The normal symbol on Riemannian manifolds

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Introduction

It is a well-known fact that on Euclidean space one can construct canonical linear isomorphisms between symbol spaces and corresponding spaces of pseudo-differential operators. Furthermore one has natural formulas which represent a pseudo-differential operator in terms of its symbol resp. which give an expression for the symbol of a pseudo-differential operator. By using symbols one gets much insight in the structure of pseudo-differential operators on Euclidean space. For example symbols give the means to construct a (pseudo) inverse of an elliptic differential operators on $\mathbb{R}^d$.

Compared to the well-established theory of pseudo-differential operators on manifolds, it seems that besides the definition of a global principal symbol, little has been done with respect to defining a complete and global symbol calculus for pseudo-differential operators on manifolds. An exception is the work of WIDOM [8, 9], where a rather general notion of a phase-function is used to construct a symbol map, i.e. a map associating a symbol to a pseudo-differential operator. Unfortunately WIDOM could only give an abstract argument for the existence of an inverse of the symbol map (modulo smoothing operators) and no formula for it.

As the pure local consideration of symbols does not reveal the geometry and topology of the manifold one is working on, we regard it as very desirable to build up a general theory of global symbol calculi on manifolds. In fact deep geometric results can be received by using such a global symbol calculus. For example one can see this from the proof by GETZLER [3] of the Atiyah-Singer index theorem for the Dirac operator on a spin manifold. GETZLER in particular applied the methods of WIDOM.

In this paper we want to introduce a symbol calculus for which the inverse of the symbol map can explicitly be constructed. We achieve this by using a natural notion of a phase-function which is different from WIDOM'S. In particular we therefore succeed in giving a concrete integral representation of the inverse of our symbol map. This integral representation then gives us the means to write down a formula for the symbol of the adjoint of a pseudo-differential operator and for the symbol of the product of two pseudo-differential operators. The product formula is similar to a result obtained by WIDOM, but we provide a new proof.

Another application of symbol calculus on manifolds lies in quantization theory. There one is interested in a quantization map associating pseudo-differential operators to certain functions on a cotangent bundle in a way that Diracs quantization condition is fulfilled.

But the inverse of a symbol map does exactly this, so it is a quantization map. See PFLAUM [6] for details.

Let us mention at the end of this introduction that we regard our paper as a starting point for more general aspects of symbols on manifolds. Besides the normal symbol - where the term normal can be interpreted in the sense of normal ordering in Quantum Filed Theory or in the sense of "being canonical" - it seems possible to extend the notion of mixed, Weyl and Berezin symbols to manifolds, too. Work on these more complex type of symbols is in progress.

1 Symbols

First we will define the notion of a symbol on a Riemannian vector bundle. Let us assume once and for all in this article that $\mu, \rho, \delta \in \mathbb{R}$ are real numbers such that $0 \leq \delta < \rho \leq 1$ and $1 \leq \rho + \delta$. The same shall hold for triples $\tilde{\mu}, \tilde{\rho}, \tilde{\delta} \in \mathbb{R}$.

We sometimes use properties of symbols on $\mathbb{R}^d \times \mathbb{R}^N$. As these are well-described in the mathematics literature we only refer the interested reader to HÖRMANDER [5], SHUBIN [7] or GRIGIS, SJÖSTRAND [4] for a general introduction to symbol theory on $\mathbb{R}^d \times \mathbb{R}^N$ and for proofs.

Definition 1.1 *Let $\pi : E \to X$ be a smooth Riemannian vector bundle. Then an element $a \in \mathcal{C}^\infty(E|_U)$ with $U \subset X$ open is a called a* **symbol** *on E over U of* **order** *μ and* **type** *(ρ, δ) if for every trivialization $(x, \zeta) : E|_V \to \mathbb{R}^d \times \mathbb{R}^N$ with $V \subset U$ open the following condition is satisfied:*

(Sy) For every $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^N$ and every $K \subset V$ compact there exists $C = C_{K,\alpha,\beta} > 0$ such that for every $e \in E|_K$ the inequality

$$\left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial^{|\beta|}}{\partial \zeta^\beta} a(e) \right| \leq C \, (1 + |e|)^{\mu + \delta|\alpha| - \rho|\beta|} \tag{1}$$

is valid.

The elements of $\mathrm{S}^\mu_{\rho,\delta}(U, E)$ are the sections of the sheaf $\mathrm{S}^\mu_{\rho,\delta}(\cdot, E)$ of symbols on E of order μ and type (ρ, δ).

By defining $\mathrm{S}^{-\infty}(U, E) = \bigcap_{\mu \in \mathbb{R}} \mathrm{S}^\mu_{\rho,\delta}(U, E)$ and $\mathrm{S}^\infty_{\rho,\delta}(U, E) = \bigcup_{\mu \in \mathbb{R}} \mathrm{S}^\mu_{\rho,\delta}(U, E)$ we get the sheaf $\mathrm{S}^{-\infty}(\cdot, E)$ of smoothing symbols resp. the sheaf $\mathrm{S}^\infty_{\rho,\delta}(\cdot, E)$ of symbols of type (ρ, δ).

In case E is the cotangent bundle T^*X of a Riemannian manifold X we write $\mathrm{S}^\mu_{\rho,\delta}(\cdot, X)$, $\mathrm{S}^\infty_{\rho,\delta}(\cdot, X)$ resp. $\mathrm{S}^{-\infty}(\cdot, X)$ or even only $\mathrm{S}^\mu_{\rho,\delta}(\cdot)$, $\mathrm{S}^\infty_{\rho,\delta}(\cdot)$ and $\mathrm{S}^{-\infty}(\cdot)$ for the corresponding symbol sheaves $\mathrm{S}^\mu_{\rho,\delta}(\cdot, T^*X)$, $\mathrm{S}^\infty_{\rho,\delta}(\cdot, T^*X)$ and $\mathrm{S}^{-\infty}(\cdot, T^*X)$.

It is possible to extend this definition to one of symbols over conic manifolds but we do not need this generality in the following and refer the interested reader to DUISTERMAAT [2] and PFLAUM [6] for more information on symbols over conic manifolds.

We want to give the symbol spaces $\mathrm{S}^\mu_{\rho,\delta}(U, E)$ a topological structure. So first choose a compact set $K \subset U$, a (not necessarily disjunct) partition $K = \bigcup_{\iota \in J} K_\iota$ of K into compact subsets together with local trivializations $(x_\iota, \zeta_\iota) : E|_{V_\iota} \to \mathbb{R}^d \times \mathbb{R}^N$ such that

$K_\iota \subset V_\iota \subset V$ and V_ι open. Then we can attach for any $\alpha \in \mathbb{R}^d$ and $\beta \in \mathbb{R}^N$ a seminorm $p = p_{(K_\iota,(x_\iota,\zeta_\iota)),\alpha,\beta} : \mathrm{S}^\mu_{\rho,\delta}(U,E) \to \mathbb{R}^+ \cup \{0\}$ to the symbol spaces $\mathrm{S}^\mu_{\rho,\delta}(U,E)$ by

$$p(a) \;=\; \sup_{\iota\in J} \left\{ \frac{\left|\frac{\partial^{|\alpha|}}{\partial {x_\iota}^\alpha}\frac{\partial^{|\beta|}}{\partial {\zeta_\iota}^\beta} a(e)\right|}{(1+|e|)^{\mu+|\alpha|\delta-|\beta|\rho}} \;:\; e \in E|_{K_\iota} \right\}. \tag{2}$$

The system of these seminorms gives $\mathrm{S}^\mu_{\rho,\delta}(U,E)$ the structure of a Fréchet space such that the restriction morphisms $\mathrm{S}^\mu_{\rho,\delta}(U,E) \to \mathrm{S}^\mu_{\rho,\delta}(V,E)$ for $V \subset U$ open are continuous. Additionally we have natural and continuous inclusions $\mathrm{S}^\mu_{\rho,\delta}(U,E) \subset \mathrm{S}^{\tilde\mu}_{\tilde\rho,\tilde\delta}(U,E)$ for $\tilde\mu \geq \mu$, $\tilde\rho \leq \rho$ and $\tilde\delta \geq \delta$. Like in the case of symbols on $\mathbb{R}^d \times \mathbb{R}^N$ one can show that pointwise multiplication of symbols is continuous with respect to the Fréchet topology on $\mathrm{S}^\mu_{\rho,\delta}(U,E)$.

Proposition 1.2 *Let $a \in \mathcal{C}^\infty(E|_U)$. If there exists an open covering of U by patches V_ι with local trivializations $(x_\iota,\zeta_\iota) : E|_{V_\iota} \to \mathbb{R}^d \times \mathbb{R}^N$ such that for every (x_ι,ζ_ι) the above condition (Sy) holds, then a is a symbol of order μ and type (ρ,δ).*

Proof: Let $(x,\zeta) : E|_V \to \mathbb{R}^d \times \mathbb{R}^N$, $V \subset U$ open be a trivialization. Then we can write on $V_\iota \cap V$

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha}\frac{\partial^{|\beta|}}{\partial \zeta^\beta} \;=\; \sum_{\tilde\alpha+\alpha'\leq\alpha} (f_{\tilde\alpha,\alpha',\beta} \circ \pi)\; {\zeta_\iota}^{\alpha'}\; \frac{\partial^{|\tilde\alpha|}}{\partial {x_\iota}^{\tilde\alpha}}\frac{\partial^{|\beta+\alpha'|}}{\partial {\zeta_\iota}^{\beta+\alpha'}}\,, \tag{3}$$

where $f_{\tilde\alpha,\alpha',\beta} \in \mathcal{C}^\infty(V_\iota \cap V)$. As $\rho+\delta \geq 1$ holds and (Sy) is true for (x_ι,ζ_ι), the claim now follows. □

Example 1.3 (*i*) Let X be a Riemannian manifold. Then every smooth function $a : T^*U \to \mathbb{C}$ which is a polynomial function of order $\leq \mu$ on every fiber lies in $\mathrm{S}^\mu_{\rho,\delta}(U,T^*X)$. In particular we have an embedding $\mathcal{D}_0 \to \mathrm{S}^\infty_{\rho,\delta}(\cdot,T^*X)$, where $\mathcal{D}_0$ is the sheaf of polynomial symbols on X, i. e. for every $U \subset X$ open $\mathcal{D}_0(U) = \{f : T^*U \to \mathbb{C} :\; f|_{T^*_xX}$ is a polynomial for every $x \in U\}$.

(*ii*) Again let X be Riemannian. Then the mapping $l : T^*X \to \mathbb{C}$, $\xi \mapsto |\xi|^2$ is a symbol of order 2 and type $(1,0)$ on X, but not one of order $\mu < 2$. Next regard the function $a : T^*X \to \mathbb{C}$, $\xi \mapsto \frac{1}{1+|\xi|^2}$. This function is a symbol of order -2 and type $(1,0)$ on X, but not one of order $\mu < -2$.

The following theorem is an essential tool for the use of symbols in the theory of partial differential equations and extends a well-known result for the case of $E = \mathbb{R}^d \times \mathbb{R}^N$ to arbitrary Riemannian vector bundles.

Theorem 1.4 *Let $a_j \in \mathrm{S}^{\mu_j}_{\rho,\delta}(U,E)$, $j \in \mathbb{N}$ be symbols such that $(\mu_j)_{j\in\mathbb{N}}$ is a decreasing sequence with $\lim_{j\mapsto\infty} \mu_j = -\infty$. Then there exists a symbol $a \in \mathrm{S}^{\mu_0}_{\rho,\delta}(U,E)$ unique up to smoothing symbols, such that $a - \sum_{j=0}^{k} a_j \in \mathrm{S}^{\mu_k}_{\rho,\delta}(U,E)$ for every $k \in \mathbb{N}$.*

This induces a locally convex Hausdorff topology on the vector space $\mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}(U,E)$, which is called the **topology of asymptotic convergence**.

PROOF: It is a well-known that the claim holds for $U \subset \mathbb{R}^d$ and $E = \mathbb{R}^d \times \mathbb{R}^N$ (see [5, 7, 4]). Covering U by trivializations $(z_\iota, \zeta_\iota) : E|_{V_\iota} \to \mathbb{R}^d \times \mathbb{R}^N$ we can find a partition (φ_ι) of unity subordinate to the covering (V_ι) and for every index ι a symbol $b_\iota \in \mathrm{S}^{\mu}_{\rho,\delta}(V_\iota)$ such that $b_\iota = \sum_{j=0}^{k} a_j|_{V_\iota} \in \mathrm{S}^{\mu_k}_{\rho,\delta}(V_\iota)$. Now define $a = \sum_\iota (\varphi_\iota \circ \pi)\, b_\iota$ and check that this a satisfies the claim. Uniqueness of a up to smoothing symbols is clear again from a local consideration. □

The polynomial symbols affected by smoothing symbols. The following proposition gives the precise statement.

Proposition 1.5 *Let X be a Riemannian manifold, and $\mathcal{D}_0$ the sheaf of polynomial symbols on X. Then by composition with the projection $\mathrm{S}^{\infty}_{\rho,\delta}(\cdot,X) \to \mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}(\cdot,X)$ the canonical embedding*

$$\begin{array}{rcl} \mathcal{D}_0 & \to & \mathrm{S}^{\infty}_{\rho,\delta}(\cdot,X) \\ \mathcal{D}_0(U) \ni f & \mapsto & f \in \mathrm{S}^{\infty}_{\rho,\delta}(U,X), \qquad U \subset X \text{ open} \end{array} \tag{4}$$

gives rise to a monomorphism $\delta : \mathcal{D}_0 \to \mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}(\cdot,X)$ of sheaves of algebras.

PROOF: If $f, g \in \mathcal{D}_0(U)$ are polynomial symbols such that $f - g \in \mathrm{S}^{-\infty}(U,X)$, then $f - g$ is a bounded polynomial function on each fiber of T^*U. Therefore $f - g$ is constant on each fiber, hence $f - g = 0$ by $f - g \in \mathrm{S}^{-\infty}(U,X)$. □

2 Fourier transform and pseudo-differential operators on manifolds

Let a be a polynomial symbol defined on the (trivial) cotangent bundle $T^*U = U \times \mathbb{R}^d$ of an open set $U \subset \mathbb{R}^d$ of Euclidean space. Then one can write $a = \sum_\alpha (a_\alpha \circ \pi)\zeta^\alpha$, where $\zeta : T^*U \to \mathbb{R}^d$ is the projection on the "cotangent vectors" and the a_α are smooth functions on U. According to standard results of partial differential equations one knows that the symbol a defines the differential operator $A = \mathrm{Op}(a)$:

$$\mathcal{C}^\infty(U) \ni f \mapsto\mapsto \sum_\alpha a_\alpha (-i)^{|\alpha|} \frac{\partial^{|\alpha|}}{\partial z^\alpha} f \in \mathcal{C}^\infty(U). \tag{5}$$

Sometimes A is called the quantization of a. In case $f \in \mathcal{D}(U)$, the Fourier transform of f is well-defined and provides the following integral representation for Af:

$$Af = \mathrm{Op}(a)(f) = \int_{\mathbb{R}^d} e^{i<\xi,\cdot>} a(\xi)\, \hat{f}(\xi)\, d\xi. \tag{6}$$

It would be very helpful for structural and calculational considerations to extend this formula to Riemannian manifolds. To achieve this it is necessary to have an appropriate notion of Fourier transform on manifolds. In the following we are going to define such a Fourier transform and will later get back to the problem of an integral representation for the "quantization map" Op on Riemannian manifolds.

Assume X to be Riemannian of dimension d and consider the exponential function exp with respect to the Levi-Civita connection on X. Choose an open neighborhood $W \subset TX$ of the zero section in TX such that $(\pi, \exp) : W \to X \times X$ maps W diffeomorphically onto an open neighborhood of the diagonal Δ of $X \times X$. Then there exists a smooth function $\psi : TX \to [0,1]$ called a **cut-off function** such that $\psi|_{\check{W}} = 1$ and $\mathrm{supp}(\psi) \subset W$ for an open neighborhood $\check{W} \subset W$ of the zero section in TX. Now the following microlocal lift $\mathcal{M}_\psi$ is well-defined:

$$\mathcal{M}_\psi : \ \mathcal{C}^\infty(U) \to \mathrm{S}^{-\infty}(U, TX), \quad f \mapsto f^\psi, \tag{7}$$

$$f^\psi(v) = \begin{cases} \psi(v) f(\exp v)) & \text{for } v \in W \\ 0 & \text{else.} \end{cases} \tag{8}$$

$\mathcal{M}_\psi$ is a linear but not multiplicative map between function spaces.

Over the tangent bundle TX we can define the **Fourier transform** as the following sheaf morphism:

$$\mathcal{F} : \mathrm{S}^{-\infty}(U, TX) \to \mathrm{S}^{-\infty}(U, T^*X), \quad a \mapsto \hat{a}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{T_{\pi(\xi)}X} e^{-i<\xi,v>}\, a(v)\, dv. \tag{9}$$

We also have a **reverse Fourier transform**:

$$\mathcal{F}^{-1} : \mathrm{S}^{-\infty}(U, T^*X) \to \mathrm{S}^{-\infty}(U, TX), \quad b \mapsto \check{b}(v) = \frac{1}{(2\pi)^{n/2}} \int_{T^*_{\pi(v)}X} e^{i<\xi,v>}\, b(\xi)\, d\xi. \tag{10}$$

It is easy to check that $\mathcal{F}^{-1}$ and $\mathcal{F}$ are inverse to each other, indeed.

Composing $\mathcal{M}_\psi$ and $\mathcal{F}$ gives rise to the **Fourier transform** $\mathcal{F}_\psi = \mathcal{F} \circ \mathcal{M}_\psi$ on the Riemannian manifold X. $\mathcal{F}_\psi$ has the left inverse

$$\mathrm{S}^{-\infty}(U, T^*X) \to \mathcal{C}^\infty(U), \quad a \mapsto \mathcal{F}^{-1}(a)|_{0_U}, \tag{11}$$

where $|_{0_U}$ means the restriction to the natural embedding of U into T^*X as zero section.

By definition $\mathcal{M}_\psi$ and $\mathcal{F}_\psi$ do depend on the smooth cut-off function ψ, but this arbitrariness will only have minor effects on the study of pseudo-differential operators defined through $\mathcal{F}_\psi$. We will get back to this point later on in this section.

The exponential function exp gives rise to normal coordinates $z_x : V_x \to \mathbb{R}^d$, where $x \in V_x \subset X$, V_x open and

$$\exp_x^{-1}(y) \;=\; \sum_{k=1}^{d} z_x^k(y) \cdot e_k(y) \tag{12}$$

for all $y \in V_x$ and an orthonormal frame $(e_1, ..., e_d)$ of TV_x. Furthermore we receive bundle coordinates $(z_x, \zeta_x) : T^*V_x \to \mathbb{R}^{2d}$ and $(z_x, v_x) : TV_x \to \mathbb{R}^{2d}$. Note that for fixed $x \in X$ the map

$$V_x \times T^*V_x \ni (y, \xi) \mapsto (z_y(\pi(\xi)), \zeta_y(\xi)) \in \mathbb{R}^{2d} \tag{13}$$

is smooth and that by the Gauß Lemma

$$z_x^k(y) = -z_y^k(x) \tag{14}$$

for every $y \in V_x$.

Now let a be a smooth function defined on T^*U and polynomial in the fibers. Then locally $a = \sum_\alpha (a_{x,\alpha} \circ \pi)\, \zeta_x$ with respect to a normal coordinate system at $x \in U$ and functions $a_{x,\alpha} \in \mathcal{C}^\infty(U)$. Define the operator $A : \mathcal{D}(U) \to \mathcal{C}^\infty(U)$ by

$$f \mapsto Af \;=\; \mathrm{Op}_\psi(a)(f) \;=\; \left(U \ni x \mapsto \frac{1}{(2\pi)^{n/2}} \int_{T_x^*X} a(\xi)\, \hat{f}^\psi(\xi)\, d\xi \in \mathbb{C} \right) \tag{15}$$

and check that

$$Af\,(x) \;=\; \sum_{\alpha} a_{x,\alpha}(x)\,(-i)^{|\alpha|}\,\frac{\partial^{|\alpha|}}{\partial {\xi_x}^{\alpha}}\left[\psi\,(f\circ\exp\,)\right](0_x) \;=\; \sum_{\alpha} a_{x,\alpha}(x)\,(-i)^{|\alpha|}\,\frac{\partial^{|\alpha|}}{\partial {z_x}^{\alpha}} f\,(x) \quad (16)$$

for every $x \in U$. Therefore Af is a differential operator independent of the choice of ψ.

However, if a is an arbitrary element of the symbol space $\mathrm{S}^{\infty}_{\rho,\delta}(U)$, Eq. (15) still defines a continuous operator $A_\psi = \mathrm{Op}_\psi(a) : \mathcal{D}(U) \to \mathcal{C}^\infty(U)$, which is a pseudo-differential operator but independent of the cut-off function ψ only up to smoothing operators. Let us show this in more detail.

First choose $\mu \in \mathbb{R}$ such that $a \in \mathrm{S}^{\mu}_{\rho,\delta}(U)$, and consider the kernel distribution $\mathcal{D}(U \times U) \to \mathbb{C}$, $(f \otimes g) \mapsto < A_\psi f, g > = \int_X g(x)\,A_\psi f(x)\,dx$ of A_ψ. It can be written in the following way:

$$\begin{aligned} &< g, A_\psi f > = \\ &= \frac{1}{(2\pi)^n}\int_X \int_{T^*_x X}\int_{T_x X} \psi(v)\,g(x)\,f(\exp v)\,e^{-i<\xi,v>}\,a(\xi)\,dv\,d\xi\,dx \\ &= \frac{1}{(2\pi)^n}\int_X \quad \int_{T_x X}\int_{T^*_x X} \psi(v)\,g(x)\,f(\exp v)\,e^{-i<\xi,v>}\,a(\xi)\,d\xi\,dv\,dx; \end{aligned} \quad (17)$$

where the first integral is an iterated one, the second one an oscillatory integral. To check that Eq. (17) is true use a density argument or in other words approximate the symbol a by elements $a_k \in \mathrm{S}^{-\infty}(U)$ (in the topology of $\mathrm{S}^{\tilde\mu}_{\rho,\delta}(U)$ with $\tilde\mu > \mu$) and prove the statement for the a_k. The claim then follows from continuity of both sides of Eq. (17) with respect to a. Let $\tilde\psi : TX \to [0,1]$ be another cut-off function such that $\tilde\psi|_{\tilde O} = 1$ on a neighborhood $W' \subset W$ of the zero section of T^*X. We can assume $W' = \tilde W$ and claim the operator $A_\psi - A_{\tilde\psi}$ to be smoothing. For the proof it suffices to show that the oscillatory integral

$$K_{A_\psi - A_{\tilde\psi}}(v) \;=\; \frac{1}{(2\pi)^n}\int_{T^*_{\pi(v)}X} \left(\psi(v) - \tilde\psi(v)\right) e^{-i<\xi,v>}\,a(\xi)\,d\xi, \quad v \in TU, \quad (18)$$

which gives an integral representation for the kernel of $A_\psi - A_{\tilde\psi}$ defines a smooth function $K_{A_\psi - A_{\tilde\psi}} \in \mathcal{C}^\infty(TU)$. The phase function $T^*_{\pi(v)}X \ni \xi \mapsto -\xi(v) \in \mathbb{R}$ has only critical points for $v = 0$. Hence for $v \neq 0$ there exists a (-1)-homogeneous vertical vector field L on T^*X such that $L\,e^{-i<\cdot,v>} = e^{-i<\cdot,v>}$. The adjoint $L^\dagger$ of L satisfies $\left(L^\dagger\right)^k a \in \mathrm{S}^{\mu-\rho k}_{\rho,\delta}(U)$, where $k \in \mathbb{N}$. As the amplitude $(v,\xi) \mapsto (\psi(v) - \tilde\psi(v))\,a(\xi)$ vanishes on $W' \times_U T^*U$, the equation

$$K_{A_\psi - A_{\tilde\psi}}(v) = \frac{1}{(2\pi)^n}\int_{T^*_{\pi(v)}X} e^{-i<\xi,v>}\,\left(L^\dagger\right)^k a(\xi)\,d\xi \quad (19)$$

holds for any integer k fulfilling $\mu+\rho k < -\dim X$. Note that the integral in Eq. (19) unlike the one in Eq. (18) is to be understood in the sense of Lebesgue. Hence $K_{A_\psi - A_{\tilde\psi}} \in \mathcal{C}^\infty(TU)$ and $A_\psi - A_{\tilde\psi}$ is smoothing.

Let us now prove that A_ψ lies in the space $\Psi^\mu_{\rho,\delta}(U)$ of pseudo-differential operators of order μ and type (ρ,δ) over U. We have to check that A_ψ is pseudolocal and with respect to some local coordinates looks like a pseudo-differential operator on an open set of $\mathbb{R}^d$.

(i) A_ψ is pseudolocal: Let $u, v \in \mathcal{D}(U)$ and $\mathrm{supp}(u) \cap \mathrm{supp}(v) = \emptyset$. Then the integral kernel K of $\mathcal{C}^\infty(U) \ni f \mapsto v\, A_\psi(uf) \in \mathcal{C}^\infty(U)$ has the form

$$K(v) \;=\; \frac{1}{(2\pi)^n} \int_{T^*_{\pi(v)}X} \psi(v)\, a(\xi)\, v(\pi(v))\, u(\exp v)\, e^{-i<\xi,v>}\, d\xi. \tag{20}$$

There exists an open neighborhood $W' \subset W$ of the zero section in TX such that the amplitude $\psi\,(v\circ\pi)\,(u\circ\exp)\,a$ vanishes on $W' \times_U T^*X$. As for all nonzero $v \in TX$ the phase function $\xi \mapsto -<\xi, v>$ is noncritical, an argument similar to the one above for $K_{A_\psi - A_{\tilde\psi}}$ shows that K is a smooth function on TU, so $\mathcal{C}^\infty(U) \ni f \mapsto v\, A_\psi(uf) \in \mathcal{C}^\infty(U)$ is smoothing.

(ii) A_ψ is a pseudo-differential operator in appropriate coordinates: Choose for $x \in U$ an open neighborhood $U_x \subset U$ so small that any two points in U_x can be connected by a unique geodesic. In particular we assume that $U_x \times U_x$ lies in the range of $(\pi, \exp) : W \to X \times X$. Furthermore let us suppose that $\psi \circ z_y(\tilde y) = 1$ for all $y, \tilde y \in U_x$. Then we have for $f \in \mathcal{D}(U_x)$:

$$\begin{aligned}
A_\psi f\,(y) \;&=\; \\
&=\; \frac{1}{(2\pi)^{n/2}} \int_{T^*_y X} a(\xi)\, \hat{f^\psi}(\xi)\, d\xi \\
&=\; \frac{1}{(2\pi)^{n/2}} \int_{T^*_x X} a\Big(\underbrace{T^*_{z_x(y)} \exp_x^{-1}}_{=\, d_x(y)}(\xi)\Big)\, \Big|\det d_x(y)\Big|\, \mathcal{F}(f\circ\exp)(d_x(y)\,(\xi))\, d\xi \\
&=\; \frac{1}{(2\pi)^{n}} \int_{T^*_x X} \int_{T_y X} a\Big(d_x(y)\,(\xi)\Big)\, \Big|\det d_x(y)\Big|\, (f\circ\exp_y)(v)\, e^{-i<d_x(y)\,(\xi),v>}\, dv\, d\xi \\
&=\; \frac{1}{(2\pi)^{n}} \int_{T^*_x X} \int_{T_x X} a\Big(d_x(y)\,(\xi)\Big)\, \Big|\det d_x(y)\Big|\, \Big|\underbrace{\det\,(T\,(z_y\circ z_x^{-1})}_{=\, Z_{y,x}}(v)\Big| \\
&\qquad (f\circ z_x^{-1})(v)\, e^{-i<d_x(y)\,(\xi), z_y\circ z_x^{-1}(v)>}\, dv\, d\xi
\end{aligned} \tag{21}$$

Now the phase function $(v,\xi) \mapsto -<d_x(y)(\xi), z_y\circ z_x^{-1}(v)>$ vanishes for $\xi \neq 0$ if and only if $v = z_x(y)$. Hence after possibly shrinking U_x the Kuranishi trick gives a smooth function

$G_x : U_x \times T_xX \to \mathrm{Iso}\left(T_x^*X, T_y^*X\right) \subset \mathrm{Lin}\left(T_x^*X, T_y^*X\right)$ such that $< d_x(y)(\xi), z_y \circ z_x^{-1}(v) > = < G_x(y,v)(\xi), v - z_x(y) >$ for all $(y,v) \in U_x \times T_xX$ and $\xi \in T_x^*X$. A change of variables in Eq. (21) then implies:

$$\begin{aligned} A_\psi f(y) \;&= \\ &= \frac{1}{(2\pi)^n} \int_{T_x^*X} \int_{T_xX} a\Big(\Big[d_x(y)\, G_x^{-1}(y,v)\Big](\xi)\Big) \Big|\det d_x(y)\Big| \Big|Z_{y,x}(v)\Big| \Big|\det G_x^{-1}(y,v)\Big| \\ &\quad (f \circ z_x^{-1})(v)\, e^{-i<\xi, v - z_x(y)>}\, dv\, d\xi \\ &= \frac{1}{(2\pi)^n} \int_{T_x^*X} \int_{T_xX} \tilde{a}_x(y,v,\xi)\, (f \circ z_x^{-1})(v)\, e^{-i<\xi, v - z_x(y)>}\, dv\, d\xi, \end{aligned} \tag{22}$$

where $(y,v,\xi) \mapsto \tilde{a}_x(y,v,\xi) = a\Big(\Big[d_x(y)\, G_x^{-1}(y,v)\Big](\xi)\Big) \Big|\det d_x(y)\Big| \Big|Z_{y,x}(v)\Big| \Big|\det G_x^{-1}(y,v)\Big|$ is a symbol lying in $\mathrm{S}^\mu_{\rho,\delta}(U_x \times U_x \times T_xX)$, as $\rho + \delta \geq 1$ and $\rho > \delta$. Hence A is a pseudo-differential operator with respect to normal coordinates over U_x of order μ and type (ρ, δ).

Our considerations now lead us to

Theorem 2.1 *Let X be Riemannian and* exp *the exponential function corresponding to the Levi-Civita connection on X. Then any smooth cut-off function $\psi : TX \to [0,1]$ gives rise to a linear sheaf morphism $\mathrm{Op}_\psi : \mathrm{S}^\infty_{\rho,\delta}(\cdot, X) \to \Psi^\infty_{\rho,\delta}(\cdot, X)$, $a \mapsto A_\psi$ defined by*

$$A_\psi f\,(x) \;=\; \frac{1}{(2\pi)^{n/2}} \int_{T_x^*X} a(\xi)\, \hat{f^\psi}(\xi)\, d\xi, \tag{23}$$

where $a \in \mathrm{S}^\infty_{\rho,\delta}(U, X)$, $f \in \mathcal{D}(U)$, $x \in U$ and $U \subset X$ open. This morphism preserves the natural filtrations of $\mathrm{S}^\infty_{\rho,\delta}(\cdot, X)$ and $\Psi^\infty_{\rho,\delta}(\cdot, X)$. In particular it maps the subsheaf $\mathrm{S}^{-\infty}(\cdot, X)$ of smoothing symbols to the subsheaf $\Psi^{-\infty}(\cdot, X) \subset \Psi^\infty_{\rho,\delta}(\cdot, X)$ of smoothing pseudo-differential operators.

The quotient morphism $\overline{\mathrm{Op}} : (\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty})(\cdot, X) \to (\Psi^\infty_{\rho,\delta}/\Psi^{-\infty})(\cdot, X)$ is an isomorphism and independent of the choice of the cut-off function ψ.

PROOF: The first part of the theorem has been shown above. So it remains to prove that $\overline{\mathrm{Op}} : (\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty})(U, X) \to (\Psi^\infty_{\rho,\delta}/\Psi^{-\infty})(U, X)$ is bijective for all open $U \subset X$. Let us postpone this till Theorem 3.3, where we will show that an explicit inverse of $\overline{\mathrm{Op}}$ is given by the complete symbol map introduced in the following section. □

By the above Theorem the effect of ψ for the operator Op_ψ is only a minor one, so from now on we will write Op instead of Op_ψ.

3 The symbol map

In the sequel denote by $\varphi : X \times T^*X|_O \to \mathbb{C}$ the smooth phase function defined by

$$X \times T^*X|_O \ni (x,\xi) \mapsto < \xi, \exp^{-1}_{\pi(\xi)}(x) > = < \xi, z_{\pi(\xi)}(x) > \in \mathbb{C}, \tag{24}$$

where O is the range of $(\pi, \exp) : W \to X \times X$.

Theorem and Definition 3.1 *Let $A \in \Psi^\mu_{\rho,\delta}(U,X)$ be a pseudo-differential operator on a Riemannian manifold X and $\psi : T^*X \to [0,1]$ a cut-off function. Then the* **ψ-cutted symbol** *of A with respect to the Levi-Civita connection on X is the function*

$$\sigma_\psi(A) = \sigma_{\psi,A} : T^*U \to \mathbb{C}, \quad \xi \mapsto \left[A\left(\psi_{\pi(\xi)}(\cdot)\, e^{i\varphi(\cdot,\xi)}\right)\right](\pi(\xi)), \tag{25}$$

where for every $x \in X$ $\psi_x = \psi \circ \exp_x^{-1} = \psi \circ z_x$. $\sigma_\psi(A)$ is an element of $\mathrm{S}^\mu_{\rho,\delta}(U,X)$. The corresponding element $\overline{\sigma}(A)$ in the quotient $(\mathrm{S}^\mu_{\rho,\delta}/\mathrm{S}^{-\infty})(U,X)$ is called the **normal symbol** *of A. It is independant of the choice of ψ.*

Remark 3.2 Before starting with the proof of the theorem let us mention that in the above theorem and definition we depart from the consideration of WIDOM in [9] insofar, as we have chosen a different (notion of a) phase function. Therefore our symbols are not the same as the ones in WIDOM'S article.

PROOF: Let us first check that $\sigma_{\psi,A}$ lies in $\mathrm{S}^\mu_{\rho,\delta}(U)$. We can find a sequence $(x_\iota)_{\iota\in\mathbb{N}}$ of points in U and coordinate patches $U_\iota \subset U$ with $x_\iota \in U_\iota$ such that there exist normal coordinates $z_\iota = z_{x_\iota} : U_\iota \to T_{x_\iota}X$ on the U_ι. We can even assume that the operator $A_\iota : \mathcal{D}(U_\iota) \to \mathcal{C}^\infty(U_\iota)$, $u \mapsto Au|_{U_\iota}$ induces a pseudo-differential operator on $T_{x_\iota}X$. In other words there exist symbols $a_\iota \in \mathrm{S}^\mu_{\rho,\delta}(O_\iota \times O_\iota, T^*_{x_\iota}X)$ with $O_\iota = z_\iota(U_\iota)$ such that A_ι is given by the following oscillatory integral:

$$A_\iota u(y) \;=\; \frac{1}{(2\pi)^n} \int\limits_{T^*_{x_\iota}X} \int\limits_{T_{x_\iota}X} a_\iota(z_\iota(y), v, \xi)\, (u \circ z_\iota^{-1})(v)\, e^{-i<\xi, v - z_\iota(y)>}\, dv d\xi. \tag{26}$$

Now let (ϕ_ι) be a partition of unity subordinate to the covering (U_ι) of U and for every index ι $\phi_{\iota,1}, \phi_{\iota,2} \in \mathcal{C}^\infty(U)$ such that $\mathrm{supp}(\phi_{\iota,1}) \subset U_\iota$, $\mathrm{supp}(\phi_\iota) \cap \mathrm{supp}(\phi_{\iota,2}) = \emptyset$ and $\phi_{\iota,1} + \phi_{\iota,2} = 1$. Then the symbol $\sigma_{\psi,A}$ can be written in the following form:

$$\begin{aligned}
\sigma_{\psi,A}(\zeta) &= \left[A\Big(\psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\Big)\right](\pi(\zeta)) \\
&= \frac{1}{(2\pi)^n} \sum_\iota \left[\phi_{\iota,1}\, A_\iota\Big(\phi_\iota(\cdot)\, \psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\Big)\right](\pi(\zeta)) \\
&\quad + K\Big(\psi_{\pi(\zeta)}(\cdot)\, e^{i\varphi(\cdot,\zeta)}\Big)(\pi(\zeta))
\end{aligned} \tag{27}$$

where

$$K : \mathcal{D}(U) \to \mathcal{C}^\infty(U) \quad f \mapsto \sum_\iota \phi_{\iota,2}\, A_\iota(\phi_\iota\, f), \quad \zeta \in T^*X_\iota \tag{28}$$

is a smoothing pseudo-differential operator. By the Kuranishi trick we can assume that there exists a smooth function $G : O_\iota \times O_\iota \to \mathrm{Lin}\left(T^*_{x_\iota}X, T^*_y X\right)$ such that $G(v,w)$ is invertible for every $v, w \in O_\iota$ and

$$< G\left(z_\iota(y), T(z_\iota \circ z_y^{-1})(v)\right)(\xi), v > = < \xi, z_\iota \circ z_y^{-1}(v) - z_\iota(y) > \tag{29}$$

for $y \in U_\iota$, $v \in T_yX$ appropriate and $\xi \in T^*_{x_\iota}X$. Several changes of variables then give the following chain of oscillatory integrals for $\zeta \in T^*U_\iota$:

$$\begin{aligned}
&\left[A_\iota\left(\phi_\iota(\cdot)\,\psi_{\pi(\zeta)}(\cdot)\,e^{i\varphi(\cdot,\zeta)}\right)\right](y) \\
&= \frac{1}{(2\pi)^n} \int\limits_{T^*_{x_\iota}X} \int\limits_{T_{x_\iota}X} \phi_{\iota,1}(y)\, a_\iota\left(z_\iota^{-1}(y), v, \xi\right)\, \phi_\iota(z_\iota^{-1}(v))\, \psi(z_y \circ z_x^{-1}(v)) \\
&\quad e^{i<\zeta, z_y \circ z_\iota^{-1}(v)>}\, e^{-i<\xi, v - z_\iota(y)>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n} \int\limits_{T^*_{x_\iota}X} \int\limits_{T_yX} \phi_{\iota,1}(y)\, a_\iota\left(z_\iota^{-1}(y), z_\iota \circ z_y^{-1}(v), \xi\right)\, \phi_\iota(z_y^{-1}(v))\, \psi(v) \\
&\quad e^{i<\zeta, v>}\, e^{-i<\xi, z_\iota \circ z_y^{-1}(v) - z_\iota(y)>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n} \int\limits_{T^*_{x_\iota}X} \int\limits_{T_yX} \phi_{\iota,1}(y)\, a_\iota\left(z_\iota^{-1}(y), z_\iota \circ z_y^{-1}(v), \xi\right)\, \phi_\iota(z_y^{-1}(v))\, \psi(v) \\
&\quad e^{i<\zeta, v>}\, e^{-i<G(z_\iota(y), T(z_\iota \circ z_y^{-1})(v))(\xi), v>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n} \int\limits_{T^*_yX} \int\limits_{T_yX} \phi_{\iota,1}(y)\, a_\iota\left(z_\iota^{-1}(y), z_\iota \circ z_y^{-1}(v), G^{-1}\left(z_\iota(y), T(z_\iota \circ z_y^{-1})(v)\right)(\xi)\right) \\
&\quad \phi_\iota(z_y^{-1}(v))\, \psi(v) \left|\det G^{-1}\left(z_\iota(y), T(z_\iota \circ z_y^{-1})(v)\right)\right| e^{i<\zeta, v>}\, e^{-i<\xi, v>}\, dv\, d\xi \\
&= \frac{1}{(2\pi)^n} \int\limits_{T^*_{x_\iota}X} \int\limits_{T_{x_\iota}X} b_\iota(z_\iota(y), v, \xi)\, e^{-i<\xi - T^*(z_y \circ z_\iota^{-1})(\zeta), v - z_\iota(y)>}\, dv\, d\xi,
\end{aligned} \tag{30}$$

where the smooth function b_ι on $O_\iota \times O_\iota \times T^*_{x_\iota}X$ is defined by

$$\begin{aligned}
&b_\iota(z_\iota(y), v, \xi) = \\
&= \phi_{\iota,1}(y)\, a_\iota\left(z_\iota^{-1}(y), z_\iota \circ z_y^{-1} \circ T(z_y \circ z_\iota^{-1})(v), G^{-1}\left(z_\iota(y), v\right) \circ \left(T^*(z_\iota \circ z_y^{-1})(\xi)\right)\right) \\
&\quad \phi_\iota(z_y^{-1}(T(z_y \circ z_\iota^{-1})(v)))\, \psi(T(z_y \circ z_\iota^{-1})(v)) \left|\det G^{-1}(z_\iota(y), v)\right|
\end{aligned} \tag{31}$$

and lies in $\mathrm{S}^\mu_{\rho,\delta}(O_\iota \times O_\iota, T^*_{x_\iota}X)$. Let $B_\iota \in \Psi^\mu_{\rho,\delta}(O_\iota, T_{x_\iota}X)$ be the pseudo-differential operator on O_ι corresponding to the symbol b_ι. The symbol σ_ι defined by $\sigma_\iota(v,\xi) =$

$e^{-i<\xi,v>}\, B_\iota\left(e^{i<\xi,\cdot>}\right)$, where $v \in O_\iota$ and $\xi \in T^*_{x_\iota}X$, now is an element of $\mathrm{S}^\mu_{\rho,\delta}(O_\iota, T^*_{x_\iota}X)$ as one knows by the general theory of pseudo-differential operators on real vector spaces (see [5, 7, 4]). But we have

$$\left[\phi_{\iota,1}\, A_\iota\Big(\phi_\iota(\cdot)\,\psi_{\pi(\zeta)}(\cdot)\,e^{i\varphi(\cdot,\zeta)}\Big)\right](\pi(\zeta)) \;=\; \sigma_\iota\left(z_\iota(\pi(\zeta)), T^*\left(z_{\pi(\zeta)}\circ z_\iota^{-1}\right)(\zeta)\right), \tag{32}$$

hence Eq. (27) shows $\sigma_{\psi,A} \in \mathrm{S}^\mu_{\rho,\delta}(U,X)$ if we can prove that $\sigma_{\psi,K} \in \mathrm{S}^{-\infty}(U,X)$ for $\sigma_{\psi,K}(\zeta) = K\Big(\psi_{\pi(\zeta)}(\cdot)\,e^{i\varphi(\cdot,\zeta)}\Big)(\pi(\zeta))$. So let $k : U\times U \to \mathbb{C}$ be the kernel function of K and write $\sigma_{\psi,K}$ as an integral:

$$\sigma_{\psi,K}(\zeta) = \int\limits_U k(\pi(\zeta),y)\,\psi_{\pi(\zeta)}(y)\,e^{-i\varphi(y,\zeta)}\,dy. \tag{33}$$

Let further $\Xi_1,...,\Xi_k$ be differential forms over U and $\mathcal{V}_1,...,\mathcal{V}_k$ the vertical vector fields on T^*U corresponding to $\Xi_1,...,\Xi_k$. Differentiating Eq. (33) under the integral sign and using an adjointness relation then gives

$$\begin{aligned}
&|\zeta|^{2m}\,\mathcal{V}_1\cdot...\cdot\mathcal{V}_k\,\sigma_{\psi,K}\,(\zeta) \;=\\
&\quad= \int\limits_U k(\pi(\zeta),y)\,\psi_{\pi(\zeta)}(y) \;<\Xi_1, z_{\pi(\zeta)}(y)> \;...\; <\Xi_k, z_{\pi(\zeta)}(y)>\\
&\qquad i^{2mn+k}\frac{\partial^{2mn}}{\partial z^{(2m,...,2m)}_{\pi(\zeta)}}e^{-i<\zeta,z_{\pi(\zeta)}(\cdot)>}(y)\,dy\\
&\quad= \int\limits_U \left(\frac{\partial^{2mn}}{\partial z^{(2m,...,2m)}_{\pi(\zeta)}}\right)^\dagger \Big(k(\pi(\zeta),\cdot)\,\psi_{\pi(\zeta)}(\cdot) \;<\Xi_1, z_{\pi(\zeta)}(\cdot)> \;...\; <\Xi_k, z_{\pi(\zeta)}(\cdot)>\Big)(y)\\
&\qquad i^{2mn+k}e^{-i<\zeta,z_{\pi(\zeta)}(y)>}\,dy.
\end{aligned} \tag{34}$$

Therefore $|\zeta|^{2m}\,\mathcal{V}_1\cdot...\cdot\mathcal{V}_k\,\sigma_{\psi,K}\,(\zeta)$ is uniformly bounded as long as $\pi(\zeta)$ varies in a compact subset of U. A similar argument for horicontal derivatives of $\sigma_{\psi,K}$ finally proves $\sigma_{\psi,K} \in \mathrm{S}^{-\infty}(U,X)$.

Next we have to show that $\sigma_{\psi,A} - \sigma_{\tilde\psi,A}$ is a smooothing symbol for any second cut-off function $\tilde\psi : T^*X \to [0,1]$. It suffices to prove that $\tilde\sigma_\iota$ with

$$\tilde\sigma_\iota(\zeta) = A_\iota\,\left(\phi_\iota\,(\psi-\tilde\psi)\circ z_{\pi(\zeta)}\,e^{-i\varphi(\cdot,\zeta)}\right) \tag{35}$$

is smoothing for every $\phi_\iota \in \mathcal{D}(U_\iota)$. We can find a (-1)-homogeneous first order differential operator L on the vector bundle $T^*_{x_\iota}X \times T_{x_\iota}X \to T_{x_\iota}X$ such that for all $v \neq z_\iota(\pi(\zeta))$ and $\xi \in T^*_{x_\iota}X$ the equation

$$Le^{-i<\cdot,v-z_\iota(\pi(\zeta))>}\,(\xi) \;=\; e^{-i<\xi,v-z_\iota(\pi(\zeta))>} \tag{36}$$

holds (see for example GRIGIS, SJÖSTRAND [4] Lemma 1.12 for a proof). Because $(\psi - \tilde{\psi}) \circ z_{\pi(\zeta)} \circ z_\iota^{-1}$ vanishes on a neighborhood of $z_\iota(\pi(\zeta))$, the following equational chain of iterated integrals is also true:

$$\begin{aligned}
\tilde{\sigma}_\iota(\zeta) \;=& \\
=\;& \frac{1}{(2\pi)^n} \int\limits_{T^*_{x_\iota}X} \int\limits_{T_{x_\iota}X} a_\iota\Big(z_\iota^{-1}(y), v, \xi\Big)\; \phi_\iota(z_\iota^{-1}(v))\; \Big(\psi - \tilde{\psi}\Big)\,(z_{\pi(\zeta)} \circ z_x^{-1}(v)) \\
& e^{i<\zeta, z_{\pi(\zeta)} \circ z_\iota^{-1}(v)>}\; e^{-i<\xi, v - z_\iota(\pi(\zeta))>}\, dv\, d\xi \\
=\;& \frac{1}{(2\pi)^n} \int\limits_{T^*_{x_\iota}X} \int\limits_{T_{x_\iota}X} a_\iota\Big(z_\iota^{-1}(y), v, \xi\Big)\; \phi_\iota(z_\iota^{-1}(v))\; \Big(\psi - \tilde{\psi}\Big)\,(z_{\pi(\zeta)} \circ z_x^{-1}(v)) \\
& e^{i<\zeta, z_{\pi(\zeta)} \circ z_\iota^{-1}(v)>}\; L^k\, e^{-i<\xi, v - z_\iota(\pi(\zeta))>}\, dv\, d\xi \\
=\;& \frac{1}{(2\pi)^n} \int\limits_{T^*_{x_\iota}X} \int\limits_{T_{x_\iota}X} \Big(\big(L^\dagger\big)^k\, a_\iota\Big)\Big(z_\iota^{-1}(y), v, \xi\Big)\; \phi_\iota(z_\iota^{-1}(v))\, \Big(\psi - \tilde{\psi}\Big)\,(z_{\pi(\zeta)} \circ z_x^{-1}(v)) \\
& e^{i<\zeta, z_{\pi(\zeta)} \circ z_\iota^{-1}(v)>}\; e^{-i<\xi, v - z_\iota(\pi(\zeta))>}\, dv\, d\xi. \qquad (37)
\end{aligned}$$

Note that the last integral of this chain is to be understood in the sense of Lebesgue if $k \in \mathbb{N}$ is large enough. The same argument which was used for proving that the symbol $\sigma_{\psi,K}$ from Eq. (33) is smoothing now shows $\sigma_\iota \in \mathrm{S}^{-\infty}(U_\iota, X)$. □

After having defined the notion of a complete symbol, we will now give its essential properties in the following theorem.

Theorem 3.3 *Let $A \in \Psi^\mu_{\rho,\delta}(U)$ be a pseudo-differential operator on a Riemannian manifold X and $a = \sigma_{\psi,A} \in \mathrm{S}^\mu_{\rho,\delta}(U)$ be its ψ-cutted symbol with respect to the Levi-Civita connection. Then A and the pseudo-differential operator $\mathrm{Op}_\psi(a)$ defined by Eq. (23) are equal modulo smoothing operators. Moreover the sheaf morphisms $\overline{\sigma} : (\Psi^\infty_{\rho,\delta}/\Psi^{-\infty})(\cdot, X) \to (\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty})(\cdot, X)$ and $\overline{\mathrm{Op}} : (\mathrm{S}^\infty_{\rho,\delta}/\mathrm{S}^{-\infty})(\cdot, X) \to (\Psi^\infty_{\rho,\delta}/\Psi^{-\infty})(\cdot, X)$ are inverse to each other.*

Before proving the theorem let us first state a lemma.

Lemma 3.4 *The operator*

$$K : \mathcal{D}(U) \to \mathcal{C}^\infty(U), \quad f \mapsto \Big(U \ni x \mapsto A[(1 - \psi_x)f]\,(x) \in \mathbb{C}\Big) \qquad (38)$$

is smoothing for any cut-off function $\psi : TX \to \mathbb{C}$.

PROOF: Consider the operators $K_{a,\iota} : \mathcal{D}(U) \to \mathcal{C}^\infty(U)$ where $K_{a,\iota} f\,(x) = \phi_\iota(x) A[(1 - \psi_a)f](x)$ for $x \in U$, $a \in U_\iota$. The open covering (U_ι) of U and a subordinate partition of

unity (ϕ_ι) are chosen such that for every point $a \in U_\iota$ there exist normal coordinates z_a on U_ι. We can even assume that $\psi(z_a(y)) = 1$ for all $a, y \in U_\iota$. Hence $\mathrm{supp}(1-\psi_a) \cap \mathrm{supp}(\phi_\iota) = \emptyset$, and $K_{a,\iota}$ is smoothing, because A is pseudolocal. This yields a family of smooth functions $k_\iota : U \times U \to \mathbb{C}$ such that

$$K_{a,\iota} f\,(x) \;=\; \int\limits_U k_\iota(x,y)\,(1-\psi_a(y))\,f(y)\,dy. \tag{39}$$

Now the smooth function $k \in \mathcal{C}^\infty(U \times U)$ with $k(x,y) \;=\; \sum_\iota k_\iota(x,y)\,(1-\psi_x(y))$ is the kernel function of K. □

Proof (of Theorem 3.3): Check that for an arbitrary cut-off function $\tilde{\psi} : TX \to \mathbb{C}$ one can find a cut-off function $\psi : TX \to \mathbb{C}$ such that the equation

$$\psi_x(y)\,f(y) \;=\; \frac{1}{(2\pi)^{n/2}} \int\limits_{T_x^*X} \psi_x(y)\,e^{i<\xi, z_x(y)>}\,\hat{f^{\tilde{\psi}}}(\xi)\,d\xi \tag{40}$$

holds for $x, y \in U$. Approximating the integral in Eq. (40) by Riemann sums, we can find a sequence $(i_k)_{k\in\mathbb{N}}$ of natural numbers, a family $(\xi_{k,\iota})_{k\in\mathbb{N}, 1\le\iota\le\iota_k}$ of elements $\xi_{k,\iota} \in T_x^*X$ and a sequence $(\epsilon_k)_{k\in\mathbb{N}}$ of positive numbers such that the following equations are satisfied:

$$\lim_{k\to\infty} \epsilon_k \;=\; 0 \tag{41}$$

$$[(\psi_x)f]\,(y) \;=\; \lim_{k\to\infty} \tfrac{\epsilon_k^n}{(2\pi)^{n/2}} \sum_{1\le i\le i_k} \psi_x(y)\,e^{i<\xi_{k,\iota},\,z_x(y)>}\,\hat{f^{\tilde{\psi}}}(\xi_{k,\iota}). \tag{42}$$

By an appropriate choice of the $\xi_{k,\iota}$ one can even assume that the series on the right hand side of the last equation converges as a function of $y \in U$ in the topology of $\mathcal{D}(U)$. As A is continuous on $\mathcal{D}(U)$, we now have:

$$\begin{aligned}
A[(\psi_x)f](x) \;&=\; \\
&=\; \lim_{k\to\infty} \frac{\epsilon_k^n}{(2\pi)^{n/2}} \sum_{1\le i\le i_k} A\left[\psi_x(\cdot)\,e^{i<\xi_{k,\iota},\,z_x(\cdot)>}\right](x)\,\hat{f^{\tilde{\psi}}}(\xi_{k,\iota}) \\
&=\; \lim_{k\to\infty} \frac{\epsilon_k^n}{(2\pi)^{n/2}} \sum_{1\le i\le i_k} \sigma_A(\xi_{k,\iota})\,\hat{f^{\tilde{\psi}}}(\xi_{k,\iota}) \\
&=\; \frac{1}{(2\pi)^{n/2}} \int\limits_{T_x^*X} \sigma_A(\xi)\,\hat{f^{\tilde{\psi}}}(\xi)\,d\xi \\
&=\; A_{\tilde{\psi}} f.
\end{aligned} \tag{43}$$

On the other hand Lemma 3.4 provides a smoothing operator K such that for $f \in \mathcal{D}(U)$ and $x \in U$:

$$Af\,(x) \;=\; A[(\psi_x)f]\,(x) + Kf\,(x) \;=\; A_{\tilde{\psi}} f\,(x) + Kf\,(x). \tag{44}$$

Thus the first part of the theorem follows.

Now let $a \in \mathrm{S}^{\mu}_{\rho,\delta}(U, T^*X)$ be a symbol, and regard the corresponding pseudo-differential operator $A = A_\psi \in \Psi^{\mu}_{\rho,\delta}(U, X)$. We can assume after possibly altering ψ that there exists a second cut-off function $\tilde{\psi} : T^*X \to \mathbb{C}$ such that $\tilde{\psi}|_{\mathrm{supp}(\psi)} = 1$. Then we have for $\zeta \in T^*_x X$:

$$\begin{aligned}
\sigma_{A,\psi}(\zeta) \;&= \\
&= \frac{1}{(2\pi)^{n/2}} \int\limits_{T^*_x X} a(\xi)\, \mathcal{F}\left(\left(\psi_x(\cdot) e^{i\varphi(\cdot,\zeta)}\right)^{\tilde{\psi}}\right)(\xi)\, d\xi \\
&= \frac{1}{(2\pi)^{n/2}} \int\limits_{T^*_x X} a(\xi)\, \mathcal{F}\left(\psi_x(\cdot) e^{i\varphi(\cdot,\zeta)}\right)(\xi)\, d\xi \\
&= \frac{1}{(2\pi)^{n}} \int\limits_{T^*_x X} \int\limits_{T_x X} a(\xi)\, \psi(v)\, e^{-i<\xi,v>}\, e^{i<\zeta,v>}\, dv\, d\xi, \qquad (45)
\end{aligned}$$

where the last integral is an iterated one. Next choose a smooth function $\phi : \mathbb{R} \to [0,1]$ such that $\mathrm{supp}(\phi) \subset [-2,2]$ and $\phi|_{[-1,1]} = 1$ and define $a_k \in \mathrm{S}^{-\infty}(U)$ by $a_k(\xi) = \phi\left(\frac{|\xi|}{k}\right)$. Then obviously $a_k \to a$ in $\mathrm{S}^{\tilde{\mu}}_{\rho,\delta}(U)$ for $\tilde{\mu} > \mu$. We can now write:

$$\begin{aligned}
\sigma_{A,\psi}(\zeta) \;&= \\
&= \lim_{k\to\infty} \frac{1}{(2\pi)^{n}} \int\limits_{T^*_x X} \int\limits_{T_x X} a_k(\xi)\, \psi(v)\, e^{-i<\xi-\zeta,v>}\, dv\, d\xi \\
&= \lim_{k\to\infty} a_k(\zeta) - \frac{1}{(2\pi)^{n/2}} \int\limits_{T_x X} (1-\psi(v)) \left[\mathcal{F}^{-1} a_k\right](v)\, e^{i<\zeta,v>}\, dv \\
&= a(\zeta) - \lim_{k\to\infty} \frac{1}{(2\pi)^{n/2}} \int\limits_{T_x X} (1-\psi(v)) \left[\mathcal{F}^{-1}\left(\left(L^{\dagger}\right)^{m} a_k\right)\right](v)\, e^{i<\zeta,v>}\, dv \\
&= a(\zeta) + \lim_{k\to\infty} K_m(\zeta, a_k), \qquad (46)
\end{aligned}$$

where $m \in \mathbb{N}$, L is a smooth (-1)-homogeneous first order differential operator on the vector bundle $T^*X \times_U \dot{T}U \to \dot{T}U$ such that $L e^{i<\cdot,v>}(\xi) = e^{i<\xi,v>}$ and

$$K_m(\zeta, a_k) \;=\; -\frac{1}{(2\pi)^{n/2}} \int\limits_{T_x X} (1-\psi(v)) \left[\mathcal{F}^{-1}\left(\left(L^{\dagger}\right)^{m} a_k\right)\right](-v)\, e^{i<\zeta,v>}\, dv. \qquad (47)$$

Note that $\dot{T}U$ consists of all nonzero vectors $v \in TU$ and that $\left(L^{\dagger}\right)^{m} a \in \mathrm{S}^{\tilde{\mu}-\rho m}_{\rho,\delta}$ for every $\tilde{\mu} \geq \mu$. Now fixing $\tilde{\mu} \geq \mu$ and choosing m large enough that $\rho m > \tilde{\mu} + 2(\dim X - 1)$ yields

$$K_m(\zeta, a) \;=:\; \lim_{k\to\infty} K_m(\zeta, a_k) \;=$$

$$= \lim_{k\to\infty} -\frac{1}{(2\pi)^n} \int\limits_{T_xX} \int\limits_{T_x^*X} \left(\left(L^\dagger\right)^m a_k\right)(\xi)\,(1-\psi(v))\,e^{-i<\xi-\zeta,v>}\,d\xi\,dv$$

$$= -\frac{1}{(2\pi)^n} \int\limits_{T_xX} \int\limits_{T_x^*X} \left(\left(L^\dagger\right)^m a\right)(\xi)\,(1-\psi(v))\,e^{i<\zeta,v>}\,e^{-i<\xi,v>}\,d\xi\,dv \qquad (48)$$

in the sense of Lebesgue integrals. As the phase function $\xi \mapsto -<\xi, v>$ is noncritical for every nonzero v and $1-\psi(v)$ vanishes in an open neighborhood of the zero section of TU, a standard consideration already carried out in preceeding proofs entails that $K_m(\cdot, a)$ is a smooothing symbol. Henceforth $\sigma_{A,\tilde{\psi}}$ and a differ by the smoothing symbol $K_m(\cdot, a)$. This gives the second part of the theorem. □

Example 3.5 Consider the symbols $l : T^*X \to \mathbb{C}$, $\xi \mapsto |\xi|^2$ and $a : T^*X \to \mathbb{C}$, $\xi \mapsto \frac{1}{1+|\xi|^2}$ of Example 1.3 (ii). Then the pseudo-differential operator corresponding to l is minus the Laplacian: $\mathrm{Op}(l) = -\Delta$. As one can now calculate directly, $\mathrm{Op}(a)$ is an inverse in $\Psi^\infty_{1,0}(X)$ of the differential operator $1-\Delta$ having symbol $T^*X \ni \xi \mapsto 1+|\xi|^2 \in \mathbb{C}$.

So with the help of a symbol calculus one can construct inverses to certain differential operators in the algebra of pseudo-differential operators. This is the main reason why a symbol calculus is so important for the theory of partial differential equations. The normal symbol on Riemannian manifolds introduced in this section furthermore avoids cumbersome considerations in local coordinates but gives a natural geometric tool for handling pseudo-differential operators on manifolds.

4 The symbol of the adjoint

In the following we want to derive an expression for the normal symbol $\sigma_A^* = \sigma_{A^*}$ of the adjoint of a pseudodifferential operator A in terms of the symbol σ_A.

For that let us first define the function $\rho : O \to \mathbb{R}^+$ by

$$\rho(x,y) \cdot \nu_x = \nu(y) \circ (T_v \exp_x \times ... \times T_v \exp_x), \quad x, y \in O, \quad v = \exp_x^{-1}(y). \tag{49}$$

Here ν is the canonical volume density on T_xX and ν_x its restriction to a volume density on T_xX. Now we claim that the operator $\mathrm{Op}^*(a) : \mathcal{D}(U) \to \mathcal{C}^\infty(U)$ defined by the iterated integral

$$\mathrm{Op}^*(a)f(x) \;=\; \frac{1}{(2\pi)^n} \int_{T_x^*X} \int_{T_xX} e^{-i<\xi,v>}\, \overline{a}\left(T_v^* \exp_x^{-1}(\xi)\right)\, f^{\tilde{\psi}}(v)\, \rho^{-1}(\exp_x(v), x)\, dv\, d\xi \tag{50}$$

is the (formal) adjoint of $\mathrm{Op}(a)$, where $a \in \mathrm{S}^\mu_{\rho,\delta}(U)$, $U \subset X$ open and $\tilde{\psi}$ is the cut-off function $TX \ni v \mapsto \psi\left(\exp^{-1}_{\exp_x v}(x)\right) \in [0,1]$. Assuming $f \in \mathcal{D}(V)$, $g \in \mathcal{D}(U)$ with $V \subset U$ open $V \times V \subset O$ and μ to be sufficiently small the following chain of (proper) integrals holds:

$$\int_X f(x)\overline{\mathrm{Op}^*(a)g}\,(x)\, d\nu(x) \;=$$

ω_x, ω – canonical symplectic forms on $T_xX \times T_x^*X$ resp. T^*X

$$= \frac{1}{(2\pi)^n} \int_X \int_{T_xX \times T_x^*X} e^{i<\xi,v>}\, a\left(T_v^* \exp_x^{-1}(\xi)\right)\, \rho^{-1}(\exp_x(v), x)\, f(x)\, \overline{g}^{\tilde{\psi}}(v)\, d\omega_x^n(v,\xi) d\nu(x)$$

---

by $T^* \exp_x : T^*V \to T_xX \times T_x^*X$ symplectic, $\quad x \in V$

$$= \frac{1}{(2\pi)^n} \int_X \int_{T^*V} e^{i<T^* \exp_x(\xi), \exp_x^{-1}(\pi(\xi))>}\, a(\xi)\, \rho^{-1}(\pi(\xi), x)$$
$$\psi(\exp^{-1}_{\pi(\xi)}(x))\, f(x)\, \overline{g}(\pi(\xi))\, d\omega^n(\xi)\, d\nu(x)$$

Gauß Lemma and Fubini's Theorem

$$= \frac{1}{(2\pi)^n} \int_{T^*V} \int_X e^{-i<\xi, \exp^{-1}_{\pi(\xi)}(x)>}\, a(\xi)\, \rho^{-1}(\pi(\xi), x)\, \psi(\exp^{-1}_{\pi(\xi)}(x))\, f(x)\, \overline{g}(\pi(\xi))\, d\nu(x)\, d\omega^n(\xi)$$

coordinate transformation $V \ni x \mapsto w = \exp_{\pi(\xi)}^{-1}(x) \in T_{\pi(\xi)}X$

$$
\begin{aligned}
&= \frac{1}{(2\pi)^n} \int\limits_{T^*V} \int\limits_{T_{\pi(\xi)}X} e^{-i<\xi,w>}\, a(\xi)\, \psi(\exp_{\pi(\xi)}^{-1}(x))\, f(x)\, \overline{g}(\pi(\xi))\, dw\, d\omega^n(\xi) \\
&= \frac{1}{(2\pi)^n} \int\limits_{X} \int\limits_{T_xX \times T_x^*X} e^{-i<\xi,w>}\, a(\xi)\, \psi(w) f(\exp w)\, \overline{g}(x)\, dw\, d\xi\, d\nu(x) \\
&= \int\limits_X \mathrm{Op}(a) f(x)\, \overline{g}(x)\, d\nu(x) \qquad (51)
\end{aligned}
$$

Recall that $\mathrm{S}^{-\infty}(U)$ is dense in any $\mathrm{S}^{\mu}_{\rho,\delta}(U)$, $\mu \in \mathbb{R}$ with respect to the Fréchet topology of $\mathrm{S}^{\tilde{\mu}}_{\rho,\delta}(U)$ if $\mu < \tilde{\mu}$. By using a partition of unity and the continuity of $\mathrm{S}^{\tilde{\mu}}_{\rho,\delta}(U) \ni a \mapsto \mathrm{Op}(a)f \in \mathcal{C}^\infty(U)$ resp. $\mathrm{S}^{\tilde{\mu}}_{\rho,\delta}(U) \ni a \mapsto \mathrm{Op}^*(a)f \in \mathcal{C}^\infty(U)$ for any $f \in \mathcal{D}(U)$ this and Eq. (51) now imply that

$$
\int\limits_X f(x) \overline{\mathrm{Op}^*(a)g}\,(x)\, d\nu(x) \;=\; \int\limits_X \mathrm{Op}(a) f(x)\, \overline{g}(x)\, d\nu(x) \qquad (52)
$$

holds for every $f, g \in \mathcal{D}(U)$ and $a \in \mathrm{S}^{\mu}_{\rho,\delta}(U)$ with $\mu \in \mathbb{R}$. Thus $\mathrm{Op}^*(a)$ is the formal adjoint of $\mathrm{Op}(a)$ indeed.

The normal symbol σ_A^* of $A^* = \mathrm{Op}^*(a)$ now is given by

$$
\begin{aligned}
&\sigma_A^*(\xi) \;\sim\; [A^* f(\cdot,\xi)]\,(\pi(\xi)) \;= \\
&\quad= \frac{1}{(2\pi)^n} \int\limits_{T_xX} \int\limits_{T_x^*X} e^{-i<\zeta,v>}\, \overline{a}\left(T_v^* \exp_x^{-1}(\zeta)\right)\, e^{i<\xi,v>} \rho^{-1}(\exp_x(v), x)\, \tilde{\psi}(v)\, \psi(v)\, dv\, d\zeta,
\end{aligned}
\qquad (53)
$$

where $\xi \in T^*X$, $x = \pi(\xi)$ and $f(y,\xi) = \psi\left(\exp_{\psi(\xi)}^{-1}(y)\right)\, e^{i<\xi, \exp_{\pi(\xi)}^{-1}(y)>}$ for $y \in X$. Thus by Theorem 3.4 of Grigis, Sjöstrand [4] the following asymptotic expansion holds:

$$
\begin{aligned}
\sigma_A^*(\xi) \;&\sim\; \sum_{\alpha \in \mathbb{N}^d} \frac{(-i)^{|\alpha|}}{\alpha!}\, \frac{\partial^{|\alpha|}}{\partial v^\alpha}\bigg|_{v=0} \frac{\partial^{|\alpha|}}{\partial \zeta^\alpha}\bigg|_{\zeta=\xi} \left[\overline{a}\left(T_v^* \exp_x^{-1}(\zeta)\right)\, \rho^{-1}(\exp_x(v), x)\right] \;= \\
&= \sum_{\alpha \in \mathbb{N}^d} \frac{(-i)^{|\alpha|}}{\alpha!}\, \frac{\partial^{|\alpha|}}{\partial z_x{}^\alpha}\bigg|_x \frac{\partial^{|\alpha|}}{\partial \zeta_x{}^\alpha}\bigg|_\xi \left(\overline{a}\, \rho^{-1}(\pi(-), x)\right) \\
&= \sum_{\alpha,\beta \in \mathbb{N}^d} \frac{(-i)^{|\alpha+\beta|}}{\alpha!\, \beta!} \left[\frac{\partial^{|\alpha|}}{\partial z_x{}^\alpha}\bigg|_x \frac{\partial^{|\alpha+\beta|}}{\partial \zeta_x{}^{\alpha+\beta}}\bigg|_\xi \overline{a}\right] \left[\frac{\partial^{|\alpha|}}{\partial z_x{}^\alpha}\bigg|_x \rho^{-1}(\cdot, x)\right]. \qquad (54)
\end{aligned}
$$

Next we will search for an expansion of ρ and its derivatives. For that first write

$$
\frac{\partial}{\partial z_x^k} = \sum_{k,l} \theta_k^l(x,\cdot)\, e_l, \qquad (55)
$$

where $(e_1, ..., e_d)$ is the orthonormal frame of Eq. (12), and the θ_k^l are smooth functions on an open neighborhood of the diagonal of $X \times X$. Then we have the following relation (see for example BERLINE, GETZLER, VERGNE [1]):

$$\theta_l^k(x,y) \;=\; \delta_l^k - \frac{1}{6}\sum_{m,n} R^k{}_{mln}(y)\, z_x^m(y)\, z_x^n(y) + O(|z_x(y)|^3), \tag{56}$$

where $R^k{}_{mln}(y)$ resp. their "lowered" versions $R_{kmln}(y)$ are given by the coefficients of the curvature tensor with respect to normal coordinates at $y \in X$, i.e. $R\left(\frac{\partial}{\partial z_y^m}\right) = \sum_{k,l,n} R^k{}_{mln}(y) \frac{\partial}{\partial z_y^k} \otimes dz_y^l \otimes dz_y^n$. But this implies

$$\begin{aligned}
\rho(x,y) &= \left|\det\left(\theta_k^l(x,y)\right)\right| \\
&= 1 - \frac{1}{6}\sum_k \sum_{m,n} R^k{}_{mkn}(x)\, z_x^m(y)\, z_x^n(y) + O(|z_x(y)|^3 \\
&= 1 - \frac{1}{6}\sum_{m,n} \mathrm{Ric}_{mn}(x)\, z_x^m(y)\, z_x^n(y) + O\left(|z_x(y)|^3\right).
\end{aligned} \tag{57}$$

Using Eq. (14) $z_x^k(y) = -z_y^k(x)$ for x and y close enough we now get

$$\left.\frac{\partial}{\partial z_x^k}\right|_y \rho^{-1}(\cdot, x) \;=\; \frac{1}{3}\sum_l \mathrm{Ric}_{kl}(x)\, z_x^l(y) + O\left(|z_x(y)|^2\right), \tag{58}$$

$$\left.\frac{\partial^2}{\partial z_x^k \partial z_x^l}\right|_y \rho^{-1}(\cdot, x) \;=\; \frac{1}{3}\mathrm{Ric}_{kl}(x) + O\left(|z_x(y)|\right). \tag{59}$$

The above results are summerized by the following theorem.

Theorem 4.1 *The formal adjoint A^* of a pseudodifferential operator $A \in \mathrm{S}^\mu_{\rho,\delta}(U)$ on an open subset U of a Riemannian manifold X is given by*

$$\begin{aligned}
A^* f\,(x) &= \mathrm{Op}^*(a) f\,(x) = \\
&= \frac{1}{(2\pi)^n} \int_{T_x^* X} \int_{T_x X} e^{-i<\xi, v>}\, \overline{a}\left(T_v^* \exp_x^{-1}(\xi)\right)\, f^{\tilde{\psi}}(v)\, \rho^{-1}(\exp_x(v), x)\, dv\, d\xi
\end{aligned} \tag{60}$$

where $x \in X$ and $a = \sigma_A$ is the symbol of A. The symbol $\sigma_A^* = \sigma_{A^*}$ of A^* has asymptotic expansion

$$\sigma_A^*(\xi) \;\sim\; \overline{a}(\xi) \;+\; \sum_{\substack{\alpha \in \mathbb{N}^d \\ |\alpha| \geq 1}} \left.\frac{\partial^{|\alpha|}}{\partial z_{\pi(\xi)}{}^\alpha}\right|_{\pi(\xi)} \left.\frac{\partial^{|\alpha|}}{\partial \zeta_{\pi(\xi)}{}^\alpha}\right|_\xi \overline{a} \;-$$

$$
\begin{aligned}
&- \frac{1}{6} \sum_{\alpha \in \mathbb{N}^d} \sum_{k,l} (-i)^{|\alpha|} \left[\left. \frac{\partial^{|\alpha|}}{\partial {z_{\pi(\xi)}}^{\alpha}} \right|_{\pi(\xi)} \left. \frac{\partial^{|\alpha|}}{\partial {\zeta_{\pi(\xi)}}^{\alpha}} \right|_{\xi} \left. \frac{\partial^2}{\partial \zeta^k_{\pi(\xi)} \partial \zeta^l_{\pi(\xi)}} \right|_{\xi} \overline{a} \right] \mathrm{Ric}_{kl}(\pi(\xi)) \; + \\
&+ \sum_{\substack{\alpha,\beta \in \mathbb{N}^d \\ |\beta| \geq 3}} \frac{(-i)^{|\alpha+\beta|}}{\alpha!\,\beta!} \left[\left. \frac{\partial^{|\alpha|}}{\partial {z_{\pi(\xi)}}^{\alpha}} \right|_{\pi(\xi)} \left. \frac{\partial^{|\alpha+\beta|}}{\partial {\zeta_{\pi(\xi)}}^{\alpha+\beta}} \right|_{\xi} \overline{a} \right] \left[\left. \frac{\partial^{|\alpha|}}{\partial {z_{\pi(\xi)}}^{\alpha}} \right|_{\pi(\xi)} \rho^{-1}(\cdot, x) \right] \\
= \;& \overline{a}(\xi) \; - \; i \sum_k \left. \frac{\partial}{\partial z^k_{\pi(\xi)}} \right|_{\pi(\xi)} \left. \frac{\partial}{\partial \zeta^k_{\pi(\xi)}} \right|_{\xi} \overline{a} \; - \; \frac{1}{2} \sum_{k,l} \left. \frac{\partial^2}{\partial z^k_{\pi(\xi)} \partial z^l_{\pi(\xi)}} \right|_{\pi(\xi)} \left. \frac{\partial^2}{\partial \zeta^k_{\pi(\xi)} \partial \zeta^l_{\pi(\xi)}} \right|_{\xi} \overline{a} \; - \\
&- \frac{1}{6} \sum_{k,l} \left[\left. \frac{\partial^2}{\partial \zeta^k_{\pi(\xi)} \partial \zeta^l_{\pi(\xi)}} \right|_{\xi} \overline{a} \right] \mathrm{Ric}_{kl}(\pi(\xi)) \; + \; r(\xi) \qquad (61)
\end{aligned}
$$

with $r \in \mathrm{S}^{\mu-2(\rho-\delta)}_{\rho,\delta}(U)$.

5 Product expansions

Most considerations of elliptic partial differential operators as well as applications of our normal symbol to quantization theory (see PFLAUM [6]) require an expression for the product of two pseudo-differential operators A, B in terms of the symbols of its components. In the flat case on $\mathbb{R}^d$ it is a well known fact that σ_{AB} has an asymptotic expansion of the form

$$\sigma_{AB}(x,\xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} \frac{\partial^{|\alpha|}}{\partial \zeta^{\alpha}} \sigma_A(x,\xi) \, \frac{\partial^{|\alpha|}}{\partial x^{\alpha}} \sigma_B(x,\xi). \tag{62}$$

Note that precisly speaking, the product AB of pseudo-differential operators is only well-defined, if at least one of them is properly supported. But this is only a minor set-back, as any pseudo-differential operator is properly supported modulo a smoothing operator and the above asymptotic expansion also describes a symbol only up to smoothing symbols.

In the following we want to derive a similar but more complicated formula for the case of pseudo-differential operators on manifolds. To achieve this let us first state a proposition.

Proposition 5.1 *Let $A \in \Psi^{\mu}_{\rho,\delta}(U,X)$ be a pseudo-differential operator on the Riemannian manifold X and $a = \sigma(A)$ its symbol. Then for any $f \in \mathcal{C}^{\infty}(U)$ the function*

$$\sigma_{A,f} : T^*U \to \mathbb{C}, \quad \xi \mapsto \left[A \left(\psi_{\pi(\xi)} f \, e^{i\varphi(\cdot,\xi)} \right) \right] (\pi(\xi)) \tag{63}$$

is a symbol of order μ and type (ρ,δ) and has an asymptotic expansion of the form

$$\sigma_{A,f}(\xi) \sim \sum_{\alpha} \frac{1}{i^{|\alpha|}\alpha!} \left[\left. \frac{\partial^{|\alpha|}}{\partial \zeta_{\pi(\xi)}{}^{\alpha}} \right|_{\xi} a \right] \left[\left. \frac{\partial^{|\alpha|}}{\partial z_{\pi(\xi)}{}^{\alpha}} \right|_{\pi(\xi)} f \right]. \tag{64}$$

PROOF: Let $x = \pi(\xi)$, $N \in \mathbb{N}$ and $\psi : T^*X \to \mathbb{C}$ a cut-off function. Then the following chain of equations holds:

$$\begin{aligned}
\sigma_{A,f}(\xi) &= \left[A \left(\psi_{\pi(\xi)} f \, e^{i\varphi(\cdot,\xi)} \right) \right] (x) \\
&= \frac{1}{(2\pi)^{n/2}} \int_{T^*_x X} a(\zeta) \, \left[\mathcal{F} \left(f^{\psi} \, e^{i<\xi,\cdot>} \right) \right] (\zeta) \, d\zeta \\
&= \frac{1}{(2\pi)^{n}} \int_{T^*_x X} \int_{T_x X} a(\zeta) \, \psi(v) \, f(\exp v) \, e^{i<\xi-\zeta,v>} \, dv \, d\zeta \\
&= \frac{1}{(2\pi)^{n}} \int_{T^*_x X} \int_{T_x X} a(\xi+\zeta) \, \psi(v) \, f(\exp v) \, e^{-i<\zeta,v>} \, dv \, d\zeta
\end{aligned}$$

$$= \frac{1}{(2\pi)^n} \sum_{|\alpha|\leq N} \frac{1}{\alpha!} \int_{T_x^*X} \int_{T_xX} \frac{\partial^{|\alpha|}}{\partial {\zeta_x}^\alpha} a(\xi)\, \zeta^\alpha\, \psi(v)\, f(\exp v)\, e^{-i<\zeta,v>}\, dv\, d\zeta + r_N(\xi)$$

$$= \sum_{|\alpha|\leq N} \frac{1}{i^{|\alpha|}\,\alpha!} \left[\left. \frac{\partial^{|\alpha|}}{\partial {\zeta_x}^\alpha} \right|_\xi a \right] \cdot \left[\left. \frac{\partial^{|\alpha|}}{\partial {z_x}^\alpha} \right|_x f \right] + r_N(\xi). \tag{65}$$

Note that in this equation all integrals are iterated ones and check that with Taylor's formula r_N is given by

$$r_N(\xi) \;=\; \frac{1}{(2\pi)^n} \sum_{|\beta|=N+1} \frac{N+1}{\beta!} \int_{T_x^*X} \int_0^1 (1-t)^N \frac{\partial^{|\beta|}}{\partial {\zeta_x}^\beta} a(\xi + t\zeta)\, dt\, \zeta^\beta \left[\mathcal{F}\left(f^\psi \right) \right](\zeta)\, d\zeta. \tag{66}$$

Choosing N large enough, the Schwarz inequality and Plancherel's formula now entail

$$|r_N(\xi)| \;\leq\; \frac{1}{(2\pi)^{n/2}} \sum_{|\beta|=N+1} \frac{N+1}{\beta!} \left|\left| \int_0^1 (1-t)^N \frac{\partial^{|\beta|}}{\partial {\zeta_x}^\beta} a(\xi + t(\cdot))dt \right|\right|_{L^2(T_x^*X)} \left|\left| \left. \frac{\partial^{|\beta|}}{\partial {v_x}^\beta}\, f^\psi \right|_{T_xX} \right|\right|_{L^2(T_xX)}. \tag{67}$$

But for $K \subset U$ compact and $t \in [0,1]$ there exist boundaries $D_{N,K}, \tilde{D}_{N,K} > 0$ such that

$$\left| \frac{\partial^{|\beta|}}{\partial {\zeta_x}^\beta} a(\xi + t\zeta) \right| \leq D_{N,K} (1 + |\xi| + |\zeta|)^{\mu - \rho(N+1)} \tag{68}$$

and

$$\begin{aligned} \left|\left| \int_0^1 (1-t)^k \frac{\partial^{|\beta|}}{\partial {\zeta_x}^\beta} a(\xi + t(\cdot))dt \right|\right|_{L^2(T_x^*X)} &\leq D_{N,K} \left|\left| (1 + |\xi| + |\,\cdot\,|)^{\mu-\rho(N+1)} \right|\right|_{L^2(T_x^*X)} \\ &= \tilde{D}_{N,K}\, (1 + |\xi|)^{n+\mu-\rho(N+1)} \end{aligned} \tag{69}$$

for all $x \in K$, $\xi, \zeta \in T_x^*X$ and all indices $\beta \in \mathbb{N}^d$ with $|\beta| = N+1$. Inserting this in (67) we can find a $C_{N,K} > 0$ such that for all $x \in K$ and $\xi \in T_x^*X$

$$|r_N(\xi)| \leq C_{N,K} \left(1 + |\xi|\right)^{n+\mu-\rho(N+1)} \sum_{|\gamma|\leq N+1} \sup \left\{ \left| \frac{\partial^{|\gamma|}}{\partial {z_x}^\gamma} f(y) \right| \,:\, y \in \mathrm{supp}\left(\psi_{\pi(\xi)}\right) \right\}. \tag{70}$$

Because of $\lim_{N\to\infty} \dim X + \mu + \rho(N+1) = -\infty$ and a similar consideration for the derivatives of $\sigma_{A,f}$ the claim now follows. □

Next let us consider two pseudo-differential operators $A, B \in \Psi^\infty_{\rho,\delta}(U,X)$. Modulo smoothing operators we can assume one of them to be properly supported, so that $AB \in \Psi^\infty_{\rho,\delta}(U,X)$ exists. Defining the extended symbol $b^{ext} = \sigma^{ext}_{B,\psi} : U \times T^*U \to \mathbb{C}$ of B by

$$b^{ext}(x,\xi) \;=\; \psi_{\pi(\xi)}(x)\, e^{-i\varphi(x,\xi)} \left[B\left(\psi_{\pi(\xi)}(\cdot)\, e^{i\varphi(\cdot,\xi)} \right) \right](x) \tag{71}$$

the symbol σ_{AB} of the composition AB is now given modulo $\mathrm{S}^{-\infty}(U)$ by

$$\sigma_{AB}(\xi) \;=\; \left[AB\left(\psi_{\pi(\xi)}e^{i\varphi(\cdot,\xi)}\right)\right](\pi(\xi)) \;=\; \left[A\left(\psi_{\pi(\xi)}\,e^{i\varphi(\cdot,\xi)}\,\sigma^{ext}_{B,\psi}(\cdot,\xi)\right)\right](\pi(\xi)). \tag{72}$$

By Proposition 5.1 this entails that σ_{AB} has an asymptotic expansion of the form

$$\sigma_{AB}(\xi) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{i^{|\alpha|}\alpha!}\left[\left.\frac{\partial^{|\alpha|}}{\partial\zeta_{\pi(\xi)}{}^{\alpha}}\right|_{\xi}\sigma_A\right]\left[\left.\frac{\partial^{|\alpha|}}{\partial z_{\pi(\xi)}{}^{\alpha}}\right|_{\pi(\xi)}\sigma^{ext}_B(\cdot,\xi)\right]. \tag{73}$$

As we want to find an even more detailed expansion for σ_{AB}, let us calculate the derivatives $\left.\frac{\partial^{|\alpha|}}{\partial z_{\pi(\xi)}{}^{\alpha}}\right|_{\pi(\xi)}\sigma^{ext}_B(\cdot,\xi)$ in the following proposition.

Proposition 5.2 *Let $A\in\Psi^{\mu}_{\rho,\delta}(U,X)$ be a pseudo-differential operator on the Riemannian manifold X and $a=\sigma(A)$ its symbol. Then for any $f\in\mathcal{C}^{\infty}(U)$ the smooth function*

$$\sigma^{ext}_{A,f}: U\times T^*U\to\mathbb{C},\quad (x,\xi)\mapsto \psi_{\pi(\xi)}(x)\,e^{-i\varphi(x,\xi)}\left[A\left(\psi_{\pi(\xi)}f\,e^{i\varphi(\cdot,\xi)}\right)\right](x) \tag{74}$$

is a symbol of order μ and type (ρ,δ) on $U\times T^*U$ and has an asymptotic expansion of the form

$$\begin{aligned}\sigma^{ext}_{A,f}(x,\xi) \;\sim\; \sum_{k\in\mathbb{N}}\sum_{\substack{\beta,\beta_0,\ldots,\beta_k\in\mathbb{N}^n\\ \beta_0+\ldots+\beta_k=\beta\\ |\beta_1|,\ldots,|\beta_k|\geq 2}} &\frac{i^{k-|\beta|}}{k!\beta_0!\cdot\ldots\cdot\beta_k!}\,\psi_{\pi(\xi)}(x)\left[\left.\frac{\partial^{|\beta|}}{\partial\zeta_x{}^{\beta}}\right|_{d_x\varphi(\,\cdot\,,\xi)}a\right]\\ &\left[\left.\frac{\partial^{|\beta_0|}}{\partial z_x{}^{\beta_0}}\right|_x\psi_{\pi(\xi)}f\right]\,\left[\left.\frac{\partial^{|\beta_1|}}{\partial z_x{}^{\beta_1}}\right|_x\varphi(\cdot,\xi)\right]\cdot\ldots\cdot\left[\left.\frac{\partial^{|\beta_k|}}{\partial z_x{}^{\beta_k}}\right|_x\varphi(\cdot,\xi)\right].\end{aligned} \tag{75}$$

Furthermore the derivative $T^*U\ni\xi\mapsto\left[\frac{\partial^{|\alpha|}}{\partial z_{\pi(\xi)}{}^{\alpha}}\sigma^{ext}_{A,f}(\cdot,\xi)\right](\pi(\xi))\in\mathbb{C}$ with $\alpha\in\mathbb{N}^n$ is a symbol of order $\mu+\delta|\alpha|$ and type (ρ,δ) on T^*U and has an asymptotic expansion of the form

$$\begin{aligned}\left[\frac{\partial^{|\alpha|}}{\partial z_{\pi(\xi)}{}^{\alpha}}\sigma^{ext}_{A,f}(\cdot,\xi)\right](\pi(\xi)) \;\sim\; \sum_{k\in\mathbb{N}}\sum_{\substack{\tilde\alpha,\alpha_0,\ldots,\alpha_k\in\mathbb{N}^n\\ \tilde\alpha+\alpha_0+\ldots+\alpha_k=\alpha}}\sum_{\substack{\beta,\beta_0,\ldots,\beta_k\in\mathbb{N}^n\\ \beta_0+\ldots+\beta_k=\beta\\ |\beta_1|,\ldots,|\beta_k|\geq 2}}&\frac{i^{k-|\beta|}\alpha!}{k!\tilde\alpha!\alpha_0!\cdot\ldots\cdot\alpha_k!\beta_0!\cdot\ldots\cdot\beta_k!}\\ \left\{\left.\frac{\partial^{|\tilde\alpha|}}{\partial z_{\pi(\xi)}{}^{\tilde\alpha}}\right|_{\pi(\xi)}\left[\left.\frac{\partial^{|\beta|}}{\partial\zeta_{(-)}{}^{\beta}}\right|_{d_{(-)}\varphi(\,\cdot\,,\xi)}a\right]\right\}&\left\{\left.\frac{\partial^{|\alpha_0|}}{\partial z_{\pi(\xi)}{}^{\alpha_0}}\right|_{\pi(\xi)}\left[\left.\frac{\partial^{|\beta_0|}}{\partial z_{(-)}{}^{\beta_0}}\right|_{(-)}f\right]\right\}\cdot\\ \left\{\left.\frac{\partial^{|\alpha_1|}}{\partial z_{\pi(\xi)}{}^{\alpha_1}}\right|_{\pi(\xi)}\left[\left.\frac{\partial^{|\beta_1|}}{\partial z_{(-)}{}^{\beta_1}}\right|_{(-)}\varphi(\cdot,\xi)\right]\right\}&\cdot\ldots\cdot\left\{\left.\frac{\partial^{|\alpha_k|}}{\partial z_{\pi(\xi)}{}^{\alpha_k}}\right|_{\pi(\xi)}\left[\left.\frac{\partial^{|\beta_k|}}{\partial z_{(-)}{}^{\beta_k}}\right|_{(-)}\varphi(\cdot,\xi)\right]\right\}.\end{aligned} \tag{76}$$

Note 5.3 The differential operators of the form $\frac{\partial^{|\alpha|}}{\partial z_{\pi(\xi)}{}^{\alpha}}$ act on the variables denoted by $(-)$, the differential operators of the form $\frac{\partial^{|\beta|}}{\partial z_{(-)}{}^{\beta}}$ on the variables $(\cdot)$.

PROOF: Let $\eta \in \mathcal{C}^{\infty}(U \times U \times T^*U)$ be a smooth function such that

$$\eta(x,y,\xi) \;=\; \varphi(y,\xi) - \varphi(x,\xi) - < d_x\varphi(\ \ ,\xi), z_x(y) > \tag{77}$$

for $x,y \in \mathrm{supp}\left(\psi_{\pi(\xi)}\right)$ and such that $T^*U \ni \xi \mapsto \eta(x,y,\xi) \in \mathbb{C}$ is linear for all $x,y \in U$. Furthermore denote by $F \in \mathcal{C}^{\infty}(U \times U \times T^*U)$ the function $(x,y,\xi) \mapsto \psi_{\pi(\xi)}(y) f(y) e^{i\eta(x,y,\xi)}$. By the Leibniz rule and the definition of η we then have

$$\begin{aligned}\left[\left.\frac{\partial^{|\beta|}}{\partial z_x{}^{\beta}}\right|_y F(x,\cdot,\xi)\right] \;&=\; \sum_{0\le k\le \frac{|\beta|}{2}} \sum_{\substack{\beta_0,\ldots,\beta_k\in\mathbb{N}^n \\ \beta_0+\ldots+\beta_k=\beta \\ |\beta_1|,\ldots,|\beta_k|\ge 2}} \frac{i^k}{k!\beta_0!\cdot\ldots\cdot\beta_k!} \left[\left.\frac{\partial^{|\beta_0|}}{\partial z_x{}^{\beta_0}}\right|_y \psi_{\pi(\xi)} f\right] \cdot \\ &\qquad \left[\left.\frac{\partial^{|\beta_1|}}{\partial z_x{}^{\beta_1}}\right|_y \eta(x,\cdot,\xi)\right] \cdot \ldots \cdot \left[\left.\frac{\partial^{|\beta_k|}}{\partial z_x{}^{\beta_k}}\right|_y \eta(x,\cdot,\xi)\right] e^{i\eta(x,y,\xi)}. \end{aligned} \tag{78}$$

As $(x,y,\xi) \mapsto \left[\left.\frac{\partial^{|\beta_j|}}{\partial z_x{}^{\beta_j}}\right|_y \eta(x,\cdot,\xi)\right]$, $0 \le j \le k$ is smooth and linear with respect to ξ, the preceeding equation entails that $(x,y,\xi) \mapsto \left[\left.\frac{\partial^{|\beta|}}{\partial z_x{}^{\beta}}\right|_y F(x,\cdot,\xi)\right]$ is absolutly bounded by a function $(x,y,\xi) \mapsto C_{\beta}(x,y)\,|\xi|^{\frac{|\beta|}{2}}$, where $C_{\beta} \in \mathcal{C}^{\infty}(U \times U)$ and $C_{\beta} \ge 0$. On the other hand we have

$$\sigma^{ext}_{A,f}(x,\xi) \;=\; \psi_{\pi(\xi)}(x) \left[A\left(\psi_{\pi(\xi)}(\cdot) f(\cdot) e^{i\eta(\cdot,x,\xi)}\, e^{i<d_x\varphi(\ \ ,\xi), z_x(\cdot)>}\right)\right](x) \tag{79}$$

and by the proof of Proposition 5.1

$$\begin{aligned}\sigma^{ext}_{A,f}(x,\xi) \;&=\; \psi_{\pi(\xi)}(x) \sum_{|\beta|\le N} \frac{i^{|\beta|}}{\beta!} \left[\left.\frac{\partial^{|\beta|}}{\partial \xi_x{}^{\beta}}\right|_{d_x\varphi(\ \ ,\zeta)} a\right] \left[\left.\frac{\partial^{|\beta|}}{\partial z_x{}^{\beta}}\right|_x F(x,\cdot,\xi)\right] + r_N(x,\xi) \\ &=\; \psi_{\pi(\xi)}(x) \sum_{|\beta|\le N} \sum_{0\le k\le \frac{|\beta|}{2}} \sum_{\substack{\beta_0,\ldots,\beta_k\in\mathbb{N}^n \\ \beta_0+\ldots+\beta_k=\beta \\ |\beta_1|,\ldots,|\beta_k|\ge 2}} \frac{i^{k-|\beta|}}{k!\beta_0!\cdot\ldots\cdot\beta_k!} \left[\left.\frac{\partial^{|\beta|}}{\partial \xi_x{}^{\beta}}\right|_{d_x\varphi(\ \ ,\zeta)} a\right] \cdot \\ &\qquad \left[\left.\frac{\partial^{|\beta_0|}}{\partial z_x{}^{\beta_0}}\right|_x \psi_{\pi(\xi)} f\right] \cdot \left[\left.\frac{\partial^{|\beta_1|}}{\partial z_x{}^{\beta_1}}\right|_x \varphi(\cdot,\xi)\right] \cdot \ldots \cdot \left[\left.\frac{\partial^{|\beta_k|}}{\partial z_x{}^{\beta_k}}\right|_x \varphi(\cdot,\xi)\right] + r_N(x,\xi), \end{aligned} \tag{80}$$

where

$|r_n(x,\xi)|$

$$
\begin{aligned}
&\leq C_{N,K}(1+|d_x\varphi(\ ,\xi)|)^{n+\mu-\rho(N+1)} \sum_{|\gamma|\leq N+1} \sup\left\{ \left| \left[\frac{\partial^{|\gamma|}}{\partial {z_x}^\gamma}\Big|_y F(x,\cdot,\xi)\right]\right| \,:\, y\in \mathrm{supp}(\psi_x)\right\} \\
&\leq D_{N,K}(1+|\xi|)^{n+\mu-(\rho-1/2)(N+1)} \qquad (81)
\end{aligned}
$$

holds for compact $K\subset U$, $x\in K$, $\xi\in T^*X|_K$ and constants $C_{N,K}, D_{N,K}>0$. Because $\rho>\frac{1}{2}$, we have $\lim_{N\to\infty} \dim X+\mu-(\rho-\frac{1}{2})(N+1)=-\infty$. By Eqs. (80) and (81) and similar relations for the derivatives of $\sigma^{ext}_{A,f}$ we therefore conclude that Eq. (75) is true. The rest of the claim easily follows from Leibniz rule. □

The last proposition and Eq. (73) imply the next theorem.

Theorem 5.4 *Let $A,B\in\Psi^\infty_{\rho,\delta}(U,X)$ be two pseudo-differential operators on the Riemannian manifold X, one of them properly supported. Then the symbol σ_{AB} of the product AB has the following asymptotic expansion.*

$$
\begin{aligned}
\sigma_{AB}(\xi) \;\sim\; & \sum_{\alpha\in\mathbb{N}^d} \frac{i^{-|\alpha|}}{\alpha!} \left[\frac{\partial^{|\alpha|}}{\partial {\zeta_{\pi(\xi)}}^\alpha}\Big|_\xi \sigma_A\right]\left[\frac{\partial^{|\alpha|}}{\partial {z_{\pi(\xi)}}^\alpha}\Big|_{\pi(\xi)}\sigma_B\right] + \\
&+\sum_{k\geq 1} \sum_{\substack{\alpha,\tilde\alpha,\alpha_1,\ldots,\alpha_k\in\mathbb{N}^n \\ \tilde\alpha+\alpha_1+\ldots+\alpha_k=\alpha}} \sum_{\substack{\beta,\beta_1,\ldots,\beta_k\in\mathbb{N}^n \\ \beta_1+\ldots+\beta_k=\beta \\ |\beta_1|,\ldots,|\beta_k|\geq 2}} \frac{i^{k-|\alpha|-|\beta|}}{k!\cdot\tilde\alpha!\cdot\alpha_1!\cdot\ldots\cdot\alpha_k!\beta_1!\cdot\ldots\cdot\beta_k!} \\
&\left[\frac{\partial^{|\alpha|}}{\partial {\zeta_{\pi(\xi)}}^\alpha}\Big|_\xi \sigma_A\right]\left\{\frac{\partial^{|\tilde\alpha|}}{\partial {z_{\pi(\xi)}}^{\tilde\alpha}}\Big|_{\pi(\xi)}\left[\frac{\partial^{|\beta|}}{\partial {\zeta_{(-)}}^\beta}\Big|_{d_{(-)}\varphi(\ ,\xi)}\sigma_B\right]\right\}\cdot \\
&\left\{\frac{\partial^{|\alpha_1|}}{\partial {z_{\pi(\xi)}}^{\alpha_1}}\Big|_{\pi(\xi)}\left[\frac{\partial^{|\beta_1|}}{\partial {z_{(-)}}^{\beta_1}}\Big|_{(-)}\varphi(\cdot,\xi)\right]\right\}\cdot\ldots\cdot\left\{\frac{\partial^{|\alpha_k|}}{\partial {z_{\pi(\xi)}}^{\alpha_k}}\Big|_{\pi(\xi)}\left[\frac{\partial^{|\beta_k|}}{\partial {z_{(-)}}^{\beta_k}}\Big|_{(-)}\varphi(\cdot,\xi)\right]\right\}. \qquad (82)
\end{aligned}
$$

A somewhat more explicit expansion up to second order is given by the following corollary.

Corollary 5.5 *If σ_A is a symbol of order μ and σ_B a symbol of order $\tilde\mu$ the coefficients in the asymptotic expansion of σ_{AB} are given up to second order by the following formula:*

$$
\begin{aligned}
&\sigma_{AB}(\xi) \;= \\
&\quad \sigma_A(\xi)\cdot\sigma_B(\xi) \;-\; i\sum_l \frac{\partial\sigma_A}{\partial\zeta_{\pi(\xi),l}}(\xi)\,\frac{\partial\sigma_B}{\partial z^l_{\pi(\xi)}}(\xi) \;-\; \frac{1}{2}\sum_{k,l}\frac{\partial^2\sigma_A}{\partial\zeta_{\pi(\xi),k}\partial\zeta_{\pi(\xi),l}}(\xi)\,\frac{\partial^2\sigma_B}{\partial z^k_{\pi(\xi)}\partial z^l_{\pi(\xi)}}(\xi) \\
&\quad -\frac{1}{12}\sum_{k,l,m,n}\frac{\partial\sigma_A}{\partial\zeta_{\pi(\xi),n}}(\xi)\,\frac{\partial^2\sigma_B}{\partial\zeta_{\pi(\xi),l}\partial\zeta_{\pi(\xi),m}}(\xi)\,R^k{}_{mln}(\pi(\xi))\,\zeta_{\pi(\xi),k}(\xi) \;+\; r(\xi), \qquad (83)
\end{aligned}
$$

where $\xi \in T^*U$, $x = \pi(\xi)$, $r \in \mathrm{S}^{\mu+\tilde{\mu}-3}_{\rho,\delta}(U)$ *and the* $R^m{}_{nkl}(y)$ *are the coefficients of the curvature tensor with respect to normal coordinates at* $y \in X$, *i.e.* $R\left(\frac{\partial}{\partial z_y^m}\right) = \sum_{k,l,n} R^k{}_{mln}(y) \frac{\partial}{\partial z_y^k} \otimes dz_y^l \otimes dz_y^n$.

PROOF: First define the order of a summand $s = s^{\beta,\beta_1,\ldots,\beta_k}_{\alpha,\tilde{\alpha},\alpha_1,\ldots,\alpha_k}$ in Eq. (82) by $\mathrm{ord}(s) = |\alpha - \tilde{\alpha} + \beta - k|$. Then it is obvious that $s \in \mathrm{S}^{\mu+\tilde{\mu}+\mathrm{ord}(s)\,(\rho-\delta)}_{\rho,\delta}(U)$. Therefore we have to calculate only the summands s with $\mathrm{ord}(s) \leq 2$. To achieve this let us calculate some coordinate changes and their derivatives. Recall the matrix-valued function $\left(\theta_l^k\right)$ of Eq. (55) and construct its inverse $\left(\bar{\theta}_l^k\right)$. In other words

$$\sum_k \theta_k^l(x,y)\,\bar{\theta}_m^k(x,y) = \delta_m^l \tag{84}$$

holds for every $l, m = 1, ..., d$. We can now write

$$\left.\frac{\partial}{\partial z_y^k}\right|_z \;=\; \sum_{l,m} \theta_k^l(y,z)\,\bar{\theta}_l^m(x,z)\;\left.\frac{\partial}{\partial z_x^m}\right|_z \tag{85}$$

and receive the following formulas:

$$\begin{aligned}\frac{\partial z_x^k}{\partial z_y^l}(z) \;&=\; \delta_l^k \;-\; \frac{1}{6}\sum_{m,n} R^k{}_{mln}(y)\,z_x^m(z)\,z_x^n(z) \\ &\quad + \frac{1}{6}\sum_{m,n} R^k{}_{mln}(x) z_x^m(z)\,z_x^n(z) \;+\; O\left(|z_x(z)+z_y(z)|^3\right), \end{aligned} \tag{86}$$

$$\begin{aligned}\frac{\partial^2 z_x^k}{\partial z_y^l \partial z_y^m}(z) \;&=\; -\frac{1}{6}\sum_n \left(R^k{}_{mln}(y) + R^k{}_{nlm}(y)\right)\, z_x^n(z) \\ &\quad + \frac{1}{6}\sum_n \left(R^k{}_{mln}(x) + R^k{}_{nlm}(x)\right) z_x^n(z) \\ &\quad + O\left(|z_x(z)+z_y(z)|^2\right), \end{aligned} \tag{87}$$

$$\left.\frac{\partial}{\partial z_x^n}\right|_{y=z} \frac{\partial^2 z_x^k}{\partial z_y^l \partial z_y^m}(y) \;=\; \frac{1}{6}\left(R^k{}_{mln}(x) + R^k{}_{nlm}(x)\right) \;+\; O\left(|z_x(z)|\right). \tag{88}$$

Next consider the functions

$$\varphi_{\alpha\beta} : T^*X \to \mathbb{R}, \qquad \xi \mapsto \left.\frac{\partial^{|\alpha|}}{\partial {z_{\pi(\xi)}}^{\alpha}}\right|_{y=\pi(\xi)} \frac{\partial^{|\beta|}\varphi(\cdot,\xi)}{\partial^{\beta} {z_y}^{\beta}}(y)$$

which appear in the summands s of Eq. (82). One can write the $\varphi_{\alpha\beta}$ in the form

$$\varphi_{\alpha\beta}(\xi) \;=\; \sum_k \zeta_{\pi(\xi),k}(\xi)\;\left.\frac{\partial^{|\alpha|}}{\partial {z_{\pi(\xi)}}^{\alpha}}\right|_{y=\pi(\xi)} \frac{\partial^{|\beta|} z^k_{\pi(\xi)}}{\partial {z_y}^{\beta}}(y) \tag{89}$$

Thus by Eq. (87)

$$\varphi_{0\beta}(\xi) \;=\; 0 \tag{90}$$

holds for every $\beta \in \mathbb{N}^d$ with $|\beta| \geq 2$. Furthermore check that

$$d_y\varphi(\cdot,\xi) \;=\; \sum_k \zeta_{\pi(\xi),k}(\xi)\; dz^k_{\pi(\xi)}\Big|_y \tag{91}$$

is true.

Now we are ready to calculate the summands s of order $\mathrm{ord}(s) \leq 2$ by considering the following cases.

(i) $\mathrm{ord}(s) = 0$:

$$s(\xi) \;=\; \sigma_A(\xi)\,\sigma_B(\xi). \tag{92}$$

(ii) $\mathrm{ord}(s) = 1$, $|\alpha| = 1$, $k = 0$, $\tilde{\alpha} = \alpha$:

$$s(\xi) \;=\; -\,i \sum_l \frac{\partial \sigma_A}{\partial \zeta_{\pi(\xi),l}}(\xi)\; \frac{\partial \sigma_B}{\partial z^l_{\pi(\xi)}}(\xi). \tag{93}$$

(iii) $\mathrm{ord}(s) = 1$, $|\alpha| = 0$, $k = 1$, $|\beta| = 2$: In this case $s = 0$ because of Eq. (90).

(iv) $\mathrm{ord}(s) = 2$, $|\alpha| = 2$, $k = 0$, $\tilde{\alpha} = \alpha$:

$$s(\xi) \;=\; -\,\frac{1}{2} \sum_{k,l} \frac{\partial^2 \sigma_A}{\partial \zeta_{\pi(\xi),k} \partial \zeta_{\pi(\xi),l}}(\xi)\; \frac{\partial^2 \sigma_B}{\partial z^k_{\pi(\xi)} \partial z^l_{\pi(\xi)}}(\xi). \tag{94}$$

(v) $\mathrm{ord}(s) = 2$, $|\alpha| = 1$, $k = 1$, $|\beta| = 2$: In this case $\tilde{\alpha} = 0$ by Eq. (90). Hence by Eq. (88)

$$s(\xi) \;=\; -\,\frac{1}{12} \sum_{k,l,m,n} \frac{\partial \sigma_A}{\partial \zeta_{\pi(\xi),n}}(\xi)\; \frac{\partial^2 \sigma_B}{\partial \zeta_{\pi(\xi),l} \partial \zeta_{\pi(\xi),m}}(\xi)\; R^k{}_{mln}(\pi(\xi))\; \zeta_{\pi(\xi),k}(\xi). \tag{95}$$

Summing up these summands we receive the expansion of σ_{AB} up to second order. This proves the claim. □

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