On complex Bloch-spaces of periodic Schrödinger operators

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Abstract

The band structure of a periodic Schrödinger operator is the common spectrum of this operator and the commuting shift operators corresponding to the period vectors. The complex continuation of this common spectrum is called complex Bloch-variety. This variety may be considered as the spectrum of the Schrödinger operator on the torus, which is a holomorphic function from the complex manifold of quasi periodic boundary conditions on the torus into the operators of the Hilbert space of the torus. We construct a complex space, whose holomorphic functions are the eigenvalues of holomorphic operator valued functions, which commute with this function. The resulting space is called complex Bloch-space. We generalize a result of Knörrer and Trubowitz, that the infinite energy limit of the complex Bloch-variety of a two-dimensional periodic Schrödinger operator contains the complex Bloch-varieties of the averaged one-dimensional potentials to arbitrary dimensions. In fact, a completion of the complex Bloch-space contains in the infinite energy limit all Bloch-spaces of the non-trivial averaged potentials. Finally we give a characterization of all complex Bloch-spaces, whose completions are compact complex spaces. The corresponding periodic Schrödinger operators are exactly the so-called algebraically integrable Schrödinger operators, which roughly speaking have a maximal number of commuting differential operators.

Contents

1 Introduction 1

2 The resolvent 2

3 The complex Bloch-space 6

4 An embedding into the projective Hilbert space 10

5 A completion of the complex Bloch-variety 17

6 The dual eigen bundle 24

7 Compact Bloch-spaces 27

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1 Introduction

The Schrödinger operator with a periodic potential commutes with all shifts of the period lattice. All this operators may be diagonalized simultaneously. The Bloch-theory is concerned with the common spectrum of all these operators. The spectrum of the shifts is equal to the dual group $\hat{\Gamma}$ of the period lattice Γ and the common spectrum of all this operators is a subspace of the product of this dual group with the real[2] numbers, which is called Bloch-variety. The inverse spectral problem for periodic Schrödinger operators contains essentially two problems:

The isospectral problem: A parameterization of all periodic potentials is sought, whose Bloch-varieties are equal to the Bloch-variety of a fixed periodic potential (these sets are called isospectral sets).

The moduli problem: A parameterization of all possible Bloch-varieties is sought (this set is called moduli space).

In the one-dimensional case, both problems are well understood (see e.g. [No, MK-M, MK-T, Kr, N-M-P-Z, D-K-N, G-S]). The solution of the first part depends on establishing a one-to-one correspondence between the isospectral potentials and some holomorphic line bundles — or equivalently some divisors — on the complexification of the Bloch-variety. Roughly speaking, the isospectral sets are generically parameterized by some infinite-dimensional compact tori. The main tool of the second part is perturbation theory. The tangent space along the moduli space at any fixed Bloch-variety may be identified with the space of holomorphic 1-forms (see e.g. [Sch]). Thus, if we fix for any Bloch-variety a holomorphic 1-form on this spectral curve, we obtain a vector field on the moduli space. Finally, the parameterization of the moduli space is the result of an appropriate choice of such vector fields commuting with each other (see [G-S]).

In higher dimensions, the first problem is more or less solved. In fact for generic potentials, the isospectral sets are equal to the orbits under the action of the group of all space symmetries (see [E-R-T, G-K-T-1, G-K-T-2]). Nevertheless, we shall generalize the one-to-one correspondence between the isospectral potentials and some holomorphic line bundles on the complex Bloch-variety to the higher dimensional case to solve the non-generic case. This would imply an equality of the dimension of the isospectral sets and the n-th cohomology group of the structure sheaf of the complex Bloch-variety. In Section 7 this is shown for those complex Bloch-varieties, which may be completed to compact complex spaces (let us call them compact Bloch-spaces). The second problem seems to be much more difficult in dimensions $n > 1$. In fact in dimension $n = 1$, the moduli space contains a dense subset, which is the union of subspaces of the Teichmüller spaces of compact Riemann surfaces of genus g over all finite genera. In dimension $n > 1$, however, the compact Bloch-spaces are not dense (see [F-K-T]). But the deformation theory may be generalized to the higher dimensional case (see [S-V]). In Section 7 we will give a complete characterization of compact Bloch-spaces. This is of course only a first step in solving the moduli problem in dimension $n > 1$.

Let us describe the tools and methods in some more detail. The complex continuation of the Bloch-variety is called complex Bloch-variety. This complex continuation may be con-

[2] This is only true if the potential is real.

sidered as the spectrum of an holomorphic function from the complexification of the dual group $\hat{\Gamma}$ into the operators of the Hilbert space of the space torus $L^2(\mathbb{R}^n/\Gamma)$. In fact, the complexification of the dual group parameterizes the quasi-periodic boundary conditions of the Schrödinger operator on this Hilbert space. We want to call this function Schrödinger operator on the torus. The main focus of this article is a complex space, whose holomorphic functions are in one-to-one correspondence with holomorphic operator-valued functions, which commute with this Schrödinger operator on the torus. This complex Bloch-space forms a parameterization of the common spectrum, which is a slightly finer than the complex Bloch-variety.

In the one-dimensional case there exists a dense family of potentials, whose complex Bloch-spaces are compact Riemann surfaces with one point removed. This point is the only pole (of second order) of the energy function, which parameterizes the eigenvalue of the Schrödinger operator. These so called finite-gap potentials are characterized by the property that their spectra have finite numbers of connected components. McKean and Trubowitz [MK-T] succeed in generalizing the powerful tool of the theory of compact Riemann surfaces to general one-dimensional smooth periodic potentials. In [Sch] techniques have been developed, which allow for the addition of the point of infinite energy to the Riemann surface, even in the case of infinite gaps. In the two-dimensional case, Knörrer and Trubowitz have shown in [K-T] that the infinite energy limit of the complex Bloch-variety contains the Bloch-varieties of all one-dimensional potentials, which are obtained by averaging the two-dimensional potential in some directions of the period lattice. In this paper we apply some of the ideas developed in [Sch] to higher-dimensional periodic Schrödinger operators. In Section 5 we add the Bloch-varieties of all potentials, which are obtained by averaging the n-dimensional potential over sub-lattices of the period lattice, to the complex Bloch-variety. The resulting space is a completion of the complex Bloch-variety.

Besides the complex Bloch-variety and the complex Bloch-space, we shall consider in Section 4 the subspace of the projective space of the Hilbert space, whose elements are the eigenvectors of the Schrödinger operator. This parameterization is even finer than the complex Bloch-space, because the eigenvalues of the Schrödinger operator are generically non-degenerated. In an appropriate setting, this space by itself contains the corresponding spaces for all non-trivial averaged potentials.

In Section 6 we consider the sheaf of sections of the dual eigen bundle. We show that it uniquely determines the potential and prove some properties, which for compact Bloch-spaces characterizes all sheaves of this form. In the last section, we give a complete characterization of compact Bloch-spaces. We shall propose a condition, which in the one-dimensional case is always fulfilled, but not in dimensions $n > 1$. Along the way, we shall see that for compact Bloch-spaces, the isospectral sets are again isomorphic to open subsets of a connected component of the Picard-group of a projective variety.

2 The resolvent

Let Γ be a lattice in $\mathbb{R}^n$ and q a periodic function on $\mathbb{R}^n/\Gamma$. In dimensions less than four q may be any element of $L^2(\mathbb{R}^n/\Gamma)$. In general, q is assumed to be an element of the Banach space $L^p(\mathbb{R}^n/\Gamma)$ with $p > n/2$ and $p \geq 2$. The Euclidean space $\mathbb{R}^n$ is endowed with a positive definite scalar product $g(\cdot,\cdot)$. This metric induces a flat Riemannian metric on the torus

$\mathbb{R}^n/\Gamma$. Let us now introduce general boundary conditions for the Laplace-operator: Suppose $\chi : \gamma \to \mathbb{C}$ is a one-dimensional representation of the lattice Γ, then Δ_χ is defined as the unique closure of the unbounded operator of $L^2(\mathbb{R}^n/\Gamma)$ with the domain of smooth continuous functions ψ, so that

$$\psi(x+\gamma) = \chi(\gamma)\psi(x) \text{ for all } x \in \mathbb{R}^n \text{ and all } \gamma \in \Gamma.$$

In order to diagonalize this operator, we extend our metric to a complex bilinear form on the complexification $\mathbb{C}^n$ of our Euclidean space and introduce the dual ore reciprocal lattice $\Gamma^\sharp$, which is a lattice in the dual space of our Euclidean space $\mathbb{R}^n$:

$$\Gamma^\sharp := \{\kappa \in \mathbb{R}^n | g(\gamma,\kappa) \in \mathbb{Z} \text{ for all } \gamma \in \Gamma\}\,.$$

The eigen functions of the Laplace-operator Δ_χ are given by

$$\psi_k(x) = \exp(2\pi\sqrt{-1}g(x,k)) \text{ with eigenvalue } \Delta_\chi\psi_k = -4\pi^2 g(k,k)\psi_k.$$

Here k is any element of $\mathbb{C}^n$, such that $\chi(\gamma) = \exp(2\pi\sqrt{-1}g(\gamma,k))$ for all $\gamma \in \Gamma$. It is obvious that such an element k exists, and that all of these elements differ by an element κ of the dual lattice $\Gamma^\sharp$. So let us identify the one-dimensional representations χ with the corresponding classes[3] $[k] \in \mathbb{C}^n/\Gamma^\sharp$. The unitary representations then may be considered as elements of $\mathbb{R}^n/\Gamma^\sharp$.

If χ is not unitary, then Δ_χ is not self-adjoint and the corresponding eigen functions are not orthogonal. But with respect to the following hermitian scalar product $\langle\cdot,\cdot\rangle_\chi$, the eigen functions are orthogonal:

$$\langle\psi,\phi\rangle_\chi := \int\limits_{\mathbb{R}^n/\Gamma} \overline{\psi(x)}\phi(x)\exp(4\pi g(x,\mathrm{Im}(k)))d^n x.$$

Here k is again an element of $\mathbb{C}^n$, such that $\chi(\gamma) = \exp(2\pi\sqrt{-1}g(k,\gamma)$ for all $\gamma \in \Gamma$. This scalar product depends only on the representation $\chi = [k] \in \mathbb{C}^n/\Gamma^\sharp$. For the trivial representation $\chi = 1$, it is the standard scalar product of $L^2(\mathbb{R}^n/\Gamma)$. Moreover, any norm corresponding to one of these scalar products may be estimated by any other of this norms. Finally let us introduce the Sobolev-space $L^2_4(\mathbb{R}^n/\Gamma)$. This is the Hilbert space completion of $C^\infty(\mathbb{R}^n/\Gamma)$ with respect to the hermitian scalar product

$$\langle\psi,\phi\rangle_4 := \int\limits_{\mathbb{R}^n/\Gamma} \overline{\psi(x)}(1\!\!1 - \Delta_1)^2\phi(x)d^n x.$$

Now we can state the main theorem of this section:

Theorem 2.1 *Let $p > n/2$ and $p \geq 2$. The map $(q,\chi,\lambda) \mapsto (\lambda 1\!\!1 + \Delta_\chi - q)^{-1}$ is a meromorphic map from $L^p(\mathbb{R}^n/\Gamma) \times \mathbb{C}^n/\Gamma^\sharp \times \mathbb{C}$ into the set of compact operators on $L^2(\mathbb{R}^n/\Gamma)$. More precisely, for each fixed $(q_0,\chi_0) \in L^p(\mathbb{R}^n/\Gamma) \times \mathbb{C}^n/\Gamma^\sharp$ there exists a discrete subset $S \subset \mathbb{C}$, such that the map $\lambda \mapsto (\lambda 1\!\!1 + \Delta_\chi - q)^{-1}$ is holomorphic on $\mathbb{C} \setminus S$. If $\lambda_0 \in S$ be an arbitrary pole of this map, then there exists an open neighborhood $W \subset L^p(\mathbb{R}^n/\Gamma) \times \mathbb{C}^n/\Gamma^\sharp$ of (q_0,χ_0) and a finite number of holomorphic maps $v_1,\ldots,v_l$ and anti-holomorphic maps $w_1,\ldots,w_l$ from W into $L^2(\mathbb{R}^n/\Gamma)$, and a neighborhood $V \subset \mathbb{C}$ of λ_0, so that the following conditions hold:*

[3] We may choose as well a basis $\gamma_1,\ldots,\gamma_n$ of the lattice $\Gamma^\sharp$. Since all representations are uniquely determined by their values $\chi(\gamma_1),\ldots,\chi(\gamma_n)$, we may identify the space of the one-dimensional representations with $(\mathbb{C}^*)^n$, and the unitary one-dimensional representations with $U(1)^n$.

(i) *For all* $(q,\chi) \in W$*, the operator* $P(q,\chi) : \psi \mapsto \sum_{i=1}^{l} \langle w_i(q,\chi), \psi \rangle v_i(q,\chi)$ *is a projection, which commutes with* $-\Delta_\chi + q$*.*

(ii) *The map* $(q,\chi,\lambda) \mapsto (\mathbb{1} - P(q,\chi)) \circ (\lambda \mathbb{1} + \Delta_\chi - q)^{-1} = (\lambda \mathbb{1} + \Delta_\chi - q)^{-1} \circ (\mathbb{1} - P(q,\chi))$ *is a holomorphic map from* $W \times V$ *into the set of the compact operators on* $L^2(\mathbb{R}^n/\Gamma)$*.*

(iii) *For all* $1 \leq i,j \leq l$*, the functions* $(q,\chi,\lambda) \mapsto \langle w_j(q,\chi), (\lambda \mathbb{1} + \Delta_\chi - q)^{-1} v_i(q,\chi) \rangle$ *are meromorphic functions on* $W \times V$*.*

For the proof we need the following lemma (compare [R-S-II, Theorem IX.28]).

Lemma 2.2 *If* p *is the same as in Theorem 2.1, then there exists a positive number* $C > 0$ *such that for all* $\psi \in L^2_4(\mathbb{R}^n/\Gamma)$*, the following estimate holds:*

$$\|\psi\|_{\frac{2p}{p-2}} \leq C \|(\mathbb{1} - \Delta_1)\psi\|.$$

Proof: Let $\hat{\psi}$ denote the Fourier transform of ψ. Due to the Hausdorff-Young inequality (the proof is the same as in the case of functions on $\mathbb{R}^n$ [R-S-II, Theorem IX.8]), a constant $C > 0$ exists such that

$$\|\psi\|_{\frac{2p}{p-2}} \leq C \left(\sum_{\kappa \in \Gamma^\sharp} |\hat{\psi}(\kappa)|^{\frac{2p}{p+2}} \right)^{\frac{p+2}{2p}}.$$

Due to Hölders inequality ([R-S-I, Theorem III.1]), we have

$$\sum_{\kappa \in \Gamma^\sharp} |\hat{\psi}(\kappa)|^{\frac{2p}{p+2}} \leq \left(\sum_{\kappa \in \Gamma^\sharp} |1 + 4\pi^2 g(\kappa,\kappa)|^{-\frac{2p}{p+2}\frac{p+2}{2}} \right)^{\frac{2}{p+2}} \left(\sum_{\kappa \in \Gamma^\sharp} |(1 + 4\pi^2 g(\kappa,\kappa))\hat{\psi}(\kappa)|^{\frac{2p}{p+2}\frac{p+2}{2}} \right)^{\frac{p}{p+2}}.$$

Due to the assumption $p > n/2$, the first factor is finite, and the second factor is proportional to $\|(\mathbb{1} - \Delta_1)\psi\|^{2p/(p+2)}$. This proves the lemma. □

Proof of Theorem 2.1: First we claim that the multiplication by an element q of $L^p(\mathbb{R}^n/\Gamma)$ defines a bounded operator from $L^2_4(\mathbb{R}^n/\Gamma)$ into $L^2(\mathbb{R}^n/\Gamma)$. In fact, as a result of Hölders inequality ([R-S-I, Theorem III.1]), we are presented with the estimate

$$\|q\psi\| \leq \|q\|_p \|\psi\|_{\frac{2p}{p-2}}.$$

The previous lemma now proves the claim. Hence there exists a bounded map from $L^p(\mathbb{R}^n/\Gamma)$ into $B(L^2_4(\mathbb{R}^n/\Gamma), L^2(\mathbb{R}^n/\Gamma))$.

Now let us fix a wave vector $k \in \mathbb{C}^n$ and an eigenvalue λ, which does not belong to the spectrum of $-\Delta_{[k]}$. We claim that $q \circ (\lambda \mathbb{1} + \Delta_{[k]})^{-1}$ defines a bounded operator from $L^2(\mathbb{R}^n/\Gamma)$ into itself. Moreover, the norm of this operator may be estimated by

$$\|q \circ (\lambda \mathbb{1} + \Delta_{[k]})^{-1}\| \leq C([k],\lambda) \|q\|_p,$$

with a constant $C([k],\lambda)$ depending only on λ and the representation $[k]$. It suffices to show the claim for all q in a dense subset on $L^p(\mathbb{R}^n/\Gamma)$. Hence we may assume that the multiplication by q commutes with the multiplication by ψ_{-k}. The multiplication by the function ψ_k is a bounded operator $M(k)$ of $L^2(\mathbb{R}^n/\Gamma)$. The inverse operator $M(-k)$ multiplies by ψ_{-k} and is also bounded. Thus it is sufficient to estimate the norm of the operator

$$M(-k)\circ q\circ(\lambda\mathbb{1}+\Delta_{[k]})^{-1}\circ M(k)=q\circ M(-k)\circ(\lambda\mathbb{1}+\Delta_{[k]})^{-1}\circ M(k)$$

in terms of $\|q\|_p$. By definition, $\mathbb{1}-\Delta_1$ is a unitary operator from $L^2_4(\mathbb{R}^n/\Gamma)$ into $L^2(\mathbb{R}^n/\Gamma)$. Hence the norm of the operator $M(-k)\circ(\lambda\mathbb{1}+\Delta_{[k]})^{-1}\circ M(k)$ from $L^2(\mathbb{R}^n/\Gamma)$ into $L^2_4(\mathbb{R}^n/\Gamma)$ is equal to the norm of the operator $(\mathbb{1}-\Delta_1)\circ M(-k)\circ(\lambda\mathbb{1}+\Delta_{[k]})^{-1}\circ M(k)$ from $L^2(\mathbb{R}^n/\Gamma)$ into itself. This norm is equal to the supremum over the absolute values of the eigenvalues:

$$\left\|(\mathbb{1}-\Delta_1)\circ M(-k)\circ(\lambda\mathbb{1}+\Delta_{[k]})^{-1}\circ M(k)\right\|=\sup_{\kappa\in\Gamma^\sharp}\left|\frac{1+4\pi^2g(\kappa,\kappa)}{\lambda-4\pi^2g(k+\kappa,k+\kappa)}\right|.$$

Hence this operator is bounded and the second claim follows from the first claim. Moreover, this formula shows that for all representations $\chi_0\in\mathbb{C}^n/\Gamma^\sharp$ and all $\epsilon>0$, there exists a complex number λ_1 and an open neighborhood $U\subset\mathbb{C}^n/\Gamma^\sharp$ of χ_0, so that $C(\chi,\lambda_1)$ is bounded by ϵ for all $\chi\in U$. Thus for all $(q_0,\chi_0)\in L^p(\mathbb{R}^n/\Gamma)\times\mathbb{C}^n/\Gamma^\sharp$, there exists a complex number λ_1 and an open neighborhood $W\subset L^p(\mathbb{R}^n/\Gamma)\times\mathbb{C}^n/\Gamma^\sharp$ of (q_0,χ_0), so that the norm of the operator

$$\mathbb{1}-q\circ(\lambda_1+\Delta_\chi)^{-1}=(\lambda_1\mathbb{1}+\Delta_\chi-q)\circ(\lambda_1+\Delta_\chi)^{-1}$$

is bounded and has a bounded inverse. Since the compact operators form a two-sided ideal (see e.g. [R-S-I, Theorem VI.12]), the operator

$$(\lambda_1\mathbb{1}+\Delta_\chi-q)^{-1}=(\lambda_1\mathbb{1}+\Delta_\chi)^{-1}\circ\left((\lambda_1\mathbb{1}+\Delta_\chi-q)\circ(\lambda_1\mathbb{1}+\Delta_\chi)^{-1}\right)^{-1}$$

is compact. Since the map $\chi\mapsto(\lambda_1\mathbb{1}+\Delta_\chi)^{-1}$ is holomorphic, the map $(q,\chi)\mapsto(\lambda_1\mathbb{1}+\Delta_\chi-q)^{-1}$ is a holomorphic map from W into the set of compact operators on $L^2(\mathbb{R}^n/\Gamma)$.

Thus, due to the Riesz-Schauder Theorem ([R-S-I, Theorem VI.15]) the spectrum of the operator $(\lambda_1\mathbb{1}+\Delta_\chi-q)^{-1}$ is discrete and the only limit point of the spectrum is zero. Now let $S\subset\mathbb{C}$ be the set

$$S=\left\{\lambda_1-\frac{1}{\lambda}\,\middle|\,\lambda\text{ is an eigenvalue of }(\lambda_1\mathbb{1}+\Delta_{\chi_0}-q_0)^{-1}\right\}.$$

Then for all $\lambda\in\mathbb{C}\setminus S$, the operator

$$(\lambda\mathbb{1}+\Delta_{\chi_0}-q_0)^{-1}=\frac{1}{\lambda_1-\lambda}(\lambda_1\mathbb{1}+\Delta_{\chi_0}-q_0)^{-1}\circ\left(\frac{\mathbb{1}}{\lambda_1-\lambda}-(\lambda_1\mathbb{1}+\Delta_{\chi_0}-q_0)^{-1}\right)^{-1}$$

is compact and depends holomorphically on λ.

Now let $\lambda_0\in S$ be a pole of this map and choose a positive number $\epsilon>0$, such that there exists no other pole in the disc $\{\lambda\mid|\lambda-\lambda_0|<2\epsilon\}$. Then for all $(q,\chi)\in L^p(\mathbb{R}^n/\Gamma)\times\mathbb{C}^n/\Gamma^\sharp$, such that

$$\left\|(\lambda_1\mathbb{1}+\Delta_\chi-q)^{-1}-(\lambda_1\mathbb{1}+\Delta_{\chi_0}-q_0)^{-1}\right\|<$$

$$<\sup_{\{\lambda\mid|\lambda-\lambda_0|=\epsilon\}}\left\|\left(\frac{\mathbb{1}}{\lambda_1-\lambda}-(\lambda_1\mathbb{1}+\Delta_{\chi_0}-q_0)^{-1}\right)^{-1}\right\|^{-1},$$

the operator $(\lambda_1 \mathbb{1} + \Delta_\chi - q)^{-1}$ has no eigenvalue on the circle $\{1/(\lambda_1 - \lambda) | \, |\lambda - \lambda_0| = \epsilon\}$. Thus there exists an open neighborhood $W \subset L^p(\mathbb{R}^n/\Gamma) \times \mathbb{C}^n/\Gamma^\sharp$ of (q_0, χ_0) such that $(\lambda \mathbb{1} + \Delta_\chi - q)^{-1}$ is compact for all $(q, \chi) \in W$ and all λ on the circle $\{\lambda | \, |\lambda - \lambda_0| = \epsilon\}$. Now we define a map P from W into the set of compact operators on $L^2(\mathbb{R}^n/\Gamma)$ by

$$P(q, \chi) = -\frac{1}{2\pi\sqrt{-1}} \oint_{\{\lambda| \, |\lambda - \lambda_0| = \epsilon\}} (\lambda \mathbb{1} + \Delta_\chi - q)^{-1} d\lambda.$$

The combination of the results from [R-S-I, VI.5] and [R-S-IV, Appendix to XII.1] implies that for all $(q, \chi) \in W$, $P(q, \chi)$ is a finite-dimensional projector, which commutes with $-\Delta_\chi + q$. And the eigenvalues of the restriction to the corresponding finite-dimensional subspace of $L^2(\mathbb{R}^n/\Gamma)$ are elements of the disc $\{\lambda | \, |\lambda - \lambda_0| < \epsilon\}$. Now let $\phi_1, \ldots, \phi_l$ be a basis of the subspace corresponding to $P(q_0, \chi_0)$, and $\psi_1, \ldots, \psi_l$ a basis of the subspace corresponding to $P^*(q_0, \chi_0)$. Then a neighborhood $W \subset L^p(\mathbb{R}^n/\Gamma) \times \mathbb{C}^n/\Gamma^\sharp$ exists, such that for all $(q, \chi) \in W$, the vectors $P(q, \chi)\phi_1, \ldots, P(q, \chi)\phi_l$ form a basis of the subspace corresponding to $P(q, \chi)$, and $P^*(q, \chi)\psi_1, \ldots, P^*(q, \chi)\psi_l$ form a basis of the subspace corresponding to $P^*(q, \chi)$. Thus the matrix

$$A(q, \chi)_{ij} = \langle P^*(q, \chi)\psi_j, (-\Delta_\chi + q)p(q, \chi)\phi_i \rangle$$

is holomorphic in W and all eigenvalues of this matrix are elements of the disc $\{\lambda | \, |\lambda - \lambda_0| < \epsilon\}$. In particular, the matrix-valued function $(q, \chi, \lambda) \mapsto (\lambda \mathbb{1} - A)^{-1}$ is meromorphic on $W \times \mathbb{C}$. This completes the proof of the theorem. □

In a grater portion of the investigation which follows, we will only consider potentials q, which are elements of the Banach space of all continuous functions on $\mathbb{R}^n/\Gamma$.

Corollary 2.3 *The restriction of the map $(q, \chi, \lambda) \mapsto (\lambda \mathbb{1} + \Delta_\chi - q)^{-1}$ described in Theorem 2.1 to $C(\mathbb{R}^n/\Gamma) \times \mathbb{C}^n/\Gamma^\sharp \times \mathbb{C}$ is a meromorphic map into the compact operators on $C(\mathbb{R}^n/\Gamma)$. Moreover, if λ be an eigenvalue of an operator $-\Delta_\chi + q$, with $q \in C(\mathbb{R}^n/\Gamma)$ and $\chi \in \mathbb{C}^n/\Gamma^\sharp$, then an eigenvalue λ_0 of the operator $-\Delta_\chi$ exists, such that $|\lambda - \lambda_0| \leq \|q\|$.*

Proof: The proof of the first statement is similar to the proof of Theorem 2.1. To prove the second statement, we make use of the Hilbert space $L^2(\mathbb{R}^n/\Gamma)$ with the scalar product $\langle \cdot, \cdot \rangle_\chi$. As a consequence, all eigen spaces of the operator $-\Delta_\chi$ are orthogonal. Moreover, the norm of the operator, which multiplies by q, is equal to the supremum norm $\|q\|$. If λ is an eigenvalue of the operator $-\Delta_\chi + q$, there exists an eigenvector ψ: $(\lambda \mathbb{1} + \Delta_\chi)\psi = q\psi$. The norm of the resolvent satisfies the inequality $\|(\lambda + \Delta_\chi)^{-1}\| \geq 1/\|q\|$. Since $-\Delta_\chi$ is an operator with pure point spectrum, whose eigen spaces are orthogonal, this implies that there exists an eigenvalue λ_0 of $-\Delta_\chi$, such that $1/|\lambda - \lambda_0| \geq 1/\|q\|$. This proves the corollary. □

3 The complex Bloch-space

For a given periodic potential q, the corresponding Schrödinger operator commutes with all shifts by the lattice vectors. Hence these shifts and the Schrödinger operator may be diagonalized simultaneously. The common spectrum of all these operators is called Bloch-variety. In [R-S-IV, Theorem XIII.97] it is shown that the spectrum of a Schrödinger operator with periodic potential is the direct integral over the spectra of the operators $-\Delta_\chi + q$, where

χ runs through all unitary one-dimensional representations of Γ. The Bloch-variety may be considered as the points $(\chi, \lambda) \in \mathbb{R}^n/\Gamma^\sharp \times \mathbb{R}$, so that λ is an eigenvalue of $-\Delta_\chi + q$. In [K-T] the collection of all points $(k, \lambda) \in \mathbb{C}^n \times \mathbb{C}$, so that λ is an eigenvalue of $-\Delta_{[k]} + q$, is called complex Bloch-variety. Let us denote this variety by $B_{\text{variety}}(q)$. The action of the dual lattice $\Gamma^\sharp$ on $\mathbb{C}^n$ given by $\kappa.k = \kappa + k$ for all $\kappa \in \Gamma^\sharp$ and all $k \in \mathbb{C}^n$, lifts to an action of the dual lattice $\Gamma^\sharp$ on this subspace $B_{\text{variety}}(q)$ of $\mathbb{C}^n \times \mathbb{C}$. The set of orbits of this action is equal to the subspace $B_{\text{variety}}(q)/\Gamma^\sharp$ of all elements $(\chi, \lambda) \in \mathbb{C}^n/\Gamma^\sharp \times \mathbb{C}$, so that λ is an eigenvalue of the operator $-\Delta_\chi + q$. Let us call this variety, which is locally biholomorphic to $B_{\text{variety}}(q)$, also complex Bloch-variety. It will be clear from the context, whether we should consider $B_{\text{variety}}(q)$ or $B_{\text{variety}}(q)/\Gamma^\sharp$. Later we will introduce the complex Bloch-space as a slightly different complex space.

Proposition 3.1 *The complex Bloch-variety $B_{\text{variety}}(q)/\Gamma^\sharp$ is an irreducible reduced analytic subset of $\mathbb{C}^n/\Gamma^\sharp \times \mathbb{C}$ in the sense of [Gr-Re, Chapter IV. §1.1.]. The map $(\chi, \lambda) \mapsto \chi$ defines an infinitely-sheeted, but locally finite-sheeted analytic covering map from $B_{\text{variety}}(q)/\Gamma^\sharp$ onto the analytic space $\mathbb{C}^n/\Gamma^\sharp$, which is denoted by χ, and the map $(\chi, \lambda) \mapsto \lambda$ defines a morphism from $B_{\text{variety}}(q)/\Gamma^\sharp$ onto $\mathbb{C}$, which is denoted by λ, respectively.*

Proof: Due to Theorem 2.1, all points $(\chi_0, \lambda_0) \in B_{\text{variety}}(q)/\Gamma^\sharp$ have an open neighborhood $U \subset \mathbb{C}^n/\Gamma^\sharp \times \mathbb{C}$, such that $U \cap B_{\text{variety}}(q)/\Gamma^\sharp$ may be described by the zeros of the holomorphic function

$$(\chi, \lambda) \mapsto \det(\lambda \mathbb{1} - A(q, \chi)),$$

where $A(q, \chi)$ is the restriction of $-\Delta_\chi + q$ to a finite-dimensional invariant subspace of $L^2(\mathbb{R}^n/\Gamma)$. We will prove in Corollary 5.4 that the complex manifold of all regular points of $B_{\text{variety}}(q)/\Gamma^\sharp$ is connected. Hence $B_{\text{variety}}(q)/\Gamma^\sharp$ is irreducible. By definition, the complex Bloch-variety is a Weierstraß-covering, and therefore a Cohen-Macaulay-space (see [G-P-R, Chapter II. §5.]). Since in Lemma 3.2 we shall prove that the complex Bloch-variety is generically reduced, we may apply [G-P-R, Chapter II. Corollary 5.10] and conclude that the complex Bloch-variety is reduced. $\square$

In the last section, we showed with the help of the resolvent that the map $\chi \mapsto -\Delta_\chi + q$ defines a closed-unbounded-operator-valued holomorphic function on $\mathbb{C}^n/\Gamma^\sharp$. Now let $\mathcal{O}_q$ be the sheaf of bounded-operator-valued holomorphic functions on $\mathbb{C}^n/\Gamma^\sharp$, which commute pointwise with $-\Delta_\chi + q$. Clearly this is a sheaf of algebras over $\mathbb{C}/\Gamma^\sharp$, and each local section of this sheaf can be diagonalized simultaneously with the corresponding operators $-\Delta_\chi + q$. The complex Bloch-space is then defined as a topological space $B_{\text{complex space}}(q)/\Gamma^\sharp$ together with a structure sheaf $\mathcal{O}_{B_{\text{complex space}}(q)/\Gamma^\sharp}$, which is locally isomorphic to the eigenvalues of $\mathcal{O}_q$. So let us now construct this complex Bloch-space in three steps:

1. The first step is the observation contained in Theorem 2.1 that for any eigenvalue λ_0 of the operator $-\Delta_{\chi_0} + q$, there exists a neighborhood $U \subset \mathbb{C}^n/\Gamma^\sharp$ of χ_0 and a finite number of holomorphic maps $v_1, \ldots, v_l$ from U into the Hilbert space $L^2(\mathbb{R}^n/\Gamma)$, so that for all $\chi \in U$ the span of $v_1(\chi), \ldots, v_l(\chi)$ is an invariant subspace of the operator $-\Delta_\chi + q$ and contains all eigenvectors, whose eigenvalues are in a small neighborhood of λ_0. Moreover, the neighborhood U and the vectors $v_1, \ldots, v_l$ may be chosen in such a way that the span of $v_1, \ldots, v_l$ would be an invariant subspace of all sections of $\mathcal{O}_q$ over U. This shows that $\mathcal{O}_q$ may be locally considered as a sheaf of all holomorphic $l \times l$-matrix-valued functions, which commute with one such function.

2. In the second step, we shall use the following

Lemma 3.2 *In each neighborhood of any $\chi_0 \in \mathbb{C}^n/\Gamma^\sharp$, there exists an element χ such that all eigenvalues of $-\Delta_\chi + q$ are different.*

Proof: In the proof of Corollary 5.4, we will use the fact that there exists an open subset of $B_{\text{variety}}(q)/\Gamma^\sharp$, for whose elements (χ, λ) the multiplicity of the eigenvalue λ of $-\Delta_\chi + q$ is equal to one. Thus due to Corollary 5.4, the set of all points $(\chi, \lambda) \in B_{\text{variety}}(q)/\Gamma^\sharp$, whose multiplicity is equal to one, is a connected submanifold of the regular part of $B_{\text{variety}}(q)/\Gamma^\sharp$. The covering map $\chi : B_{\text{variety}}(q)/\Gamma^\sharp \to \mathbb{C}^n/\Gamma^\sharp$ has countable sheets. Hence in any open subset of $\mathbb{C}^n/\Gamma^\sharp$, there exists an element, with no branchpoint over this point. □

This lemma implies that $\mathcal{O}_q$ is a sheaf of commutative algebras.

3. In the third step we construct the spectral complex space of a holomorphic function f from an open connected subset $U \subset \mathbb{C}^n$ into the $l \times l$-matrices. We assume that all eigenvalues of f are different at some point of U. The discriminant of the characteristic polynomial of f is a holomorphic function. Hence all the eigenvalues of f are different for an open dense subset of U. Again we define $\mathcal{O}_f$ as the sheaf of all holomorphic $l \times l$-matrix-valued functions, which commute with f. Due to our assumption that all eigenvalues of f are different at some point of U, this is a sheaf of commutative algebras over $\mathbb{C}$. Again we define the spectral variety as

$$X_{\text{variety}}(f) := \left\{(u, \lambda) \in U \times \mathbb{C} \middle| \ \det(\lambda \mathbb{1} - f(u)) = 0\right\}.$$

For each point $(u_0, \lambda_0) \in X_{\text{variety}}(f)$, a neighborhood $V \subset U$ of u_0 and a holomorphic function (see e.g. [R-S-IV, Appendix to XII.1])

$$P : V \to l \times l\text{-matrices}, v \mapsto -\frac{1}{2\pi\sqrt{-1}} \oint\limits_{\{\lambda | \, |\lambda - \lambda_0| = \epsilon\}} (\lambda \mathbb{1} - f(v))^{-1} d\lambda$$

exists, so that P is a projector in $\mathcal{O}_f$ over V and all eigenvalues of the restriction of $f(u_0)$ to the subspace of $\mathbb{C}^l$ corresponding to $P(u_0)$ are equal. This is equivalent to the statement that the polynomial $\det(\lambda \mathbb{1} - f(u_0))$ has no other zeros than $\lambda = \lambda_0$ inside the disc $\{\lambda | \, |\lambda - \lambda_0| \leq \epsilon\}$, which may always be attained by diminishing ϵ.

Lemma 3.3 *The subset of all points $(u_0, \lambda_0) \in X_{\text{variety}}(f)$, in which the germ of the projector $P(u_0)$ described above is not a minimal projector of the stalk of $\mathcal{O}_f$ at the point u_0, is closed.*

Proof: Let $p_{f(u)}(T) = \det(t\mathbb{1} - f(u))$ be the characteristic polynomial of $f(u)$. For all $(u, \lambda) \in X_{\text{variety}}(f)$, the rational function $p_{f(u)}(T)/(T - \lambda)$ may be considered as a polynomial with respect to T of degree $l - 1$. It is quite obvious that the subspace of the lemma is given by all irreducible components of dimension $n - 1$ of the subvariety of $X_{\text{variety}}(f)$, which is defined by

$$\left\{(u, \lambda) \in X_{\text{variety}}(f) \,\middle|\, \frac{p_{f(u)}(f(u))}{f(u) - \lambda \mathbb{1}} = 0\right\}.$$

□

Since $\mathcal{O}_f$ is a sheaf of commutative algebras, each stalk has only a finite number of minimal projectors. Now let $p_1, \ldots, p_k$ be all minimal projectors of the stalk of $\mathcal{O}_f$ at some point $u_0 \in U$. The $p_1, \ldots, p_k$ extend to some sections of $\mathcal{O}_f$ over a neighborhood $U_1 \subset U$ of

u_0. Hence we may decompose the vector space $\mathbb{C}^l$ over U_1 into a direct sum of vector spaces and the function f into a direct sum of matrices $f = f_1 \oplus \ldots \oplus f_k$. Due to the preceding lemma, there exists a neighborhood $U_2 \subset U_1 \subset U$ of u_0, such that the corresponding subset of the spectral varieties $X_{\text{variety}}(f_1), \ldots, X_{\text{variety}}(f_k)$ of $f_1, \ldots, f_k$ over U_2 described in the lemma is empty. The spectral complex space of f in the neighborhood U_2 of u_0 is now defined as the disjoint union of the spectral varieties

$$X_{\text{variety}}(f_1) \cup \ldots \cup X_{\text{variety}}(f_k).$$

The whole spectral complex space is then defined by gluing together the local spectral varieties in the natural way. We will denote in the sequel the elements of the spectral complex space $X_{\text{complex space}}(f)$ by pairs (u, λ). Such a pair does not uniquely determine an element of the spectral complex space, but locally it does. Due to Proposition 3.1, the spectral complex space $X_{\text{complex space}}(f)$ is an analytic covering (see e.g. [G-P-R, Chapter I. §12.]) of the spectral variety, which itself is an analytic covering of U. The composition of these analytic coverings is denoted by $\pi : X_{\text{complex space}}(f) \to U$. As a set the spectral complex space $X_{\text{complex space}}(f)$ may be identified with all minimal projectors of $\mathcal{O}_f$ in all stalks over U. So let us call the natural homomorphism from $\mathcal{O}_f$ into the direct image sheaf $\pi_*(\mathcal{O}_{X_{\text{complex space}}(f)})$ of the structure sheaf of the spectral complex space eigenvalue homomorphism.

Lemma 3.4 *The eigenvalue homomorphism is an isomorphism of the sheaves*

$$\mathcal{O}_f \simeq \pi_*(\mathcal{O}_{X_{\text{complex space}}(f)}).$$

Proof: For each $u_0 \in U$, the sum over all minimal projections of the stalk of $\mathcal{O}_f$ at the point u_0 is equal to $\mathbb{1}$. And the product of two different minimal projections is zero. This proves the claim for each stalk. $\square$

The projection $P(f)$ onto the eigen space of f with eigenvalue λ can be written as a matrix, whose entries are rational functions in terms of the entries of f and the eigenvalue λ. Hence $P(f)$ defines a meromorphic $l \times l$-matrix-valued function on the spectral complex space.

Lemma 3.5 **(i)** *The divisor of $P(f)$ is equal to the inverse of the branching divisor of the natural analytic covering $\pi : X_{\text{complex space}}(f) \to U$.*

(ii) *The sum over all sheets of the analytic covering $\pi : X_{\text{complex space}}(f) \to U$ of $P(f)$ is equal to $\mathbb{1}$.*

Proof: Based on the assumption that all eigenvalues of f are different at some point of u, the trace of $P(f)$ is equal to 1. But this implies that the set of singularities of $P(f)$ is equal to the set of branchpoints of the covering map $\pi : X_{\text{complex space}}(f) \to U$ of $P(f)$. This proves (i). The second statement is obvious at all points $u_0 \in U$, at which f has l different eigenvalues. Now continuity implies (ii). $\square$

Finally we want to remark that, in general, the branch locus of the covering map $\pi : X_{\text{complex space}}(f) \to U$ of $P(f)$ is contained in the branch locus of the covering map mentioned above $X_{\text{variety}}(f) \to U$. This completes the third step.

The first step reduces the problem of the construction of the complex Bloch-space to the construction of the spectral complex space of a holomorphic matrix-valued function. In the second step the assumption of the third step is fulfilled. Finally in the third step we construct this spectral complex space. All of these steps together imply

Theorem 3.6 *The complex Bloch-space is an infinitely-sheeted, locally finite-sheeted analytic covering of the complexification of the dual group of the lattice* Γ*:*

$$\chi : B_{\text{complex space}}(q)/\Gamma^\sharp \to \mathbb{C}^n/\Gamma^\sharp.$$

In particular, each element $\gamma \in \Gamma$ *defines a holomorphic function*

$$\chi(\gamma) : B_{\text{complex space}}(q)/\Gamma^\sharp \to \mathbb{C}^*.$$

Moreover, the eigenvalue λ *of the operator* $-\Delta_\chi + q$ *may also be considered as a holomorphic function of* $B_{\text{complex space}}(q)/\Gamma^\sharp$*. Finally, the projection* $P(q)$ *on the eigen space of* $-\Delta_\chi+q$ *with eigenvalue* λ *is a meromorphic function of the complex Bloch-space into the set of bounded operators of* $L^2(\mathbb{R}^n/\Gamma)$*, which satisfies the following conditions:*

(i) *The trace of* $P(q)$ *and hence also the dimension of the eigen space is equal to* 1*.*

(ii) $P(q) \circ (-\Delta_\chi + q) = (-\Delta_\chi + q) \circ P(q) = \lambda \cdot P(q)$.

(iii) *The inverse of the divisor of* $P(q)$ *is equal to the branching divisor of the analytic covering* $\chi : B_{\text{complex space}}(q)/\Gamma^\sharp \to \mathbb{C}^n/\Gamma^\sharp$*.*

(iv) *The sum over all sheets of the analytic covering* $\chi : B_{\text{complex space}}(q)/\Gamma^\sharp \to \mathbb{C}^n/\Gamma^\sharp$ *of* $P(q)$ *converges in the strong operator topology to* $1\!\!1$ *.*

□

4 An embedding into the projective Hilbert space

Due to Theorem 3.6, there exists an holomorphic map of the regular part of $B_{\text{complex space}}(q)$ into the projective space of $L^2(\mathbb{R}^n/\Gamma)$:

$$\text{regular part of } B_{\text{complex space}}(q) \to \mathbb{P}L^2(\mathbb{R}^n/\Gamma), (k,\lambda) \mapsto \text{the image of } P(k,\lambda).$$

In this section, we renormalize this map, such that it may be extended to the high energy limit, and then determine the closure of the image of this map. It will turn out that this map is an embedding, and we will give an explicit description of the closure of this image.

In the limit $k \to \infty$, the character $[k]$ and therefore also the eigen functions have an essential singularity. In order to remove this singularity we make the following ansatz for the eigen functions:

$$\psi = \exp\left(2\pi\sqrt{-1}g(x,k)\right) \sum_{\kappa\in\Gamma^\sharp} \tilde{\psi}(\kappa) \exp\left(2\pi\sqrt{-1}g(x,\kappa)\right).$$

For all $k \in \mathbb{C}^n$ the map $\psi \mapsto (\tilde{\psi}(\kappa))_{\kappa\in\Gamma^\sharp}$ is an bounded operator from $L^2(\mathbb{R}^n/\Gamma)$ onto $\ell^2(\Gamma^\sharp)$ with bounded inverse. Hence the eigen space defines a holomorphic map from the regular part of $B_{\text{complex space}}(q)$ into the projective space of $\ell^2(\Gamma^\sharp)$. The dual lattice acts on the projective space of the Hilbert space $\ell^2(\Gamma^\sharp)$ by translation:

$$\kappa.\left(\tilde{\psi}(\kappa')\right)_{\kappa'\in\Gamma^\sharp} = \left(\tilde{\psi}(\kappa'-\kappa)\right)_{\kappa'\in\Gamma^\sharp}.$$

And the embedding of the regular part of $B_{\text{complex space}}(q)$ into the projective space is invariant under the action of $\Gamma^\sharp$ on both spaces. In order to investigate this map we use the Fourier transform $\hat{q}$ of q:

$$q(x) = \sum_{\kappa\in\Gamma^\sharp} \hat{q}(\kappa)\exp\left(2\pi\sqrt{-1}g(x,\kappa)\right).$$

The eigenvalue equation may be written as a set of equations labeled by $\kappa \in \Gamma^\sharp$:

$$\left(\lambda - 4\pi^2 g(k,k) - 8\pi^2 g(k,\kappa)\right)\tilde{\psi}(\kappa) = 4\pi^2 g(\kappa,\kappa)\tilde{\psi}(\kappa) + \sum_{\kappa'\in\Gamma^\sharp} \hat{q}(\kappa-\kappa')\tilde{\psi}(\kappa').$$

For all finite sequences $\kappa_0,\ldots,\kappa_m$ of pairwise different elements of $\Gamma^\sharp$ and some non-zero numbers $c_0,\ldots,c_m$ with the properties

$$c_0\kappa_0 + \ldots + c_m\kappa_m = 0 \text{ and } c_0 + \ldots + c_m = 0,$$

we get an equation of the form

$$4\pi^2\left(\sum_{i=0}^m c_i g(\kappa_i,\kappa_i)\right)\left(\prod_{i=0}^m \tilde{\psi}(\kappa_i)\right) + \sum_{i=0}^m c_i \prod_{j\neq i}\tilde{\psi}(\kappa_j)\sum_{\kappa\in\Gamma^\sharp}\hat{q}(\kappa_i-\kappa)\tilde{\psi}(\kappa) = 0.$$

Each element κ_i of the sequence may be written as $\kappa_i = \sum_{j\neq i} -c_j/c_i\ \kappa_j$, with $\sum_{j\neq i} -c_j/c_i = 1$. Hence all elements of the sequence belong to one affine subspace of dimension at most $m-1$. If the dimension is less than $m-1$, the two subsequences $\kappa_0,\ldots,\kappa_{m-1}$ and $\kappa_1,\ldots,\kappa_m$ satisfy the same conditions with some constants $c'_0,\ldots,c'_{m-1}$ and $c''_1,\ldots,c''_m$, respectively. Furthermore these constants may be chosen in such a way that $c_0 = c'_0, c_1 = c'_1 + c''_1,\ldots,c_{m-1} = c'_{m-1} + c''_{m-1}, c_m = c''_m$. The sum of the equation corresponding to the sequence $\kappa_0,\ldots,\kappa_{m-1}$ multiplied by $\tilde{\psi}(\kappa_m)$ and the equation corresponding to the sequence $\kappa_1,\ldots,\kappa_m$ multiplied by $\tilde{\psi}(\kappa_0)$ gives the equation corresponding to the whole sequence $\kappa_0,\ldots,\kappa_m$. Hence all equations corresponding to sequences $\kappa_0,\ldots,\kappa_m$, in which all κ are elements of a $(m-1)$-dimensional affine subspace imply the equations corresponding to arbitrary sequences. Let us call m elements of a vector space affine independent, if they generate a $(m-1)$-dimensional affine subspace. Moreover, if

> all subsequences of $\kappa_0,\ldots,\kappa_m$ of length m are affine independent and generate the same $m-1$-dimensional affine subspace, **(sequence condition)**

then unique non-zero numbers $c_0 = 1, c_1,\ldots,c_m$ would be, so that

$$c_0\kappa_0 + \ldots + c_m\kappa_m = 0 \text{ and } c_0 + \ldots + c_m = 0.$$

Now we define $\mathcal{B}(q)$ as the closed subspace of the projective space of $\ell^2(\Gamma^\sharp)$, defined by the equations

$$4\pi^2\left(\sum_{i=0}^m c_i g(\kappa_i,\kappa_i)\right)\left(\prod_{i=0}^m \tilde{\psi}(\kappa_i)\right) + \sum_{i=0}^m c_i \prod_{j\neq i}\tilde{\psi}(\kappa_j)\sum_{\kappa\in\Gamma^\sharp}\hat{q}(\kappa_i-\kappa)\tilde{\psi}(\kappa) = 0.$$

for all sequences $\kappa_0,\ldots,\kappa_m$ in $\Gamma^\sharp$ obeying the sequence condition and the corresponding numbers $c_0,\ldots,c_m$. Due to the sequence condition, the length of all these sequences is bounded by $m+1 \leq n+2$. On the other hand, sequences of length 2, which obey the sequence condition, have to consist of two equal elements. The corresponding equation would be trivial. Hence we may assume $1 < m \leq n+1$.

We equip $\mathcal{B}(q)$ with the quotient sheaf of the sheaf of holomorphic functions of the projective space of $\ell^2(\Gamma^\sharp)$ restricted to $\mathcal{B}(q)$ divided by the subsheaf of all holomorphic functions, which vanish on $\mathcal{B}(q)$.

Let K be a subset of $\mathbb{R}^n$, such that $\Gamma^\sharp \cap K$ is a lattice in K of the same dimension as K. The potential

$$q_K(x) := \sum_{\kappa \in K \cap \Gamma^\sharp} \exp(2\pi\sqrt{-1}g(x,\kappa))\hat{q}(\kappa)$$

is constant on the orthogonal complement of K. In fact, it is the average of the potential q over the orthogonal complement of K. So let us consider q_K as a potential on K, which is periodic with respect to the dual lattice of $\Gamma^\sharp \cap K$. By definition, $\mathcal{B}(q_K)$ is a subspace of $\ell^2(\Gamma^\sharp \cap K)$. The intersection of any sequence $\kappa_0, \ldots, \kappa_m$ in $\Gamma^\sharp$, which obey the sequence condition, with K either contains less than m elements or the whole sequence. Hence the map

$$\mathcal{B}(q_K) \hookrightarrow \mathcal{B}(q), \left(\tilde{\psi}_K(\kappa)\right)_{\kappa \in \Gamma^\sharp \cap K} \mapsto \left(\tilde{\psi}(\kappa)\right)_{\kappa \in \Gamma^\sharp}, \text{ with } \tilde{\psi}(\kappa) = \begin{cases} \tilde{\psi}_K(\kappa) & \text{if } \kappa \in K, \\ 0 & \text{if } \kappa \notin K. \end{cases}$$

defines an embedding of $\mathcal{B}(q_K)$ into $\mathcal{B}(q)$.

Let $\mathcal{I}_m$ be the set of all m-dimensional real subspaces K of the dual space of our Euclidean space $\mathbb{R}^n$, so that the intersection of K with the dual lattice $\Gamma^\sharp$ is an m-dimensional lattice in K. This implies that the intersection of Γ with the orthogonal complement $K^\perp$ of K is an $(n-m)$-dimensional lattice in $K^\perp$. Hence the quotient lattice $\Gamma_K := \Gamma/(\Gamma \cap K^\perp)$ may be considered as an m-dimensional lattice in K, which is the dual lattice of the lattice $K \cap \Gamma^\sharp$. Now let $\mathcal{I}$ denote the union $\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_1 \cup \ldots \cup \mathcal{I}_n$. The complexification of an element $K \in \mathcal{I}$ is denoted by $\mathbb{C}K$.

Not all of the potentials $q_K, K \in \mathcal{I}$ have to be different. It may happen that the whole potential is equal to an averaged potential q_K, with $\dim K < n$. In this case there exists an unique $K_{\text{effective}}$, which is the intersection of all K, so that $q = q_K$. And the Schrödinger operator commutes with all differential operators corresponding to elements of the orthogonal complement $K^\perp_{\text{effective}}$ of $K_{\text{effective}}$. Hence the complex Bloch-space is the product of the complex Bloch-space of $q_{\text{effective}} = q_{K_{\text{effective}}}$ times the Bloch-space of the zero potential on $K^\perp_{\text{effective}}/(K^\perp_{\text{effective}} \cap \Gamma)$. Here the total energy is the sum of the two energies $\lambda = \lambda_{K_{\text{effective}}} + \lambda_{K^\perp_{\text{effective}}}$.

Theorem 4.1 *If $\dim K_{\text{effective}}$ is equal to n, this means that q is an effective n-dimensional potential, the subspace $\mathcal{B}_0(q)$ of $\mathcal{B}(q)$, containing all elements $\left(\tilde{\psi}(\kappa)\right)_{\kappa \in \Gamma^\sharp}$, so that there exists $n+1$ affine independent elements $\kappa_0, \ldots, \kappa_n$ of $\Gamma^\sharp$, with $\tilde{\psi}(\kappa_0) \neq 0, \ldots, \tilde{\psi}(\kappa_n) \neq 0$, is a complex space. If in addition the potential q is continuous, there exists a morphism from this complex space onto the complex Bloch-space $B_{\text{complex space}}(q)$, such that the restriction of this morphism to the preimage of the regular part of $B_{\text{complex space}}(q)$ is the inverse of the natural embedding from the regular part of $B_{\text{complex space}}(q)$ into the projective space of $\ell^2(\Gamma^\sharp)$. If $n > \dim K_{\text{effective}} > 0$, $\mathcal{B}(q)$ is equal to the orbit of $\mathcal{B}(q_{\text{effective}})$ in the projective space of $\ell^2(\Gamma^\sharp)$ under the action of $\Gamma^\sharp$.*

Proof: Let $\left(\tilde{\psi}(\kappa)\right)_{\kappa\in\Gamma^\sharp}$ be an element of $\mathcal{B}(q)$, such that there exists $n+1$ affine independent elements $\kappa_0,\ldots,\kappa_n$ of $\Gamma^\sharp$, with $\tilde{\psi}(\kappa_0)\neq 0,\ldots,\tilde{\psi}(\kappa_n)\neq 0$, then the equation

$$\left(\lambda - 4\pi^2 g(k,k) - 8\pi^2 g(k,\kappa)\right)\tilde{\psi}(\kappa) = 4\pi^2 g(\kappa,\kappa)\tilde{\psi}(\kappa) + \sum_{\kappa'\in\Gamma^\sharp}\hat{q}(\kappa-\kappa')\tilde{\psi}(\kappa'),$$

with $\kappa=\kappa_0,\ldots,\kappa_n$ uniquely determine the complex functions λ and k in terms of the components of $\left(\tilde{\psi}(\kappa)\right)_{\kappa\in\Gamma^\sharp}$. Now the defining equations of $\mathcal{B}(q)$ guarantees that this equation is fulfilled for all $\kappa\in\Gamma^\sharp$. Hence (k,λ) defines a map from the subspace $\mathcal{B}_0(q)$ into $B_{\text{variety}}(q)/\Gamma^\sharp$. Moreover, we conclude from the definition of $B_{\text{complex space}}(q)$ that (k,λ) defines a map into $B_{\text{complex space}}(q)$. In the last section, we showed that locally the operator $-\Delta_{[k]}+q$ may be restricted to a finite-dimensional subspace of $L^2(\mathbb{R}^n/\Gamma)$, which depend holomorphically on k. Hence near all elements $\left(\tilde{\psi}(\kappa)\right)_{\kappa\in\Gamma^\sharp}$ of $\mathcal{B}(q)_0$, all components of the eigen functions $\left(\tilde{\psi}(\kappa)\right)_{\kappa\in\Gamma^\sharp}$ generate the holomorphic functions of a complex space, and there exists a morphism from this complex space into $B_{\text{complex space}}(q)$. Next this morphism is shown to be onto, if the potential q is continuous. Let $\left(\tilde{\psi}(\kappa)\right)_{\kappa\in\Gamma^\sharp}$ be an element of $\mathcal{B}(q)$ such that for all affine independent lattice vectors $\kappa_0,\ldots,\kappa_n$, there exists one κ_i with $\tilde{\psi}(\kappa_i)=0$. Hence there exists a minimal affine subspace K of dimension $m<n$, which contains all κ with $\tilde{\psi}(\kappa)\neq 0$. Since $\mathcal{B}(q)$ is invariant under the action of $\Gamma^\sharp$, we may assume that this affine subspace is a linear subspace. The element $\left(\tilde{\psi}(\kappa)\right)_{\kappa\in\Gamma^\sharp}$ is in the image of an embedding $\mathcal{B}(q_K)\hookrightarrow\mathcal{B}(q)$, and there exists affine independent elements $\kappa_0,\ldots,\kappa_m$ of $\Gamma^\sharp$, so that $\tilde{\psi}(\kappa_0)\neq 0,\ldots,\tilde{\psi}(\kappa_m)\neq 0$. Now we apply the construction above to the potential q_K and obtain a unique element $(k_K,\lambda_K)\in B_{\text{complex space}}(q_K)$, such that

$$\psi_K = \exp\left(2\pi\sqrt{-1}g(x,k_K)\right)\sum_{\kappa\in\Gamma^\sharp\cap K}\tilde{\psi}(\kappa)\exp\left(2\pi\sqrt{-1}g(x,\kappa)\right)$$

defines an eigen function in $L^2(K/(K\cap\Gamma^\sharp))$ of the Schrödinger operator with potential q_K and eigenvalue λ_K. Now we assume that there exists $(k,\lambda)\in B_{\text{complex space}}(q)$, such that the function

$$\psi = \exp\left(2\pi\sqrt{-1}g(x,k)\right)\sum_{\kappa\in\Gamma^\sharp}\tilde{\psi}(\kappa)\exp\left(2\pi\sqrt{-1}g(x,\kappa)\right)$$

defines an eigen function of $-\Delta_{[k]}+q$ with eigenvalue λ. The equation

$$\left(\lambda - 4\pi^2 g(k,k) - 8\pi^2 g(k,\kappa)\right)\tilde{\psi}(\kappa) = 4\pi^2 g(\kappa,\kappa)\tilde{\psi}(\kappa) + \sum_{\kappa'\in\Gamma^\sharp}\hat{q}(\kappa-\kappa')\tilde{\psi}(\kappa'),$$

with $\kappa\notin K$, show that

$$\sum_{\kappa'\in\Gamma^\sharp}\hat{q}(\kappa-\kappa')\tilde{\psi}(\kappa') = 0 \text{ for all } \kappa\notin K.$$

If $\dim K=0$, this would be a contradiction to $\dim K_{\text{effective}}=n$. If $\dim K>0$, it would imply that the eigen function $\psi_K(x)$ is equal to zero for all $x\in K$, so that there exists an element $x^\perp\in K^\perp$ with $q_K(x)\neq q(x+x^\perp)$. Due to the assumption $\dim K_{\text{effective}}=n$, for all elements x of an open subset of K, there exists some $x^\perp$ with with $q_K(x)\neq q(x+x^\perp)$. Hence the eigen function $\psi_K(x)$ is zero for all elements x of an open subset of K. This is a contradiction to [R-S-IV, Theorem XIII.63]. We conclude that if $\dim K_{\text{effective}} = n$, then all sequences $\left(\tilde{\psi}(\kappa)\right)_{\kappa\in\Gamma^\sharp}$ in $\mathcal{B}(q)$, which corresponds to an eigenvector of $-\Delta_{[k]}+q$ with eigenvalue $\lambda\in\mathbb{C}$ and $k\in\mathbb{C}^n$, contains $l+1$ non-zero elements $\tilde{\psi}(\kappa_0)\neq 0,\ldots,\tilde{\psi}(\kappa_n)\neq 0$, whose indices $\kappa_0,\ldots,\kappa_n$ are affine independent. This proves the claim. The restriction of the morphism constructed above to the preimage of the regular part of $B_{\text{complex space}}(q)$ is by construction the inverse of the natural holomorphic map from the regular part of $B_{\text{complex space}}(q)$ into the projective space of $\ell^2(\Gamma^\sharp)$, where (k,λ) is mapped onto the elements $\left(\tilde{\psi}(\kappa)\right)_{\kappa\in\Gamma^\sharp}$ of $\mathcal{B}(q)$, so that

$$\psi = \exp\left(2\pi\sqrt{-1}g(x,k)\right)\sum_{\kappa\in\Gamma^\sharp}\tilde{\psi}(\kappa)\exp\left(2\pi\sqrt{-1}g(x,\kappa)\right)$$

defines an eigen function of $-\Delta_{[k]}+q$ with eigenvalue λ. In particular this map is an embedding.

Now let us assume that $n>\dim K_{\text{effective}}>0$. Moreover, let us assume for a moment an element $\left(\tilde{\psi}(\kappa)\right)_{\kappa\in\Gamma^\sharp}$ of $\mathcal{B}(q)$ and affine independent elements $\kappa_0,\ldots,\kappa_m$, so that

(i) $\tilde{\psi}(\kappa_0)\neq 0,\ldots,\tilde{\psi}(\kappa_m)\neq 0$,

(ii) the span K of $\kappa_1-\kappa_0,\ldots,\kappa_m-\kappa_0$ is not contained in $K_{\text{effective}}$,

(iii) and $\tilde{\psi}(\kappa+\kappa_0)\neq 0$ implies $\kappa\in K$.

We may apply the construction above and obtain an unique $(k,\lambda_K)\in B_{\text{complex space}}(q_K)$, such that

$$\begin{aligned}\psi_K &= \exp\left(2\pi\sqrt{-1}g(x,k)\right)\sum_{\kappa\in\Gamma^\sharp\cap K}\tilde{\psi}(\kappa+\kappa_0)\exp\left(2\pi\sqrt{-1}g(x,\kappa)\right)\\ &= \exp\left(2\pi\sqrt{-1}g(x,k-\kappa_0)\right)\sum_{\kappa\in\Gamma^\sharp}\tilde{\psi}(\kappa+\kappa_0)\exp\left(2\pi\sqrt{-1}g(x,\kappa)\right)\end{aligned}$$

defines an eigen function in $L^2(K/(K\cap\Gamma^\sharp)$ of the Schrödinger operator with potential q_K and eigenvalue λ_K. K is not contained in $K_{\text{effective}}$, and the potential q_K is equal to $q_{K\cap K_{\text{effective}}}$. But for all eigen functions of such a potential there exists an affine proper subspace of K, which contains all κ with $\tilde{\psi}(\kappa)\neq 0$. if κ. This contradicts (i) and proves the last statement of the theorem. $\square$

In the case $\dim K_{\text{effective}}=0$, the potential is constant and $\mathcal{B}(q)$ degenerates to the subspace

$$\left\{\left(\tilde{\psi}(\kappa)\right)_{\kappa\in\Gamma^\sharp}\middle|\text{ there exists }\kappa_1,\kappa_2,\text{ so that }\tilde{\psi}(\kappa)\neq 0\Rightarrow\kappa\in\{\kappa_1,\kappa_2\}\right\}.$$

But the image of the embedding of $B_{\text{complex space}}(q)$ into the projective space is equal to the subspace

$$\left\{\left(\tilde{\psi}(\kappa)\right)_{\kappa\in\Gamma^\sharp}\middle|\text{ there exists }\kappa_1,\text{ such that }\tilde{\psi}(\kappa)\neq 0\Rightarrow\kappa=\kappa_1\right\}.$$

So let us change in this case the definition of $\mathcal{B}(q)$ to the second subspace, which is the orbit of one point under the action of $\Gamma^\sharp$.

For a given potential q we may restrict the set of components $\mathcal{I}$ to the set

$$\mathcal{I}_{\text{effective}}:=\mathcal{I}_{\text{effective},0}\cup\ldots\cup\mathcal{I}_{\text{effective},n}\text{ with }\mathcal{I}_{\text{effective},0}=\mathcal{I}_0\text{ and}$$

$$\mathcal{I}_{\text{effective},m} := \{K \in \mathcal{I}_m \mid \text{ there exists no proper subspace } K' \in \mathcal{I}_{m-1} \text{ with } q_K = q_{K'}.\}$$

for all $0 < m \leq n$. The combination of the embeddings $\mathcal{B}(q_K) \hookrightarrow \mathcal{B}(q)$ and the preceding theorem shows that as a set $\mathcal{B}(q)$ may be identified with

$$\mathcal{B}(q) = \bigcup_{K \in \mathcal{I}_{\text{effective}}} \Gamma^\sharp.\mathcal{B}_0(q_K).$$

We will prove in the next section that $\mathcal{B}_0(q)$ is dense in $\mathcal{B}(q)$. Moreover, all the maps

$$\mathcal{B}_0(q_K) \to B_{\text{complex space}}(q_K) \to \mathbb{C}K$$

together form the map

$$\mathcal{B}(q) \to \bigcup_{K \in \mathcal{I}_{\text{effective}}} \Gamma^\sharp.\mathcal{B}_0(q_K) \to \bigcup_{K \in \mathcal{I}_{\text{effective}}} \mathbb{C}K.$$

Let us now assume that $\mathcal{I}_{\text{effective}}$ is finite and that $\dim K_{\text{effective}} = n$. Then $\mathcal{I}_{\text{effective},1}$ contains at least n elements $K_1, \ldots, K_N$. So let us choose generators $\kappa_1, \ldots, \kappa_N$ of the one-dimensional lattices $K_1 \cap \Gamma^\sharp, \ldots, K_N \cap \Gamma^\sharp$, respectively. For all minimal subsets $R \subset \{1, \ldots, N\}$, so that the vectors $(\kappa_i)_{i \in R}$ are linear dependent, there exist a relation of the form

$$\sum_{i \in R} n_{R,i} \kappa_i = 0 \text{ with } n_{R,i} \in \mathbb{Z} \setminus \{0\} \text{ for all } i \in R.$$

The integers $(n_{R,i})_{i \in R}$ may be chosen to be relatively prime. Now let $X(\mathcal{I}_{\text{effective},1})$ be the subvariety of all elements

$$((x_1, y_1), \ldots, (x_N, y_N)) \in \mathbb{P}^N,$$

which satisfies the following relations

$$\sum_{i \in R} n_{R,i} x_i \prod_{j \in R \setminus \{i\}} y_j = 0$$

for all minimal subsets $R \subset \{1, \ldots, N\}$, so that the vectors $(\kappa_i)_{i \in R}$ are linear dependent. For all subsets $S \subset \{1, \ldots, N\}$, the condition $y_i \neq 0$ for all $i \in S$ implies that $y_j \neq 0$ for all j, so that κ_j is an element of the $\mathbb{R}$-linear hull of $(\kappa_i)_{i \in S}$. In fact, if such a y_j were zero the relations imply that also x_j were zero, which is impossible. Let S_K denote the set of all indices i, so that $\kappa_i \in K$. Now it is obvious that we may identify $X(\mathcal{I}_{\text{effective},1})$ as a set with the disjoint union of $\mathbb{C}K$ over all $K \in \mathcal{I}_{\text{effective}}$:

$$X(\mathcal{I}_{\text{effective},1}) = \bigcup_{K \in \mathcal{I}_{\text{effective}}} \mathbb{C}K.$$

In fact, for all elements $((x_1, y_1), \ldots, (x_N, y_N)) \in X(\mathcal{I}_{\text{effective},1})$, there exists a unique $K \in \mathcal{I}_{\text{effective}}$ such that $y_i \neq 0$ for all $i \in S_K$ and $y_j = 0$ for all $j \notin S_K$, and a unique $k \in \mathbb{C}K$ such that

$$g(k, \kappa_i) = x_i / y_i \text{ for all } i \in S_K.$$

Lemma 4.2 *The singular locus of the compact complex space $X(\mathcal{I}_{\text{effective},1})$ is contained in the union of all components $\mathbb{C}K$ with $\dim K \leq n-2$. Moreover, $X(\mathcal{I}_{\text{effective},1})$ is a Cohen-Macaulay-space (see [G-P-R, Chapter II. §5.]) and therefore a normal complex space.*

Proof: It is obvious that the singular locus of $X(\mathcal{I}_{\text{effective},1})$ is contained in the union of all components $\mathbb{C}K$ with $\dim K \leq n-1$. Now let K be any element of $\mathcal{I}_{\text{effective}}$ of dimension $n-1$. In a neighborhood of any point of the component $\mathbb{C}K$ we may fix $y_i = 1$ for all $i \in S_K$ and $x_j = 1$ for all $j \notin S_K$. For any pair $j, j' \notin S_K$ there exists a relation involving the vectors κ_j and $\kappa_{j'}$. This yields to a relation of the form

$$n_j y_{j'} + n_{j'} y_j + y_j y_{j'} \cdot \text{a polynomial of } (x_i)_{i \in S_K} = 0, \text{ with } n_j, n_{j'} \in \mathbb{Z} \setminus \{0\}.$$

Hence for any index $j \notin S_K$, the parameters $(x_i)_{i \in S_K}$ and y_j are local coordinates near a point of the component $\mathbb{C}K$. This proves the first statement.

Now let k be an element of the complexification of a component $K \in \mathcal{I}_{\text{effective}}$. Again we may fix $y_i = 1$ for all $i \in S_K$ and $x_j = 1$ for all $j \notin S_K$ in a neighborhood of this point. Obviously there exists two finite subsets $S_1 \subset S_K$ and $S_2 \subset \mathcal{I}_{\text{effective},1} \setminus S_K$, so that all κ_i, with $i \in S_1 \cup S_2$, are linear independent and all κ_i, with $i \in S_1$, span K. Thus the sequence $(x_i - x_i(k))$, with $i \in S_1$, together with the sequence y_j, with $j \in S_2$ is a M-sequence of the local ring at k of length n. This proves the second statement. Due to [G-P-R, Chapter II. Corollary 5.10] the last statement is a consequence of the other statements. □

Let us now treat the case, where $\mathcal{I}_{\text{effective},1}$ is infinite. If $\mathcal{J}_1 \supset \mathcal{J}_2$ are two finite subsets of $\mathcal{I}_{\text{effective},1}$ there exists a natural morphism of complex spaces $X(\mathcal{J}_1) \twoheadrightarrow X(\mathcal{J}_2)$. Let $X(\mathcal{I}_{\text{effective},1})$ be the projective limit of all these complex spaces $X(\mathcal{J})$, where $\mathcal{J}$ runs through the finite subsets of $\mathcal{I}_{\text{effective},1}$. As a set $X(\mathcal{I}_{\text{effective},1})$ is equal to

$$X(\mathcal{I}_{\text{effective},1}) = \bigcup_{K \in \mathcal{I}_{\text{effective}}} \mathbb{C}K.$$

And a basis of the topology is given by the inverse images of all open sets of the complex spaces $X(\mathcal{J})$ under the mappings $X(\mathcal{I}_{\text{effective},1}) \twoheadrightarrow X(\mathcal{J})$. The sections on these open sets uniquely determines the sheaf of holomorphic[4] functions. On these open sets the holomorphic functions are defined as the pullback of the holomorphic functions of the corresponding open sets of $X(\mathcal{J})$ under the mappings $X(\mathcal{I}_{\text{effective},1}) \twoheadrightarrow X(\mathcal{J})$. This makes $X(\mathcal{I}_{\text{effective},1})$ into a ringed space.

The proof of Theorem 4.1 shows that the map mentioned above gives a morphism of ringed spaces from $\mathcal{B}(q)$ onto $X(\mathcal{I}_{\text{effective},1})$. Let us now consider the quotient $\mathcal{B}(q)/\Gamma^\sharp$. The action of the dual lattice $\Gamma^\sharp$ on $\mathbb{C}^n$ does lift to a continuous action on $X(\mathcal{I}_{\text{effective},1})$, but the quotient is not a Hausdorff-space. However, due to the following lemma, the quotient $\mathcal{B}(q)/\Gamma^\sharp$ is a Hausdorff-space.

Lemma 4.3 *For all non-zero elements $\left(\tilde{\psi}(\kappa)\right)_{\kappa \in \Gamma^\sharp} \in \ell^2(\Gamma^\sharp)$, an open neighborhood U of $\left(\tilde{\psi}(\kappa)\right)_{\kappa \in \Gamma^\sharp}$ exists, whose shifted sets $\kappa.U$, with $\kappa \in \Gamma^\sharp$, do not intersect each other.*

Proof: For each non-zero element $\left(\tilde{\psi}(\kappa)\right)_{\kappa \in \Gamma^\sharp} \in \ell^2(\Gamma^\sharp)$, there exists an $\epsilon > 0$ and a $\delta > 0$, so that a finite number of the absolute values $|\tilde{\psi}(\kappa)|$ are equal to $\epsilon + \delta$ and all other absolute values $|\tilde{\psi}(\kappa)|$ are elements of the set $[0, \delta]$. Hence for all $\kappa \in \Gamma^\sharp$ there exists a $\kappa' \in \Gamma^\sharp$ so that $|\tilde{\psi}(\kappa' - \kappa) - \tilde{\psi}(\kappa')| \geq \epsilon$. This shows that the open ball of radius $\epsilon/2$ around $\left(\tilde{\psi}(\kappa)\right)_{\kappa \in \Gamma^\sharp}$ gives an open neighborhood with the desired properties. □

[4]The space $X(\mathcal{I}_{\text{effective},1})$ is not a complex spaces. Nevertheless the sections of the structure sheaf are termed here holomorphic.

The lemma shows that the quotient of the projective space of $\ell^2(\Gamma^\sharp)$ under the action of $\Gamma^\sharp$ is a complex Hilbert manifold, which is locally homeomorphic to the projective space of $\ell^2(\Gamma^\sharp)$. Thus the quotient $\mathcal{B}(q)/\Gamma^\sharp$ is a well defined ringed space, which is locally isomorphic to the ringed space $\mathcal{B}(q)$. As a set this space may be identified with the disjoint union

$$\mathcal{B}(q)/\Gamma^\sharp = \bigcup_{K\in\mathcal{I}_{\text{effective}}} \mathcal{B}_0(q_K)/(K\cap\Gamma^\sharp).$$

5 A completion of the complex Bloch-variety

In the last section, we saw that the complex Bloch-variety should contain in the infinite energy limit the complex Bloch-varieties of the averaged potentials q_K, with $K \in \mathcal{I}_{\text{effective}}$. In this section, we will construct a completion of the complex Bloch-variety, which contains in the infinite energy the complex Bloch-varieties of all averaged potentials q_K, with $K\in\mathcal{I}$. This has the advantage of providing stability under small perturbations of the potential.

If $K_0 \subset K$ are two elements of $\mathcal{I}$ and $\chi_0 \in \mathbb{C}K_0/(\mathbb{C}K_0\cap\Gamma^\sharp)$, $\chi\in\mathbb{C}K/(\mathbb{C}K\cap\Gamma^\sharp)$ two representations of the lattices Γ_{K_0} and Γ_K, respectively, then $K_0^\perp\cap\Gamma_K$ is a lattice in $K\cap K_0^\perp$ and Γ_{K_0} is isomorphic to $\Gamma_K/(K_0^\perp\cap\Gamma_K)$. Hence χ_0 may be considered as a one-dimensional representation of Γ_K, which is trivial on $K_0^\perp\cap\Gamma_K$, and χ may be restricted to a one-dimensional representation χ_{K_0} of $K_0^\perp\cap\Gamma_K$.

For different $K_0\subset K$ the averaged potentials q_K and q_{K_0} act on different Hilbert spaces $L^2(K_0/\Gamma_{K_0})$ and $L^2(K/\Gamma_K)$, respectively. The orthogonal projection $K \twoheadrightarrow K_0$ induces an embedding of the Hilbert spaces $L^2(K_0/\Gamma_{K_0})\hookrightarrow L^2(K/\Gamma_K)$. If this embedding is multiplied by the inverse of square root of the volume of $K\cap K_0^\perp/(K_0^\perp\cap\Gamma_K)$, we obtain an isometry

$$I(K_0,K): L^2(K_0/\Gamma_{K_0})\hookrightarrow L^2(K/\Gamma_K), \text{ with } I^*(K_0,K)\circ I(K_0,K) = 1\!\!1_{L^2(K_0/\Gamma_{K_0})}.$$

If K_0 is the zero-subspace, we identify $L^2(K_0/\Gamma_{K_0})$ with the Hilbert space $\mathbb{C}$, and $I(K_0,K)$ is given by the natural isometry onto the constant elements of $L^2(K/\Gamma_K)$.

The pullback of the infinitely-sheeted analytic covering $\chi: B_{\text{variety}}(q)/\Gamma^\sharp\to\mathbb{C}^n/\Gamma^\sharp$ under the natural morphism of complex spaces $\mathbb{C}^n\to\mathbb{C}^n/\Gamma^\sharp$ defines the infinitely sheeted analytic covering[5] $k: B_{\text{variety}}(q)\to\mathbb{C}^n$. They fit together to the following commutative diagram:

$$\begin{array}{ccc} B_{\text{variety}}(q) & \longrightarrow & B_{\text{variety}}(q)/\Gamma^\sharp \\ \Big\downarrow k & & \Big\downarrow \chi \\ \mathbb{C}^n & \longrightarrow & \mathbb{C}^n/\Gamma^\sharp \end{array}$$

In the last section, we constructed a candidate for the completion of $\mathbb{C}^n$, the complex space $X(\mathcal{I}_1)$. We want the completion of the complex Bloch-variety $\overline{B_{\text{variety}}(q)}$ to be locally a finitely-sheeted analytic covering of $X(\mathcal{I}_1)$. But for this purpose, the topology of $X(\mathcal{I}_1)$ is not fine enough. Thus we define another topology on the set $X(\mathcal{I}_1)$. The corresponding ringed space will be denoted by $X(\mathcal{I}_1)_{\text{fine}}$.

[5] $B_{\text{variety}}(q)$ is the variety of all pairs $(k,\lambda)\in\mathbb{C}^n\times\mathbb{C}$, so that λ is an eigenvalue of $-\Delta_{[k]}+q$)

For all $K_0 \subset K \in \mathcal{I} \times \mathcal{I}$, all open sets $U \subset \mathbb{C}K_0$ and all $\epsilon > 0$, let (K_0, K, U, ϵ) be the set

$$\text{interior of}\Big\{k = k_0 + k^{\perp} \in \mathbb{C}K \;\Big|\; (k_0, k^{\perp}) \in U \times (\mathbb{C}K \cap \mathbb{C}K_0^{\perp}) \\ \text{and } 4\pi^2 |g(\kappa - 2k, \kappa)| > 1/\epsilon \text{ for all } \kappa \in \Gamma^{\sharp} \cap K \setminus (\Gamma^{\sharp} \cap K_0)\Big\}.$$

For all $K_0 \in \mathcal{I}$ and all open subsets $U \subset \mathbb{C}K_0$ we define a family of open sets (K_0, U, ϵ) of $X(\mathcal{I}_1)_{\text{fine}}$ indexed by $\epsilon > 0$:

$$(K_0, U, \epsilon) := \bigcup_{K \in \mathcal{I} \text{ so that } K_0 \subset K} (K_0, K, U, \epsilon).$$

Here the sets (K_0, K, U, ϵ) are considered as open subsets of the components $\mathbb{C}K$ of $X(\mathcal{I}_1)_{\text{fine}}$. Now we take these subsets as a base of open sets of the topology of $X(\mathcal{I}_1)_{\text{fine}}$. The restriction of any open subset of $X(\mathcal{I}_1)_{\text{fine}}$ to any component $\mathbb{C}K$ is open. The structure sheaf $\mathcal{O}_{X(\mathcal{I}_1)_{\text{fine}}}$ of holomorphic[6] functions is defined as the sheaf of all functions, which obey the following conditions:

holomorphic (i) They are continuous.

holomorphic (ii) Their restrictions to any component $\mathbb{C}K$, with $K \in \mathcal{I}$, are holomorphic.

In what follows, we will use the following notation: For all elements of (K_0, U, ϵ) denoted by $\overline{k}$, k denotes the corresponding value in $\mathbb{C}^n$, where all components $\mathbb{C}K$ are identified with the corresponding subspace of $\mathbb{C}^n$. Moreover, if $\overline{k}$ is an element of the component $\mathbb{C}K$, k_0 will denote the orthogonal projection of k onto the subspace $\mathbb{C}K_0$ and $k^{\perp}$ the element $k^{\perp} = k - k_0 \in (\mathbb{C}K \cap \mathbb{C}K_0^{\perp})$.

Lemma 5.1 *If $\dim K = \dim K_0 + 1$ and $k = k_0 + k^{\perp}$ is an element of (K_0, K, U, ϵ) such that $k^{\perp} \neq 0$ is a real element of $(K \cap K_0^{\perp})$, there exists a positive $\alpha > 0$ and $\beta > 0$, which can be chosen to depend continuously on k_0 and $k^{\perp}$, so that (K_0, K, U, ϵ) contains the set*

$$\Big\{(k_0 + zk^{\perp} \in \mathbb{C}K \;\Big|\; |Re(z)| < \alpha Im^2(z) - \beta\Big\}.$$

Moreover, for $n = 1$ the direct image sheaf of the structure sheaf of $X(\mathcal{I}_1)_{\text{fine}}$ under the natural homomorphism of ringed spaces $X(\mathcal{I}_1)_{\text{fine}} \to X(\mathcal{I}_1)$ is equal to the structure sheaf of $X(\mathcal{I}_1)$. Finally, for $n \leq 2$ all global holomorphic functions of $X(\mathcal{I}_1)_{\text{fine}}$ are constant.

Proof: Let $\kappa^{\perp} \in (\Gamma^{\sharp} \cap K)$ be an element, such that the corresponding element of $(\Gamma^{\sharp} \cap K)/(\Gamma^{\sharp} \cap K_0) \simeq \mathbb{Z}$ is a primitive element. For all $\kappa \in \Gamma^{\sharp} \cap K_0$, the condition

$$4\pi^2 |g(\kappa + n\kappa^{\perp} - 2(k_0 + zk^{\perp}), \kappa + n\kappa^{\perp})| > 1/\epsilon$$

is then equivalent to the condition

$$\left| z - \frac{g(\kappa + n\kappa^{\perp} - 2k_0, \kappa + n\kappa^{\perp})}{2ng(k^{\perp}, \kappa^{\perp})} \right| > \frac{1}{8\pi^2 \epsilon |ng(k^{\perp}, \kappa^{\perp})|}.$$

[6] The spaces $X(\mathcal{I}_1)_{\text{fine}}$ are not complex spaces. Nevertheless holomorphic is again used to refer to the sections of the structure sheaf.

Now the expression $g(\kappa+n\kappa^\perp,\kappa+n\kappa^\perp)-2g(k_0,\kappa+n\kappa^\perp)$ is an element of a domain of the form

$$\left\{z\in\mathbb{C}\;\middle|\;2g(\mathrm{Im}(k_0),\mathrm{Im}(k_0))\left(\mathrm{Re}(z)+g(\mathrm{Re}(k_0),\mathrm{Re}(k_0))\right)\geq\left(|\mathrm{Im}(z)|-|g(\mathrm{Im}(k_0),\mathrm{Re}(k_0))|\right)^2\right\}.$$

This is a subset of the domain

$$\left\{z\in\mathbb{C}\;\middle|\;4g(\mathrm{Im}(k_0),\mathrm{Im}(k_0))\left(\mathrm{Re}(z)+g(\mathrm{Re}(k_0),\mathrm{Re}(k_0))\right)+2|g(\mathrm{Im}(k_0),\mathrm{Re}(k_0))|^2\geq\mathrm{Im}^2(z)\right\}.$$

Hence all elements of the domain

$$\left\{z\in\mathbb{C}\;\middle|\;|\mathrm{Re}(z)|<\alpha\mathrm{Im}^2(z)-\beta\right\}$$

with a positive $\alpha>0$ and a positive $\beta>0$ satisfies the condition above for all $\kappa\in(\Gamma^\sharp\cap K_0)$ and $n\in\mathbb{Z}\setminus\{0\}$. This implies the first statement of the lemma. Let us now proof the second statement of the lemma in the case of $n=1$. In this case, $\Gamma^\sharp\simeq\mathbb{Z}$. So let us fix a primitive $\kappa\in\Gamma^\sharp$. Then for every local section f of the direct image sheaf of the structure sheaf of $X(\mathcal{I}_1)_{\mathsf{fine}}$ under the natural morphism $X(\mathcal{I}_1)_{\mathsf{fine}}\to\mathbb{P}$ the function $z\mapsto f(z\kappa)$ is holomorphic on a domain of $\mathbb{C}$ and bounded on a domain

$$\left\{z\;\middle|\;\left|z-\frac{n}{2}\right|>\frac{1}{8|n|\pi^2\epsilon g(\kappa,\kappa)}\text{ for all }n\in\mathbb{Z}\setminus\{0\}\right\}.$$

Thus due to the Maximum principle (see e.g. [Ah, Chapter 4. Theorem 12.]), this function is holomorphic on the corresponding domain of $\mathbb{P}$. This proves the second statement of the lemma and the third statement in the case of $n=1$. Finally, we shall prove the third statement in the case of $n=2$. Let κ_0 and κ_1 be two generators of $\Gamma^\sharp$ and let k_0 and k_1 be the two elements of $\mathbb{C}^2$, which are dual with respect to the metric $g(\cdot,\cdot)$, i.e. $g(k_i,\kappa_j)=\delta_{ij}$ for all $0\leq i,j\leq 1$. For any complex number α and every holomorphic function f of $X(\mathcal{I}_1)_{\mathsf{fine}}$ the restriction to $z\mapsto f(zk_0+z\alpha k_1)$ defines a holomorphic function of $\mathbb{C}$, which is bounded on a domain

$$\left\{z\;\middle|\;\left|z-\frac{g(n_0\kappa_0+n_1\kappa_1,n_0\kappa_0+n_1\kappa_1)}{2(n_0+\alpha n_1)}\right|>\frac{1}{8\pi^2\epsilon|n_0+\alpha n_1|}\text{ for all }(n_0,n_1)\in\mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}\right\}.$$

If the imaginary part of α is not zero, again the Maximum principle shows that the restricted f is holomorphic on $\mathbb{P}$ and therefore constant. Hence f is constant on an open subset of $\mathbb{C}^2$. This completes the proof of the lemma. □

The completion of the complex Bloch-variety will be again a topological space $\overline{B_{\mathsf{variety}}(q)}$ together with a structure sheaf $\mathcal{O}_{\overline{B_{\mathsf{variety}}(q)}}$. For all elements $\overline{k}$ of the component $\mathbb{C}K$ of $X(\mathcal{I}_1)_{\mathsf{fine}}$, there exists an open neighborhood U of $\overline{k}$, whose shifted sets $\kappa.U$, with $\kappa\in K\cap\Gamma^\sharp$, do not intersect each other. Hence the disjoint union

$$\Gamma^\sharp\tilde{\times}X(\mathcal{I}_1)_{\mathsf{fine}}:=\bigcup_{K\in\mathcal{I}}\Gamma^\sharp\times_{K\cap\Gamma^\sharp}\mathbb{C}K$$

fits together to a ringed space, which is locally isomorphic to $X(\mathcal{I}_1)_{\mathsf{fine}}$. Here $\Gamma^\sharp\times_{K\cap\Gamma^\sharp}\mathbb{C}K$ denotes the set of all equivalence classes of $\Gamma^\sharp\times\mathbb{C}K$ with respect to the equivalence relation $(\kappa+\kappa',k)\sim(\kappa,\kappa'+k)$ for all $\kappa'\in K\cap\Gamma^\sharp$. The equivalence class of an element $(\kappa,\overline{k})\in\Gamma^\sharp\times$

$X(\mathcal{I}_1)_{\text{fine}}$ is denoted by $[\kappa,\overline{k}]$. Let $\overline{B_{\text{variety}}(q)}$ be the topological subspace of $\Gamma^\sharp\tilde{\times}X(\mathcal{I}_1)_{\text{fine}}\times\mathbb{C}$ of all elements $([\kappa,\overline{k}],\lambda_{\text{renormalized}})$, so that $(k,\lambda_{\text{renormalized}}+4\pi^2g(k+\kappa,k+\kappa))$ is an element of $B_{\text{variety}}(q_K)$, whenever $\overline{k}$ is an element of the component $\mathbb{C}K$ of $X(\mathcal{I}_1)_{\text{fine}}$. Since $B_{\text{variety}}(q_K)$ is invariant under the action of $K\cap\Gamma^\sharp$, this condition is equivalent for all representatives $(\kappa,\overline{k})$ of the equivalence class $[\kappa,\overline{k}]$. In spite of $\Gamma^\sharp\tilde{\times}X(\mathcal{I}_1)_{\text{fine}}$ not being a Hausdorff space, the completion $\overline{B_{\text{variety}}(q)}$ is a Hausdorff space. The structure sheaf $\mathcal{O}_{\overline{B_{\text{variety}}(q)}}$ of holomorphic[7] functions is again defined as all functions, which obey the following conditions:

holomorphic (i) They are continuous.

holomorphic (ii) Their restrictions to any component $\mathbb{C}K$, with $K\in\mathcal{I}$, are holomorphic.

As a set $\overline{B_{\text{variety}}(q)}$ is equal to

$$\overline{B_{\text{variety}}(q)}:=\bigcup_{K\in\mathcal{I}}\Gamma^\sharp\times_{K\cap\Gamma^\sharp}B_{\text{variety}}(q_K).$$

We will see in the next proposition that locally, $\overline{B_{\text{variety}}(q)}$ is a finitely-sheeted analytic covering of $\Gamma^\sharp\tilde{\times}X(\mathcal{I}_1)_{\text{fine}}$.

Proposition 5.2 *For each $q_0\in C(\mathbb{R}^n/\Gamma)$ and each $([\kappa_0,k_0],\lambda_0)\in\Gamma^\sharp\times_{K_0\cap\Gamma^\sharp}B_{\text{variety}}(q_{0,K_0})\subset\overline{B_{\text{variety}}(q_0)}$ there exists a neighborhood of the form $W\times(K_0,U,\epsilon)\subset C(\mathbb{R}^n/\Gamma)\times X(\mathcal{I}_1)_{\text{fine}}$ of (q_0,k_0) and a holomorphic map P from $W\times(K_0,U,\epsilon)$ into the set of bounded operators of $L^2(\mathbb{R}^n/\Gamma)$, which has the following properties:*

(i) *For all elements $(q,\overline{k})$ of the component $W\times(K_0,K,U,\epsilon)$ of the neighborhood $W\times(K_0,U,\epsilon)$, $P(q,\overline{k})$ is a finite-dimensional projector, which commutes with*[8]

$$M(-k)\circ I(K,\mathbb{R}^n)\circ(-\Delta_{[k]}+q_K)\circ I^*(K,\mathbb{R}^n)\circ M(k).$$

(ii) *The following map from $W\times(K_0,K,U,\epsilon)$ into the set of bounded operators of $L^2(\mathbb{R}^n/\Gamma)$: $(q,\overline{k})\mapsto$*

$$P(q,\overline{k})\circ M(-k)\circ I(K,\mathbb{R}^n)\circ(-\Delta_{[k]}+q_K-4\pi^2g(k+\kappa_0,k+\kappa_0)1\!\!1\,)\circ I^*(K,\mathbb{R}^n)\circ M(k)\circ P(q,\overline{k})$$

forms together a holomorphic map from $W\times(K_0,U,\epsilon)$ into the set of bounded operators of $L^2(\mathbb{R}^n/\Gamma)$.

(iii) *For all fixed $q\in W$, the collection of all $([\kappa_0,\overline{k}],\lambda_{\text{renormalized}})\in\overline{B_{\text{variety}}(q_0)}$, so that $(q,\overline{k})\in W\times(K_0,U,\epsilon)$ and $\lambda=\lambda_{\text{renormalized}}+4\pi^2g(k+\kappa_0,k+\kappa_0)$ is an eigenvalue, whose eigenvector belongs to the finite-dimensional invariant subspace of the projector $P(q,\overline{k})$, is an open subset of $\overline{B_{\text{variety}}(q_0)}$, which is a finitely-sheeted analytic covering of $\{[\kappa_0,\overline{k}]\mid\overline{k}\in(K_0,U,\epsilon)\}$. Moreover, if $q=q_0$, this subset contains the point $([\kappa_0,k_0],\lambda_0)$.*

The following lemma is necessary to prove Proposition 5.2:

[7]The spaces $\overline{B_{\text{variety}}(q)}$ are not complex spaces. Nevertheless, holomorphic is again used to refer to the sections of the structure sheaf.

[8]Here $M(k)$ denotes again the operator on $L^2(\mathbb{R}^n/\Gamma)$, which multiplies by the function $\psi_k(x)=\exp(2\pi\sqrt{-1}g(x,k))$.

Perturbation Lemma 5.3 *Let $V_A \oplus V_B \oplus V_C$ be a direct sum of three Banach spaces and*

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & E \\ F & G & H \end{pmatrix}$$

two operators of this Banach space, so that the following conditions are fulfilled:

(i) *The operator*[9] $B(V_A, V_B) \to B(V_A, V_B), X \mapsto B \circ X - X \circ A$ *is invertible and the inverse is bounded by* δ.

(ii) *The operator* $B(V_A, V_C) \to B(V_A, V_C), X \mapsto C \circ X - X \circ A$ *is invertible and the inverse is bounded by* $\epsilon \leq \delta$.

(iii) *The operators* D, E, F, G, H *are bounded by* 1.

Then the perturbation ansatz

$$\left(\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} + z \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & E \\ F & G & H \end{pmatrix} \right) \circ \begin{pmatrix} \mathbb{1}_{V_A} \\ X(z) \\ Y(z) \end{pmatrix} = \begin{pmatrix} \mathbb{1}_{V_A} \\ X(z) \\ Y(z) \end{pmatrix} \circ A(z)$$

defines unique holomorphic operator-valued functions $X(z) \in B(V_A, V_B), Y(z) \in B(V_A, V_C)$ *and* $A(z) \in B(V_A, V_A)$ *on the domain*

$$\left\{ z \,\middle|\, |z| \leq \frac{\sqrt{16\delta/\epsilon - 7} - 1}{8\delta} \right\} \cap \left\{ z \,\middle|\, |z| \leq \frac{1}{\epsilon(1 + 2\sqrt{2}} \right\}$$

with $X(0) = 0 = Y(0)$ *and* $A(0) = A$. *Moreover, on this domain these operators are bounded by*

$$\|X(z)\| \leq \delta \frac{1 - \epsilon|z| - \sqrt{1 - 2\epsilon|z| - 7\epsilon^2|z|^2}}{\epsilon}$$

$$\|Y(z)\| \leq \frac{1 - \epsilon|z| - \sqrt{1 - 2\epsilon|z| - 7\epsilon^2|z|^2}}{2\epsilon|z|}$$

$$\|A(z) - A\| \leq \frac{1 - \epsilon|z| - \sqrt{1 - 2\epsilon|z| - 7\epsilon^2|z|^2}}{2\epsilon}$$

Proof: Let us expand the functions $X(z), Y(z)$ and $A(z)$ as power series:

$$X(z) = \sum_{i \geq 0} X_i z^i, \; Y(z) = \sum_{i \geq 0} Y_i z^i \text{ and } A(z) = A + \sum_{i \geq 1} A_i z^i.$$

Then the perturbation ansatz gives the following set of equations:

$$D \circ Y_{i-1} = A_i \text{ for } i \geq 1,$$

[9]Here, as usual, $B(V_A, V_B)$ denotes the Banach space of all bounded operators from V_A into V_B.

$$B \circ X_i - X_i \circ A = -E \circ Y_{i-1} + \sum_{j=1}^{i} X_{i-j} \circ D \circ Y_{j-1} \text{ for } i \geq 1,$$
$$C \circ Y_1 - Y_1 \circ A = F \text{ and}$$
$$C \circ Y_i - Y_i \circ A = -G \circ X_{i-1} - H \circ Y_{i-1} + \sum_{j=1}^{i} Y_{i-j} \circ D \circ Y_{j-1} \text{ for } i \geq 2.$$

Together with the conditions $X_0 = 0 = Y_0$ and $A_0 = A$, this inductively determines all the coefficients of the power series. Next we define real-valued power series in order to estimate the norms of the power series above. Let $x(t) = \sum_{i\geq 0} x_i t^i$ and $y(t) = \sum_{i \geq 0} y_i t^i$ be defined by

$$x_i = \delta \left(y_{i-1} + \sum_{j=1}^{i} x_{i-j} y_{j-1} \right) \text{ for } i \geq 1,$$

$$y_1 = \epsilon \text{ and } y_i = \epsilon \left(x_{i-1} + y_{i-1} + \sum_{j=1}^{i} y_{i-j} y_{j-1} \right) \text{ for } i \geq 2,$$

and the conditions $x_0 = 0 = y_0$. Then the assumptions (i), (ii) and (iii) imply the estimates $\|X(z)\| \leq x(|z|), \|Y(z)\| \leq y(|z|)$ and $\|A(z)\| \leq y(|z|)|z|$. With these definitions, one gets two equations for the functions $x(t)$ and $y(t)$:

$$x(t) = \delta t(y(t) + x(t)y(t)) \text{ and } y(t) = \epsilon t(x(t) + y(t) + y^2(t) + 1).$$

Both can be solved immediately:

$$x(t) = \frac{\delta t y(t)}{1 - \delta t y(t)} \text{ and } y(t) = \frac{1 - \epsilon t - \sqrt{1 - 2\epsilon t - (3 + 4x(t))\epsilon^2 t^2}}{2\epsilon t}.$$

The sign in front of the square root is uniquely determined by the condition $y(0) = 0$. If we assume that $y(t)$ satisfies the inequality $0 \leq \delta t y(t) \leq 1/2$, then $x(t)$ must satisfy the inequality $0 \leq x(t) \leq 1$. If we insert this into the second equation we obtain the inequality

$$\frac{1 - \epsilon t - \sqrt{1 - 2\epsilon t - 3\epsilon^2 t^2}}{2\epsilon t} \leq y(t) \leq \frac{1 - \epsilon t - \sqrt{1 - 2\epsilon t - 7\epsilon^2 t^2}}{2\epsilon t}.$$

Hence our assumption is fulfilled, if the inequality $1 - \epsilon t - \sqrt{1 - 2\epsilon t - 7\epsilon^2 t^2} \leq \epsilon/\delta$ holds. This is true on the domain

$$0 \leq t \leq \frac{\sqrt{16\delta/\epsilon - 7} - 1}{8\delta}.$$

Moreover, on the domain

$$0 \leq t \leq \frac{1}{\epsilon(1 + 2\sqrt{2})}$$

all functions are real and positive. Finally the assumption $\delta t y(t) \leq 1/2$ implies the estimate $x(t) \leq 2\delta t y(t)$. This completes the proof of the Perturbation Lemma. □

Proof of Proposition 5.2: Let us fix a potential $q_0 \in C(\mathbb{R}^n/\Gamma)$, and a $([\kappa_0, k_0], \lambda_0) \in \Gamma^\sharp \times_{K_0 \cap \Gamma^\sharp} B_{\text{variety}}(q_{K_0}) \subset \overline{B_{\text{variety}}(q)}$. Due to Theorem 2.1, there exists an open neighborhood $W \times U \times V \subset C(\mathbb{R}^n/\Gamma) \times \mathbb{C}K_0 \times \mathbb{C}$ of (q_0, k_0, λ_0), such that for all $q \in W$, $V(q) := B_{\text{variety}}(q_{K_0}) \cap U \times V \subset$

$\mathbb{C}K_0 \times \mathbb{C}$ is a l-sheeted analytic covering of U. Moreover, we may assume that there exists a positive $\delta > 0$, such that for all $(q,k) \in W \times U$ the difference of any eigenvalue of $-\Delta_{[k]} + q_{K_0}$, which belongs to $V(q)$, minus any other eigenvalue, which does not belong to $V(q)$, is greater than $1/\delta$. Now let us fix a $K \supset K_0$. We may consider the potential q_{K_0} as a potential on K, which is constant on $K \cap K_0^{\perp}$. So the potential q_K is a sum of q_{K_0} and the perturbation

$$q_K^{\perp} = \sum_{\kappa \in (\Gamma^{\sharp} \cap K) \setminus (\Gamma^{\sharp} \cap K_0)} \exp(2\pi\sqrt{-1}g(x,\kappa))\hat{q}(\kappa).$$

For each fixed $(q,k) \in W \times (K_0, K, U, \epsilon)$, the Hilbert space $L^2(K/\Gamma_K)$ may be decomposed into a direct sum of topological vector spaces $V_A \oplus V_B \oplus V_C$: V_A is the image under $I(K_0, K)$ of the finite-dimensional subspace of $L^2(K_0/\Gamma_{K_0})$, which corresponds to all eigenvalues λ of $-\Delta_{[k_0]} + q_{K_0}$, with $(k_0, \lambda) \in V(q)$. And V_B is the image under $I(K_0, K)$ of the subspace of $L^2(K_0/\Gamma_{K_0})$, which corresponds to all other eigenvalues. Hence $V_A \oplus V_B$ is equal to the image of $L^2(K_0/\Gamma_{K_0})$ under $I(K_0, K)$. Finally V_C is the orthogonal complement of this subspace. Now let $M(k^{\perp})$ denote the operator on $L^2(K/\Gamma_K)$, which multiplies by the function $\exp(2\pi\sqrt{-1}g(x,k^{\perp}))$. Then the operators $M(-k^{\perp}) \circ (-\Delta_{[k]} + q_{K_0}) \circ M(k^{\perp})$ and $M(-k^{\perp}) \circ q_K^{\perp} \circ M(k^{\perp}) = q_K^{\perp}$ are of the form

$$\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & D \\ 0 & 0 & E \\ F & G & H \end{pmatrix}$$

with respect to this decomposition, respectively. Hence we may apply the Perturbation Lemma. The eigenvalues of the operator A are equal to the sum of $4\pi^2 g(k^{\perp}, k^{\perp})$ and the eigenvalues of $-\Delta_{[k_0]} + q_{K_0}$, which belong to V, and the eigenvalues of B are equal to the sum of $4\pi^2 g(k^{\perp}, k^{\perp})$ and the eigenvalues of $-\Delta_{[k_0]} + q_{K_0}$, which do not belong to V. Finally, the eigenvalues of C are equal to the sum of $4\pi^2 g(k^{\perp} + \kappa, k^{\perp} + \kappa)$ and the eigenvalues of $-\Delta_{[k_0]} + q_{K_0}$, where κ runs through $(\Gamma^{\sharp} \cap K) \setminus (\Gamma^{\sharp} \cap K_0)$. Hence the difference of an eigenvalue of A minus an eigenvalue of B is greater than $1/\delta$. And due to Corollary 2.3, the difference of an eigenvalue of A minus an eigenvalue of C is greater than $1/\epsilon - 2\|q_{K_0}\|$. Then similar arguments as in the proof of Theorem 2.1 show that for all eigenvalues λ of A, the restrictions of the resolvent $(\lambda \mathbb{1} + \Delta_{[k]} - q_{K_0})^{-1}$ to the subspaces V_B and V_C are bounded by C/δ and C/ϵ, respectively, with a constant depending only on the neighborhood V and on the norm of the potential $\|q_{K_0}\|$. Hence the inverse operators, described in (i) and (ii) are bounded by C/δ and C/ϵ, respectively. In spite of the unboundedness of the operators B and C, the essential parts of the assumptions (i) and (ii) are fulfilled. In order to verify condition (iii) we divide the perturbation $q_K^{\perp}$ by $\|q_K^{\perp}\|$. If ϵ is small enough the estimates of the Perturbation Lemma are valid for $z = \|q_K^{\perp}\|$, i.e. the perturbed operator $M(-k^{\perp}) \circ (-\Delta_{[k]} + q_K) \circ M(k^{\perp})$. This shows the statement of the proposition for all components $\mathbb{C}K$ with $K \supset K_0$. $\square$

It is quite obvious that the dual lattice $\Gamma^{\sharp}$ acts on $\overline{B_{\text{variety}}(q)}$ and that the quotient $\overline{B_{\text{variety}}(q)}/\Gamma^{\sharp}$ is a ringed space, which is locally isomorphic to $\overline{B_{\text{variety}}(q)}$. Moreover, as a set, this space may be identified as

$$\overline{B_{\text{variety}}(q)}/\Gamma^{\sharp} = \bigcup_{K \in \mathcal{I}} B_{\text{variety}}(q_K)/(\Gamma^{\sharp} \cap K).$$

Corollary 5.4 *The regular part of $B_{\text{variety}}(q)/\Gamma^{\sharp}$ is a connected n-dimensional complex manifold.*

Proof: The completion $\overline{B_{\text{variety}}(q)}/\Gamma^\sharp$ of the complex Bloch-variety always contains one point corresponding to the complex Bloch-variety of the zero-dimensional, completely averaged potential $q_{\{0\}}$. Due to the preceding proposition, there always exists a neighborhood of this point, which contains only regular points. Let us choose a sequence $\{0\} = K_0 \subset K_1 \subset \ldots \subset K_{n-1} \subset K_n = \mathbb{C}^n$ of elements of $\mathcal{I}$, so that $\dim K_i = i$ for all $i = 0, \ldots, n$. And let $k_1, \ldots, k_n$ be real elements of $K_1, K_2 \cap K_1^\perp, \ldots, K_n \cap K_{n-1}^\perp$, which are not zero. Then due to Lemma 5.1, every neighborhood of the point mentioned above contains a set of the form

$$\left\{ z_1 k_1 + \ldots + z_n k_n \;\middle|\; \mathrm{Im}(z_i) > \sqrt{\frac{|\mathrm{Re}(z_i)| + \beta_i}{\alpha_i}} \text{ for all } i = 1, \ldots, n \right\},$$

where the α_i and β_i depend continuously on $z_1, \ldots z_{i-1}$ in a way described in Lemma 5.1. These sets are obviously connected, and every element of $B_{\text{variety}}(q)/\Gamma^\sharp$ may be connected by a path with this set. In fact, due to Corollary 2.3, for all $(k, \lambda) \in B_{\text{variety}}(q)$, we may choose a $\kappa \in \Gamma^\sharp$, such that $|\lambda - 4\pi^2 g(k + \kappa, k + \kappa)| \leq \|q\|$. If we enlarge the imaginary parts of the coordinates $(z_1, \ldots z_n)$ of any $k = z_1 k_1 + \ldots + z_n k_n + \kappa$ in the order of their indices, then κ will become unique, and we will always reach this set. The regular part of $B_{\text{variety}}(q)/\Gamma^\sharp$ is dense and open. Therefore we may arrange these paths within the regular part of $B_{\text{variety}}(q)/\Gamma^\sharp$. This completes the proof. □

With the help of the preceding proposition we may modify the sheaf $\mathcal{O}_q$ into the sheaf $\tilde{\mathcal{O}}_q$ on $X(\mathcal{I}_1)_{\text{fine}}$ of all $L^2(\mathbb{R}^n/\Gamma)$-operator-valued holomorphic functions, which commute on the component $\mathbb{C}K$ of $X(\mathcal{I}_1)_{\text{fine}}$ with

$$M(-k) \circ I(K, \mathbb{R}^n) \circ (-\Delta_{[k]} + q_k) \circ I^*(K, \mathbb{R}^n) \circ M(k).$$

The modification of the complex Bloch-variety into the complex Bloch-space, described in Section 3 within three steps, carries over to a modification of the completion of the complex Bloch-variety into the completion of the complex Bloch-space $\overline{B_{\text{complex space}}(q)}/\Gamma^\sharp$. In general, the completion of the complex Bloch-space as a set is not equal to the union of all complex Bloch-spaces $B_{\text{complex space}}(q_K)/(\Gamma^\sharp \cap K)$. In fact, the results of [F-K-T] show that singularities, which may be neglected for the one-dimensional averaged potentials, in general, cannot be neglected for the two-dimensional potentials.

6 The dual eigen bundle

Due to Theorem 3.6, the one-dimensional eigen space, which is equal to the image of the meromorphic projection P, defines a holomorphic line bundle on the regular part of the complex Bloch-space $B_{\text{complex space}}(q)/\Gamma^\sharp$. In this section, we will define a locally finite generated subsheaf $\mathcal{E}(q)$ of the sheaf of meromorphic functions on the completion of the complex Bloch-variety, whose sections are isomorphic to the sections of the dual eigen bundle. We will see that the Bloch-variety together with this sheaf $\mathcal{E}(q)$ completely determines the potential q. This means that, in general, several potentials have the same Bloch-variety, but only one has, in addition, the same sheaf of sections of the dual eigen bundle.

Let us fix the potential $q \in C(\mathbb{R}^n/\Gamma)$ and let $\psi(q)$ denote the unique meromorphic function from $B_{\text{variety}}(q)/\Gamma^\sharp$ into $C(\mathbb{R}^n)$, such that for all $(\chi, \lambda) \in B_{\text{variety}}(q)/\Gamma^\sharp$,

eigen function (i) $\psi(q)(\chi, \lambda)$ is an eigen function of $(-\Delta_\chi + q)$ with eigenvalue λ and

eigen function (ii) the evaluation of $\psi(q)(\chi,\lambda)$ at $x=0$ is equal to 1.

In order to investigate this function $\psi(q)$ and the eigen sheaf it is convenient to consider the pullback of this function under the natural map $B_{\text{variety}}(q)\to B_{\text{variety}}(q)/\Gamma^\sharp$, which is also denoted by $\psi(q)$. Again we shall remove the essential singularities of this function in the limit $k\to\infty$. For all $\kappa\in\Gamma^\sharp$, let $\tilde{\psi}(q,\kappa)$ be defined by

$$\psi(q)=\exp\left(2\pi\sqrt{-1}g(x,k)\right)\sum_{\kappa\in\Gamma^\sharp}\tilde{\psi}(q,\kappa)\exp\left(2\pi\sqrt{-1}g(x,\kappa)\right).$$

Here k denotes the analytic covering $k:B_{\text{variety}}(q)\to\mathbb{C}^n$.

In [Ku, Theorem 4.4.2] it has been proven that the complex Bloch-variety $B_{\text{variety}}(q)$ is the zero-set of an entire function of finite order. A holomorphic function on $B_{\text{variety}}(q)$ is referred to as being of finite order, if the absolute value may be estimated by the absolute value of an entire function of finite order on $\mathbb{C}^n\times\mathbb{C}$.

Lemma 6.1 *All the functions $\tilde{\psi}(q,\kappa)$ may be written as the quotients of holomorphic functions of finite order. Moreover, the denominator may be chosen to be the same for all these functions.*

Proof: Let us fix an element (k_0,λ_0) of the complex Bloch variety, such that the eigenvalue λ_0 of the Schrödinger operator $-\Delta_{[k_0]}+q$ has multiplicity one, and let λ_1 be equal to $\lambda_1=-4\pi^2\|k_0\|^2-\|q\|-1$. Here we use the norm $\|k_0\|^2=g(\mathrm{Re}(k_0),\mathrm{Re}(k_0))+g(\mathrm{Im}(k_0),\mathrm{Im}(k_0))$ and the supremum norm on $C(\mathbb{R}^n/\Gamma)$. Due to the proof of Theorem 2.1, the norm of the resolvent $\|M(-k_0)\circ(\lambda_1\mathbb{1}+\Delta_{[k_0]})^{-1}\circ M(k_0)\|$ is bounded by $1/(\|q\|+1)$. Thus we may estimate the norm of the resolvent $R(\lambda_1)=M(-k_0)\circ(\lambda_1\mathbb{1}+\Delta_{[k_0]}-q)^{-1}\circ M(k_0)$ of the Schrödinger operator by the bound:

$$\|R(\lambda_1)\|=\|M(-k_0)\circ(\lambda_1\mathbb{1}+\Delta_{[k_0]}-q)^{-1}\circ M(k_0)\|\leq\left(\left(1-\frac{\|q\|}{\|q\|+1}\right)(\|q\|+1)\right)^{-1}\leq 1.$$

Let $S(\lambda_0)$ be the set of all eigenvalues of $-\Delta_{[k_0]}+q$ in the disc $|\lambda_0-\lambda'|\leq 6\|q\|$, with the exception of λ_0. The number of elements $\kappa\in\Gamma^\sharp$, so that $|\lambda_0-4\pi^2g(k_0+\kappa,k_0+\kappa)|\leq 7\|q\|$, is bounded by $\|q\|(|\lambda_0|+4\pi^2\|k_0\|^2+7\|q\|)^{(n-1)/2}$ times a constant depending on the lattice only, because $4\pi^2g(\mathrm{Re}(k_0)+\kappa,\mathrm{Re}(k_0)+\kappa)$ is bounded by $|\lambda_0|+4\pi^2\|k_0\|^2+7\|q\|$. Due to Corollary 2.3, the order of $S(\lambda_0)$ is also bounded by this estimate. Hence, in the limit $|\lambda_0|+4\pi^2\|k_0\|^2\to\infty$ we may estimate the following polynomial of the resolvent:

$$\left\|\prod_{\lambda'\in S(\lambda_0)}(\lambda_1-\lambda_0)\left((\lambda_1-\lambda')R(\lambda_1)-\mathbb{1}\right)\right\|\leq|\lambda_1-\lambda_0|^{\#S(\lambda_0)}\prod_{\lambda'\in S(\lambda_0)}(|\lambda_1-\lambda'|+1)\leq$$

$$\leq(|\lambda_0|+4\pi^2\|k_0\|^2+7\|q\|+1)^{2\#S(\lambda_0)}\leq\exp\left((|\lambda_0|+4\pi^2\|k_0\|^2)^{n-1+\epsilon}\right)$$

with an $\epsilon>0$. The part of this polynomial, which acts on the eigen spaces corresponding to all eigenvalues of $S(\lambda_0)\cup\{\lambda_0\}$ is equal to

$$M(-k_0)\circ P(k_0,\lambda_0)\circ M(k_0)\prod_{\lambda'\in S(\lambda_0)}(\lambda_0-\lambda').$$

Now we shall prove that the norm of this operator may be estimated by three times the bound of the resolvent:

$$\left\| M(-k_0)\circ P(k_0,\lambda_0)\circ M(k_0)\prod_{\lambda'\in S(\lambda_0)}(\lambda_0-\lambda')\right\| \leq 3\left\|\prod_{\lambda'\in S(\lambda_0)}(\lambda_1-\lambda_0)\left((\lambda_1-\lambda')R(\lambda_1)-1\!\!1\right)\right\|.$$

Let us decompose the Hilbert space $L^2(\mathbb{R}^n/\Gamma)$ into the direct sum of all Fourier modes corresponding to $\kappa\in\Gamma^\sharp$, with $|\lambda_0-4\pi^2 g(k_0+\kappa,k_0+\kappa)|\leq 3\|q\|$, and the orthogonal complement of this subspace. The part corresponding to the first summand will be denoted by the index $\|$ and the other part by the index $\perp$. The eigenvalue equation shows that the decomposition of an eigenvector ψ of $-\Delta_{[k_0]}+q$ with eigenvalue λ satisfies the following inequalities:

$$\|\psi_\|\|\geq 2\|\psi_\perp\| \text{ if } \lambda=\lambda_0 \text{ and } 2\|\psi_\|\|\leq\|\psi_\perp\| \text{ if } |\lambda_0-\lambda|>6\|q\|.$$

The action of the polynomial above vanishes on all other eigenvectors. If $\psi+\psi'$ is the sum of two eigenvectors with eigenvalues $\lambda=\lambda_0$ and $\lambda'\notin S(\lambda_0)\cup\{\lambda_0\}$, respectively, then we may estimate

$$\|\psi_\|\|\leq\|(\psi+\psi')_\|\|+\frac{\|\psi'_\perp\|}{2}\leq\|(\psi+\psi')_\|\|+\frac{\|(\psi+\psi')_\perp\|}{2}+\frac{\|\psi_\|\|}{4}.$$

We conclude that $\|\psi\|$ is bounded by $3\|\psi+\psi'\|$, which proves the claim.

In [Ku, Theorem4.4.2] it has been proven that the Bloch-variety is the zero-set of an entire function of finite order on $\mathbb{C}^n\times\mathbb{C}$. Hence there exists an entire function of finite order, whose zero-set contains the branching divisor of the analytic covering $k:B_{\text{variety}}(q)\to\mathbb{C}^n$. For fixed $k=k_0$, the quotient of this function divided by the polynomial $\prod_{\lambda'\in S(\lambda_0)}(\lambda-\lambda')$ is an entire function with respect to $\lambda\in\mathbb{C}$, which for $|\lambda|\geq|\lambda_0|+6\|q\|+1$ is bounded by the absolute value of this entire function. This proves that the meromorphic projector-valued function $(k,\lambda)\mapsto M(-k)\circ P(k,\lambda)\circ M(k)$ may be written as the quotient of holomorphic functions of finite order. This proves the lemma. $\square$

The sheaf of meromorphic functions on $\overline{B_{\text{variety}}(q)}$ and $\overline{B_{\text{variety}}(q)}/\Gamma^\sharp$ is defined as the sheaf of all quotients of holomorphic functions, so that the restriction of the denominator to any component $\Gamma^\sharp\times_{\Gamma^\sharp\cap K}B_{\text{variety}}(q_K)$ and $B_{\text{variety}}(q_K)/(\Gamma^\sharp\cap K)$ is not identically zero[10], respectively. Let us remark that the global meromorphic functions build not a field. Due to Proposition 5.2, all the functions $\tilde\psi(q,\kappa)$ extend to global meromorphic functions $\overline{\psi}(q,\kappa)$ on $\overline{B_{\text{variety}}(q)}$. Let $\mathcal{E}(q)$ denote the subsheaf of the meromorphic $\mathcal{M}$ functions on $\overline{B_{\text{variety}}(q)}$, which is generated by $\left(\tilde\psi(q,\kappa)\right)_{\kappa\in\Gamma^\sharp}$ as a module of holomorphic functions. Since $\overline{B_{\text{variety}}(q)}/\Gamma^\sharp$ is as a ringed space locally isomorphic to $\overline{B_{\text{variety}}(q)}$, the sheaf $\mathcal{E}(q)$ also may be considered as a sheaf on $\overline{B_{\text{variety}}(q)}/\Gamma^\sharp$. In the following theorem, we will make use of the subsheaf of the direct image sheaf $\overline{k}_*(\mathcal{E}(q))$ under the infinitely-sheeted covering map $\overline{k}:\overline{B_{\text{variety}}(q)}\to X(\mathcal{I}_1)_{\text{fine}}$, which is locally the direct sum of the corresponding sheaves over all sheets of this infinitely-sheeted analytic covering, and not the direct product, as the direct image sheaves. It will be denoted by $\overline{k}_\oplus(\mathcal{E}(q))$.

[10]Probably it would be appropriate to consider more general meromorphic functions, which include, for example, the rational functions of λ and k. But in any case, we should pay attention to the possibility of holomorphic functions, which have zeros of infinite order at the exceptional points (compare [Sch, Section 3.]).

Lemma 6.2 *The sheaf $\mathcal{E}(q)$ is a locally finite generated subsheaf of the sheaf of meromorphic functions on $\overline{B_{\text{variety}}(q)}$. Moreover, the map $(f_\kappa)_{\kappa\in\Gamma^\sharp} \mapsto \sum_{\kappa\in\Gamma^\sharp} f_\kappa\tilde{\psi}(q,\kappa)$ from the sheaf of all holomorphic $\ell^2(\Gamma^\sharp)$-valued functions into the direct image sheaf $\overline{k}_*(\mathcal{E}(q))$ is injective and the image of this map contains the sheaf $\overline{k}_\oplus(\mathcal{E}(q))$.*

Proof: The first statement follows from Proposition 5.2. Let $\mathcal{F}\circ\tilde{\mathcal{O}}_q\circ\mathcal{F}^{-1}$ be the sheaf on $X(\mathcal{I}_1)_{\text{fine}}$ of all $\ell^2(\Gamma^\sharp)$-operator-valued holomorphic functions, which commute with

$$\mathcal{F}\circ M(-k)\circ I(K,\mathbb{R}^n)\circ(-\Delta_{[k]}+q_k)\circ I^*(K,\mathbb{R}^n)\circ M(k)\circ\mathcal{F}^{-1},$$

where $\mathcal{F}$ is the unitary Fourier transform from $L^2(\mathbb{R}^n/\Gamma)$ onto $\ell^2(\Gamma^\sharp)$. Here for all $\overline{k}\in X(\mathcal{I}_1)_{\text{fine}}$, k is the corresponding element of $\mathbb{C}^n$, where all the components $\mathbb{C}K$ of $X(\mathcal{I}_1)_{\text{fine}}$ are identified with the subspaces $\mathbb{C}K\subset\mathbb{C}^n$. By definition, the image of the map $(f_\kappa)_{\kappa\in\Gamma^\sharp}\mapsto\sum_{\kappa\in\Gamma^\sharp} f_\kappa\tilde{\psi}(q,\kappa)$ from the sheaf of all holomorphic $\ell^2(\Gamma^\sharp)$-valued functions into the direct image sheaf $\overline{k}_*(\mathcal{E}(q))$ is an $\mathcal{F}\circ\tilde{\mathcal{O}}_q\circ\mathcal{F}^{-1}$-module. Thus, due to Lemma 3.4, this map is injective and the image is a module of the subsheaf $\overline{k}_\oplus(\mathcal{O}_{\overline{B_{\text{variety}}(q)}})$ of the direct image sheaf of the structure sheaf of $\overline{B_{\text{variety}}(q)}$. But the sections $\tilde{\psi}(q,\kappa)$ generate the sheaf $\overline{k}_\oplus(\mathcal{E}(q))$ as a $\overline{k}_\oplus(\mathcal{O}_{\overline{B_{\text{variety}}(q)}})$-module. This proves the second claim. □

In general, the sheaf $\mathcal{E}(q)$ is not locally free and therefore not isomorphic to the sheaf of sections of a line bundle. But the restriction of the natural line bundle of the projective space of $\ell^2(\Gamma^\sharp)$ to the subspace $\mathcal{B}(q)$ defines a line bundle on this subspace, which we want to call the eigen bundle. Moreover, the preceding theorem shows that the sheaf $\mathcal{E}(q)$ is a candidate for the direct image sheaf of the sheaf of sections of the dual eigen bundle under the natural map $\mathcal{B}(q)\to\overline{B_{\text{variety}}(q)}$ which, however, is not continuous.

Let $\hat{\mathcal{S}}$ be the sheaf of sections of the induced Hilbert bundle

$$\ell^2(\Gamma^\sharp)\times_{\Gamma^\sharp}\overline{B_{\text{variety}}(q)}.$$

These sections may be considered $\ell^2(\Gamma^\sharp)$-valued functions $(f(\kappa))_{\kappa\in\Gamma^\sharp}$, which satisfy

$$f(\kappa)(\kappa'.y)=f(\kappa-\kappa')(y)\text{ for all }\kappa,\kappa'\in\Gamma^\sharp\text{ and all }y\in\overline{B_{\text{variety}}(q)}.$$

Roughly speaking, the sheaves of the form $\mathcal{E}(q)$ may be characterized by two properties: They are coherent sheaves of rang 1, and the tensor product with $\hat{\mathcal{S}}$ is free. In the next section, we will see that this is a precise statement for compact Bloch-spaces.

7 Compact Bloch-spaces

In the one-dimensional case the complex Bloch-spaces of finite-gap potentials are compact Riemann surfaces with one point removed, which forms the only pole of the energy function λ. These potentials are well understood (see e.g. [No, MK-M, Kr, N-M-P-Z]). They are dense in all one-dimensional periodic potentials. This section is concerned with possible candidates for compact complex Bloch-spaces. In Section 4, we saw how the pole divisor of the meromorphic energy function λ contains twice the sum of all complex Bloch-varieties of the averaged potentials q_K with $K\in\mathcal{I}_{\text{effective}}$. Therefore, the completion of the complex Bloch-space have the possibility of being a compact complex space, when $\mathcal{I}_{\text{effective},1}$ is finite.

For the reason, let $\mathcal{I}_{\text{effective},1}$ be a finite subset of $\mathcal{I}_1$, such that the corresponding set $\mathcal{I}_{\text{effective}}$ contains the whole Euclidean space $\mathbb{R}^n$. Let $\pi : Y \to X(\mathcal{I}_{\text{effective},1})$ be an infinitely-sheeted, but locally finite-sheeted analytic covering, which obeys the following conditions:

compact Bloch-space (i) For each $K \in \mathcal{I}_{\text{effective}}$, the subspace $\pi^{-1}(\mathbb{C}K)$ has one component Y_K. This complex space Y_K has a global holomorphic function λ_K, such that it is locally biholomorphic to the complex space defined by an equation of the form

$$\lambda_K^l + \lambda_K^{l-1} e_1 + \ldots + e_l = 0,$$

where $e_1, \ldots, e_l$ are holomorphic functions on an open subset of $\mathbb{C}K$.

compact Bloch-space (ii) For the zero-space $K = \{0\}$, the space $Y_{\{0\}}$ is the one point complex space.

compact Bloch-space (iii) For each $K \in \mathcal{I}_{\text{effective}}$, the action of the dual lattice $\Gamma^\sharp \cap K$ on $\mathbb{C}K$ lifts to an action on Y_K, which leaves λ_K invariant. In addition, the subspace $\pi^{-1}(\mathbb{C}K)$ is equal to $\Gamma^\sharp \times_{\Gamma^\sharp \cap K} Y_K$. This complex space is defined as all equivalence classes of the product of the discrete complex space $\Gamma^\sharp$ with Y_K with respect to the equivalence relation $(\kappa + \kappa', y) \sim (\kappa, \kappa'.y)$ for all $\kappa' \in \Gamma^\sharp \cap K$. Finally, the action of $\Gamma^\sharp$, which on all components $\Gamma^\sharp \times_{\Gamma^\sharp \cap K} Y_K$ is given by the right action on the first factor, fit together to form a holomorphic action on Y.

compact Bloch-space (iv) For all $\kappa \in \Gamma^\sharp$, the union $\bigcup_{K \in \mathcal{I}_{\text{effective}}} \kappa.Y_K$ is an open subset of Y. Moreover the functions $\lambda_{\text{renormalized},\kappa} = \lambda_K - 4\pi^2 g(k+\kappa, k+\kappa)$ fit together to form a holomorphic function on this open subset. Here k denotes for all $\overline{k} \in X(\mathcal{I}_{\text{effective},1})$, the corresponding element of $\mathbb{C}^n$, where all the components $\mathbb{C}K$ are considered as subsets of $\mathbb{C}^n$. In particular, Y is locally biholomorphic to the complex space defined by an equation of the form

$$\lambda_{\text{renormalized},\kappa}^l + \lambda_{\text{renormalized},\kappa}^{l-1} e_1 + \ldots + e_l = 0,$$

where $e_1, \ldots, e_l$ are holomorphic functions on an open subset of $X(\mathcal{I}_{\text{effective},1})$.

compact Bloch-space (v) There exists a coherent subsheaf $\mathcal{E}$ of the sheaf $\mathcal{M}_{Y/\Gamma^\sharp}$ of meromorphic functions of $Y/\Gamma^\sharp$, such that for all $K \in \mathcal{I}_{\text{effective}}$ the dimensions of the homology groups of the restriction of $\mathcal{E}$ to $\overline{Y_K}/(\Gamma^\sharp \cap K)$ are equal to

$$\dim H^i\left(\overline{Y_K}/(\Gamma^\sharp \cap K), \mathcal{E}|_{\overline{Y_K}/(\Gamma^\sharp \cap K)}\right) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

The case $K = \{0\}$ should be understood as the assumption that $\mathcal{E}$ is equal to $\mathcal{O}_{Y/\Gamma^\sharp}$ on a neighborhood of the unique point of $Y_{\{0\}} \subset Y/\Gamma^\sharp$.

The following are some of the consequences of these conditions:

1. Due to assumption (i) and (ii), $\lambda_{\{0\}}$ is a complex number.
2. For each $K \in \mathcal{I}_{\text{effective}}$ and each point $y \in Y_K$, there exists an open neighborhood $U \subset Y_K$ of y, whose shifted sets $\kappa.U$, with $\kappa \in \Gamma^\sharp \cap K$, do not intersect each other. Hence $Y_K/(\Gamma^\sharp \cap K)$ is a complex space, which is locally biholomorphic to Y_K. Thus, all sheaves on $Y_K/(\Gamma^\sharp \cap K)$ may be considered sheaves on Y_K.

3. For each point y of Y, there exists an open neighborhood $U \subset Y$, of which the shifted sets $\kappa.U$ with $\kappa \in \Gamma^\sharp$ are disjoint for all. Hence $Y/\Gamma^\sharp$ is a complex space, which is locally biholomorphic to Y. All sheaves on $Y/\Gamma^\sharp$ may therefore be considered also as sheaves on Y.

4. As a set, the complex space $Y/\Gamma^\sharp$ may be identified with the disjoint union
$$Y/\Gamma^\sharp = \bigcup_{K\in\mathcal{I}_{\text{effective}}} Y_K/(\Gamma^\sharp \cap K).$$

5. For each $\mathcal{I}_{\text{effective}} \ni K \neq \{0\}$, the closure $\overline{Y_K}$ of Y_K is an infinitely-sheeted, locally finite-sheeted analytic covering of $X(\{K' \in \mathcal{I}_{\text{effective},1} | \, K' \subset K\})$, which also satisfies the conditions on compact Bloch-spaces (i) – (iv), where $\mathcal{I}_{\text{effective}}$ is replaced by $\{K' \in \mathcal{I}_{\text{effective}} | \, K' \subset K\}$ and $\Gamma^\sharp$ is replaced by $\Gamma^\sharp \cap K$.

6. For each $K \in \mathcal{I}_{\text{effective}}$, the complex space $\overline{Y_K}/(\Gamma^\sharp \cap K)$ may be considered as a closed subspace of $Y/\Gamma^\sharp$.

Theorem 7.1 *Let Y satisfy conditions (i)–(iv) on compact Bloch-spaces. Then the quotient $Y/\Gamma^\sharp$ is a projective variety and the condition (v) on compact Bloch-spaces is equivalent to the existence of N analytic functions $q_1, \ldots, q_N$ from $\mathbb{R}/\mathbb{Z}$ into $\mathbb{C}$, so that Y is a completion of the complex Bloch-space of the potential*
$$q : \mathbb{R}^n \to \mathbb{C}, x \mapsto \lambda_{\{0\}} + \sum_{i=1}^{N} q_i\left(g(x, \kappa_i)\right).$$
Here again $\kappa_1, \ldots, \kappa_N$ are some generators of the one-dimensional lattices $\Gamma^\sharp \cap K$, with $K \in \mathcal{I}_{\text{effective},1}$.

Proof: Let Y obey conditions (i) – (iv) on compact Bloch-spaces, and let $\mathcal{U}$ be an open covering of $Y/\Gamma^\sharp$. This complex space is, as a set, equal to the disjoint union
$$Y/\Gamma^\sharp = \bigcup_{K\in\mathcal{I}_{\text{effective}}} Y_K/(\Gamma^\sharp \cap K).$$
Each component $Y_K/(\Gamma^\sharp \cap K)$ is a locally finite-sheeted analytic covering of the complex space $\mathbb{C}K/(\Gamma^\sharp \cap K)$. More precisely, the number of sheets is equal to the order of $\Gamma^\sharp \cap K$. One element $U_0 \in \mathcal{U}$ contains the point $Y_{\{0\}}$. Now we isolate a one-dimensional $K \in \mathcal{I}_{\text{effective},1}$. If U is an open set of $X(\{K\})$ containing the only point of the component $\{0\}$, then for all elements κ of the complement of a finite subset of $\Gamma^\sharp \cap K$, the shifted sets $\kappa.U$ cover the complement $X(\{K\}) \setminus U$ of U. This implies that the intersection of the complement of U_0 and the component $Y_K//(\Gamma^\sharp \cap K)$ is compact. Moreover, for all such K, there exists a finite subset $\mathcal{U}_K \subset \mathcal{U}$, such that $\{U_0\} \cup \mathcal{U}_K$ covers the component $Y_K/(\Gamma^\sharp \cap K)$. But when we isolate an arbitrary $K \in \mathcal{I}_{\text{effective}}$. and an open set U of $X(\{K' \in \mathcal{I}_{\text{effective},1} | \, K' \subset K\})$, which covers all components $\mathbb{C}K_0$ of $X(\{K' \in \mathcal{I}_{\text{effective},1} | \, K' \subset K\})$, with $K_0 \subsetneq K$, then again for all elements κ of the complement of a finite subset of $\Gamma^\sharp \cap K$, the shifted sets $\kappa.U$ covers the complement $X(\{K' \in \mathcal{I}_{\text{effective},1} | \, K' \subset K\}) \setminus U$ of U. The inductive use of this argumentation shows that for all $K \in \mathcal{I}_{\text{effective}}$, there exists a finite subset $\mathcal{U}_K \subset \mathcal{U}$, such that $\bigcup_{K_0 \subset K} \mathcal{U}_{K_0}$

is a finite subcovering of the component $\overline{Y_K}/(\Gamma^\sharp \cap K)$. This proves that $Y/\Gamma^\sharp$ is compact. Next we shall present an invertible sheaf on $Y/\Gamma^\sharp$, which is positive in the sense of [G-P-R, Chapter V. §2.4.]. For any Weil-divisor D on $X(\mathcal{I}_{\text{effective},1})$ of the form

$$D = \sum_{K \in \mathcal{I}_{\text{effective},n-1}} n_K \cdot \mathbb{C}K \text{ with } n_K \geq 1,$$

let $\pi^*(D)$ be the corresponding Weil-divisor on $Y/\Gamma^\sharp$:

$$\pi^*(D) = \sum_{K \in \mathcal{I}_{\text{effective},n-1}} n_K \cdot \overline{Y_K}/(\Gamma^\sharp \cap K).$$

In general the complex space $Y/\Gamma^\sharp$ is not normal and Weil-divisors are not defined on $Y/\Gamma^\sharp$ (see [G-P-R, Chapter II. §5.4.] and [Ha, Chapter II. 6]), but the sheaves corresponding to these divisors are well defined. In fact they are locally equal to the pullback of the sheaves $\mathcal{O}_D$ on $X(\mathcal{I}_{\text{effective},1})$ under the morphism π. All divisors of this from $\pi^*(D)$, which correspond to invertible sheaves, are positive. In fact, since the complex spaces Y_K are Stein-spaces, the inductive application of [Gr-2, §3. Satz 4.] proves the claim. Since for any $K_1 \in \mathcal{I}_{\text{effective},1}$ the Weil-divisor

$$\sum_{K \in \mathcal{I}_{\text{effective},n-1}, K_1 \not\subset K} \mathbb{C}K$$

corresponds to an invertible sheaf, a lot of these divisors correspond to invertible sheaves. Finally, [Gr-2, §3. Satz 2.] (see also [G-P-R, Chapter V. Corollary 4.5.]) implies that $Y/\Gamma^\sharp$ is a projective variety.

Let us now assume that condition (v) is fulfilled. Let $\mathcal{S}$ denote the invertible sheaf of the complex space $Y/\Gamma^\sharp \times \mathbb{C}^n$, whose sections may be represented as holomorphic functions f on $Y \times \mathbb{C}^n$, so that

$$f(\kappa.y, x) = \exp\left(-2\pi\sqrt{-1}g(x,\kappa)\right) f(y,x) \text{ for all } \kappa \in \Gamma^\sharp, y \in Y \text{ and } x \in \mathbb{C}^n.$$

Due to the first claim, the projection $\overline{Y_K}/(\Gamma^\sharp \cap K) \times \mathbb{C}^n \twoheadrightarrow \mathbb{C}^n$ onto the second factor is, for each $K \in \mathcal{I}_{\text{effective}}$, a proper map and the sheaf $\mathcal{S}|_{\overline{Y_K}/(\Gamma^\sharp \cap K) \times \mathbb{C}^n}$ is flat with respect to this map. Moreover, the sheaf

$$\left(\mathcal{E}|_{\overline{Y_K}/(\Gamma^\sharp \cap K)} \times \mathcal{O}_{\mathbb{C}^n}\right) \otimes \mathcal{S}|_{\overline{Y_K}/(\Gamma^\sharp \cap K) \times \mathbb{C}^n}$$

is also flat with respect to this proper map. Hence we may apply here Grauerts Semi-Continuity Theorem [Gr-1] (compare [G-P-R, Chapter III. Theorem 4.7]). For all $x \in \mathbb{C}^n$, let $\mathcal{S}(x)$ denote the sheaf $\mathcal{S}|_{Y/\Gamma^\sharp \times \{x\}}$ on $Y/\Gamma^\sharp$. Thus there exists a subvariety $\mathbb{C}^n \supset S \not\ni 0$, such that for all $x \in \mathbb{C}^n \setminus S$, the homology groups of the sheaf $\mathcal{E}|_{\overline{Y_K}/(\Gamma^\sharp \cap K)} \otimes \mathcal{S}(x)|_{\overline{Y_K}/(\Gamma^\sharp \cap K)}$ are equal to

$$\dim H^i\left(\overline{Y_K}/(\Gamma^\sharp \cap K), \mathcal{E}|_{\overline{Y_K}/(\Gamma^\sharp \cap K)} \otimes \mathcal{S}(x)|_{\overline{Y_K}/(\Gamma^\sharp \cap K)}\right) = \begin{cases} 1 & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

Hence for all $x \in \mathbb{C}^n \setminus S$ and all $K \in \mathcal{I}_{\text{effective}}$, there exists unique functions ψ_K on $\Gamma^\sharp.Y_K$, which satisfy the following conditions:

Bloch-function (i) The functions ψ_K are invariant under the action of $\Gamma^\sharp$.

Bloch-functions (ii) For $K = \{0\}$, the function $\psi_{\{0\}}$ is equal to 1.

Bloch-function (iii) For all $\kappa \in \Gamma^\sharp$, the functions $\exp\left(-2\pi\sqrt{-1}g(x,k+\kappa)\right)\psi_K$ together form a section of $\mathcal{E}\otimes\mathcal{S}(x)$ on the open sets $\bigcup\limits_{K\in\mathcal{I}_{\text{effective}}} \kappa.Y_K$. Here the notation is the same as in condition (iv) on compact Bloch-spaces.

These functions have three additional properties:

Bloch-function (iii)' Since the functions $\exp\left(-2\pi\sqrt{-1}g(x,k+\kappa)\right)\psi_K$ together form a section of $\mathcal{E}\otimes\mathcal{S}$ on the open set $\bigcup\limits_{K\in\mathcal{I}_{\text{effective}}} \kappa.Y_K \times \mathbb{C}^n \setminus S$, the functions ψ_K depend holomorphically on x.

Bloch-function (iv) Since for all $K \in \mathcal{I}_{\text{effective}}$, the sheaf $\mathcal{S}(x)|_{\overline{Y_K}/(\Gamma^\sharp\cap K)}$ depends only on the orthogonal projection of x onto $\mathbb{C}K$, the function ψ_K depends also only on the orthogonal projection of x onto $\mathbb{C}K$.

Bloch-function (v) Since for all $\gamma\in\Gamma$ and all $x\in\mathbb{C}^n\setminus S$ the sheaf $\mathcal{S}(x+\gamma)$ is equal to $\mathcal{S}(x)$, the function ψ_K corresponding to $x+\gamma$, for all $K\in\mathcal{I}_{\text{effective}}$, is equal to $\exp\left(2\pi\sqrt{-1}g(x,k)\right)$ times the function ψ_K corresponding to x.

Next the function $(\Delta+\lambda_K)\psi_K$ is claimed, for all $x\in\mathbb{C}^n\setminus S$ and all $K\in\mathcal{I}_{\text{effective}}$, to be equal to ψ_K multiplied by a complex number $q_K(x)$. In fact, it follows inductively from properties (iii)' and (iv) and condition (iv) on compact Bloch-spaces that, for all $K\in\mathcal{I}_{\text{effective}}$, the function $\exp\left(-2\pi\sqrt{-1}g(x,k+\kappa)\right)(\Delta+\lambda_K)\psi_K$ extends to a section of $\left(\mathcal{E}|_{\overline{\kappa.Y_K}}\times\mathcal{O}_{\mathbb{C}^n}\right)\otimes\mathcal{S}|_{\overline{\kappa.Y_K}\times\mathbb{C}^n}$ on the set $\overline{\kappa.Y_K}\times\mathbb{C}^n\setminus S$. In a neighborhood of the unique point of $Y_{\{0\}}$, we may expand the function

$$\exp\left(-2\pi\sqrt{-1}g(x,k)\right)\psi_K = 1 + \sum_{i\in S_K}\frac{Q_i(x)}{g(k,\kappa_i)} + \text{ higher order terms},$$

with some holomorphic functions $Q_1,\ldots,Q_N:\mathbb{C}^n\setminus S\to\mathbb{C}$. Due to property (iv), for all $i=1,\ldots,N$, the function Q_i depends only on $g(x,\kappa_i)$. A simple calculation shows that for all $K\in\mathcal{I}_{\text{effective}}$ the potential $q_K(x)$ is equal to

$$q_K(x) = \lambda_{\{0\}} + \sum_{i\in S_K} q_i\left(g(x,\kappa_i)\right),$$

where the functions $q_1,\ldots,q_N$ are equal to

$$q_i(g(x,\kappa_i)) = 4\pi\sqrt{-1}\frac{dQ_i(x)}{dg(x,\kappa_i)}.$$

The theory of one-dimensional finite-gap potentials shows that these functions $q_1,\ldots,q_N$ are meromorphic functions on $\mathbb{C}$ (see e.g. [N-M-P-Z]) and due to property (v), periodic with period 1. Hence there exists a shift, such that the shifted one-dimensional potentials are analytic functions on $\mathbb{R}/\mathbb{Z}$. This proves that condition (v) on compact Bloch-spaces implies the existence of a potential with the stated properties.

Now let us assume that there exists analytic functions $q_1,\ldots,q_N$ from $\mathbb{R}/\mathbb{Z}$ into $\mathbb{C}$, so that for all $K\in\mathcal{I}_{\text{effective}}$, the complex Bloch-space of the potential

$$q_K(x) = \lambda_{\{0\}} + \sum_{i\in S_K} q_i\left(g(x,\kappa_i)\right)$$

is equal to Y_K. Hence the eigen function $\left(\tilde{\psi}(q,\kappa)\right)_{\kappa\in\Gamma^\sharp}$ extends to a meromorphic $\ell^2(\Gamma^\sharp)$-valued function of Y, which is equivariant under the action of $\Gamma^\sharp$ on Y and $\ell^2(\Gamma^\sharp)$. These functions generate a subsheaf $\mathcal{E}(q)$ of the sheaf $\mathcal{M}_Y$ of meromorphic functions on Y. Due to the equivariance of the function $\left(\tilde{\psi}(q,\kappa)\right)_{\kappa\in\Gamma^\sharp}$, this sheaf is invariant under the action of $\Gamma^\sharp$ on Y. Hence this sheaf may also be considered as a sheaf of $Y/\Gamma^\sharp$. Moreover, due to assumption (iv) on compact Bloch-spaces, this sheaf $\mathcal{E}(q)$ is a coherent subsheaf of $\mathcal{M}_{Y/\Gamma^\sharp}$ (compare [G-P-R, Chapter I. (7.8)]). Due to Lemma 6.2, the subsheaf $\pi_\oplus\left(\mathcal{E}(q)\right)$ of the direct image sheaf of the sheaf $\mathcal{E}(q)$ on Y is a subsheaf of the free sheaf of $\ell^2(\Gamma^\sharp)$-valued holomorphic functions on $X(\mathcal{I}_{\text{effective},1})$. Hence we may expect that the cohomology groups $H^i\left(Y/\Gamma^\sharp,\mathcal{E}(q)\right)$ are isomorphic to the cohomology groups $H^i\left(X(\mathcal{I}_{\text{effective},1}),\mathcal{O}\right)$. This is proven in Lemma 7.2. The argumentation is the same for the restriction of the sheaf $\mathcal{E}(q)$ to any component $\overline{Y_K}/(\Gamma^\sharp\cap K)$.
□

Lemma 7.2 *If D is a Weil-divisor on $X(\mathcal{I}_{\text{effective},1})$ of the form*

$$D=\sum_{K\in\mathcal{I}_{\text{effective},n-1}} n_K\cdot\mathbb{C}K \quad \textit{with } n_K\geq -1$$

and $\pi^*(D)$ the corresponding Weil-divisor[11] *on $Y/\Gamma^\sharp$*

$$\pi^*(D)=\sum_{K\in\mathcal{I}_{\text{effective},n-1}} n_K\cdot\overline{Y_K}/(\Gamma^\sharp\cap K),$$

then the dimensions of the following cohomology groups are equal to:

$$\dim H^i\left(Y/\Gamma^\sharp,\mathcal{E}(q)\otimes\mathcal{O}_{\pi^*(D)}\right)=\begin{cases}\dim H^i\left(X(\mathcal{I}_{\text{effective},1}),\mathcal{O}_D\right) & \textit{if } i=0\\ 0 & \textit{if } i>0.\end{cases}$$

Proof: We use an inductive argument with respect to the dimension n. For $n=0$, there is nothing to prove. Next let us assume that the statement is correct in dimension less than n. Let K be an element of $\mathcal{I}_{\text{effective},n-1}$ and D and $D'=D+\mathbb{C}K$ two divisors on $X(\mathcal{I}_{\text{effective},1})$ of the form mentioned above. The short exact sequence of sheaves

$$0\to\mathcal{E}(q)\otimes\mathcal{O}_{\pi^*(D)}\to\mathcal{E}(q)\otimes\mathcal{O}_{\pi^*(D')}\to\mathcal{E}(q)\otimes\mathcal{O}_{\pi^*(D)}/\left(\mathcal{E}(q)\otimes\mathcal{O}_{\pi^*(D')}\right)\to 0$$

induces a long exact sequence of the corresponding cohomology groups. The support of the sheaf $\mathcal{E}(q)\otimes\mathcal{O}_{\pi^*(D)}/\left(\mathcal{E}(q)\otimes\mathcal{O}_{\pi^*(D')}\right)$ is equal to the divisor $\pi^*(K)=\overline{Y_K}/(\Gamma^\sharp\cap K)$ and on this subspace the sheaf is equal to a sheaf of the form mentioned in the lemma corresponding to the potential q_K. If some components Y_K are not biholomorphic to the complex Bloch-space of the potential q_K, then the complex Bloch-space of q_K is a covering of Y_K, and the restriction of the sheaf $\mathcal{E}(q)$ to Y_K is isomorphic to the direct image of the sheaf $\mathcal{E}(q_K)$ under the covering map of the complex Bloch-space of the potential q_K onto Y_K. Thus the cohomology groups of the sheaf $\mathcal{E}(q)|_{\overline{Y_K}/(\Gamma^\sharp\cap K)}$ on $\overline{Y_K}/(\Gamma^\sharp\cap K)$ are equal to the cohomology groups of the sheaf $\mathcal{E}(q_K)$ on some completion of the complex Bloch-space of the potential

[11] In general the complex space $Y/\Gamma^\sharp$ is not normal and the Weil-divisors on $Y/\Gamma^\sharp$ are not defined (see [G-P-R, Chapter II. §5.4.] and [Ha, Chapter II. 6]), but the sheaves corresponding to these divisors are well defined. In fact they are locally equal to the pullback of the sheaves $\mathcal{O}_D$ on $X(\mathcal{I}_{\text{effective},1})$ under the morphism π.

q_K. Due to the assumption that the statement of the lemma is correct in dimension less than n, we may calculate the cohomology groups of the sheaf $\mathcal{E}(q) \otimes \mathcal{O}_{\pi^*(D)} / \left(\mathcal{E}(q) \otimes \mathcal{O}_{\pi^*(D')}\right)$. They are equal to $\dim H^i\left(Y/\Gamma^\sharp, \mathcal{E}(q) \otimes \mathcal{O}_{\pi^*(D)} / \left(\mathcal{E}(q) \otimes \mathcal{O}_{\pi^*(D')}\right)\right) =$

$$= \begin{cases} \dim H^i\left(X(\mathcal{I}_{\text{effective},1}), \mathcal{O}_{D'}\right) - \dim H^i\left(X(\mathcal{I}_{\text{effective},1}), \mathcal{O}_D\right) & \text{if } i = 0 \\ 0 & \text{if } i > 0. \end{cases}$$

Each global section of $\mathcal{O}_{D'}$, which is not a global section of $\mathcal{O}_D$ is a polynomial with respect to the entries of k. The action of the corresponding differential operator on the Bloch-function $\psi(q)$ at the point $x = 0$ defines a global section of $\mathcal{E}(q) \otimes \mathcal{O}_{\pi^*(D')}$, which is not a global section of $\mathcal{E}(q) \otimes \mathcal{O}_{\pi^*(D)}$. Hence the the dimensions of the corresponding cohomology groups satisfy the following inequality:

$$\dim H^0\left(Y/\Gamma^\sharp, \mathcal{E}(q) \otimes \mathcal{O}_{\pi^*(D')}\right) - \dim H^0\left(Y/\Gamma^\sharp, \mathcal{E}(q) \otimes \mathcal{O}_{\pi^*(D)}\right) \geq$$

$$\geq \dim H^0\left(X(\mathcal{I}_{\text{effective},1}), \mathcal{O}_{D'}\right) - \dim H^0\left(X(\mathcal{I}_{\text{effective},1}), \mathcal{O}_D\right).$$

We conclude with the help of the long exact sequence of cohomology groups, that if for all $i > 0$, the dimensions of the cohomology groups $H^i\left(Y/\Gamma^\sharp, \mathcal{E}(q) \otimes \mathcal{O}_{\pi^*(D')}\right)$ are zero, then the same is true for $H^i\left(Y/\Gamma^\sharp, \mathcal{E}(q) \otimes \mathcal{O}_{\pi^*(D)}\right)$. Due to [Gr-2, §3. Hilfssatz 1.] (see also [G-P-R, Chapter V. Theorem 4.3.]), for some integer l and all $i > 0$, the dimensions of the cohomology groups $H^i\left(Y/\Gamma^\sharp, \mathcal{E}(q) \otimes \mathcal{O}_{\pi^*(l \cdot D)}\right)$ are zero, where $\mathcal{O}_{\pi^*(D)}$ is a positive sheaf on $Y/\Gamma^\sharp$. This proves the statement of the lemma for all cohomology groups in dimensions $i > 0$. Moreover, the proof would be complete, if we can show that the cohomology group $H^0\left(Y/\Gamma^\sharp, \mathcal{E}(q) \otimes \mathcal{O}_{\pi^*(-\infty)}\right)$ is zero, where ∞ is the following Weil-divisor on $X(\mathcal{I}_{\text{effective},1})$:

$$\infty = \sum_{K \in \mathcal{I}_{\text{effective},n-1}} \mathbb{C}K.$$

The complex Bloch-space of the transposed potential $q^t(x) = q(-x)$ is the same as the complex Bloch-space of q. In fact, the projection $P(q)$ onto the eigenspaces may be expressed as

$$P(q) : L^2(\mathbb{R}^n/\Gamma) \to L^2(\mathbb{R}^n/\Gamma), \phi \mapsto \psi(q) \frac{\int_{\mathbb{R}^n/\Gamma} \psi(q^t)(-x)\phi(x)d^n x}{\int_{\mathbb{R}^n/\Gamma} \psi(q^t)(-x)\psi(q)(x)d^n x},$$

where $\psi(q)$ is the Bloch-function of the potential q and $\psi(q^t)$ is the Bloch-function of the transposed potential. Due to Theorem 3.6 the sheaf of regular n-forms of $B_{\text{complex space}}(q)$ in the sense of [Kun] are equal to $\mathcal{E}(q) \otimes \mathcal{E}(q^t) \otimes k^*(\omega_{\mathbb{C}^n})$, where $\omega_{\mathbb{C}^n}$ is the sheaf of holomorphic n-forms on $\mathbb{C}^n$. From Lemma 4.2 and [G-P-R, Chapter II. Lemma 5.30] we conclude that the dualizing sheaf (see [G-P-R, Chapter II. §5.3.]) $\omega_{X(\mathcal{I}_{\text{effective},1})}$ on $X(\mathcal{I}_{\text{effective},1})$ is equal to $\mathcal{O}_{-2\cdot\infty}$. Thus the dualizing sheaf on $Y/\Gamma^\sharp$ is equal to $\mathcal{E}(q) \otimes \mathcal{E}(q^t) \otimes \mathcal{O}_{\pi^*(-2\cdot\infty)}$. Since $Y/\Gamma^\sharp$ is a Cohen-Macaulay-space, we may apply Serre-Duality (see e.g. [G-P-R, Chapter III. Theorem 4.15a]). This shows that the dimension of $H^0\left(Y/\Gamma^\sharp, \mathcal{E}(q) \otimes \mathcal{O}_{\pi^*(-\infty)}\right)$ is equal to the dimension of $H^n\left(Y/\Gamma^\sharp, \mathcal{E}(q^t) \otimes \mathcal{O}_{\pi^*(-\infty)}\right)$, which is equal to zero. □

In the case of $n = 1$, the conditions (i) – (iv) on compact complex spaces imply condition (v). In this case Theorem 7.1 is well known (see e.g. [Kr]). The results of the papers [C-V, C-S-V] show that there exist non trivial examples of compact complex Bloch-spaces and complex spaces, which satisfies conditions (i) – (iv) on compact Bloch-spaces, but not condition (v). These examples are singular complex spaces, whose normalizations are biholomorphic to some completions of the complex Bloch-space of the zero potential. It could be possible that this theorem here may be used to classify all compact complex Bloch-spaces, whose normalizations are biholomorphic to a completion of the complex Bloch-space of the zero potential. In [S-V] this theorem is used to determine a compact complex Bloch-space of a two-dimensional potential, which is equal to the sum of three Weierstrass functions.

The final proof of this paper concerns a theorem, which essentially is known to the experts, but to our knowledge not published. First we have to introduce some terminology. A smooth potential q on the Euclidean space $\mathbb{R}^n$ is called integrable if there exists n commuting differential operators $L_1 = -\Delta + q, L_2, \ldots, L_n$, with constant algebraically independent highest symbols $P_1(k) = 4\pi^2 g(k,k), P_2(k), \ldots, P_n(k)$. Moreover, a potential is called algebraically integrable if there exists $n+1$ commuting differential operators $L_1 = -\Delta + q, L_2, \ldots, L_{n+1}$, with constant highest symbols $P_1(k) = 4\pi^2 g(k,k), P_2(k), \ldots, P_{n+1}(k)$, so that all components $k_1, \ldots, k_n$ may be written as rational functions in terms of the highest symbols $P_1(k) = 4\pi^2 g(k,k), P_2(k), \ldots, P_{n+1}(k)$ (compare with [C-V, C-S-V]).

Theorem 7.3 *The following conditions on a periodic potential $q \in C(\mathbb{R}^n/\Gamma)$ are equivalent:*

(i) *$\mathcal{B}(q)$ is a complex space.*

(ii) *$\mathcal{I}_{\text{effective},1}$ is finite and there exists a completion of the complex Bloch-space of q, which satisfies the conditions (i) – (iv) on compact Bloch-spaces.*

(iii) *The periodic potential q is smooth and algebraically integrable.*

(iv) *$\mathcal{B}(q)/\Gamma^\sharp$ is a projective variety*[12].

Moreover, all these equivalent statements imply that $\mathcal{I}_{\text{effective}}$ is a finite set, that q is an analytic potential and that there exists a morphism of the projective variety $\mathcal{B}(q)/\Gamma^\sharp$ onto the compact complex Bloch-space $Y/\Gamma^\sharp$.

Proof: (i) $\Rightarrow$ (ii): Let $\mathcal{B}(q)$ be a complex space. First let us prove that $\mathcal{I}_{\text{effective},1}$ is a finite set. Due to Theorem 4.1, we may assume that $\dim K_{\text{effective}}$ is equal to n. Again due to Theorem 4.1, the union of all components $\Gamma^\sharp.\mathcal{B}(q_K)$ with $\dim K < n$ is equal to the polar set of the meromorphic energy function λ and therefore an analytic subset of $\mathcal{B}(q)$ (compare [G-P-R, Chapter I. §7.3.]). For any $K \in \mathcal{I}_{\text{effective},n-1}$ let γ_K be a vector in Γ, which is perpendicular to K. Then the polar set of the meromorphic function $g(k, \gamma_K)$ is equal to $\Gamma^\sharp.\mathcal{B}(q_K)$. The following element of $\mathcal{B}(q)$

$$\left(\tilde{\psi}(\kappa)\right)_{\kappa\in\Gamma^\sharp}, \text{ with } \tilde{\psi}(\kappa) = \begin{cases} 1 & \text{if } \kappa = 0, \\ 0 & \text{if } \kappa \neq 0. \end{cases}$$

is contained in all components $\mathcal{B}(q_K)$ of the polar set of λ. Due to the Local Decomposition Lemma ([G-P-R, Chapter I. 4.10]), locally the number of irreducible components of the polar

[12] The weaker condition (iii)' that $\mathcal{B}(q)/\Gamma^\sharp$ is compact can be proven to imply statement (i) and therefore all other statements. Firstly, $\mathcal{I}_{\text{effective},1}$ is proven to be finite. Secondly, one shows that the topology of $X(\mathcal{I}_{\text{effective},1})$ is the finest topology, such that the natural map $\mathcal{B}(q) \to X(\mathcal{I}_{\text{effective},1})$ is continuous. Finally, the Riemann Extension Theorem is used to show that the functions $\lambda_{\text{renormalized},\kappa}$ define a locally finite-sheeted analytic covering of $X(\mathcal{I}_{\text{effective},1})$.

set of λ has to be finite. This shows that $\mathcal{I}_{\text{effective},n-1}$ is finite and an inductive argument proves the first claim. For all $K \in \mathcal{I}_{\text{effective}}$, the energy functions λ_K are holomorphic functions of $\mathcal{B}_0(q_K)$ and there exists a holomorphic map from $\mathcal{B}(q)$ onto $X(\mathcal{I}_{\text{effective},1})$. Let Y be the unique complex space, such that there exists a holomorphic map from $\mathcal{B}(q)$ onto Y and whose holomorphic functions are locally generated by the holomorphic functions of $X(\mathcal{I}_{\text{effective},1})$ and the functions λ_K. Due to Theorem 4.1, this complex space Y is a completion of the complex Bloch-space, which satisfies conditions (i) – (iv) on compact Bloch-spaces.

(ii) $\Rightarrow$ (iii): Due to the theory of finite-gap potentials (see [No, Kr, N-M-P-Z, D-K-N]), for all $K \in \mathcal{I}_{\text{effective},1}$ the one-dimensional potentials q_K are algebraically integrable and analytic. Thus the whole potential is analytic just as the eigen functions are analytic with respect to $x \in \mathbb{R}^n$. In the proof of Theorem 7.1 actually we showed that the subspace $Y_{\mathbb{R}^n}/\Gamma^\sharp$ of the projective variety $Y/\Gamma^\sharp$ is an affine subvariety. Furthermore, due to Lemma 7.2, all meromorphic functions on $Y/\Gamma^\sharp$, which are holomorphic on this affine subvariety, acts on the Bloch-function like partial differential operators commuting with each other. This proves (iii).

(iii) $\Rightarrow$ (iv): Let us first prove two lemmata:

Lemma 7.4 *Let q be an algebraically integrable periodic potential. Then the eigenvalues $\lambda_1 = \lambda, \lambda_2, \ldots, \lambda_{n+1}$ satisfy some polynomial equation. Moreover, the complex Bloch-space $B_{\text{complex space}}(q)/\Gamma^\sharp$ is locally biholomorphic to a local component of the subvariety of $\mathbb{C}^{n+1}$ defined by this equation.*

Proof: Let $L_1, \ldots, L_n$ be differential operators on $\mathbb{R}^n$, with constant and algebraically independent highest symbols $P_1(k), \ldots, P_n(k)$. Let $\underline{\psi}(x)$ be a vector, whose entries are space-like partial differentials at a point $x \in \mathbb{R}^n$ of the eigen function ψ, so that all partial differentials of ψ at the point x may be written as a polynomial in terms of this collection of partial differentials, the eigenvalues $\lambda_1, \ldots, \lambda_n$ and the potentials and their derivatives of $L_1, \ldots, L_n$ at x. Since $L_1, \ldots, L_n$ are integrable, this vector may be chosen to be of finite length l. Hence there exists n $l \times l$-matrices $U_1(\lambda_1, \ldots, \lambda_n, x), \ldots, U_n(\lambda_1, \ldots, \lambda_n, x)$, so that the compatibility equations of the partial differential equations

$$\left(\frac{\partial}{\partial x_1} + U_1(\lambda_1, \ldots, \lambda_n, x)\right)\underline{\psi}(x) = 0, \ldots, \left(\frac{\partial}{\partial x_n} + U_n(\lambda_1, \ldots, \lambda_n, x)\right)\underline{\psi}(x) = 0$$

are equivalent to the commutativity of the operators $L_1, \ldots, L_n$. Any other differential operator L_{n+1} acts on $\underline{\psi}$ as a matrix $W(\lambda_1, \ldots, \lambda_n, x)$, whose entries are again polynomials with respect to the eigenvalues and the potentials and their derivatives of $L_1, \ldots, L_n$ at x. If L_{n+1} commutes with the other differential operators $L_1, \ldots, L_n$, we obtain the partial differential equations

$$\begin{aligned}
\frac{\partial W(\lambda_1, \ldots, \lambda_n, x)}{\partial x_1} + [U_1(\lambda_1, \ldots, \lambda_n, x), W(\lambda_1, \ldots, \lambda_n, x)] &= 0 \\
&\vdots \\
\frac{\partial W(\lambda_1, \ldots, \lambda_n, x)}{\partial x_n} + [U_n(\lambda_1, \ldots, \lambda_n, x), W(\lambda_1, \ldots, \lambda_n, x)] &= 0.
\end{aligned}$$

This implies that the coefficients of the characteristic polynomial of $W(\lambda_1, \ldots, \lambda_n, x)$ does not depend on x. Hence it defines some polynomial equation and the first part of the lemma is proven. All the eigenvalues $\lambda_1,\ldots,\lambda_{n+1}$ are holomorphic functions on the complex Bloch-space. Hence we have a map from the complex Bloch-space into the subvariety of $(\lambda_1,\ldots,\lambda_{n+1})\in\mathbb{C}^{n+1}$ defined by this equation. Due to Corollary 5.4, this subvariety is connected. Moreover, due to the assumption of algebraically integrability and Proposition 5.2, the eigenvalues of $W(\lambda_1,\ldots,\lambda_n,x)$ are all different for some points near the limit $\lambda_1\to\infty,\ldots,\lambda_n\to\infty$. Thus all local sections of the sheaf $\mathcal{O}_q$ give rise to some local sections of the sheaf $\mathcal{O}_W$ (compare Section 3) and vice versa. Hence the complex Bloch-space is biholomorphic to the spectral complex space of the holomorphic matrix function $\mathbb{C}^n\ni(\lambda_1,\ldots,\lambda_n)\mapsto W(\lambda_1,\ldots,\lambda_n,x)$. This proves the lemma. □

Lemma 7.5 *Let q be an algebraically integrable periodic potential. Then for all $K\in\mathcal{I}_{\text{effective}}$ the averaged potential q_K is algebraically integrable.*

Proof: An inductive argument reduces the proof to the case $\dim K=n-1$. So let us fix a $K\in\mathcal{I}_{\text{effective},n-1}$ and denote $\partial_\perp$ the partial differential operator along the vector field corresponding to a primitive lattice vector $\gamma^\perp$ in $\Gamma\cap K^\perp$. Any differential operator L may be written as a polynomial in $\partial_\perp$, whose coefficients are smooth functions from $K^\perp$ into the differential operators on K. Due to Lemma 3.2, any differential operator L with constant highest symbol, which commutes with $-\Delta+q$, is also periodic with respect to Γ. Hence these coefficients are smooth functions from $K^\perp/(\Gamma\cap K^\perp)$ into the differential operators on K. Let us parameterize $K^\perp/(\Gamma\cap K^\perp)$ by

$$\mathbb{R}/\mathbb{Z}\ni x^\perp\mapsto x^\perp\gamma\in K^\perp/(\Gamma\cap K^\perp).$$

For the corresponding two highest coefficients

$$L=\partial_\perp^l L_l(x^\perp)+\partial_\perp^{l-1}L_{l-1}(x^\perp)+\ldots$$

the commutativity implies two equations

$$-2g(\gamma^\perp,\gamma^\perp)L_l'(x^\perp)=0 \text{ and } \left[-\Delta_K+q,L_l(x^\perp)\right]-g(\gamma^\perp,\gamma^\perp)(L_l''(x^\perp)+2L_{l-1}'(x^\perp))=0.$$

The first equation shows that L_l does not depend on $x^\perp$, and the integral over one period of the second equation shows that L_l commutes with the Schrödinger operator of the averaged potential q_K. Moreover, if $\tilde{L}$ is a differential operator, which commutes with $-\Delta+q$ and L, the corresponding highest coefficient $\tilde{L}_{\tilde{l}}$ commutes with L_l. Now the lemma follows from the definition of algebraically integrable. □

Let $\underline{\tilde{\psi}}(x)$ denote the vector of all partial differentials of the eigen function ψ at a point $x\in\mathbb{R}^n$ up to maximal order of the partial differential operators $L_1,\ldots,L_{n+1}$. For any fixed $x\in\mathbb{R}^n$ these vector $\underline{\tilde{\psi}}(x)$ define a holomorphic map from $\mathcal{B}(q)/\Gamma^\sharp$ into some projective space $\mathbb{P}_L$. The eigenvalues $\lambda_1,\ldots,\lambda_{n+1}$ of the operators $L_1,\ldots,L_{n+1}$ are meromorphic functions in terms of the entries of $\underline{\tilde{\psi}}(x)$ and the potentials and their derivatives of $L_1,\ldots,l_{n+1}$ at x. Due to Lemma 7.4, the image of this map is a projective subvariety of $\mathbb{P}_L$, and due to Lemma 7.5 this map is an holomorphic embedding. In fact, due to the equations

$$\left(\frac{\partial}{\partial x_1}+U_1(\lambda_1,\ldots,\lambda_n,x)\right)\underline{\psi}(x)=0,\ldots,\left(\frac{\partial}{\partial x_n}+U_n(\lambda_1,\ldots,\lambda_n,x)\right)\underline{\psi}(x)=0,$$

the eigen function ψ is uniquely determined by the corresponding partial differentials $\underline{\psi}(x)$, which has to be an eigenvector of the matrix $W(\lambda_1,\ldots,\lambda_n,x)$. This proves (iv).

(iv) $\Rightarrow$ (i): Due to Lemma 4.3, (iv) implies that $\mathcal{B}(q)$ is a complex space.

The last statement of the theorem follows from Theorem 7.1 and Theorem 4.1. □

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