{Aut}_\Psi\mathcal{D}$ *onto* $\mathrm{Aut}\Psi(\mathcal{D})$*. The automorphism* g *is unique up to multiplication with an element of* $\mathrm{Ker}\psi$*.*

Proof: Let $\tilde{T}$ be a Euclidean motion, which leaves the image $\Psi(\mathcal{D})$ invariant. Let us choose two arbitrary points z_0 and z_1 in $\mathcal{D}$, s.t. $\Psi(z_1) = \tilde{T}\Psi(z_0)$. Then, since Ψ is locally injective and conformal, for each such pair (z_0, z_1) there exists an open neighbourhood U_0 of z_0 and an open neighbourhood U_1 of z_1, s.t. $\tilde{T} \circ \Psi(z) = \Psi(h(z))$, $z \in U_0$, defines an orientation preserving isometry $h : U_0 \to U_1$.

Since M is complete, also $\mathcal{D}$ is complete [13, Prop. I.10.6]. Thus, $\mathcal{D}$ is an analytic, complete, simply connected manifold. Therefore, by [13, Sect. I.11], the local isometry h can be extended to a unique, global, orientation preserving self-isometry $g \in \mathrm{Aut}\mathcal{D}$ of $\mathcal{D}$, s.t. $g|_{U_0} = h$. By the definition of h we have $\Psi(g(z)) = \tilde{T}\Psi(z)$ on U_0. Since all occuring maps are analytic, we get $\tilde{T} \circ \Psi = \Psi \circ g$ on $\mathcal{D}$. Uniqueness of g up to an element of $\mathrm{Ker}\psi$ is trivial. □

The only group still to be discussed is $\mathrm{Aut}\Psi(\mathcal{D})$. Unfortunately, in general $\mathrm{Aut}\Psi(\mathcal{D})$ doesn't seem to be closed in $\mathrm{OAff}(\mathbb{R}^3)$. In Section 2.13 we will give a simple condition on (M, Φ), under which $\mathrm{Aut}\Psi(\mathcal{D})$ can be shown to be closed.

However, here we are able to conclude:

Corollary: *If* (M, Φ) *is complete, then* $\psi : \mathrm{Aut}_\Psi\mathcal{D} \to \mathrm{Aut}\Psi(\mathcal{D})$ *is surjective. In particular,* $\mathrm{Aut}\Psi(\mathcal{D}) \cong \mathrm{Aut}_\Psi\mathcal{D}/\mathrm{Ker}\psi$ *is a Lie group.*

Theorem 2.10 and Corollary 2.10 have well known equivalents for the map π. The arguments leading to Eq. (2.3.4) prove the following

Proposition: *Let* M *be a Riemann surface with simply connected cover* $\pi : \mathcal{D} \to M$*. With the notation as above we have:*

a) For every $\hat{g} \in \mathrm{Aut}M$ *there exists a* $g \in \mathrm{Aut}_\pi\mathcal{D}$*, s.t.* $\pi \circ g = \hat{g} \circ \pi$*.*

b) The map $\bar{\pi} : \mathrm{Aut}_\pi \mathcal{D} \to \mathrm{Aut} M$ is surjective.

2.11 We want to investigate, how the groups defined in Section 2.6 are related to each other by the maps $\bar{\pi}$, ϕ and ψ.

For every $g \in \bar{\pi}^{-1}(\mathrm{Aut}_\Phi M)$ we have

$$\Psi \circ g = \Phi \circ \pi \circ g = \Phi \circ \bar{\pi}(g) \circ \pi = \phi(\bar{\pi}(g)) \circ \Psi, \tag{2.11.1}$$

hence

$$\bar{\pi}^{-1}(\mathrm{Aut}_\Phi M) \subset \mathrm{Aut}_\Psi \mathcal{D} \tag{2.11.2}$$

and $\psi = \phi \circ \bar{\pi}$ on $\bar{\pi}^{-1}(\mathrm{Aut}_\Phi M)$.

For $g \in \mathrm{Ker}\bar{\pi}$ we have $\Psi \circ g = \Phi \circ \pi \circ g = \Phi \circ \pi = \Psi$. Therefore,

$$\mathrm{Ker}\bar{\pi} \subset \mathrm{Ker}\psi. \tag{2.11.3}$$

We also recall, that $\mathrm{Ker}\bar{\pi} = \Gamma$, the Fuchsian group of M.

Lemma: *Let (M, Φ) be a CMC-immersion with $\mathrm{Ker}\psi = \mathrm{Ker}\bar{\pi}$. Then the following holds:*

a) $(\bar{\pi})^{-1}(\mathrm{Aut}_\Phi M) = \mathrm{Aut}_\Psi \mathcal{D}$.

b) $\phi : \mathrm{Aut}_\Phi M \longrightarrow \mathrm{Aut}\Psi(\mathcal{D})$ *is an injective group homomorphism.*

If, in addition, (M, Φ) is complete, then:

c) *The action of $\tilde{T} \in \mathrm{Aut}\Psi(\mathcal{D})$ can be lifted to an action on M, i.e., ϕ is surjective.*

d) $\phi : \mathrm{Aut}_\Phi M \to \mathrm{Aut}\Psi(\mathcal{D})$ *is a group isomorphism.*

Proof: a) Since $\mathrm{Ker}\psi = \mathrm{Ker}\bar{\pi}$, we have that $\mathrm{Aut}_\Psi \mathcal{D}$ is in the normalizer of $\mathrm{Ker}\bar{\pi} = \Gamma$, whence $\mathrm{Aut}_\Psi \mathcal{D} \subset \mathrm{Aut}_\pi \mathcal{D}$, by (2.3.3). Therefore, for $g \in \mathrm{Aut}_\Psi \mathcal{D}$ we have

$$\Phi \circ \bar{\pi}(g) \circ \pi = \Phi \circ \pi \circ g = \Psi \circ g = \psi(g) \circ \Psi = \psi(g) \circ \Phi \circ \pi, \tag{2.11.4}$$

and $\bar{\pi}(g) \in \mathrm{Aut}_\Phi M$ follows, i.e., $\mathrm{Aut}_\Psi \mathcal{D} \subset \bar{\pi}^{-1}(\mathrm{Aut}_\Phi M)$. Now a) follows from Eq. (2.11.2).

b) From Eq. (2.11.1) and a) it follows, that $\psi = \phi \circ \bar{\pi}$ on $\mathrm{Aut}_\Psi \mathcal{D}$, which implies $\mathrm{Ker}\phi = \{\mathrm{id}\}$ and therefore b).

c) Let $\tilde{T} \in \mathrm{Aut}\Psi(\mathcal{D})$. Since (M, Φ) is complete, there exists, by Theorem 2.10, $g \in \mathrm{Aut}_\Psi \mathcal{D}$, s.t. $\tilde{T} = \psi(g)$. From a) it follows, that there exists $\hat{g} \in \mathrm{Aut}_\Phi M$, s.t. $\hat{g} = \bar{\pi}(g)$, whence $\tilde{T} = \phi(\hat{g})$. The map ϕ is therefore surjective.

d) follows from b) and c). □

Proposition:

a) $\mathrm{Ker}\psi$ is a discrete subgroup of $\mathrm{Aut}_\Psi \mathcal{D}$ and acts freely and discontinuously on $\mathcal{D}$.

b) $M' = \mathrm{Ker}\psi \setminus \mathcal{D}$ is a Riemann surface.

c) Let $\pi' : \mathcal{D} \to M'$ denote the natural projection. Then there exists an immersion Φ' of M' into $\mathbb{R}^3$, s.t. the following diagram commutes:

$$\begin{tikzcd}
& M = \mathrm{Ker}\bar{\pi} \setminus \mathcal{D} \arrow[dr, "\Phi"] \arrow[dd] & \\
\mathcal{D} \arrow[ur, "\pi"] \arrow[rr, "\Psi"] \arrow[dr, "\pi'"'] & & \mathbb{R}^3 \\
& M' = \mathrm{Ker}\psi \setminus \mathcal{D} \arrow[ur, "\Phi'"'] &
\end{tikzcd} \tag{2.11.5}$$

d) For the CMC-immersion (M',Φ') as above we define $\bar{\pi}'$ and ϕ' as in Section 2.9. Then

$$\mathrm{Ker}\bar{\pi}' = \mathrm{Ker}\psi. \tag{2.11.6}$$

Proof: a) Since $\psi : \mathrm{Aut}_\Psi\mathcal{D} \to \mathrm{OAff}(\mathbb{R}^3)$ is a continuous homomorphism of Lie groups, $\mathrm{Ker}\psi$ is, with the induced topology, a Lie subgroup of $\mathrm{Aut}_\Psi\mathcal{D}$. Therefore, if $\mathrm{Ker}\psi$ were nondiscrete, it would contain a one-parameter subgroup $\gamma(t)$. Hence $\Psi(\gamma(t).z) = \Psi(z)$ for all $z \in \mathcal{D}$ and all $t \in \mathbb{R}$. This implies $\gamma(t).z = z$ for all $z \in \mathcal{D}$ and all $t \in \mathbb{R}$, whence $\gamma(t) = I$ for all t, a contradiction.

Now let us assume, that $g \in \mathrm{Ker}\psi$ has a fixed point $z_0 \in \mathcal{D}$. Then

$$\begin{aligned} \Psi(g(z)) &= \Psi(z) \qquad \text{for all } z \in \mathcal{D}, & (2.11.7)\\ g(z_0) &= z_0. & (2.11.8) \end{aligned}$$

Taking into account the injectivity of the derivative of Ψ one gets by differentiating Eq. (2.11.7) at $z = z_0$,

$$g'(z_0) = 1. \tag{2.11.9}$$

In the case of $\mathcal{D} = \mathbb{C}$ one has $g(z) = az + b$, with $a, b \in \mathbb{C}$. It follows from Eqs. (2.11.9) and (2.11.8), that $g = \mathrm{id}$.

In the case of $\mathcal{D}$ being the unit circle we can view g as an isometry w.r.t. the Bergmann metric on $\mathcal{D}$. This together with [13, Lemma I.11.2] implies again $g = \mathrm{id}$.

It remains to be proved, that $\mathrm{Ker}\psi$ acts discontinuously, i.e. that there is a point $z_0 \in \mathcal{D}$, s.t. the orbit of $\mathrm{Ker}\psi$ through z_0 is discrete. For the unit circle this follows from the discreteness of $\mathrm{Ker}\psi$ and [10, Theorem IV.5.4]. For $\mathcal{D} = \mathbb{C}$ it is trivial, since then, by the arguments above, $\mathrm{Ker}\psi$ is a discrete group of translations.

b) Since by a), $\mathrm{Ker}\psi$ is a Fuchsian or elementary group, it follows, that $M = \mathrm{Ker}\psi \setminus \mathcal{D}$ is a Riemann surface (see e.g. [10, Section IV.5]).

c) From the definition of $\mathrm{Ker}\psi$ and M' it is clear that Ψ factors through M'. This defines an immersion $\Phi' : M' \to \mathbb{R}^3$. If $\pi' : \mathcal{D} \to M'$ is the natural projection, then $\Phi' \circ \pi' = \Psi$ and (2.11.5) follows.

d) is clear from the definition of M'. □

The last lemma shows, that for our purposes it is actually enough to restrict our attention to surfaces with

$$\mathrm{Ker}\psi = \mathrm{Ker}\bar{\pi}. \tag{2.11.10}$$

For these surfaces the conclusions of Lemma 2.11 hold.

2.12 The following proposition shows, what it means for the symmetry group $\mathrm{Aut}_\Psi\mathcal{D}$ that $\mathrm{Aut}\Psi(\mathcal{D})$ is not discrete.

Proposition: *Let (M,Φ) be a CMC-immersion with simply connected cover $\mathcal{D}$, which is complete w.r.t. the induced metric and admits a one parameter group of Euclidean motions $P \subset \mathrm{Aut}\Psi(\mathcal{D})$. Then $\mathrm{Aut}_\Psi\mathcal{D}$ also contains a one parameter group.*

Proof: Let $P = \{\tilde{T}_x, x \in \mathbb{R}\}$ be a one parameter subgroup of $\mathrm{Aut}\Psi(\mathcal{D})$, where $\mathrm{Aut}\Psi(\mathcal{D})$ carries the Lie group structure stated in Corollary 2.10. Let $A \subset \mathcal{D}$ be an open subset such that Ψ is injective on A. Let $a \in A$ be arbitrary. Since $\tilde{T}_0\Psi(a) = \Psi(a) \in \Psi(A)$, there exists some $\epsilon > 0$ and an open subset $A_\epsilon \subset A$, s.t. $\tilde{T}_x\Psi(A_\epsilon) \subset \Psi(A)$ for all $|x| < \epsilon$. Therefore, by Theorem 2.10, there exists an automorphism $g_x \in \mathrm{Aut}\mathcal{D}$, $|x| < \epsilon$, satisfying $\Psi \circ g_x = \tilde{T}_x \circ \Psi$, which is unique up to multiplication with an element in Kerψ. In addition, it follows from the proof of Theorem 2.10, that we can choose g_x s.t.

$$g_x(A_\epsilon) \subset A. \tag{2.12.1}$$

Since Kerψ is discrete, the condition (2.12.1) determines g_x uniquely, if A is small enough. This shows, that $g_{x+y} = g_x g_y$ for all sufficiently small $x, y \in \mathbb{R}$. For $x \to 0$ we have $\tilde{T}_x \to \tilde{T}_0 = I$, therefore $\Psi \circ g_x \to \Psi$ uniformly on $\mathcal{D}$. This shows, that g_x converges to an element of Kerψ. Since Kerψ acts freely on $\mathcal{D}$, Eq. (2.12.1) implies $g_x \to g_0 = I$ for $x \to 0$. This shows that $\omega : x \to g_x$ is a continuous and thus analytic (see [13, Th. II.2.6]) homomorphism from some interval $(-\tilde{\epsilon}, \tilde{\epsilon})$, $\tilde{\epsilon} > 0$, into $\mathrm{Aut}_\Psi \mathcal{D}$. For an arbitrary $x \in \mathbb{R}$ we write $x = m\frac{\tilde{\epsilon}}{2} + r$, where $r \in [0, \frac{\tilde{\epsilon}}{2})$ and $m \in \mathbb{Z}$ are uniquely determined. The definition

$$g_x = g_r (g_{\frac{\tilde{\epsilon}}{2}})^m \in \mathrm{Aut}_\Psi \mathcal{D}. \tag{2.12.2}$$

extends ω to a one-parameter subgroup of $\mathrm{Aut}_\Psi \mathcal{D}$, which finishes the proof. □

2.13 It remains to be investigated under which circumstances the existence of a cluster point of $\mathrm{Aut}\Psi(\mathcal{D})$ implies $\dim \mathrm{Aut}\Psi(\mathcal{D}) \geq 1$. This is certainly the case, if $\mathrm{Aut}\Psi(\mathcal{D})$ is closed in $\mathrm{OAff}(\mathbb{R}^3)$.

To this end we introduce the notion of an admissible immersion.

Definition: Let (M, Φ) be an immersed manifold in $\mathbb{R}^3$. A point $p \in \Phi(M)$ is called *admissible*, if there is an open neighbourhood U of p in $\mathbb{R}^3$, s.t. the intersection $\Phi(M) \cap U$ is closed in U. The immersion (M, Φ) is called admissible, if $\Phi(M)$ contains at least one admissible point.

We think it is fair to say that, basically, every surface of interest is admissible. Most surfaces studied actually belong to the smaller class of locally closed surfaces (see [15, II.2]), for which each point of the image is admissible. Among the locally closed surfaces are e.g. the immersed surfaces with closed image $\Phi(M)$ in $\mathbb{R}^3$, especially compact submanifolds of $\mathbb{R}^3$, and immersed surfaces (M, Φ), for which Φ is proper (see e.g. [20, I.2.30]).

Also note that, geometrically speaking, a surface has to return infinitely often to each neighbourhood of each of its nonadmissible points. A nonadmissible surface is therefore in a sense a two-dimensional analog of a Peano curve.

Remark: It is important to note, that admissibility is a property of the image $\Phi(M)$ of the immersion Φ. We don't claim that it is preserved under isometries (see Section 2.5). In particular, for an admissible surface it may well be, that not all members of the associated family are admissible.

The definition of admissible surfaces allows us to describe a large class of surfaces, for which $\mathrm{Aut}\Psi(\mathcal{D})$ is closed.

Theorem: *If (M, Φ) is a complete, admissible surface in $\mathbb{R}^3$ and $(\mathcal{D}, \Psi)$ is the simply connected cover of M with the covering immersion $\Psi = \Phi \circ \pi$, then the group $\mathrm{Aut}\Psi(\mathcal{D})$ is closed in $\mathrm{OAff}(\mathbb{R}^3)$.*

<u>Proof:</u> Let $\tilde{T}_n \in \mathrm{Aut}\Psi(\mathcal{D})$ be a sequence of symmetry transformations of $\Psi(D)$ which converges to $\tilde{T} \in \mathrm{OAff}(\mathbb{R}^3)$. Therefore, also the sequence $\tilde{T}_n^{-1}$ converges in $\mathrm{OAff}(\mathbb{R}^3)$.

Since (M, Φ) is admissible, there exists an admissible point $p \in \Psi(D)$, together with an open ball $B(p, \epsilon)$ of radius $\epsilon < 1$ around p in $\mathbb{R}^3$, s.t. $B(p, \epsilon) \cap \Psi(\mathcal{D})$ is closed in $B(p, \epsilon)$.

W.l.o.g. we can assume that p and the whole bounded sequence $\{\tilde{T}_n^{-1}(p)\}$ lies in $B(0, \frac{1}{2})$. Otherwise we first apply a scaling transformation of $\mathbb{R}^3$, which changes neither the admissibility of (M, Φ) nor the group structure of $\mathrm{Aut}\Psi(\mathcal{D})$.

We take $N \in \mathbb{N}$ s.t. $\|\tilde{T} - \tilde{T}_n\| < \frac{\epsilon}{3}$ for $n \geq N$, where $\|\cdot\|$ denotes the operator norm.

We choose $p' = \tilde{T}_N^{-1}(p)$ and $z' \in \mathcal{D}$, s.t. $p' = \Psi(z')$. Since $\tilde{T}_N$ is a Euclidean motion we have that $\tilde{T}_N(B(p', \epsilon)) = B(p, \epsilon)$.

For all $q \in B(p', \frac{\epsilon}{3}) \cap \Psi(\mathcal{D})$ we get $|q| \leq |q - p'| + |p'| < \frac{\epsilon}{3} + \frac{1}{2} < 1$, therefore, if $n \geq N$,

$$|\tilde{T}_n(q) - p| \leq |\tilde{T}_n(q) - \tilde{T}(q)| + |\tilde{T}(q) - \tilde{T}(p')| + |\tilde{T}(p') - \tilde{T}_N(p')| < \frac{5}{6}\epsilon. \tag{2.13.1}$$

Thus we have $\tilde{T}_n(q) \in B(p, \frac{5}{6}\epsilon) \cap \Psi(\mathcal{D})$ for all $n \geq N$ and $\overline{\{\tilde{T}_n(q), n \geq N\}} \subset B(p, \epsilon)$. Now we know, that $B(p, \epsilon) \cap \Psi(\mathcal{D})$ is closed in $B(p, \epsilon)$, and that $\tilde{T}_n(q)$ converges by assumption to $\tilde{T}(q) \in \mathbb{R}^3$. Therefore, the limit $\tilde{T}(q)$ is in $\Psi(\mathcal{D})$. Since Ψ is an immersion, there exists an open neighbourhood A of z' which is mapped into $B(p', \frac{\epsilon}{3})$ by Ψ. Therefore, if we choose $z \in \mathcal{D}$, s.t. $p = \Psi(z)$, then $\tilde{T}$ induces an isometry g of A onto an open neighbourhood $B \subset \mathcal{D}$ of z, s.t. $\tilde{T} \circ \Psi = \Psi \circ g$ on A. By [13, Sect. I.11], this isometry can be extended globally to a unique automorphism $g \in \mathrm{Aut}\mathcal{D}$. Since all maps are analytic and globally defined, the relation $\tilde{T} \circ \Psi = \Psi \circ g$ holds on the whole of $\mathcal{D}$. It follows, that $g \in \mathrm{Aut}_\Psi \mathcal{D}$ and $\tilde{T} \in \mathrm{Aut}\Psi(\mathcal{D})$. □

Corollary: Let (M, Φ) and $\mathcal{D}$ be as in Theorem 2.13. If $\mathrm{Aut}\Psi(\mathcal{D})$ is nondiscrete, then also $\mathrm{Aut}_\Psi \mathcal{D}$ is nondiscrete.

Proof: By the assumptions, $\mathrm{Aut}\Psi(\mathcal{D})$ is closed in $\mathrm{OAff}(\mathbb{R}^3)$ and nondiscrete. It therefore contains a one parameter group and the corollary follows from Proposition 2.12 above. □

Finally, we state the following result for the translational parts of the elements of $\mathrm{Aut}\Psi(\mathcal{D})$:

Proposition: *If (M, Φ) is admissible and complete, and $\mathrm{Aut}\Psi(\mathcal{D})$ is discrete, then the set $\mathcal{L} = \{t | \tilde{T} = (R, t) \in \mathrm{Aut}\Psi(\mathcal{D})\}$ of translations is discrete.*

Proof: Assume, $\mathcal{L}$ is not discrete. Then there exists a $t \in \mathcal{L}$ and a sequence $\{t_n\} \subset \mathcal{L}$, $t_n \neq t$, s.t. t_n converges to t. Since the set of rotations is compact, we obtain a subsequence $\tilde{T}_n = (R_n, t_n) \in \mathrm{Aut}\Psi(\mathcal{D})$, which converges in $\mathrm{OAff}(\mathbb{R}^3)$. Since by Theorem 2.13, $\mathrm{Aut}\Psi(\mathcal{D})$ is closed, this sequence converges in $\mathrm{Aut}\Psi(\mathcal{D})$ to some $\tilde{T}$. But then $\tilde{T}_n = \tilde{T}$ for sufficiently large n, since $\mathrm{Aut}\Psi(\mathcal{D})$ is discrete. This shows, that $t_n = t$ for sufficiently large n, a contradiction. □

2.14 As examples for the discussion in this chapter, let us investigate two well known classes of CMC-surfaces, the Delaunay and the Smyth surfaces.

We recall that a Delaunay surface is defined as a complete, immersed surface of constant mean curvature which is generated in $\mathbb{R}^3$ by rotating a curve around a given axis. Clearly, every sphere and every cylinder is a Delaunay surface. We will restrict the definition to those surfaces which have mean curvature $H = -\frac{1}{2}$ and exclude the degenerate case of the sphere.

Let us translate two well known facts about Delaunay surfaces into our language:

Proposition: *1. Let (M, Φ) be a noncylindrical CMC-immersion with universal covering immersion $(\mathcal{D}, \Psi)$, s.t. $\Phi(M)$ is a Delaunay surface. Then $\Phi(M)$ is generated by rotating the roulette of an ellipse (unduloid) or a hyperbola (nodoid) along the line on which the conic rolled.*

2. Let $S \subset \mathbb{R}^3$ be a Delaunay surface. Then there exists a CMC-immersion (M, Φ) with universal covering immersion $(\mathcal{D}, \Psi)$, $\mathcal{D} = \mathbb{C}$, s.t.

- *$S = \Phi(M)$,*
- *$\mathrm{Aut}_\Psi \mathcal{D}$ contains a one parameter group $\mathcal{T}$ of translations, wich is mapped by the surjective homomorphism $\psi : \mathrm{Aut}_\Psi \mathcal{D} \to \mathrm{Aut}\Psi(\mathcal{D})$ to the group of rotations around the axis of revolution of the Delaunay surface.*

Proof: 1. well known, see e.g. [8].

2. such immersions (with $\Phi = \Psi$, $M = \mathcal{D} = \mathbb{C}$) are explicitly constructed in [19]. $\square$

For later use, we collect some definitions for surfaces of revolution (see e.g. [22]): If S is a surface of revolution, generated by rotating the plane curve $C \subset \mathbb{C}$ around the axis A, then A is called the ***axis of revolution***, the images of C under a rotation around A are called ***meridians***, and the intersection circles of S with any plane perpendicular to A are called the ***parallels*** of S.

Remark: 1. While it is clear for the unduloid, that the roulette of an ellipse gives a periodic curve along A, this is not clear for the nodoid. Here, in order to get a complete noncompact surface w.r.t. the induced metric, one has to continue the roulette periodically along A. This is possible in a unique way (see [8]) and gives a periodic curve along A.

2. Besides of being periodic, the roulette of an ellipse or hyperbola has another important property: In each of its periods there is a unique point of maximal distance from the line A on which the conic rolls. The roulette is symmetric w.r.t. the reflection at any line perpendicular to A, which passes through such a point of maximal distance from A. The Delaunay surface is therefore invariant under a 180°-rotation around any axis which is perpendicular to the axis of revolution A and passes through a parallel of maximal radius.

Let us recall again, that in the discussion of Delaunay surfaces we exclude the catenoid, the roulette of the parabola, which is a minimal surface. Since we also exclude the sphere, we get the following result, the proof of which is an exercise in elementary geometry:

Lemma: *Each Delaunay surface determines its axis of revolution uniquely.*

Proposition 2.14 gives the following

Corollary: *Let (M, Φ) be as in Proposition 2.14. Let A be the generating axis of the Delaunay surface $\Phi(M)$. Then, as a set,* $\mathrm{Aut}\Psi(\mathcal{D})$ *can be written as*

$$\mathrm{Aut}\Psi(\mathcal{D}) = \mathcal{R} \times \tilde{Q} \times \{I, \tilde{R}\}, \tag{2.14.1}$$

where $\mathcal{R}$ is the one-parameter group of rotations around A, $\tilde{Q}$ is a nontrivial discrete group of translations along A, and $\tilde{R}$ is a 180°-rotation around an axis which is perpendicular to A.

<u>Proof:</u> By the remark above, every Delaunay surface is generated by a periodic function that has only one maximum in every period interval. Let $\mathcal{P} \subset \mathbb{R}^3$ be the union of all parallels of $\Phi(M)$ which have maximal radius. Denote by $I(\mathcal{P})$ the set of all proper Euclidean motions, which leave $\mathcal{P}$ invariant. $\mathcal{P}$ consists of disjoint circles which all lie on the same cylinder around A. Let $\mathcal{C} \subset A$ denote the set of centers of these circles. Then, clearly, $I(\mathcal{P})$ is the set of all elements of $\mathrm{OAff}(\mathbb{R}^3)$, which leave A and $\mathcal{C}$ invariant. By the second part of Remark 2.14, the map $\mathcal{C}$ consists of equidistant points along A. It follows, that $I(\mathcal{P})$ is generated by

- the one-parameter group $\mathcal{R}$ of rotations around A,
- a discrete group $\tilde{Q}$ of translations along A, which is given by $\mathcal{C}$, i.e., the periodicity of the roulette,
- the 180°-rotation $\tilde{R}$ around an axis, which passes through an arbitrary fixed point on $\mathcal{P}$ and the center of the corresponding parallel.

We therefore have

$$I(\mathcal{P}) = \mathcal{R} \times \tilde{Q} \times \{I, \tilde{R}\}, \tag{2.14.2}$$

where '$\times$' denotes the product of sets. By Lemma 2.14, a Delaunay surface is left invariant precisely by those proper Euclidean motions, which preserve the axis of revolution and map a meridian into another meridian. In particular, if $p \in \Phi(M)$ and $\tilde{T} \in \mathrm{Aut}\Psi(\mathcal{D})$, then p and $\tilde{T}(p)$ have the same distance from A. The set $\mathcal{P}$ is the set of all points on $\Phi(M)$, which have maximal distance from the axis A. Therefore, every element of $\mathrm{Aut}\Psi(\mathcal{D})$ leaves also the set $\mathcal{P}$ invariant, i.e., $\mathrm{Aut}\Psi(\mathcal{D}) \subset I(\mathcal{P})$. As a surface of revolution around A, the Delaunay surface is certainly invariant under the group $\mathcal{R}$ of rotations around A. In addition, by the second part of Remark 2.14, for a Delaunay surface, the meridians are periodic along the axis of revolution and symmetric w.r.t. a $180°$-rotation around any axis which is perpendicular to A and intersects $\mathcal{P}$. Therefore, $\Phi(M) = \Psi(\mathcal{D})$ is invariant also under the group $\tilde{Q}$ and the rotation $\tilde{R}$ defined above. This gives $I(\mathcal{P}) \subset \mathrm{Aut}\Psi(\mathcal{D})$, which finishes the proof. □

Smyth [19] introduced for every integer $m \geq 0$ a one-parameter family of conformal immersions

$$\Psi_c^m : \mathbb{C} \longrightarrow \mathbb{R}^3, \qquad c \in \mathbb{C} \setminus \{0\}, \tag{2.14.3}$$

with constant mean curvature, s.t. the induced metric is complete and invariant under the one-parameter group of rotations around $z = 0$ in $\mathbb{C}$. We will call these surfaces Smyth surfaces. The Hopf differential of $(\mathbb{C}, \Psi_c^m)$ is $cz^m \mathrm{d}z^2$, and therefore each $(\mathbb{C}, \Psi_c^m)$ has an umbilic of order m at the origin. For $m = 0$ the family Ψ_c^m contains the cylinder. We will call a Smyth surface *nondegenerate*, if its image in $\mathbb{R}^3$ is not a cylinder.

Two surfaces in $\mathbb{R}^3$ will be called congruent if they are related by a proper Euclidean motion of $\mathbb{R}^3$. Recall also the definition of the associated family in Section 2.5.

The following results were proved by Smyth [19]:

Theorem: *Let (M, Φ), with covering immersion $(\mathcal{D}, \Psi)$, be a complete, immersed surface of constant mean curvature, admitting a one-parameter group of self-isometries. Then the following holds:*

1. *The simply connected cover of the Riemann surface M is $\mathcal{D} = \mathbb{C}$.*

2. *The associated family of $(\mathcal{D}, \Psi)$ contains either a Delaunay or a Smyth surface, i.e., $(\mathcal{D}, \Psi)$ is isometric to the simply connected cover of a Delaunay or a Smyth surface.*

3. *The surface $(\mathcal{D}, \Psi)$ admits a one-parameter group P of self-isometries which is*

  a) *a one-parameter group of translations ($P \cong \mathbb{R}$) in case of the Delaunay surfaces,*

  b) *a one-parameter group of rotations around a fixed point in $\mathbb{C}$ ($P \cong S^1$) in case of the Smyth surfaces.*

2.15 With the results in the previous section we can also easily derive the uniformization of Delaunay and Smyth surfaces.

Proposition: *1. Each Delaunay surface is conformally equivalent to the cylinder, i.e., its simply connected cover is $\mathbb{C}$ and the Fuchsian group is a one-parameter group of translations.*

2. Each nondegenerate Smyth surface is conformally equivalent to $\mathbb{C}$.

3. For a nondegenerate Smyth surface we have

$$\mathrm{Ker}\,\overline{\pi} = \Gamma = \{\mathrm{id}\}. \tag{2.15.1}$$

Proof: We already know from Theorem 2.14, that both, Delaunay and Smyth surfaces, have, as Riemann surfaces, the simply connected cover $\mathbb{C}$. Therefore, by Lemma 2.3, they are biholomorphically equivalent to the plane, the cylinder or a torus and the Fuchsian group is either trivial or it is a discrete group of translations.

1. Delaunay surfaces are surfaces of revolution, i.e., there exists a conformal immersion $\Psi : \mathbb{C} \to \mathbb{R}^3$, which is invariant under a discrete one-dimensional lattice Γ of translations in $\mathbb{C}$, s.t. $\Psi(\mathcal{D})$ is the Delaunay surface. The Fuchsian group of the underlying Riemann surface contains therefore a group of translations. Since Delaunay surfaces are noncompact, they are biholomorphically equivalent to the cylinder.

2. Let $\Psi : \mathbb{C} \to \mathbb{R}^3$ be an immersion s.t. $\Psi(\mathcal{D}) = \Psi_c^m(\mathcal{D})$ is a Smyth surface for some parameters $m \in \mathbb{N}$, $c \in \mathbb{C}$. Assume, that the Fuchsian group of the surface contains a nontrivial translation. Then, by Theorem 2.14, the group $\mathrm{Aut}_\Psi \mathcal{D}$ satisfies the assumptions of the first part of Lemma 2.7. Therefore, $\Psi(\mathcal{D})$ is a cylinder. For a nondegenerate Smyth surface this gives $M = \mathbb{C} = \mathcal{D}$, i.e., the Smyth surface is conformally equivalent to the complex plane.

3. From 2. it follows, that $\pi = \mathrm{id}$ and $\mathrm{Aut}_\pi \mathcal{D} = \mathrm{Aut}\mathcal{D} = \mathrm{Aut}M$, therefore $\mathrm{Ker}\bar{\pi} = \Gamma = \{\mathrm{id}\}$. □

We continue with the following

Lemma: *1. For each noncylindrical Delaunay surface $S \subset \mathbb{R}^3$, there exists a CMC-immersion (M, Φ) with $\Phi(M) = S$, s.t.*

$$\mathrm{Ker}\psi = \mathrm{Ker}\bar{\pi}, \tag{2.15.2}$$

$$\mathrm{Aut}_\Phi M \cong \mathrm{Aut}\Psi(\mathcal{D}), \tag{2.15.3}$$

and, as a product of sets, we have

$$\mathrm{Aut}_\Psi D = \mathrm{Iso}_\Psi D = \mathcal{T} \times Q \times R, \tag{2.15.4}$$

where $\mathcal{T} \subset \mathrm{Aut}\mathbb{C}$ is a one-parameter group of translations, $Q \subset \mathrm{Aut}\mathbb{C}$ is a discrete group of translations with one generator, and $R = \{I, R_\pi\} \subset \mathrm{Aut}\mathbb{C}$ is the group generated by the inversion $R_\pi : z \mapsto -z$.

2. If $(\mathbb{C}, \Psi = \Psi_c^m)$ is a nondegenerate Smyth surface, then

$$\mathrm{Ker}\psi = \mathrm{Ker}\bar{\pi} = \{\mathrm{id}\} \tag{2.15.5}$$

and

$$\mathrm{Aut}\Psi(\mathcal{D}) \cong \mathrm{Aut}_\Psi \mathcal{D} = \mathcal{R}, \tag{2.15.6}$$

where $\mathcal{R}$ is a finite group of rotations around $z = 0$ in $\mathbb{C}$.

Proof: 1. For a Delaunay surface, there exists, by Proposition 2.14, a universal covering immersion $\Psi : \mathbb{C} \to \mathbb{R}^3$, s.t. the set $\mathrm{Aut}_\Psi \mathcal{D} \subset \mathrm{Iso}_\Psi \mathcal{D}$ contains a one-parameter group $\mathcal{T} \cong \mathbb{R}$ of translations. The group $\mathcal{T}$ is then mapped by the surjective homomorphism $\psi : \mathrm{Aut}_\Psi \mathcal{D} \to \mathrm{Aut}\Psi(\mathcal{D})$, defined in Section 2.11, into the set of rotations $\mathcal{R}$ around A, the axis of revolution. By Proposition 2.11 we can further assume, that Eq. (2.15.2) holds. Therefore, $\mathcal{T}$ contains the discrete Fuchsian group Γ of the Delaunay surface. By a rotation of the coordinate system in $\mathbb{C}$, we can choose $\mathcal{T}$ as the group of translations along the imaginary axis in $\mathbb{C}$. If we identify $M = \Gamma \setminus \mathbb{C}$ with a fundamental domain of Γ in $\mathbb{C}$, then Ψ maps this region, a strip parallel to the real axis, conformally to the Delaunay surface. The meridians of $\Phi(M)$ are identified with lines in M which are parallel to the real axis, the parallels of $\Phi(M)$ are identified with the intersection of M with lines parallel to the imaginary axis. In particular, the set $\mathcal{P}$, which was defined in the proof of Corollary 2.14, is identified with the intersection of M with an equidistant set $\mathcal{L}$ of lines parallel to the imaginary axis. By a translation of the coordinate system in $\mathbb{C}$ we can, in addition, assume $0 \in \mathcal{L}$.

By Eq. (2.15.2) and Lemma 2.11, we have that $\phi : \mathrm{Aut}_\Phi M \to \mathrm{Aut}\Psi(\mathcal{D})$, defined in Section 2.11, is an isomorphism of Lie groups. Therefore, Eq. (2.15.3) holds. With the definitions in Corollary 2.14 we can descibe the group $\mathrm{Aut}_\Phi M$, and, by the identification of M with a subset of $\mathbb{C}$, also $\mathrm{Aut}_\Psi \mathcal{D}$, explicitly:

- The group $\mathcal{R} \subset \mathrm{Aut}\Psi(\mathcal{D})$ is identified in $\mathrm{Aut}M$ with the set of translations parallel to the imaginary axis in $\mathbb{C}$. We therefore have $\mathcal{T} = \psi^{-1}(\mathcal{R})$.
- The group $\tilde{Q} \subset \mathrm{Aut}\Psi(\mathcal{D})$ is identified in $\mathrm{Aut}M$ with the group Q of translations in $\mathbb{C}$ which leave $\mathcal{L}$ invariant. We therefore have $\psi^{-1}(\tilde{Q}) = Q$.
- The rotation $\tilde{R}$ is identified in $\mathrm{Aut}M$ with a $180°$-rotation around an arbitrary fixed point $z \in \mathcal{L}$. We choose $z = 0$. Then $\tilde{R}$ is identified with $R_\pi : z \to -z$ up to an automorphism in $\Gamma \subset \mathcal{T}$. We therefore have $\psi^{-1}(\mathcal{R} \times \{I, \tilde{R}\}) = \mathcal{T} \times \{I, R_\pi\}$.

Since ψ is a surjective homomorphism and since $\mathrm{Aut}\Psi(\mathcal{D}) = \mathcal{R} \times \tilde{Q} \times \{I, \tilde{R}\}$, we have

$$\mathrm{Aut}_\Psi\mathcal{D} = \psi^{-1}(\mathrm{Aut}\Psi(\mathcal{D})) = \mathcal{T} \times Q \times \{I, R_\pi\}. \tag{2.15.7}$$

Since with Theorem 2.7,

$$\mathcal{T} \times Q \times \{I, R_\pi\} \subset \mathrm{Aut}_\Psi\mathcal{D} \subset \mathrm{Iso}_\Psi\mathcal{D} \subset \mathcal{T} \times Q \times \{I, R_\pi\}, \tag{2.15.8}$$

we get $\mathrm{Iso}_\Psi\mathcal{D} = \mathcal{T} \times Q \times \{I, R_\pi\}$, and therefore Eq. (2.15.4).

2. For the immersions Ψ_c^m, the metric $\frac{1}{2}e^u dz d\bar{z}$ and, by Eq. (2.4.8), also $|E|$ is invariant under the one-parameter group of rotations around $z = 0$. Therefore, by Corollary 2.6 and Theorem 2.7, we have that either $\mathrm{Aut}_\Psi\mathcal{D}$ is contained in the group of rotations around a fixed point, or $\Psi(\mathcal{D})$ is a cylinder. Therefore, for a nondegenerate Smyth surface, $\mathrm{Ker}\psi \subset \mathrm{Aut}_\Psi\mathcal{D}$ consists only of rotations. But since, by Proposition 2.11, $\mathrm{Ker}\psi$ acts freely on $M = \mathbb{C}$, this implies that $\mathrm{Ker}\psi = \{\mathrm{id}\} = \mathrm{Ker}\bar{\pi}$ and, with Corollary 2.10, $\psi : \mathrm{Aut}_\Psi\mathcal{D} \to \mathrm{Aut}\Psi(\mathcal{D})$ is an isomorphism of Lie groups, i.e.,

$$\mathrm{Aut}\Psi(\mathcal{D}) \cong \mathrm{Aut}_\Psi\mathcal{D}. \tag{2.15.9}$$

Let $g \in \mathrm{Aut}_\Psi\mathcal{D}$. Then g is a rotation around $z = 0$, i.e., $g(z) = e^{i\theta}z$ for some $\theta \in [0, 2\pi)$. By Eq. (2.6.7) and Corollary 2.7, we get that $e^{i(m+2)\theta} = 1$ for some integer $m \geq 0$. This shows, that $\mathrm{Aut}_\Psi\mathcal{D}$ is a discrete group of rotations around $z = 0$ in $\mathbb{C}$, finishing the proof. $\square$

In Section 5.1 we will revisit the Smyth surfaces.

3 Automorphisms in the DPW approach

We are working in the framework of [6] and [3]. For notational conventions see the appendix of [3].

In this chapter we consider only complete CMC-immersions (M, Φ) which satisfy $\mathrm{Ker}\psi = \mathrm{Ker}\bar{\pi}$. Therefore, by Lemma 2.11, every $\tilde{T} \in \mathrm{Aut}\Psi(\mathcal{D})$ is associated with a $\hat{g} \in \mathrm{Aut}_\Phi M$. We have the following commutative diagram of surjective group homomorphisms

$$\begin{tikzcd} \mathrm{Aut}_\Psi\mathcal{D} = (\bar{\pi}^{-1})(\mathrm{Aut}_\Phi M) \arrow[r, "\psi"] \arrow[d, "\bar{\pi}"] & \mathrm{Aut}\Psi(\mathcal{D}) \\ \mathrm{Aut}_\Phi M \arrow[ur, "\phi"'] & \end{tikzcd}$$

and ϕ is a group isomorphism.

3.1 Let $\hat{N}$ be the Gauß map of M. We lift $\hat{N}$ to $\mathcal{D}$:

$$\begin{tikzcd} \mathcal{D} \arrow[r, "N"] \arrow[d, "\pi"] & S^2 \\ M = \Gamma \setminus \mathcal{D} \arrow[ur, "\hat{N}"'] & \end{tikzcd}$$

For $\gamma \in \Gamma$ we get $N \circ \gamma = N$ and for $g \in \mathrm{Aut}_\Psi \mathcal{D}$, $\tilde{T} = \psi(g) = (R_{\tilde{T}}, t_{\tilde{T}})$, we get

$$N \circ g = R_{\tilde{T}} \circ N. \tag{3.1.1}$$

To translate everything into the language of Lie groups we note $S^2 \cong \mathbf{SO}(3)/\mathbf{SO}(2) \cong \mathbf{SU}(2)/\mathbf{U}(1)$. From now on we use, without changing the notation, the spinor representation $J : \mathbb{R}^3 \to \mathbf{su}(2)$, of $\mathbb{R}^3$, which is defined by

$$r \in \mathbb{R}^3 \longmapsto -\frac{i}{2}\mathbf{r} \cdot \sigma, \qquad \sigma = (\sigma_1, \sigma_2, \sigma_3), \tag{3.1.2}$$

σ_i, $i = 1, 2, 3$, being the Pauli matrices.

Using the universal covering immersion $\Psi : \mathcal{D} \to \mathbb{R}^3$ we can lift the Gauß map $N : \mathcal{D} \to S^2$ to a frame $\hat{F} = (e^{-\frac{u}{2}}\Psi_x, e^{-\frac{u}{2}}\Psi_y, N) : \mathcal{D} \to \mathbf{SO}(3)$. We will also normalize the frame, s.t. $\hat{F}(0) = I$.

Consider now $g \in \mathrm{Aut}_\Psi \mathcal{D}$. Then (3.1.1) implies, that also the tangent space at $\Psi(z, \overline{z})$ is moved by $R_{\tilde{T}}$ to the tangent space at $\Psi(g(z), \overline{g(z)})$. Since $R_{\tilde{T}}$ is orientation preserving, the axes can be aligned by a rotation fixing N. Hence,

$$(\hat{F} \circ g)(z, \overline{z}) = R_{\tilde{T}} \hat{F}(z, \overline{z}) \hat{k}(g, z, \overline{z}). \tag{3.1.3}$$

Note, that here $\hat{k}$ is differentiable in $(z, \overline{z})$. Since with $g \mapsto \psi(g)$ also $g \mapsto R_{\tilde{T}}$ is a homomorphism, it is easy to verify, that $\hat{k}$ is a cocycle, i.e.,

$$\hat{k}(g_2 \circ g_1, z, \overline{z}) = \hat{k}(g_1, z, \overline{z}) \hat{k}(g_2, g_1(z), \overline{g_1(z)}). \tag{3.1.4}$$

Since $\mathcal{D}$ is simply connected, $\hat{F}$ can be lifted to a frame $\tilde{F}$ taking values in $\mathbf{SU}(2)$. This lift is unique, if we require

$$\tilde{F}(0) = I. \tag{3.1.5}$$

Taking a preimage $\tilde{\chi}(g) \in \mathbf{SU}(2)$ of $R_{\tilde{T}} \in \mathbf{SO}(3)$ and $k(g, z, \overline{z}) \in \mathbf{U}(1)$ of $\hat{k}(g, z, \overline{z}) \in \mathbf{SO}(2)$ gives

$$(\tilde{F} \circ g)(z, \overline{z}) = \tilde{\chi}(g) \tilde{F}(z, \overline{z}) k(g, z, \overline{z}). \tag{3.1.6}$$

Clearly, $\tilde{\chi}(g)$ and k are determined up to a factor $\pm I$. In particular, k is differentiable in $(z, \overline{z})$. The homomorphism property of $R_{\tilde{T}}$ and the cocycle condition for $\hat{k}$ only lift to

$$\tilde{\chi}(g_2 \circ g_1) = \pm \tilde{\chi}(g_2) \tilde{\chi}(g_1), \tag{3.1.7}$$

$$k(g_2 \circ g_1, z, \overline{z}) = \pm k(g_1, z, \overline{z}) k(g_2, g_1(z), \overline{g_1(z)}). \tag{3.1.8}$$

Although we cannot expect $\tilde{\chi}(g)$ to be a homomorphism, at least for translations in $\mathrm{Aut}_\Psi \mathcal{D}$, $\mathcal{D} = \mathbb{C}$, we have

Lemma: *Let $\Psi : \mathbb{C} \to \mathbb{R}^3$ be a CMC-immersion. Define the frame $\hat{F} = (e^{-\frac{u}{2}}\Psi_x, e^{-\frac{u}{2}}\Psi_y, N) : \mathbb{C} \to$ $\mathbf{SO}(3)$ and its lift $\tilde{F} : \mathbb{C} \to \mathbf{SU}(2)$ as above. Let T be the subgroup of translations in $\mathrm{Aut}_\Psi \mathbb{C}$. Then $\hat{k}(g, z, \overline{z}) \equiv I \in \mathbf{SO}(3)$ in Eq. (3.1.3) for all $g \in T$. If one lifts $\hat{k}(g, z, \overline{z})$ to $k(g, z, \overline{z}) \equiv I \in \mathbf{SU}(2)$ for all $g \in T$, then, after restriction to T, $\tilde{\chi}(g)$ becomes a homomorphism of groups from T into $\mathrm{Aut}\Psi(\mathbb{C})$.*

Proof: If $g \in T \subset \mathrm{Aut}_\Psi \mathbb{C}$, then $\Psi \circ g = \tilde{T} \circ \Psi$ for some $\tilde{T} = (R_{\tilde{T}}, t_{\tilde{T}}) \in \mathrm{OAff}(\mathbb{R}^3)$. From this it follows for the derivative of Ψ, since g is a translation:

$$D(\Psi \circ g)(z, \overline{z}) = R_{\tilde{T}}(D\Psi(g(z), \overline{g(z)}) \cdot Dg(z, \overline{z})) = R_{\tilde{T}}(D\Psi(z, \overline{z})). \tag{3.1.9}$$

Using Eq. (2.6.6) and $g'(z) = 1$, we get

$$\hat{F} \circ g = (e^{-\frac{u}{2}} R_{\tilde{T}} \Psi_x, e^{-\frac{u}{2}} R_{\tilde{T}} \Psi_y, R_{\tilde{T}} N) = R_{\tilde{T}} \hat{F}, \tag{3.1.10}$$

i.e., $\hat{k}(g,z,\overline{z}) \equiv I$. From this it follows, that we can lift $\hat{k}(g,z,\overline{z})$ for all $g \in T$ to $k(g,z,\overline{z}) \equiv I$. Since then, by Eq. (3.1.6),

$$(\tilde{F} \circ g)(z,\overline{z}) = \tilde{\chi}(g)\tilde{F}(z,\overline{z}), \tag{3.1.11}$$

we have that $\tilde{\chi} : T \to \mathrm{Aut}\Psi(\mathbb{C})$ is a homomorphism of groups. □

Remark: In particular, if $M = \Gamma \setminus \mathcal{D}$ is a torus or a Delaunay surface, then the group Γ consists of translations in $\mathcal{D} = \mathbb{C}$. Lemma 3.1 then shows, that we can assume $k(\gamma, z, \overline{z}) = I$ for all $\gamma \in \Gamma \subset \mathrm{Aut}_\Psi \mathbb{C}$. Thus, $\tilde{\chi}$ maps Γ and the subgroup of all translations in $\mathrm{Aut}_\Psi \mathcal{D}$ homomorphically into $\mathrm{Aut}\Psi(\mathbb{C})$.

3.2 Since $S^2 \cong \mathbf{SU}(2)/\mathbf{U}(1)$ is a compact symmetric space, we have an associated Cartan decomposition $\mathbf{su}(2) = \mathbf{k} + \mathbf{p}$, where in a suitable matrix representation $\mathbf{k}$ consists of diagonal and $\mathbf{p}$ consists of off-diagonal matrices. Then we consider the $\mathbf{su}(2)$-valued differential form

$$\tilde{\alpha} = \tilde{F}^{-1}\mathrm{d}\tilde{F}. \tag{3.2.1}$$

It decomposes relative to the Cartan decomposition $\tilde{\alpha} = \tilde{\alpha}_k + \tilde{\alpha}_p$. The $\mathbf{p}$-valued one-form $\tilde{\alpha}_p$ can be further decomposed into a holomorphic and an antiholomorphic part $\tilde{\alpha}_p = \tilde{\alpha}'_p \mathrm{d}z + \tilde{\alpha}''_p \mathrm{d}\bar{z}$, which gives

$$\tilde{\alpha} = \tilde{F}^{-1}\mathrm{d}\tilde{F} = \tilde{\alpha}'_p \mathrm{d}z + \tilde{\alpha}_k + \tilde{\alpha}''_p \mathrm{d}\bar{z}. \tag{3.2.2}$$

For each $\lambda \in S^1$ we define the following $\mathbf{su}(2)$-valued one-form:

$$\alpha = \lambda^{-1}\tilde{\alpha}'_p \mathrm{d}z + \tilde{\alpha}_k + \lambda\tilde{\alpha}''_p \mathrm{d}\bar{z}. \tag{3.2.3}$$

Because of [6] and [17] we know, that $\tilde{F}$ is the frame of a CMC-immersion iff α is integrable. In this case we can define the extended frame $F : \mathcal{D} \times S^1 \longrightarrow \mathbf{SU}(2)$ by

$$\alpha = F^{-1}\mathrm{d}F, \qquad F(0,\lambda) = I,\ \lambda \in S^1. \tag{3.2.4}$$

Then

$$F(z,\overline{z},1) = \tilde{F}(z,\overline{z}) \tag{3.2.5}$$

by the uniqueness of the initial value problem. We will interpret F in terms of loop groups in the next section.

3.3 Let $\Lambda\mathbf{G}_\sigma$, $\mathbf{G} = \mathbf{SL}(2,\mathbb{C})$ or $\mathbf{G} = \mathbf{SU}(2)$, denote the group of smooth maps $g(\lambda)$ from S^1 to $\mathbf{G}$, which satisfy the twisting condition

$$g(-\lambda) = \sigma(g(\lambda)), \tag{3.3.1}$$

where $\sigma : \mathbf{SU}(2) \to \mathbf{SU}(2)$ is the group automorphism of order 2, which is given by conjugation with the Pauli matrix

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.3.2}$$

The Lie algebras of these groups, which we denote by $\Lambda\mathbf{g}_\sigma$, where $\mathbf{g}$ is the Lie algebra of $\mathbf{G}$, consist of maps $x : S^1 \to \mathbf{g}$, which satisfy a similar twisting condition as the group elements

$$x(-\lambda) = \sigma_3 x(\lambda)\sigma_3. \tag{3.3.3}$$

In order to make these loop groups complex Banach Lie groups, we equip them, as in [6], with some H^s-topology for $s > \frac{1}{2}$. Elements of these twisted loop groups are matrices with off-diagonal entries which are odd functions, and diagonal entries which are even functions in the parameter λ. All entries are in the Banach algebra $\mathcal{A}$ of H^s-smooth functions.

Furthermore, we will use the following subgroups of $\Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$: Let $\mathbf{B}$ be a subgroup of $\mathbf{SL}(2,\mathbb{C})$ and $\Lambda_B^+\mathbf{SL}(2,\mathbb{C})_\sigma$ be the group of maps in $\Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$, which can be extended to holomorphic maps on the open unit circle and take values in $\mathbf{B}$ at $\lambda = 0$. Analogously, let $\Lambda_B^-\mathbf{SL}(2,\mathbb{C})_\sigma$ be the group of maps in $\Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$, which can be extended to the outside of the unit circle in $\mathbb{C}\mathrm{P}_1 = \mathbb{C}\cup\{\infty\}$ and take values in $\mathbf{B}$ at $\lambda = \infty$. If $\mathbf{B} = \{I\}$ (based loops) we write the subscript $*$ instead of $\mathbf{B}$, if $\mathbf{B} = \mathbf{SL}(2,\mathbb{C})$ we omit the subscript for Λ entirely.

Corresponding to these subgroups we define Lie subalgebras of $\Lambda\mathbf{sl}(2,\mathbb{C})_\sigma$: We will only consider $\Lambda^+\mathbf{sl}(2,\mathbb{C})_\sigma$, which consists of maps $x(\lambda)$ that can be continued holomorphically to the unit circle, and $\Lambda_*^-\mathbf{sl}(2,\mathbb{C})_\sigma$, which consists of maps, which can be continued holomorphically to the outside of the unit circle and become the zero matrix at $\lambda = \infty$.

We quote the following result from [6]:

(i) For each solvable subgroup $\mathbf{B}$ of $\mathbf{SL}(2,\mathbb{C})$ satisfying $\mathbf{SU}(2)\cdot\mathbf{B} = \mathbf{SL}(2,\mathbb{C})$ and $\mathbf{SU}(2)\cap\mathbf{B} = \{I\}$, multiplication

$$\Lambda\mathbf{SU}(2)_\sigma \times \Lambda_B^+\mathbf{SL}(2,\mathbb{C})_\sigma \longrightarrow \Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$$

is a diffeomorphism onto. The associated splitting

$$g = F g_+ \tag{3.3.4}$$

of an element g of $\Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$, s.t. $F \in \Lambda\mathbf{SU}(2)_\sigma$ and $g_+ \in \Lambda_B^+\mathbf{SL}(2,\mathbb{C})_\sigma$ will be called Iwasawa decomposition (relative to $\mathbf{B}$). If not specified otherwise, we will always choose $\mathbf{B}$ to be the set of upper triangular matrices in $\mathbf{SL}(2,\mathbb{C})$ which have real, positive diagonal entries.

(ii) Multiplication

$$\Lambda_*^-\mathbf{SL}(2,\mathbb{C})_\sigma \times \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma \longrightarrow \Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$$

is a diffeomorphism onto the open and dense subset $\Lambda_*^-\mathbf{SL}(2,\mathbb{C})_\sigma\cdot\Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$ of $\Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$, called the "big cell" [18]. The associated splitting

$$g = g_- g_+ \tag{3.3.5}$$

of an element g of the big cell, where $g_- \in \Lambda_*^-\mathbf{SL}(2,\mathbb{C})_\sigma$ and $g_+ \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$, will be called Birkhoff factorization.

Using the definitions above, it follows from (3.2.3), that α is a one-form taking values in $\Lambda\mathbf{su}(2)_\sigma$. By the definition (3.2.4) of the extended frame, we see, that F can be interpreted as a map from $\mathcal{D}$ into $\Lambda\mathbf{SU}(2)_\sigma$.

Theorem: *The extended frame F transforms under $g \in \mathrm{Aut}_\Psi\mathcal{D}$ as follows:*

$$(F\circ g)(z,\bar{z},\lambda) = \chi(g,\lambda)F(z,\bar{z},\lambda)k(g,z,\bar{z}), \tag{3.3.6}$$

where $\chi(g,\lambda) \in \Lambda\mathbf{SU}(2)_\sigma$. In addition, there exists a map

$$\epsilon : \mathrm{Aut}_\Psi\mathcal{D}\times\mathrm{Aut}_\Psi\mathcal{D} \to \{I,-I\} \subset \mathbf{SU}(2), \tag{3.3.7}$$

s.t. for each $\lambda \in S^1$, the map $\chi(\cdot,\lambda) : \mathrm{Aut}_\Psi\mathcal{D} \to \mathbf{SU}(2)$ is a projective group homomorphism with cocycle ϵ:

$$\chi(g_2\circ g_1,\lambda) = \epsilon(g_2,g_1)\chi(g_2,\lambda)\chi(g_1,\lambda), \tag{3.3.8}$$

and $k(\cdot,z,\bar{z})$ satisfies the projective cocycle relation

$$k(g_2\circ g_1,z,\bar{z}) = \epsilon(g_2,g_1)k(g_2,g_1(z),\overline{g_1(z)})k(g_1,z,\bar{z}). \tag{3.3.9}$$

Proof: The differential $\tilde{\alpha} = \tilde{F}^{-1}\mathrm{d}\tilde{F}$ transforms under an automorphism $g \in \mathrm{Aut}_\Psi \mathcal{D}$ as

$$g_* \tilde{\alpha} = k^{-1}\tilde{\alpha}k + k^{-1}\mathrm{d}k. \tag{3.3.10}$$

Using the Cartan decomposition above we get

$$\begin{aligned} g_* \tilde{\alpha}_k &= k^{-1}\tilde{\alpha}_k k + k^{-1}\mathrm{d}k, \\ g_* \tilde{\alpha}_p &= k^{-1}\tilde{\alpha}_p k. \end{aligned} \tag{3.3.11}$$

By further decomposing $\tilde{\alpha}_p$ into a holomorphic and an antiholomorphic part, we get

$$\begin{aligned} (\tilde{\alpha} \circ g)'_p \partial_z g &= k^{-1}\tilde{\alpha}'_p k, \\ (\tilde{\alpha} \circ g)''_p \overline{\partial_z g} &= k^{-1}\tilde{\alpha}''_p k. \end{aligned} \tag{3.3.12}$$

If we define $\check{F}(z, \overline{z}, \lambda) = F(z, \overline{z}, \lambda)k(g, z, \overline{z})$, then

$$\check{\alpha} = \check{F}^{-1}\mathrm{d}\check{F} = \lambda^{-1}k^{-1}\tilde{\alpha}'_p k \mathrm{d}z + k^{-1}\tilde{\alpha}_k k + \lambda k^{-1}\tilde{\alpha}''_p k \mathrm{d}\bar{z} + k^{-1}\mathrm{d}k. \tag{3.3.13}$$

Therefore, by Eqs. (3.3.11), (3.3.12) and the definition (3.2.3),

$$g_* \alpha = \check{\alpha} \tag{3.3.14}$$

and $\check{F}$ and $F \circ g$ are equal up to left multiplication by a unitary z-independent matrix $\chi(g, \lambda)$,

$$(F \circ g)(z, \overline{z}, \lambda) = \chi(g, \lambda)F(z, \overline{z}, \lambda)k(g, z, \overline{z}). \tag{3.3.15}$$

The matrix $\chi(g, \lambda)$ is fixed by the initial condition (3.2.5) as

$$\chi(g, \lambda) = F(g(0), \lambda)k(g, 0)^{-1} \in \Lambda \mathbf{SU}(2)_\sigma. \tag{3.3.16}$$

This shows, that $\chi(g, \lambda)k(g, 0)$ is uniquely determined. Since k satisfies (3.1.8), we can find for each pair of automorphisms g_1 and g_2 in $\mathrm{Aut}_\Psi \mathcal{D}$ an $\epsilon(g_2, g_1) \in \{I, -I\}$, s.t. Eq. (3.3.9) holds. This in turn gives Eq. (3.3.8), since

$$\begin{aligned} \chi(g_2 \circ g_1, \lambda)k(g_2 \circ g_1, 0) &= F((g_2 \circ g_1)(0)) \\ &= \chi(g_2, \lambda)\chi(g_1, \lambda)k(g_1, 0)k(g_2, g_1(0)) \\ &= \epsilon(g_2, g_1)\chi(g_2, \lambda)\chi(g_1, \lambda)k(g_2 \circ g_1, 0). \end{aligned} \tag{3.3.17}$$

□

3.4 Let us now look at Sym's formula

$$J(\Psi_\lambda) = \frac{\partial}{\partial \theta}F \cdot F^{-1} + \frac{i}{2}F\sigma_3 F^{-1}, \ \lambda = e^{i\theta}, \tag{3.4.1}$$

which gives for each extended frame F an associated family (see Section 2.5) of CMC-immersions $\Psi_\lambda : \mathcal{D} \to \mathbb{R}^3$ in the spinor representation. Remember, that we have chosen $H = -\frac{1}{2}$.
The initial condition in (3.2.4) implies (see [3, Remark A.7]) an initial condition for Ψ_λ:

$$\Psi_\lambda(z = 0) = -e_3, \tag{3.4.2}$$

where e_3 is the unit vector mapped to $-\frac{i}{2}\sigma_3$ by the spinor representation. We will in the following always assume, that the members of the associated family are normalized by (3.4.2). Note, that then Eq. (2.5.2) holds for the normalized surfaces.

Using Eq. (3.3.6), it is easy to see that the family of immersions Ψ_λ transforms under $g \in \mathrm{Aut}_\Psi \mathcal{D}$ as

$$\begin{aligned} J((\Psi_\lambda \circ g)(z)) &= \frac{\partial}{\partial\theta}\chi \cdot \chi^{-1} + \chi\frac{\partial}{\partial\theta}F \cdot F^{-1}\chi^{-1} + \frac{i}{2}\chi F\sigma_3 F^{-1}\chi^{-1} \\ &= \chi(J(\Psi_\lambda(z)))\chi^{-1} + \frac{\partial}{\partial\theta}\chi \cdot \chi^{-1}. \end{aligned} \tag{3.4.3}$$

The second term in Eq. (3.4.3) is a translation in $\mathbb{R}^3$, the first term describes a rotation of the initial surface. For $\lambda = 1$ we have $\Psi_1 = \Psi$, Ψ being the original immersion. Therefore $R_{\tilde{T}}$, $\tilde{T} = (R_{\tilde{T}}, t_{\tilde{T}}) = \psi(g)$, acts on vectors in the spinor representation by conjugation with $\chi(g, \lambda = 1)$ and the translation t_T acts by adding $\frac{\partial}{\partial\theta}|_{\theta=0}\chi \cdot \chi^{-1}(g, \lambda = 1)$.

Proposition: *$\mathrm{Aut}_{\Psi_\lambda}\mathcal{D} = \mathrm{Aut}_\Psi\mathcal{D}$ for all $\lambda \in S^1$.*

Proof: From Eq. (3.4.3) we see that for $g \in \mathrm{Aut}_\Psi\mathcal{D}$ and arbitrary $\lambda \in S^1$ we have

$$\Psi_\lambda \circ g = \tilde{T}_\lambda \circ \Psi \tag{3.4.4}$$

where $\tilde{T}_\lambda \in \mathrm{OAff}(\mathbb{R}^3)$ is given in the spinor representation by conjugation with $\chi(g,\lambda)$ and subsequent addition of $\frac{\partial}{\partial\theta}\chi\cdot\chi^{-1}(g,\lambda)$. This implies $\mathrm{Aut}_\Psi\mathcal{D} \subset \mathrm{Aut}_{\Psi_\lambda}\mathcal{D}$. Since, by Eq. (2.5.2), the surface $\Psi = \Psi_1$ is in the associated family of Ψ_λ for each $\lambda \in S^1$, we get also

$$\mathrm{Aut}_{\Psi_\lambda}\mathcal{D} = \mathrm{Aut}_{(\Psi_\lambda)_{\lambda^{-1}}}\mathcal{D} \subset \mathrm{Aut}_\Psi\mathcal{D}, \tag{3.4.5}$$

which finishes the proof. □

Using Proposition 3.4 and Lemma 2.15, we get the following

Corollary: *Let (M, Φ) be a complete, nonspherical CMC-surface with simply connected cover $(\mathcal{D}, \Psi)$. If $\mathrm{Aut}\Psi(\mathcal{D})$ contains a one-parameter group of proper Euclidean motions, then $(\mathcal{D}, \Psi)$ is in the associated family of a Delaunay surface.*

Proof: If $(\mathcal{D}, \Psi)$ is in the associated family of a Delaunay surface, then by Lemma 2.15 and Proposition 3.4, $\mathrm{Aut}_\Psi\mathcal{D}$ and therefore, by Theorem 2.9, also $\mathrm{Aut}\Psi(\mathcal{D})$ contains a one-parameter group.

Conversely, if $\mathrm{Aut}\Psi(\mathcal{D})$ contains a one-parameter group, then, by Proposition 2.12 and Corollary 2.6, the group $\mathrm{Aut}_\Psi\mathcal{D}$ also contains a one-parameter group of self-isometries of $(\mathcal{D}, \Psi)$. By Theorem 2.14, only those surfaces which are isometric to Smyth or Delaunay surfaces can have a one parameter group of self-isometries. From Lemma 2.15 we know, that for Smyth surfaces $\mathrm{Aut}\Psi(\mathcal{D})$ is discrete. This shows, that $(\mathcal{D}, \Psi)$ is isometric to a Delaunay surface, which, by Lemma 2.5, proves the claim. □

For each CMC-immersion Ψ_λ in the associated family of Ψ we get, as in Section 2.9, an analytic group homomorphism

$$\psi_\lambda : \mathrm{Aut}_{\Psi_\lambda}\mathcal{D} = \mathrm{Aut}_\Psi\mathcal{D} \to \mathrm{Aut}\Psi_\lambda(\mathcal{D}). \tag{3.4.6}$$

Since by Corollary 2.5, with $(\mathcal{D}, \Psi)$ also $(\mathcal{D}, \Psi_\lambda)$ is complete, we get, by Corollary 2.10, that ψ_λ is surjective. If we write the canonical covering $P : \mathbf{SU}(2) \to \mathbf{SO}(3)$ using the spinor representation J of $\mathbb{R}^3$,

$$P(A)(x) = \mathrm{Ad}(A)(J(x)), \qquad A \in \mathbf{SU}(2), \qquad x \in \mathbb{R}^3, \tag{3.4.7}$$

we get from Eq. (3.4.3) for fixed $\lambda \in S^1$:

$$\psi_\lambda(g) = \tilde{T}_\lambda = (R_{\tilde{T}_\lambda}, t_{\tilde{T}_\lambda}), \tag{3.4.8}$$

where for $x \in \mathbb{R}^3$,

$$J(R_{\tilde{T}_\lambda}(x)) = \mathrm{Ad}(\chi(g,\lambda))(J(x)), \qquad J(t_{\tilde{T}_\lambda}(x)) = J(x) + \frac{\partial}{\partial\theta}\chi \cdot \chi^{-1}(g,\lambda). \tag{3.4.9}$$

Since, by Theorem 2.9, ψ_λ is analytic, we get

Lemma: *For each $\lambda \in S^1$,*

$$\mathrm{Ad}(\chi(\cdot,\lambda)) : \mathrm{Aut}_\Psi \mathcal{D} \longrightarrow \mathrm{Aut}(\mathbf{su}(2)) \tag{3.4.10}$$

is a continuous group homomorphism, and the map

$$\frac{\partial}{\partial\theta}\chi \cdot \chi^{-1}(\cdot,\lambda) : \mathrm{Aut}_\Psi \mathcal{D} \longrightarrow \mathbf{su}(2),\ \lambda = e^{i\theta}, \tag{3.4.11}$$

is continuous.

Proof: Since the decomposition (2.6.1) of a proper Euclidean motion is unique, we get, that with $\tilde{T}_\lambda = \psi_\lambda(g)$ also the rotational and translational parts, $R_{\tilde{T}_\lambda}$ and $t_{\tilde{T}_\lambda}$, depend continuously on $g \in \mathrm{Aut}_{\Psi_\lambda}\mathcal{D} = \mathrm{Aut}_\Psi \mathcal{D}$. Using Eq. (3.4.9), we get the lemma. □

Remark: Let us define a Riemann surface $M_\lambda = \mathrm{Ker}\psi_\lambda \setminus \mathcal{D}$ as in Proposition 2.11. Let $\pi_\lambda : \mathcal{D} \to M_\lambda$ denote the natural covering map and let $\bar{\pi}_\lambda$ be defined as in Definition 2.9. Then we have

$$\mathrm{Ker}\bar{\pi}_\lambda = \mathrm{Ker}\psi_\lambda \tag{3.4.12}$$

for all $\lambda \in S^1$. This shows, that if we define M_λ as above, then Condition (2.11.10) is satisfied for all members of the associated family. However, it should be noted, that $\mathrm{Ker}\psi_\lambda$ and therefore also M_λ can vary in a very complicated way with λ.

3.5 Now we apply the Birkhoff splitting $F = g_- g_+$ for $z \in \mathcal{D} \setminus \mathcal{S}$, where $\mathcal{S}$ is the discrete set of points in $\mathcal{D}$, where F cannot be split. Then g_- can be continued meromorphically to $z \in \mathcal{D}$. For a detailed account see [6, Section 4], especially the remarks following Theorem 4.10.

By Eq. (3.3.6) we get for $g \in \mathrm{Aut}_\Psi \mathcal{D}$

$$(g_- \circ g)(z,\lambda) = \chi(g,\lambda) g_-(z,\lambda) p_+(g,z,\lambda), \tag{3.5.1}$$

where

$$p_+ = g_+(z,\bar{z}),\lambda) k(g,z,\bar{z})((g_+ \circ g)(z,\bar{z},\lambda))^{-1} \tag{3.5.2}$$

is determined by g up to a sign. Since k satisfies Eq. (3.3.9) we have for $p_+(g,z,\lambda)$:

$$p_+(g_2 \circ g_1, z, \lambda) = \epsilon(g_2,g_1) p_+(g_2, g_1(z),\lambda) p_+(g_1,z,\lambda), \tag{3.5.3}$$

where $\epsilon : \mathrm{Aut}_\Psi \mathcal{D} \times \mathrm{Aut}_\Psi \mathcal{D} \to \{I,-I\} \subset \mathbf{SU}(2)$ is defined as in Theorem 3.3.

The meromorphic potential ξ is defined by $\xi = g_-^{-1} \mathrm{d} g_-$. It is a $\Lambda^- \mathbf{sl}(2,\mathbb{C})_\sigma$-valued one-form with only a λ^{-1}-coefficient. Therefore, it is of the form

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix} \mathrm{d}z, \tag{3.5.4}$$

where f is a meromorphic function and $E\mathrm{d}z^2$ is the holomorphic Hopf differential. From Eq. (3.5.1) we obtain, that ξ transforms under the automorphism $g \in \mathrm{Aut}_\Psi \mathcal{D}$ as

$$g_*\xi(z,\lambda) = (\xi \circ g)(z,\lambda) g'(z) = p_+^{-1} \xi p_+ + p_+^{-1} \mathrm{d} p_+. \tag{3.5.5}$$

Elements of $\mathrm{Aut}_\Psi \mathcal{D}$ therefore act as gauge transformations on the meromorphic potential.

Remark: By the normalization of F we have

$$(F \circ g)(0,\lambda) = \chi(g,\lambda) k(g,0). \tag{3.5.6}$$

If $g(0)$ is not in the singular set $\mathcal{S}$, then, since $g_-(0,\lambda) = I$, we can evaluate (3.5.1) at $z = 0$. This shows $g_-(g(0),\lambda) = \chi(g,\lambda)p_+(g,0,\lambda)$. In particular $\chi(g,\lambda)$ is splittable. Therefore,

$$g_-(g(0),\lambda) = \chi_-(g,\lambda), \tag{3.5.7}$$

where

$$\chi = \chi_-\chi_+ \tag{3.5.8}$$

is the Birkhoff splitting of χ. From Eq. (3.5.1) it follows

$$\chi_-^{-1}(g,\lambda)(g_- \circ g)(z) = \chi_+(g,\lambda)g_-(z,\lambda)p_+(g,z,\lambda). \tag{3.5.9}$$

Now

$$g_-^{\chi}(z,\lambda) = \chi_-^{-1}(g,\lambda)(g_- \circ g)(z,\lambda) \tag{3.5.10}$$

is normalized at $z = 0$. Thus Eq. (3.5.9) can be looked at as a dressing transformation from g_- to g_-^{χ}, a new surface.

3.6 The procedure presented in [6] and used in this section characterizes CMC-surfaces in terms of the meromorphic potential ξ (for smoothness questions see also [3]). It is therefore natural to attempt to characterize $g \in \mathrm{Aut}_\Psi\mathcal{D}$ by using equations only involving ξ. The obvious equation to start with is (3.5.5). Thus, let us assume that to some $g \in \mathrm{Aut}\mathcal{D}$ we can find some $p_+ = p_+(g,z,\lambda) \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$, s.t.

$$(g_*\xi)(z,\lambda) = p_+^{-1}\xi p_+ + p_+^{-1}\mathrm{d}p_+. \tag{3.6.1}$$

Note, that every such p_+ is automatically meromorphic in z. Next we solve

$$g_-^{-1}\mathrm{d}g_- = \xi, \qquad\qquad g_-(0,\lambda) = I \tag{3.6.2}$$

and consider $h = g_-p_+$. In view of Eq. (3.6.1) and (3.6.2) it is straightforward to verify that $h^{-1}\mathrm{d}h = g_*\xi$ holds. On the other hand, $\hat{h} = g_- \circ g$ also satisfies $\hat{h}^{-1}\mathrm{d}\hat{h} = g_*\xi$. Therefore,

$$(g_- \circ g)(z,\lambda) = \sigma(g,\lambda)g_-(z,\lambda)p_+(g,z,\lambda). \tag{3.6.3}$$

Note, that here $\sigma \in \Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$ does not depend on z.

In order to be able to conclude, that $g \in \mathrm{Aut}_\Psi\mathcal{D}$, we need to obtain Eq. (3.3.6), since the discussion in Section 3.4 shows that Eq. (3.3.6) implies $g \in \mathrm{Aut}_\Psi\mathcal{D}$. The relation between (3.3.6) and (3.5.1) is obtained via the Birkhoff splitting

$$(g_- \circ g)\cdot(g_+ \circ g) = \chi g_- g_+ k. \tag{3.6.4}$$

We therefore need

$$\chi^{-1}\sigma g_- = g_- q_+, \qquad q_+ = q_+(g,z,\lambda) \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma. \tag{3.6.5}$$

But then, setting $z = 0$, we obtain

$$\chi^{-1}\sigma \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma. \tag{3.6.6}$$

Setting $h_+ = \chi^{-1}\sigma$ we thus obtain $h_+g_- = g_-q_+$. This shows

$$\sigma g_- = \chi h_+ g_- = \chi g_- q_+, \tag{3.6.7}$$

whence

$$g_- \circ g = \sigma g_- p_+ = \chi g_- q_+ p_+. \tag{3.6.8}$$

Altogether, we have shown

Proposition: *Let $g \in \mathrm{Aut}\mathcal{D}$. Then $g \in \mathrm{Aut}_\Psi\mathcal{D}$ iff there exists some $p_+ = p_+(g,z,\lambda) \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$ s.t.*

$$g_*\xi = p_+^{-1}\xi p_+ + p_+^{-1}\mathrm{d}p_+ \tag{3.6.9}$$

and the z-independent matrix function σ associated with p_+ by (3.6.3) is unitary.

Remark: We have seen in the proof that, starting from some p_+ satisfying Eq. (3.6.1), we may have to modify the corresponding σ by some h_+ s.t. $\chi = \sigma h_+^{-1}$ becomes unitary. In this case h_+ satisfies $h_+g_- = g_-q_+$, i.e., h_+ is in the isotropy group of the dressing action on g_-.

3.7 To illustrate the discussion of the last section we look at the dressing orbit of the standard cylinder.

The meromorphic potential of the cylinder can be chosen to be

$$\xi_0 = \lambda^{-1}A\mathrm{d}z, \tag{3.7.1}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.7.2}$$

is constant. We can integrate ξ_0 without problems:

$$g_-^0 = \exp\left(\lambda^{-1}zA\right), \tag{3.7.3}$$

and, since A commutes with its adjoint matrix $\bar{A} = A$, also the Iwasawa splitting doesn't pose any problems:

$$F = \exp\left((\lambda^{-1}z - \lambda\bar{z})A\right). \tag{3.7.4}$$

We will call the cylinder, which is generated by applying Sym's formula to Eq. (3.7.4), the *standard cylinder.* For surfaces in the dressing orbit of the standard cylinder we therefore have

$$g_- = (h_+g_-^0)_- = h_+g_-^0g_+^{-1}, \qquad \xi = g_-^{-1}\mathrm{d}g_-, \tag{3.7.5}$$

with $h_+ \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$. For these g_- each translation $z \mapsto z+q$, $q \in \mathbb{C}$, in $\mathcal{D}$ can be implemented by a transformation of type (3.5.1),

$$g_-(z+q,\lambda) = Q \cdot g_-(z,\lambda)r_+(q,z,\lambda), \tag{3.7.6}$$

with the z-independent matrix

$$Q = h_+e^{\lambda^{-1}qA}h_+^{-1} \tag{3.7.7}$$

and

$$r_+(q,z,\lambda) = g_+(z,\lambda)g_+^{-1}(z+q,\lambda). \tag{3.7.8}$$

To illustrate Proposition 3.6, we consider the standard cylinder itself, i.e., $h_+ = I$ and $g_- = e^{\lambda^{-1}zA}$. Then for every $q \in \mathbb{C}$, $Q = e^{\lambda^{-1}qA}$. Moreover, $Qe^{-\lambda\bar{q}A}$ is unitary and $e^{-\lambda\bar{q}A}g_- = g_-p_+$. Therefore, Proposition 3.6 shows that every translation $z \mapsto z+q$, $q \in \mathbb{C}$, is in $\mathrm{Aut}_\Psi\mathcal{D}$. But for general h_+ the matrix Q cannot be modified in the sense of Section 3.6 to yield a unitary χ.

In fact we can prove the following

Theorem: *Other than cylinders there are no surfaces with translational symmetry in the dressing orbit of the standard cylinder.*

Proof: Let (M,Φ) be a CMC-surface in the dressing orbit of the standard cylinder with simply connected cover $(\mathcal{D},\Psi)$. Let $F: \mathcal{D} \to \Lambda\mathbf{SU}(2)_\sigma$ be its extended frame and let $g_-: \mathcal{D} \to \Lambda^-\mathbf{SL}(2,\mathbb{C})_\sigma$ be defined by the Birkhoff splitting $F = g_-g_+$. We assume, that $\mathrm{Aut}_\Psi\mathcal{D}$ contains a translation $z \mapsto z+q$, $q \in \mathbb{C}^* = \mathbb{C}\setminus\{0\}$.

By Proposition 3.6, a translation $z \mapsto z+q$, $q \in \mathbb{C}^*$, is in $\mathrm{Aut}_\Psi \mathcal{D}$ iff g_- satisfies

$$g_-(z+q,\lambda) = \chi(q,\lambda) g_-(z,\lambda) p_+(q,z,\lambda) \tag{3.7.9}$$

where $\chi(q,\lambda) \in \Lambda \mathbf{SU}(2)_\sigma$. By Equations (3.7.6) and (3.7.7), this is equivalent to

$$Q(q,\lambda) g_-(z,\lambda) r_+(q,z,\lambda) = \chi(q,\lambda) g_-(z,\lambda) p_+(q,z,\lambda), \tag{3.7.10}$$

where $Q = h_+ e^{\lambda^{-1} qA} h_+^{-1}$. Setting $z=0$ in Eq. (3.7.10), it follows that $Q = \chi R_+$ with $R_+ = p_+(q,0,\lambda) r_+(q,0,\lambda)^{-1} \in \Lambda^+ \mathbf{SL}(2,\mathbb{C})$. Therefore, g_- is invariant under the dressing with R_+. This implies with Eq. (3.7.5) that for

$$w_+(q,\lambda,z) = g_+^{-1} p_+ r_+^{-1} g_+ \in \Lambda^+ \mathbf{SL}(2,\mathbb{C})_\sigma, \tag{3.7.11}$$

we get

$$R_+ = g_- p_+ r_+^{-1} g_-^{-1} = h_+ e^{\lambda^{-1} zA} w_+ e^{-\lambda^{-1} zA} h_+^{-1}, \tag{3.7.12}$$

or equivalently

$$\chi = h_+ e^{\lambda^{-1}(q+z)A} w_+^{-1} e^{-\lambda^{-1} zA} h_+^{-1}. \tag{3.7.13}$$

From Eq. (3.7.12) we get the condition

$$e^{\lambda^{-1} zA} w_+ e^{-\lambda^{-1} zA} \in \Lambda^+ \mathbf{SL}(2,\mathbb{C})_\sigma \tag{3.7.14}$$

for all $z \in \mathcal{D}$. If we conjugate the expression in Eq. (3.7.14) with

$$D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \tag{3.7.15}$$

then the exponent $DAD^{-1} = \sigma_3$ becomes diagonal and with

$$\tilde{w} = DwD^{-1} = \begin{pmatrix} \tilde{w}_a & \tilde{w}_b \\ \tilde{w}_c & \tilde{w}_d \end{pmatrix}, \tag{3.7.16}$$

Condition (3.7.14) is equivalent to

$$e^{\lambda^{-1} z\sigma_3} \tilde{w} e^{-\lambda^{-1} z\sigma_3} = \begin{pmatrix} \tilde{w}_a & e^{2\lambda^{-1} z} \tilde{w}_b \\ e^{-2\lambda^{-1} z} \tilde{w}_c & \tilde{w}_d \end{pmatrix} \tag{3.7.17}$$

having a holomorphic extension to the unit disk for all $z \in \mathcal{D} \setminus \mathcal{S}$. Since for every fixed $z \in \mathcal{D} \setminus \mathcal{S}$ the off-diagonal entries of the r.h.s. of (3.7.17) vanish identically or have an essential singularity at $\lambda = 0$, this implies that $\tilde{w}_b = \tilde{w}_c = 0$. This shows that $\tilde{w}$ commutes with σ_3. Thus w commutes with A. We therefore have

$$\chi = h_+ H h_+^{-1} \tag{3.7.18}$$

with

$$H(\lambda) = e^{\lambda^{-1} qA} w_+^{-1}(\lambda) = \alpha(\lambda) I + \beta(\lambda) A, \qquad \alpha(\lambda), \beta(\lambda) \in \mathbb{C}. \tag{3.7.19}$$

Since $\det H = 1$ we get also

$$\alpha^2 - \beta^2 = 1. \tag{3.7.20}$$

It follows from the unitarity of $\chi(q,\lambda)$ that

$$\overline{H}^\top = PH^{-1}P^{-1}, \qquad P = \overline{h_+}^\top h_+. \tag{3.7.21}$$

If we write Eq. (3.7.21) using Eq. (3.7.19), this gives

$$(\alpha - \overline{\alpha}) I = \beta PAP^{-1} + \overline{\beta} A. \tag{3.7.22}$$

By taking the trace we get

$$\alpha = \overline{\alpha}, \tag{3.7.23}$$

i.e., α is real, and

$$\overline{\beta}AP = -\beta PA. \tag{3.7.24}$$

Using (3.7.20), we get from (3.7.23)

$$\beta^2 = \overline{\beta}^2. \tag{3.7.25}$$

From this and (3.7.24) it follows, that either 1) $\beta = \overline{\beta} \equiv 0$ or 2) $\beta = -\overline{\beta} \not\equiv 0$ and $[A, P] = 0$ or 3) $\beta = \overline{\beta} \not\equiv 0$ and $AP + PA = 0$.

In the first case, we get by (3.7.20), that $\alpha = \pm 1$, i.e., $H = \chi = \pm I$. But this implies with Eq. (3.7.19), that $q = 0$, which contradicts the assumption that $q \in \mathbb{C}^*$.

In the second case, it follows from $[A, P] = 0$, that $S = h_+ A h_+^{-1}$ is unitary. The matrix $h_+ A h_+^{-1}$ can then be continued holomorphically to $\mathbb{C}\mathrm{P}_1$, which is only possible if

$$h_+ A h_+^{-1} = h_0 A h_0^{-1}, \tag{3.7.26}$$

where h_0 is a λ-independent diagonal matrix. For the diagonal matrix h_0 the unitarity of $h_0 A h_0^{-1}$ amounts to the unimodularity of the diagonal entries. Therefore, the matrix h_0 is of the form

$$h_0 = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \qquad \phi \in \mathbb{R}. \tag{3.7.27}$$

By Eq. (3.7.26) dressing of the standard cylinder with h_+ is therefore the same as dressing with a unitary matrix h_0. By writing the dressing action as an action on the extended frames, we see, that dressing with a unitary matrix amounts to a rotation in $\mathbb{R}^3$. Therefore, in the second case the surface (M, Φ) is a cylinder.

In the third case, it follows from $AP + PA = 0$, that $S = h_+ A h_+^{-1}$ satisfies $S\overline{S}^\top = -S^2 = -I$. This is impossible since $S\overline{S}^\top$ is positive definite. Therefore, the third case cannot occur. □

Corollary 1: *Let (M, Φ) be obtained from the standard cylinder (M_0, Φ_0) by dressing with $h_+ \in \Lambda^+\mathbf{SL}(2, \mathbb{C})_\sigma$. If (M, Φ) admits a translational symmetry, then (M, Φ) is a cylinder and $h_+ = h_0 H_+$, where h_0 is unitary and λ-independent, and H_+ commutes with A.*

Theorem 3.7 shows, that there are neither Delaunay surfaces nor CMC-tori in the dressing orbit of the standard cylinder. It therefore also proves:

Corollary 2: *The dressing action does not act transitively on the set of all CMC-surfaces without umbilics.*

The method developed in Section 3.6 can also be applied to the more general dressing action of [7]. In this case the orbit of the standard cylinder under this action contains all surfaces of finite type, therefore all CMC-tori (see [16]). In [4] we will use this method to derive the classification of CMC-tori given in [1] in a more geometric way.

3.8 Since we are interested in CMC-surfaces with symmetry groups, we would like to understand what it means for ξ and $g \in \mathrm{Aut}\mathcal{D}$ to admit some p_+ satisfying Eq. (3.6.9). Expanding $p_+ = p_0 + \lambda p_1 + \ldots$ into powers of λ, we see, that (3.6.9) implies

$$g_*\xi = p_0^{-1} \xi p_0. \tag{3.8.1}$$

Evaluating this with $\xi = \lambda^{-1} \begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix} \mathrm{d}z$ we obtain for every $g \in \mathrm{Aut}_\Psi \mathcal{D}$

$$(E \circ g)(z)(g'(z))^2 = E(z), \tag{3.8.2}$$

which we derived already in Lemma 2.6, and in addition

$$(f \circ g)(z)g'(z) = r^{-2}(g,z)f(z), \tag{3.8.3}$$

where $p_0 = \mathrm{diag}(r, r^{-1})$. The function $r^{-2}(g,z)$ is uniquely determined by $g \in \mathrm{Aut}_\Psi\mathcal{D}$. Using Eq. (3.8.3), it is easy to verify that r^{-2} satisfies the cocycle condition

$$r^{-2}(g_2 \circ g_1, z) = r^{-2}(g_2, g_1(z))r^{-2}(g_1, z), \tag{3.8.4}$$

for $g_1, g_2 \in \mathrm{Aut}_\Psi\mathcal{D}$ and $z \in \mathcal{D}$.

Remark: 1. If $\mathcal{D} = \mathbb{C}$ and $G \subset \mathrm{Aut}_\Psi\mathcal{D}$ is a group of (pure) translations, then $g'(z) = 1$ for all $g \in G$. Hence, E is invariant under G. E.g., for tori, E is doubly periodic and holomorphic. As a consequence, E is constant. This just reproduces the well known fact, that CMC-tori do not have any umbilics.

2. If $\mathcal{D}$ is the unit disk, then $\mathrm{Aut}\mathcal{D} \cong \mathbf{SU}(1,1)$ consists of transformations of the form $g(z) = \frac{az+b}{cz+d}$. Hence, $g'(z) = (cz+d)^{-2}$, since $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1$. Therefore, in this case E is an automorphic form of degree 4 for $\mathrm{Aut}_\Psi\mathcal{D}$.

3. For f the situation is unfortunately less straightforward. A condition on E and f can only be derived, if one is able to express r^{-2} in terms of E and f and $g \in \mathrm{Aut}_\Psi\mathcal{D}$. This will be discussed in Chapter 4.

4. We would also like to note, that for any $g \in \mathrm{Aut}\mathcal{D}$ we can use Eq. (3.8.3) as a definition of the function $s_0 = r^{-2}$.

3.9 If M is a Riemann surface with simply connected cover $\mathcal{D} = \mathbb{C}$, then by Lemma 2.3 it is conformally equivalent to the complex plane, the cylinder or a torus. Of course, if M is a torus, then M is complete. If M is the plane or a cylinder, it would be interesting to know, what it means for the meromorphic potential, that M with the induced metric is complete. For the case that the universal cover of M is the unit disk, we can show

Theorem: *Let (M, Φ) be a CMC-immersion with universal cover $\mathcal{D}$ the unit disk. Let $\xi = \lambda^{-1}\begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix} dz$ be the meromorphic potential for (M, Φ). Then if M is complete, ξ cannot be extended meromorphically to any open set containing a point of the unit circle.*

<u>Proof:</u> If ξ could be extended meromorphically to an open set U which contains a point on the unit circle, then U contains an open subset U_1, which contains a point on the unit circle, and on which ξ is holomorphic.

If we take a curve inside U_1 which connects a point inside $\mathcal{D}$ with a point on the unit circle, then ξ, and therefore also the surface, can be extended along this line. The function $u(z)$ defined in Eq. (2.4.2) satisfies [3, Sect. A.8]

$$w^2(z,\bar{z})f(z) = \frac{1}{4}e^{\frac{u(z,\bar{z})}{2}} \tag{3.9.1}$$

where $w(z,\bar{z})$ is a map which is defined and nonzero on $U_1 \cup (\mathcal{D} \setminus \mathcal{S})$, the domain of definition of ξ. Therefore, the line has finite length w.r.t. the metric on M, which contradicts the completeness of M. $\square$

4 Unitary gauge condition

In this chapter we will use the results of Chapter 3 to derive conditions on the meromorphic potential for the associated CMC-immersion to be symmetric.

4.1 Let

$$\xi = \lambda^{-1}\begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix} \mathrm{d}z, \qquad \hat{\xi} = \lambda^{-1}\begin{pmatrix} 0 & \hat{f} \\ \frac{\hat{E}}{\hat{f}} & 0 \end{pmatrix} \mathrm{d}z \tag{4.1.1}$$

be defined on $\mathcal{D}$, where $\mathcal{D}$ is the unit disk or the complex plane. We define g_- and $\hat{g}_-$ by

$$\begin{aligned} g_-' &= g_-\xi, & g_-(0,\lambda) = g_0(\lambda), \\ \hat{g}_-' &= \hat{g}_-\hat{\xi}, & \hat{g}_-(0,\lambda) = \hat{g}_0(\lambda), \end{aligned} \tag{4.1.2}$$

where we have chosen arbitrary initial conditions $g_0, \hat{g}_0 \in \Lambda^-\mathbf{SL}(2,\mathbb{C})_\sigma$. In this section we want to investigate, under which conditions g_- and $\hat{g}_-$ give the same CMC-immersion up to a proper Euclidean motion.

If we split the unitary matrix $\overline{g}_-^\top g_- = \overline{r}_+^\top r_+$ with $r_+ \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$, we have a $\mathbf{U}(1)$-freedom to choose the λ^0-factor r_0 of r_+. We can use this freedom to make r_0 real and positive.

Using this, we let now D and $\hat{D}$ be the real valued functions defined by

$$\begin{aligned} (\overline{g}_-^\top g_-)_{+0} &= \exp(D\sigma_3), \\ (\overline{\hat{g}}_-^\top \hat{g}_-)_{+0} &= \exp(\hat{D}\sigma_3), \end{aligned} \tag{4.1.3}$$

where g_{+0} denotes the real λ^0-coefficient of the positive splitting factor of $g \in \Lambda\mathbf{SL}(2,\mathbb{C})_\sigma$.

With the definitions above we get the following

Theorem: *Let ξ and $\hat{\xi}$ be meromorphic differentials which give under the DPW construction, using the initial conditions in (4.1.2), complete CMC-immersions $(\mathcal{D},\Psi)$ and $(\mathcal{D},\hat{\Psi})$ with mean curvature $H = -\frac{1}{2}$. Then the following statements are equivalent*

1. The CMC-immersions $(\mathcal{D},\Psi)$ and $(\mathcal{D},\hat{\Psi})$ differ only by a proper Euclidean motion, i.e., $\hat{\Psi} = \tilde{T}\circ\Psi$ for some $\tilde{T} \in \mathrm{OAff}(\mathbb{R}^3)$.

2. There is a gauge transformation

$$\hat{g}_- = \chi g_- p_+, \tag{4.1.4}$$

with $\chi \in \Lambda\mathbf{SU}(2)_\sigma$ z-independent, $p_+ = p_+(z,\lambda) \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$.

3. $E = \hat{E}$ and

$$\exp(D - \hat{D}) = \frac{|\hat{f}|}{|f|}. \tag{4.1.5}$$

<u>Proof:</u> Let F and $\tilde{F}$ denote the extended frames of $(\mathcal{D},\Psi)$ and $(\mathcal{D},\hat{\Psi})$ defined by the Iwasawa decomposition of g_- and $\hat{g}_-$, respectively.

1.$\Rightarrow$2.: If $(\mathcal{D},\Psi)$ and $(\mathcal{D},\hat{\Psi})$ are related by a proper Euclidean motion, then for their frames we get

$$\hat{F}(z,\overline{z},\lambda=1) = UF(z,\overline{z},\lambda=1)k(z,\overline{z}), \tag{4.1.6}$$

where $U \in \mathbf{SU}(2)$ and $k : \mathcal{D} \to \mathbf{U}(1)$ (see Section 3.1). By the same arguments as in Section 3.2 and 3.3 we get for the extended frames

$$\hat{F}(z,\overline{z},\lambda) = \chi(\lambda)F(z,\overline{z},\lambda)k(z,\overline{z}), \tag{4.1.7}$$

with $\chi \in \Lambda\mathbf{SU}(2)_\sigma$. Here it should be noted, that the initial condition in (3.2.4) is replaced by

$$F(0,\lambda) = F_0(\lambda), \qquad \hat{F}(0,\lambda) = \hat{F}_0(\lambda), \tag{4.1.8}$$

where F_0 and $\hat{F}_0$ are the unitary parts of g_0 and $\hat{g}_0$, respectively, w.r.t. the Iwasawa decomposition. Using the Birkhoff splitting, we get Eq. (4.1.4) from (4.1.7), with $p_+ \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$.

2.$\Rightarrow$3.: Starting from g_-, $\hat{g}_-$ satisfying Eq. (4.1.4) we consider the associated meromorphic potentials

$$\xi(\lambda) = \lambda^{-1} \begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix} \mathrm{d}z \qquad \text{and} \qquad \hat{\xi}(\lambda) = \lambda^{-1} \begin{pmatrix} 0 & \hat{f} \\ \frac{\hat{E}}{\hat{f}} & 0 \end{pmatrix} \mathrm{d}z. \tag{4.1.9}$$

Let $\frac{1}{2}e^{u}\mathrm{d}z\mathrm{d}\bar{z}$ and $\frac{1}{2}e^{\hat{u}}\mathrm{d}z\mathrm{d}\bar{z}$ be the metrics of the surfaces defined by ξ and $\hat{\xi}$. Then, by [3, Sect. A.8], f and $\hat{f}$ are given by

$$f = \frac{1}{4}w_0^{-2}e^{\frac{u}{2}}, \qquad \hat{f} = \frac{1}{4}\hat{w}_0^{-2}e^{\frac{\hat{u}}{2}}. \tag{4.1.10}$$

Here w_0 and $\hat{w}_0$ are defined by the Iwasawa decomposition

$$\hat{g}_- = \hat{F}\hat{h}_+, \qquad g_- = Fh_+, \tag{4.1.11}$$

which is chosen, s.t. u and $\hat{u}$ in Eq. (4.1.10) are real (see [6, 3]). They are the upper left entries of the λ^0-coefficients of h_+ and $\hat{h}_+$, respectively.

Using the definition of $\exp D$ and

$$\overline{g}_-^{\top} g_- = \overline{h}_+^{\top} h_+, \tag{4.1.12}$$

we see that

$$\exp D = w_0^2 e^{i\theta}, \ \theta(z)\,\text{real}, \tag{4.1.13}$$

and similarly

$$\exp \hat{D} = \hat{w}_0^2 e^{i\hat{\theta}}, \ \hat{\theta}(z)\,\text{real}. \tag{4.1.14}$$

Taking the quotient, we get

$$\left|\frac{\hat{f}}{f}\right| = \left|\frac{w_0}{\hat{w}_0}\right|^2 e^{\frac{\hat{u}-u}{2}} = \exp(D - \hat{D})e^{\frac{\hat{u}-u}{2}}. \tag{4.1.15}$$

This is true for any two CMC-immersions. Next we invoke Eq. (4.1.4). Since χ is unitary, an Iwasawa splitting gives $\hat{F} = \chi F k$ with some λ-independent, unitary $k = k(z, \overline{z})$. Using again that χ is unitary, this implies that the induced metrics are the same, whence (4.1.5). Moreover, from Eq. (4.1.4) we get

$$\hat{\xi} = p_+^{-1}\xi p_+ + p_+^{-1}\mathrm{d}p_+ = p_0^{-1}\xi p_0, \tag{4.1.16}$$

where $p_0(z) = p_+(z, \lambda = 0)$. From this we get

$$\hat{E} = -\lambda^2 \det(\hat{\xi}) = -\lambda^2 \det(\xi) = E. \tag{4.1.17}$$

3.$\Rightarrow$1.: If Eq. (4.1.5) holds, then, by Eq. (4.1.15), $u = \hat{u}$, i.e., the CMC-immersions $(\mathcal{D}, \Psi)$ and $(\mathcal{D}, \hat{\Psi})$ have the same metric and the same mean curvature. Since they also have the same Hopf differential, they differ, by Corollary 2.4, only by a proper Euclidean motion in $\mathbb{R}^3$. $\square$

Remark: D and $\hat{D}$ can also be defined in the Grassmannian representation. There the Birkhoff splitting is replaced by the block matrix splitting

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} a_- & 0 \\ c & d_- \end{pmatrix} \begin{pmatrix} a_+ & b \\ 0 & d_+ \end{pmatrix}, \tag{4.1.18}$$

(see [18, 5]). The upper diagonal block a_+ in the second factor can be chosen to be an upper triangular matrix with a positive real number s in the lower right corner. This fixes the splitting factors completely and an easy calculation shows that $s^2 = \exp(-D)$ and analogously for $\hat{D}$.

Let us define the associated families $\{\Psi_\lambda\}_{\lambda\in S^1}$, $\{\hat{\Psi}_\lambda\}_{\lambda\in S^1}$ of the CMC-immersions Ψ and $\hat{\Psi}$ using extended frames and Sym's formula as in Section 3.4. In particular, we normalize the immersions Ψ_λ and $\hat{\Psi}_\lambda$ for all λ by Eq. (3.4.2). From the proof of Proposition 3.4, we know, that with Ψ and $\hat{\Psi}$ also Ψ_λ and $\hat{\Psi}_\lambda$, $\lambda \in S^1$, are related by a unique proper Euclidean motion $\tilde{T}_\lambda$. It is a natural question, under what conditions on the meromorphic potentials ξ and $\hat{\xi}$, this Euclidean motion does not depend on λ. In this case, the corresponding members of the associated families differ all by the same proper Euclidean motion. This problem is treated in the following

Corollary: *Let ξ and $\hat{\xi}$ be meromorphic potentials associated with CMC-immersions. Let g_- and $\hat{g}_-$ be solutions of*

$$g_-' = g_-\xi,\ g_-(0,\lambda) = I, \qquad \hat{g}_-' = \hat{g}_-\hat{\xi},\ \hat{g}_-(0,\lambda) = I. \tag{4.1.19}$$

Assume also $E = \hat{E}$. Then the following are equivalent:

a) For the functions f and $\hat{f}$ we have

$$\exp(D - \hat{D}) = \frac{|\hat{f}|}{|f|}. \tag{4.1.20}$$

b) There exists a λ-independent, unitary matrix χ, s.t.

$$\hat{g}_- = \chi g_- \chi^{-1}. \tag{4.1.21}$$

c) There exists a constant $c \in \mathbb{C}$, $|c| = 1$, s.t. $\hat{f} = cf$.

d) There exists a λ-independent, unitary matrix $\chi \in \mathbf{U}(1)$, s.t. for the associated families Ψ_λ and $\hat{\Psi}_\lambda$ defined by ξ and $\hat{\xi}$ we have in the spinor representation

$$J(\hat{\Psi}_\lambda) = \chi J(\Psi_\lambda)\chi^{-1}. \tag{4.1.22}$$

<u>Proof:</u> a)$\Rightarrow$b) From Theorem 4.1 and Eq. (4.1.20) we get, that

$$\hat{g}_-(z,\lambda) = \chi(\lambda) g_-(z,\lambda) p_+(z,\lambda), \tag{4.1.23}$$

where $\chi \in \Lambda\mathbf{SU}(2)_\sigma$ and $p_+(z) \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$. By setting $z = 0$ in (4.1.23) and using Eq. (4.1.19), we get

$$I = \chi(\lambda) p_+(0,\lambda), \tag{4.1.24}$$

which implies that

$$\chi(\lambda) = p_+^{-1}(0,\lambda) \in \mathbf{U}(1) = \Lambda\mathbf{SU}(2)_\sigma \cap \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma \tag{4.1.25}$$

is λ-independent. Then $\chi g_- \chi^{-1} \in \Lambda_*^-\mathbf{SL}(2,\mathbb{C})_\sigma$ and the uniqueness of the Birkhoff decomposition gives (4.1.21).

b)$\Leftrightarrow$c) is trivial.

b)$\Rightarrow$d) Via Iwasawa splitting we obtain

$$\hat{F} = \chi F \chi^{-1} \tag{4.1.26}$$

for the extended frames, and thus by Sym's formula:

$$J(\hat{\Psi}_\lambda) = \chi J(\Psi_\lambda)\chi^{-1}. \tag{4.1.27}$$

d)$\Rightarrow$a) Since the surfaces coincide up to a Euclidean motion, their metrics $\frac{1}{2}e^u dz d\bar{z}$ and $\frac{1}{2}e^{\hat{u}} dz d\bar{z}$ coincide. This implies Eq. (4.1.20) by Eq. (4.1.15). $\square$

Finally, we consider a case of special interest to us.

Proposition: *Consider two CMC-immersions $\Psi : \mathcal{D} \to \mathbb{R}^3$ and $\hat{\Psi} : \mathcal{D} \to \mathbb{R}^3$ with associated meromorphic potentials ξ and $\hat{\xi}$. Assume moreover, that the metrics induced by Ψ and $\hat{\Psi}$ are complete and that there exists some proper Euclidean motion $\tilde{T}$, s.t. $\tilde{T}\Psi(\mathcal{D}) = \hat{\Psi}(\mathcal{D})$. Then*

a) *there exists a biholomorphic automorphism $g \in \mathrm{Aut}\mathcal{D}$, s.t. the diagram*

$$\begin{tikzcd} \mathcal{D} \arrow[r, "g"] \arrow[d, "\Psi"'] & \mathcal{D} \arrow[d, "\hat{\Psi}"] \\ \mathbb{R}^3 \arrow[r, "\tilde{T}"] & \mathbb{R}^3 \end{tikzcd}$$

commutes,

b) *for the maps Ψ and $\hat{\Psi} \circ g$, the associated meromorphic potentials ξ and $g_*\hat{\xi}$, and the coefficients for the Hopf differentials $E = (\hat{E} \circ g)(g')^2$, the statements of Theorem 4.1 are valid.*

Proof: a) Choose any $z_0, \hat{z}_0 \in \mathcal{D}$, satisfying $\hat{\Psi}(\hat{z}_0) = \tilde{T}\Psi(z_0)$. Then for sufficiently small neighbourhoods U and $\hat{U}$ of z_0 and $\hat{z}_0$, respectively, $\tilde{T}$ induces an isometry from U to $\hat{U}$. Therefore, from [13, Sect. I.11], existence of g follows.

b) clear. □

4.2 It remains to apply Theorem 4.1 and Corollary 4.1 to the case of symmetric CMC-immersions. First we treat the general case:

Theorem: *Let $\Psi : \mathcal{D} \longrightarrow \mathbb{R}^3$ be a CMC-immersion with meromorphic potential*

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix} \mathrm{d}z.$$

Let $g \in \mathrm{Aut}\mathcal{D}$ be such that $(E \circ g) \cdot (g')^2 = E$. Let furthermore g_- be the solution of Eq. (3.6.2) for ξ. Then the following are equivalent:

1. *The immersions Ψ and $\Psi \circ g$ are related by a proper Euclidean motion in $\mathbb{R}^3$, i.e., $g \in \mathrm{Aut}_\Psi\mathcal{D}$.*
2. *The surfaces $(\mathcal{D}, \Psi)$ and $(\mathcal{D}, \Psi \circ g)$ have the same conformal metric.*
3. *There exists a unitary z-independent $\chi(g, \lambda) \in \Lambda\mathbf{SU}(2)_\sigma$, s.t.*

$$g_-(g(z), \lambda) = \chi(g, \lambda) g_-(z, \lambda) p_+(g, z, \lambda),$$

with a meromorphic function p_+ taking values in $\Lambda^+\mathbf{SL}(2, \mathbb{C})_\sigma$.

4. *The equation*

$$\left|\frac{\hat{f}}{f}\right| = \frac{((\overline{g}_-^{\top} g_-)_{+0})_{11}}{((\overline{\hat{g}}_-^{\top} \hat{g}_-)_{+0})_{11}} \tag{4.2.1}$$

is satisfied, where $\hat{g}_- = g_- \circ g$, $\hat{f} = (f \circ g) g'$ and $(\cdot)_{11}$ denotes the upper left entry of a two by two matrix.

Proof: If we define $\hat{\xi} = g_*\xi$, $\hat{E} = (E \circ g)(g')^2$, and $\hat{\Psi} = \Psi \circ g$, then $\hat{g}_-$ and g_- satisfy Eq. (4.1.2) with $g_0 = I$, $\hat{g}_0 = g_-(g(0), \lambda)$. The equivalence 1.⇔3.⇔4. follows immediately from Theorem 4.1.

1.$\Rightarrow$2. is trivial. 2.$\Rightarrow$1. follows from $\hat{E} = E$ and the second part of Corollary 2.4, since both, Ψ and $\hat{\Psi}$, are CMC-surfaces with the same mean curvature. $\square$

Corollary 4.1 translates into

Corollary: *Let Ψ, ξ and g_- be defined as in Theorem 4.2. Let $g \in \mathrm{Aut}\mathcal{D}$ be s.t. $(E \circ g)(g')^2 = E$. We further assume, that $g_-(g(0), \lambda) = I$ for all $\lambda \in S^1$. Then the following statements are equivalent:*

a) For $\hat{f} = (f \circ g)g'$ and $\hat{g}_- = g_- \circ g$, Eq. (4.2.1) is satisfied.

b) There exists a λ-independent, unitary matrix $\chi \in \mathbf{U}(1)$, s.t.

$$g_-(g(z), \lambda) = \chi g_-(z, \lambda)\chi^{-1}. \tag{4.2.2}$$

c) There exists a constant $c \in \mathbb{C}$, $|c| = 1$, s.t. $(f \circ g)g' = f$.

d) There exists a λ-independent, unitary matrix $\chi \in \mathbf{U}(1)$, s.t. for the associated family Ψ_λ we have in the spinor representation

$$J(\Psi_\lambda \circ g) = \chi J(\Psi_\lambda)\chi^{-1}. \tag{4.2.3}$$

e) $g \in \mathrm{Aut}_\Psi\mathcal{D} = \mathrm{Aut}_{\Psi_\lambda}\mathcal{D}$ and $\psi_\lambda(g) = \psi(g)$ for all $\lambda \in S^1$.

f) The immersions Ψ_λ and $\Psi_\lambda \circ g$ are for all $\lambda \in S^1$ related by the same rotation around the e_3-axis in $\mathbb{R}^3$.

<u>Proof:</u> As in the proof of Theorem 4.2 we set $\hat{g}_- = g_- \circ g$, $\hat{\xi} = \hat{g}_-^{-1}\mathrm{d}\hat{g}_- = g_*\xi$, $\hat{\Psi} = \Psi \circ g$, and $\hat{E} = (E \circ g)(g')^2$. By assumption, $\hat{E} = E$ and $\hat{g}_-(0, \lambda) = g_-(0, \lambda) = I$. Therefore, with these definitions, the assumptions of Corollary 4.1 are satisfied, and the equivalence a)$\Leftrightarrow$b)$\Leftrightarrow$c)$\Leftrightarrow$d) follows immediately from Corollary 4.1.

d)$\Rightarrow$f) From d) it follows (see also the remark after Eq. (3.4.3)), that Ψ_λ and $\Psi_\lambda \circ g$ are related by a λ-independent rotation R around the e_3-axis in $\mathbb{R}^3$.

f)$\Rightarrow$e) By f), $g \in \mathrm{Aut}_{\Psi_\lambda}\mathcal{D}$ for all λ. If we define $\psi_\lambda : \mathrm{Aut}_{\Psi_\lambda}\mathcal{D} \to \mathrm{Aut}\Psi_\lambda(\mathcal{D})$ as in Section 3.4, then $R = \psi_\lambda(g) = \psi(g)$ for all $\lambda \in S^1$.

e)$\Rightarrow$d) By Theorem 3.3, to $g \in \mathrm{Aut}_\Psi\mathcal{D}$ there is associated a matrix $\chi(g, \lambda) \in \Lambda\mathbf{SU}(2)_\sigma$. If we set $\tilde{T}_\lambda = (R_{\tilde{T}_\lambda}, t_{\tilde{T}_\lambda}) = \psi_\lambda(g)$, then we see by Eq. (3.4.9), that for every λ, $\tilde{T}_\lambda$ determines $\chi(g, \lambda)$ up to a sign. Since $\chi(g, \lambda)$ depends continuously on λ, we have, that with $\tilde{T}_\lambda$ also $\chi(g, \lambda)$ is independent of λ. Therefore, $\chi(g, \lambda)$ is a diagonal, unitary matrix, from which, by Eq. (3.4.3), d) follows. $\square$

4.3 In the next two sections we want to summarize the previous considerations and, as a conclusion, we want to provide a recipe for the construction of CMC-immersions of a fixed Riemann surface M.

Let us start with $\mathcal{D}$ the complex plane or the unit disk, and let us choose a group Γ of biholomorpic automorphisms of $\mathcal{D}$, which acts freely and discontinuously. Then $M = \Gamma\backslash\mathcal{D}$ is a Riemann surface [10, 21].

Let us further assume, that there exists a CMC-immersion $\Phi : M \to \mathbb{R}^3$, s.t. M with the induced metric is complete. We obtain the following commutative diagram of Section 2.4:

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, "\pi"'] \arrow[dr, "\Psi"] & \\ M \arrow[rr, "\Phi"'] & & \mathbb{R}^3 \end{tikzcd}$$

We can assume, that Ψ is a conformal immersion.

We consider the frame $\hat{F}: \mathcal{D} \to \mathbf{SO}(3)$ given by

$$\hat{F} = (e^{-\frac{u}{2}}\Psi_x, e^{-\frac{u}{2}}\Psi_y, \frac{\Psi_x \times \Psi_y}{|\Psi_x \times \Psi_y|}). \tag{4.3.1}$$

We normalize Φ and Ψ by assuming $\hat{F}(0) = I$. The frame $\hat{F}$ can be lifted to a frame $\tilde{F}: \mathcal{D} \to \mathbf{SU}(2)$.

Considering the differential $\tilde{\alpha} = \tilde{F}^{-1}\mathrm{d}\tilde{F}$, we decompose $\tilde{\alpha} = \tilde{\alpha}_p + \tilde{\alpha}_k$, where $\tilde{\alpha}_k$ is the diagonal, and $\tilde{\alpha}_p$ is the off-diagonal part of $\tilde{\alpha}$. Then $\tilde{\alpha}_p$ decomposes into a $\mathrm{d}z$- and a $\mathrm{d}\bar{z}$-part: $\tilde{\alpha}_p = \tilde{\alpha}_p'\mathrm{d}z + \tilde{\alpha}_p''\mathrm{d}\bar{z}$.

We set

$$\alpha^\lambda = \lambda^{-1}\tilde{\alpha}_p'\mathrm{d}z + \tilde{\alpha}_k + \tilde{\alpha}_p''\mathrm{d}\bar{z}. \tag{4.3.2}$$

We know [6], that the Gauss map of Φ is harmonic iff α^λ is integrable, i.e., if there exists an extended frame

$$F: \mathcal{D} \times S^1 \to \mathbf{SU}(2), \qquad F = F(z, \overline{z}, \lambda), \tag{4.3.3}$$

s.t.

$$F^{-1}\mathrm{d}F = \alpha^\lambda, \qquad F(0,0,\lambda) = I. \tag{4.3.4}$$

Then, of course, $F(z, \overline{z}, \lambda = 1) = \tilde{F}(z, \overline{z})$.

We note, that for every fixed λ, Sym's formula [23, 1] gives a CMC-immersion Ψ_λ:

$$\Psi_\lambda(z, \overline{z}) = \partial_\theta F \cdot F^{-1} + \frac{i}{2}F\sigma_3 F^{-1}, \qquad \lambda = e^{i\theta}. \tag{4.3.5}$$

For $g \in \Gamma$, we have (Theorem 3.3)

$$(F \circ g)(z, \overline{z}, \lambda) = \chi(g, \lambda)F(z, \overline{z}, \lambda)k(g, z, \overline{z}). \tag{4.3.6}$$

where $\chi : \Gamma \to \Lambda\mathbf{SU}(2)_\sigma$ is a homomorphism up to a sign, and k is a cocycle up to a sign. Projecting F back into $\mathbf{SO}(3)$ then actually gives a homomorphism and a cocycle, respectively.

From (4.3.6) we know, that $\chi(g, \lambda = 1) = \pm I$ for $g \in \Gamma$.

Conversely, let F be an extended frame, i.e., F satisfies Eq. (4.3.4) with α^λ of the form (4.3.2). Assume now, that F satisfies Eq. (4.3.6) for all $g \in \Gamma$. Then, for every fixed λ, Sym's formula gives a CMC-immersion $(\mathcal{D}, \Psi_\lambda)$, s.t. $\Gamma \subset \mathrm{Aut}_{\Psi_\lambda}\mathcal{D}$.

If there exists some $\lambda_0 \in S^1$, s.t. $\chi(g, \lambda_0) = \pm I$ for each $g \in \Gamma$, i.e., such that $\Psi_{\lambda_0} \circ g = \Psi_{\lambda_0}$ for all $g \in \Gamma$, then Ψ_{λ_0} factors through $M = \Gamma \setminus \mathcal{D}$, i.e., there exists a CMC-immersion $\Phi : M \to \mathbb{R}^3$.

Therefore, in order to find CMC-immersions of $M = \Gamma \setminus \mathcal{D}$, it suffices to find an extended frame satisfying Eq (4.3.6), s.t. for some $\lambda_0 \in S^1$, the kernel equation $\chi(g, \lambda_0) = \pm I$ holds for all $g \in \Gamma$.

4.4 Assume now that the extended frame F satisfies Eq. (4.3.6). We split $F = g_- g_+$ for $z \in \mathcal{D} \setminus \mathcal{S}$, where $\mathcal{S} \subset \mathcal{D}$ is the discrete set of points where the Birkhoff splitting of F fails. For $\xi = g_-^{-1}\mathrm{d}g_-$ we then obtain:

$$\xi = \lambda^{-1}\begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix}\mathrm{d}z, \tag{4.4.1}$$

$$g_*\xi = p_+^{-1}\xi p_+ + p_+^{-1}\mathrm{d}p_+, \qquad g \in \Gamma, \tag{4.4.2}$$

where $p_+ = p_+(g, z, \lambda)$ has no negative λ-coefficients and is holomorphic on $\mathcal{D} \setminus \mathcal{S}$. Moreover, p_+ is uniquely defined by g up to a sign. We note, that the conditions (4.4.1) and (4.4.2) imply for g_-, defined by

$$g_-^{-1}\mathrm{d}g_- = \xi, \qquad g_-(0, \lambda) = I, \tag{4.4.3}$$

the relation

$$g_- \circ g = \chi g_- p_+, \tag{4.4.4}$$

where, for each $g \in \Gamma$, $\chi = \chi(g,\lambda)$ is z-independent, see Section 3.6. From this we cannot conclude the identity (4.3.6). This is the case, if actually $\chi(g,\lambda) \in \mathbf{SU}(2)$ for all $g \in \Gamma$, $\lambda \in S^1$. Below we give conditions that imply $\chi(g,\lambda) \in \mathbf{SU}(2)$.

Evaluating the λ^{-1}-coefficient of (4.4.2) we obtain (see Section 3.6)

$$(E \circ g)(z)(g'(z))^2 \quad = \quad E(z), \tag{4.4.5}$$

$$(f \circ g)(z) g'(z) \quad = \quad r^{-2}(g,z) f. \tag{4.4.6}$$

Here r^{-2} satisfies the cocycle condition

$$r^{-2}(g_2 \circ g_1, z) = r^{-2}(g_2, g_1(z)) r^{-2}(g_1, z), \; g_1, g_2 \in \Gamma. \tag{4.4.7}$$

Moreover, if $\chi(g,\lambda)$ in (4.4.4) is unitary for all $\lambda \in S^1$, then (Theorem 4.2) $|r(g,z)|$, $g \in \Gamma$, is explicitly given by

$$|r(g,z)|^2 = \frac{((\overline{g_- \circ g}^\top (g_- \circ g))_{+0})_{11}}{((\overline{g}_-^\top g_-)_{+0})_{11}}. \tag{4.4.8}$$

Conversely, let E be holomorphic on $\mathcal{D}$ and let f be meromorphic on $\mathcal{D}$, s.t.

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix} \mathrm{d}z$$

defines a smooth CMC-immersion $(\mathcal{D}, \Psi)$ with associated family Ψ_λ (see [3]). *If E and f satisfy Eqs. (4.4.5), (4.4.6), and (4.4.8) for all $g \in \Gamma \subset \mathrm{Aut}\mathcal{D}$, then (Theorem 4.2) $\Gamma \subset \mathrm{Aut}_\Psi \mathcal{D} = \mathrm{Aut}_{\Psi_\lambda} \mathcal{D}$.* In particular, the solution g_- of (4.4.3) satisfies Eq. (4.4.4) with $\chi = \chi(g,\lambda) \in \Lambda\mathbf{SU}(2)_\sigma$ and $p_+ \in \Lambda^+\mathbf{SL}(2,\mathbb{C})_\sigma$. *If we want to factor Ψ_{λ_0} for some $\lambda_0 \in S^1$ through $M = \Gamma \setminus \mathcal{D}$, then we need, in addition, a fourth condition*

$$\chi(g, \lambda_0) = \pm I, \qquad \text{for all } g \in \Gamma. \tag{4.4.9}$$

In a brief summary: Every CMC-immersion (M, Φ) (with the exception of a sphere), in particular every compact CMC-immersion, can be obtained by the following construction:

Let $\mathcal{D}$ be the complex plane or the unit disk. Let Γ be a Fuchsian or elementary group acting on $\mathcal{D}$. Let E be holomorphic on $\mathcal{D}$ and f meromorphic on $\mathcal{D}$, s.t.

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & f \\ \frac{E}{f} & 0 \end{pmatrix} \mathrm{d}z$$

defines a smooth CMC-immersion $(\mathcal{D}, \Psi)$.

If E and f satisfy Eq. (4.4.5), (4.4.6), and (4.4.8) for all $g \in \Gamma$, then Eq. (4.4.4) holds, with a unitary matrix $\chi = \chi(g,\lambda)$, for all $g \in \Gamma$. If, in addition, there exists some $\lambda_0 \in S^1$, s.t. Eq. (4.4.9) is satisfied for all $g \in \Gamma$, then $(\mathcal{D}, \Psi_{\lambda_0})$ covers a CMC-immersion (M, Φ) with Fuchsian group Γ.

It would be interesting to see the program above being carried out for compact CMC-surfaces explicitly.

5 Two examples

5.1 As an example for the case treated in Corollary 4.2, we revisit the Smyth surfaces, where the metric is rotationally symmetric.

First we characterize Smyth surfaces in terms of their meromorphic potentials.

Proposition: *The Smyth surfaces are, up to coordinate transformation, in $1-1$-correspondence to the meromorphic potentials of the form*

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ cz^m & 0 \end{pmatrix} \mathrm{d}z, \tag{5.1.1}$$

where $m = 0, 1, 2, \ldots$ is an integer and $c \neq 0$ a complex number.

Proof: Let $(\mathcal{D}, \Psi)$ be a CMC-immersion that admits a one-parameter group of rotations as self-isometries. W.l.o.g. we assume, that the center of these rotations is $z_0 = 0$. From Corollary 2.7 we thus obtain, that up to a reparametrization $E = dz^m$, $d \in \mathbb{C} \setminus 0$. Moreover, for the function f we have by Wu's formula [24]: $f(z) = \frac{1}{4} e^{u(z,0) - \frac{1}{2} u(0,0)}$. Since $u = u(r)$, we have $u(z,0) = u(0,0)$ and $f = f_0 = \text{const}$ follows. If we change coordinates by $\mathrm{d}w = f_0 \mathrm{d}z$, then in w-coordinates the meromorphic potential is of the form (5.1.1). That all c and all m actually occur, follows from [19]. $\square$

5.2 Consider now ξ as in Theorem 5.1 and the corresponding CMC-immersion $\Psi = \Psi_c^m : \mathbb{C} \to \mathbb{R}^3$, which has a single umbilic of order m at $z = 0$.

We will first describe the associated family of $(\mathbb{C}, \Psi_\lambda) = (\mathbb{C}, (\Psi_c^m)_\lambda)$ geometrically.

In the DPW construction we get Ψ_μ for fixed $\mu \in S^1$ by substituting $\lambda \to \lambda\mu$ in the formulas for g_-, F and Ψ. If we do this, we get for the meromorphic potential

$$\xi(z,\lambda) \to \tilde{\xi}(z,\lambda) = \xi(z, \mu\lambda) = \lambda^{-1} \begin{pmatrix} 0 & \mu^{-1} \\ \mu^{-1} cz^m & 0 \end{pmatrix} \mathrm{d}z. \tag{5.2.1}$$

Let us set $\mu = e^{i\theta}$, $\theta \in [0, 2\pi)$. If we rotate the coordinate system on $\mathcal{D}$ by $\alpha\theta$, $\alpha \in \mathbb{R}$, i.e., $\tilde{z} = e^{-i\alpha\theta} z$, then

$$\tilde{\xi}(\tilde{z},\lambda) = \lambda^{-1} \begin{pmatrix} 0 & e^{i(\alpha-1)\theta} \\ e^{i(\alpha(m+1)-1)\theta} c\tilde{z}^m & 0 \end{pmatrix} \mathrm{d}\tilde{z}. \tag{5.2.2}$$

Therefore, if we choose $\alpha = \frac{2}{m+2}$, then

$$\tilde{\xi}(\tilde{z},\lambda) = \chi_\mu \lambda^{-1} \begin{pmatrix} 0 & 1 \\ c\tilde{z}^m & 0 \end{pmatrix} \mathrm{d}\tilde{z} \chi_\mu^{-1}, \tag{5.2.3}$$

where conjugation with

$$\chi_\mu = \begin{pmatrix} e^{-i\frac{m}{2m+4}\theta} & 0 \\ 0 & e^{i\frac{m}{2m+4}\theta} \end{pmatrix} \tag{5.2.4}$$

describes a rotation in $\mathbb{R}^3$ about an angle $-\frac{m}{m+2}\theta$ around the e_3 axis.

This gives the following

Lemma: *For the CMC-immersions $(\mathbb{C}, \Psi_c^m)$, defined by $E = cz^m$, $f = 1$, all members of the associated family coincide up to a rigid rotation and a coordinate transformation.*

Proof: By Eq. (5.2.3) the variation of the spectral parameter amounts for the meromorphic potential to a coordinate transformation on $\mathcal{D}$ plus a rigid rotation in $\mathbb{R}^3$. Since a rigid rotation also leaves the initial condition for g_- invariant, we get with the same notation as in Eq. (5.2.3) for fixed $\mu \in S^1$

$$g_-(z, \mu\lambda) = \chi_\mu g_-(e^{-i\alpha\theta} z, \lambda) \chi_\mu^{-1}, \qquad \mu = e^{i\theta}, \tag{5.2.5}$$

where $\alpha = \frac{2}{m+2}$. This shows, that $\Psi_c^m(\mathcal{D})$ and $(\Psi_c^m)_\mu(\mathcal{D})$ are related by a rigid rotation around the e_3-axis in $\mathbb{R}^3$. The rotation angle in $\mathbb{R}^3$ is given by Eq. (5.2.4) as $-\frac{m}{m+2}\theta$. $\square$

5.3 In this section we determine the group $\mathrm{Aut}_{\Psi_c^m}\mathcal{D}$.

Proposition: *If* $(\mathbb{C},\Psi_c^m)$ *is a nondegenerate Smyth surface, then:*

$$\mathrm{Aut}\Psi_c^m(\mathcal{D}) \cong \mathrm{Aut}_{\Psi_c^m}\mathcal{D} = \{g | g(z) = e^{2\pi i \frac{k}{m+2}} z, k = 0,\dots,m+1\}. \tag{5.3.1}$$

Proof: From Theorem 5.1 we know that the Hopf differential is of the form $E = cz^m \mathrm{d}z^2$ and the metric factor e^u only depends on the radius. Therefore, Lemma 2.6 shows, that a rotation is in $\mathrm{Aut}_{\Psi_c^m}\mathbb{C}$, iff it leaves $cz^m\mathrm{d}z^2$ invariant. It is easy to see, that these are exactly the rotations $z \mapsto e^{\frac{2\pi i}{m+2}}z$. Using Lemma 2.15, we see that for nondegenerate Smyth surfaces we have $\mathrm{Aut}_{\Psi_c^m}\mathcal{D} \cong \mathrm{Aut}\Psi_c^m(\mathcal{D})$. Altogether, this proves the claim. □

5.4 Finally, we want to determine, which of the surfaces $(\mathbb{C},\Psi_c^m)$ are nondegenerate, i.e., are not cylinders in $\mathbb{R}^3$. Since m gives the order of the single umbilic of the surface, and since cylinders do not have any umbilics, we only need to consider the case $m = 0$.

We already now, that for $m = 0$, $c = 1$ we get a cylinder. For arbitrary $c \neq 0$ we can write

$$\xi(z,\lambda) = \lambda^{-1}\begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}\mathrm{d}z = \Omega\xi_0(\tilde{z},\lambda)\Omega^{-1}, \tag{5.4.1}$$

where ξ_0 is the meromorphic potential of the standard cylinder (see Section 3.7), $\tilde{z} = \sqrt{c}z$, and

$$\Omega = \begin{pmatrix} c^{-\frac{1}{4}} & 0 \\ 0 & c^{\frac{1}{4}} \end{pmatrix}. \tag{5.4.2}$$

This shows, that ξ is obtained from the standard cylinder by a coordinate transformation and subsequent dressing with $h_+ = \Omega$. Corollary 1 of Section 3.7 then shows, that $\Psi_c^0(\mathcal{D})$ is a cylinder iff Ω is unitary, i.e., iff $|c| = 1$.

We have proved the following

Proposition: *The meromorphic potential* $\xi = \lambda^{-1}\begin{pmatrix} 0 & 1 \\ cz^m & 0 \end{pmatrix}\mathrm{d}z$, $c \in \mathbb{C}\setminus\{0\}$, $m \in \mathbb{N}_0$, *gives a nondegenerate Smyth surface iff* $m = 0$ *and* $|c| \neq 1$.

5.5 Other simple examples are provided by surfaces with branchpoints. An example is the surface with the meromorphic potential

$$\xi = \lambda^{-1}\begin{pmatrix} 0 & z - z_0 \\ 1 & 0 \end{pmatrix}\mathrm{d}z, \tag{5.5.1}$$

where $f(z) = E(z) = z - z_0$ has a simple zero at $z_0 \in \mathbb{C}^*$. While this potential is holomorphic at $z = z_0$, it cannot be produced from a CMC-immersion, since f is not the square of a meromorphic function (see [3, Theorem 2.3]). However, it does give a nonsingular immersion and globally a surface $\Psi(\mathcal{D})$, $\mathcal{D} = \mathbb{C}$, in $\mathbb{R}^3$ with one branchpoint. To find the "correct" meromorphic potential of the punctured surface $\Psi(\mathbb{C}\setminus\{z_0\})$, we need to take into account the elementary group of the surface.

We define a coordinate $\tilde{w}$ on the universal cover of $\mathbb{C}^*$ by $w = z - z_0$ and $w = e^{\tilde{w}} = q(\tilde{w})$, $q : \mathbb{C} \to \mathbb{C}\setminus\{0\}$. We get $\tilde{w} = \ln w$, $\mathrm{d}\tilde{w} = \frac{1}{w}\mathrm{d}w$, $q_*(f(z)\mathrm{d}z) = e^{2\tilde{w}}\mathrm{d}\tilde{w}$, $q_*(E(z)\mathrm{d}z^2) = e^{3\tilde{w}}\mathrm{d}\tilde{w}^2$. On the universal cover we have

$$\tilde{\xi} = \lambda^{-1}\begin{pmatrix} 0 & e^{2\tilde{w}} \\ e^{\tilde{w}} & 0 \end{pmatrix}\mathrm{d}\tilde{w}. \tag{5.5.2}$$

The initial condition changes from $g_-(z = 0) = I$ to $\tilde{g}_-(\tilde{w}_0) = I$, where some $\tilde{w}_0$ with $q(\tilde{w}_0) = -z_0 \neq 0$ is fixed. In the coordinate $\tilde{w}$ on the universal cover $\mathbb{C}$, the elementary group of the surface is the translation group generated by $\tilde{w} \to \tilde{w} + 2\pi i$.

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