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On the absence of bound-state stabilization through short ultra-intense fields

A. Fring*, V. Kostrykin† and R. Schrader* *

* *Institut für Theoretische Physik*
Freie Universität Berlin, Arnimallee 14, D-14195 Berlin, Germany
† *Institut für Reine und Angewandte Mathematik*
RWTH Aachen, Templergraben 55, D-52056 Aachen, Germany

Abstract

We address the question of whether atomic bound states begin to stabilize in the short ultra-intense field limit. We provide a general theory of ionization probability and investigate its gauge invariance. For a wide range of potentials we find an upper and lower bound by non-perturbative methods, which clearly exclude the possibility that the ultra intense field might have a stabilizing effect on the atom. For short pulses we find almost complete ionization as the field strength increases.

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*e-mail address: Fring or Schrader@omega.physik.fu-berlin.de, Kostrykin@ilt.fhg.de

1 Introduction

Fermi's Golden Rule, as one of the central elements in quantum mechanics, has served for many years for the understanding of photoionization rates of atoms in weak radiation fields. Its origin is, however, perturbative and therefore when applying very intense fields (with intensities which are greater or of the order of one atomic unit $3.5 \times 10^{16}\mathrm{W/cm}^2$) one leaves its range of validity. With the advance of laser technology this high intensity region has become accessible to real experiments in the form of laser pulses of 1ps or less, at frequencies ranging from the infrared to the ultraviolet [1]. The prediction of atomic ionization rates are of practical importance for instance in the study of gas breakdown [2].

In order to treat the new regime, several alternative approximation methods have been proposed [3, 4, 5], which are essentially all based on a perturbation around the Gordon-Volkov solution [6] of the Schrödinger equation. The question of convergence of these series and their precise range of validity has not yet been put on firm grounds. Despite these problems, these methods have been applied to find numerical solutions for the ionization probabilities. There are numerous one-dimensional studies [7, 8], which are put in question in general [9], since in comparison with the full three dimensional situation, they do not account for the full angular dependence and provide misleading results. Recently there have been full three dimensional computations [10, 11, 12, 13, 14]. But the complete problem has not been solved yet and as it is pointed out in [15], "even the simplest one-electron atom in an intense laser field presents too great a challenge for truly *ab inito* numerical work, and a variety of compromises have been developed".

Several authors claim to have found the very surprising and counter-intuitive result, that the bound state of the atom stabilizes as the field strength increases [11, 12, 13, 14]. Similar results have also been obtained by many authors for the one dimensional situation [7]. In fact some authors regard these computations as so surprising, that they consider them as a challenge to the concepts of quantum mechanics itself and believe that "a dramatic shift in viewpoint is required to explain the physics of atoms in very strong laser fields " [16]. We shall comment on these results below and for the moment refer the reader to the review article on these findings by Eberly and Kulander [16] and one by Geltman [17], who takes the opposite point of view that atomic bound states do not stabilize as the field strength is increased and who asserts that the "conventional interpretation of the theory of the interaction of radiation and atoms is quite sound even in this regime".

The controversy is mainly based on numerical results, which makes a detailed theoretical analysis of the problem very suggestive. This is the main subject of our paper, intended to provide an alternative approach to the matter. We consider the Schrödinger equation for an atom in a linear polarized electric field,

$$i\frac{\partial\psi}{\partial t} = (-\Delta/2 + V + z \cdot E(t))\psi,$$

where $E(t)$ stands for the intensity of the field and is supposed to have finite duration (for instance, 1 ps = 4.17×10^4 a.u.). We do not specify $E(t)$ in more detail: it can be, for instance, a pulse which contains a number of optical periods (the frequency of which is determined by the frequency of the laser) possibly with some turn-on and turn-off parts. Such kind of pulses were used in the search for possible suppression of ionization. We note that 1 ps pulses have a duration comparable with a classical Kepler period for the highly excited Rydberg states ($T_K = 2\pi n^3$). Another example are half-cycle pulses with duration of about 500 fs, generated in the experiments of Jones et al. [18]. However, the maximal intensity reached in these experiments was about 10^{-6} a.u., so that these pulses are ultimately far from the ultra-intense limit.

We suppose furthermore that the wave function $\psi(\vec{x}, t)$ is given by the bound state wave function of $-\Delta/2 + V$ before the pulse is turned on. One can easily estimate that relativistic effects might be appreciable as soon as $E(t)$ is so strong that the classical theory predicts electron velocities approaching the speed of the light. According to the estimates in [15] this occurs for laser intensities $E \approx 10^{18}\mathrm{W/cm}^2$.

Our results below show that atoms do not become resistant to ionization when exposed to short ultra-intense laser pulses. At present there are no known experimental confirmations of this point of view and in fact the main result of this paper explains this fact. Our statements are of qualitative nature, in the sense that they provide upper and lower bounds and do not predict precise values of the ionization rates.

The paper is organized as follows. In Section 2 we formulate a general theory of ionization probability and investigate its gauge invariance. We also make contact to the various approximation methods based on perturbative expansions. In Section 3 we briefly discuss these methods in the context of quantum mechanical one-particle Stark Hamiltonians and provide the proofs for the upper and lower bounds for the ionization probability for a wide range of one-particle potentials, which in particular include all potentials appearing in atomic and molecular physics. In Section 4 we state our conclusions. In Appendix A we provide an upper bound for the Coulomb potential and in Appendix B we optimize this bound for the ground state of the hydrogen atom.

2 The general theory of ionization probability and its gauge invariance

In this section we will give a general discussion of the ionization probability and its gauge invariance. We will relate it to familiar concepts in scattering theory and explicitly discuss its relevance in the context of the Stark Hamiltonian. In the last part of this section we will show the gauge covariance of time-dependent perturbation theory. In order to convey the general ideas we will avoid bulky mathematical notations in this section.

Let $H(t)(-\infty < t < \infty)$ be a general time dependent selfadjoint Hamiltonian in some Hilbert space $\mathcal{H}$ and let $U(t,t')$ denote the resulting time evolution operator from t' to t, i.e. $U(t,t')$ satisfies

$$\begin{aligned} i\partial_t U(t,t') &= H(t)U(t,t') \\ U(t,t')U(t',t'') &= U(t,t'') \\ U(t,t) &= \mathbf{1} \end{aligned} \tag{2.1}$$

for all t, t', t''. In the context of the Stark Hamiltonians, $U(t,t')$ exists for all t, t' and is unitary (see below).

Assume now that $H(t)$ approaches an operator $H_{+\infty}$ for $t \to \infty$ and $H_{-\infty}$ for $t \to -\infty$, i.e.

$$\begin{aligned} H_{+\infty} &= \lim_{t\to\infty} H(t) \\ H_{-\infty} &= \lim_{t\to-\infty} H(t) \end{aligned} \tag{2.2}$$

holds in a suitable sense. It is important to note that we do not assume $H_{+\infty}$ to equal $H_{-\infty}$. In fact, for the Stark Hamiltonian in certain gauges, these operators will in general differ. In analogy to the scattering matrix (see below) we define the abstract S-Matrix to be the following weak limit (if it exists)

$$S = \lim_{\substack{t\to+\infty \\ t'\to-\infty}} \exp\ itH_{\infty} \cdot U(t,t') \cdot \exp -it'H_{-\infty}. \tag{2.3}$$

In particular S exists trivially and is unitary if $H(t)$ becomes stationary for all large $|t|$, i.e. if $H(t) = H_{+\infty}$ for all $t \geq t_+$ and $H(t) = H_{-\infty}$ for all $t \leq t_-$ for suitable finite t_-, t_+. In this case we will call $H(t)$ a *finitely pulsed Hamiltonian.* S then takes the form

$$S = \exp\ itH_{+\infty} \cdot U(t,t') \cdot \exp -it'H_{-\infty} \tag{2.4}$$

for all $t \geq t_+$ and all $t' \leq t_-$. In particular S is then unitary.

Let $P_{+\infty}$ be the orthogonal projection onto the subspace spanned by the bound states of $H_{+\infty}$. $P_{-\infty}$ is defined analogously in terms of $H_{-\infty}$. Then for any normalized state ψ in the range of $P_{-\infty}$ its ionization probability is defined to be

$$I(\psi) = \|(\mathbf{1} - P_{+\infty})S\psi\|^2. \tag{2.5}$$

For the case when $H_{+\infty} = H_{-\infty}$ this agrees with the definition used in [19, 20, 21]. For a finitely pulsed Hamiltonian we have

$$I(\psi) = \|(\mathbf{1} - P_{+\infty})U(t_+, t_-)\psi\|^2 \tag{2.6}$$

whenever ψ is a bound state of $H_{-\infty}$.

Abstract gauge transformations are now introduced as follows. Let $A(t)$ $(-\infty < t < \infty)$ be a one parameter family of unitary operators (suitably differentiable in t). If $\psi(t)$ is a solution of the Schrödinger equation

$$i\partial_t\psi(t) = H(t)\psi(t)$$

then $\psi'(t) = A(t)\psi(t)$ is a solution of the equation

$$\begin{aligned} i\partial_t\psi'(t) &= i\left(\partial_t A(t)\right)\psi(t) + A(t)i\partial_t\psi(t) \\ &= i(\partial_t A(t))A(t)^{-1}\psi'(t) + A(t)H(t)A(t)^{-1}\psi'(t) \\ &= H'(t)\psi'(t) \end{aligned} \tag{2.7}$$

with

$$H'(t) = i(\partial_t A(t))A(t)^{-1} + A(t)H(t)A(t)^{-1} \tag{2.8}$$

being formally selfadjoint. If $U'(t,t')$ is the time evolution operator for $H'(t)$ then obviously

$$U'(t,t') = A(t)U(t,t')A(t')^{-1}. \tag{2.9}$$

We note that the set of all gauge transformations form a non-commutative group under the obvious multiplication rule $(A_1A_2)(t) = A_1(t)A_2(t)$, with unit $\mathbf{1}(t) = \mathbf{1}$ and inverse $A^{-1}(t) = A(t)^{-1}$. The familiar interaction picture used in scattering theory is now a special case. Indeed, to be more specific assume $H(t)$ to be of the form $H(t) = H_0 + H_I(t)$, where H_0 is the "free" Hamiltonian and $H_I(t)$ the (possibly time dependent) interaction Hamiltonian. Set

$$A(t) = \exp itH_0. \tag{2.10}$$

Then

$$H'(t) = H_I'(t) \tag{2.11}$$

is the Hamiltonian in the interaction picture with

$$H_I'(t) = \exp\ itH_0 \cdot H_I(t) \cdot \exp -itH_0. \tag{2.12}$$

In this case the limit (if it exists)

$$S(H(\cdot), H_0) = \lim_{\substack{t\to+\infty \\ t'\to-\infty}} U'(t,t') = \lim_{\substack{t\to+\infty \\ t'\to-\infty}} \exp\ itH_0 \cdot U(t,t') \cdot \exp -it'H_0 \tag{2.13}$$

is called the scattering matrix (S-matrix) for the pair $(H(\cdot), H_0)$. Let us elaborate briefly in what sense there is an analogy between the quantity (2.3) and the case of ordinary potential scattering, i.e. where H_0 is $-\Delta/2$ and $H_I(t) = V$ is a potential. In the latter case $H = H_0 + V$ is compared with H_0 in spatial regions far out, i.e. where V is small and where the incoming and outgoing wave packets are located for large $|t|$. In the case we are interested in presently, $H(t)$ is compared with $H_{+\infty}$ for large positive times and with $H_{-\infty}$ for large negative times. To summarize: In one case one compares Hamiltonians for large spatial coordinates and in the other case for large time coordinates.

To apply this general concept of gauge transformations and gauge covariance to our discussion of ionization, assume now in addition that $A(t)$ approaches suitably unitary operators $A_{+\infty}$ and $A_{-\infty}$ when $t \to +\infty$ and $t \to -\infty$, respectively.

Then $H'(t)$ (see (2.8)) approaches

$$H'_{+\infty} = A_{+\infty} H_{+\infty} A_{+\infty}^{-1} \tag{2.14}$$

and

$$H'_{-\infty} = A_{-\infty} H_{-\infty} A_{-\infty}^{-1} \tag{2.15}$$

as $t \to +\infty$ and $t \to -\infty$ respectively.

By formal manipulations we therefore have

$$S' = A_{+\infty} S A_{-\infty}^{-1}. \tag{2.16}$$

In particular if $H(t)$ is finitely pulsed and if in addition $A(t)$ is stationary for all large $|t|$, then $H'(t)$ is also finitely pulsed, S' exists, is unitary and (2.16) holds.

In general, by (2.14) and (2.15), if $P'_{\pm\infty}$ are the orthogonal projections onto the space spanned by the bound states of $H'_{\pm\infty}$, we have

$$P'_{\pm\infty} = A_{\pm\infty} P_{\pm\infty} (A_{\pm\infty})^{-1}. \tag{2.17}$$

In particular $A_{-\infty}\psi$ is in the range of $P'_{-\infty}$ if ψ is in the range of $P_{-\infty}$. Inserting (2.16) and (2.17) gives the desired gauge invariance in the form

$$\begin{aligned} I'(A_{-\infty}\psi) &= \|(\mathbf{1} - P'_{+\infty}) S' A_{-\infty}\psi\|^2 \\ &= \|A_{+\infty}(\mathbf{1} - P_{+\infty}) A_{+\infty}^{-1} A_{+\infty} S A_{-\infty}^{-1} A_{-\infty}\psi\|^2 \\ &= \|(\mathbf{1} - P_{+\infty}) S\psi\|^2 = I(\psi), \end{aligned} \tag{2.18}$$

since by assumption $A_{+\infty}$ is unitary. In the example we will be interested in $A_{-\infty} = \mathbf{1}$. In this case $H'_{-\infty} = H_{-\infty}$ and (2.18) takes the simpler form $I'(\psi) = I(\psi)$. From the proof we see that gauge invariance is an important regulating principle in the following sense. One has to choose the projection $P_{+\infty}$ in (2.5) and not $P_{-\infty}$ in order to obtain gauge invariance.

We now apply these concepts to the theory of the time dependent Stark Hamiltonian. In order to make notations more transparent we choose a linearly polarized electric field, which does however not limitate our discussion since more general fields, may be simply obtained by replacing $z \to \vec{x}$ and $E(t) \to \vec{E}(t)$ ($\vec{x} \in I\!R^3, \vec{E} \in I\!R^3$), such that particular other choices, like for instance circular polarized light maybe, easily be derived from there. We do, however, assume a dipole approximation, such that the electric field becomes a function only of time and thus is independent of space. Then using atomic units $\hbar = e = m_e = c \cdot \alpha = 1$ we consider on $L^2(I\!\!R^3, d^3x)$ the 3 time-dependent Hamiltonians

$$\begin{aligned} H_1(t) &= -\frac{\Delta}{2} + V + z \cdot E(t) \\ H_2(t) &= \frac{1}{2}(-i\nabla - b(t)e_z)^2 + V \\ H_3(t) &= -\frac{\Delta}{2} + V(\cdot - c(t)e_z) \quad . \end{aligned} \tag{2.19}$$

Here V is an arbitrary potential. e_z is the unit vector in the z-direction and $V(\cdot - \vec{y})$ is the shifted potential, i.e. the multiplication operator on wave functions given as $(V(\cdot - \vec{y})\psi)(\vec{x}) = V(\vec{x} - \vec{y})\psi(\vec{x})$. Also $E(t)$ is the electric field, assumed to vanish unless $0 \leq t \leq \tau$ (i.e. $t_- = 0$ and $t_+ = \tau > 0$ in the notation above). Apart from this condition the pulse $E(t)$ may be arbitrary. We only make the mathematical restriction, that $E(t)$ is piecewise continuous, which means that the pulse may have jumps and all commonly used enveloping shapes, for instance cosine squared, smooth adiabatic turn on and off, etc., are included. Then the following quantities $a(t), b(t)$ and $c(t)$ are well defined

$$b(t) = \int_0^t E(s)ds \tag{2.20}$$

$$c(t) = \int_0^t b(s)ds = tb(t) - \int_0^t sE(s)ds \tag{2.21}$$

$$a(t) = \frac{1}{2}\int_0^t b(s)^2 ds \quad . \tag{2.22}$$

Note that $b(\tau)e_z$ describes the classical momentum transfer of the pulse, such that $\frac{1}{2}b(\tau)^2$ is the classical energy transfer. Also $c(\tau)c_z$ is the classical displacement caused by the pulse. Then $H_2(t)$ is obtained from $H_1(t)$ by the gauge transformation

$$A_{2\leftarrow 1}(t) = \exp\ ib(t)z.$$

$H_1(t)$ is obtained from $H_3(t)$ by the Kramers-Henneberger transformation [22, 23, 24].

$$A_{1\leftarrow 3}(t) = T(t) = \exp -ia(t) \cdot \exp -ib(t)z \cdot \exp ic(t)p_z. \tag{2.23}$$

Therefore we call $H_3(t)$ the Hamiltonian in the Kramers-Henneberger gauge. We note that a corresponding transformation in Quantum Electrodynamics was introduced by Pauli and Fierz already in 1938 but with a different motivation [25]. The Hamiltonian $H_1(t)$ is usually referred to as the Hamiltonian in the length- or electric field gauge, whereas $H_2(t)$ is denoted as the Hamiltonian in the velocity-, radiation- or Coulomb gauge.

As a consequence $H_2(t)$ is obtained from $H_3(t)$ by the gauge transformation

$$A_{2\leftarrow 3}(t) \;\; = A_{2\leftarrow 1}(t)A_{1\leftarrow 3}(t) = \exp -ia(t)\cdot \exp ic(t)p_z. \qquad (2.24)$$

The general gauge transformation $A_{j\leftarrow i}(t)$ for $H_i(t) \to H_j(t)$ is then obtained from the rules

$$\begin{aligned} A_{j\leftarrow i}(t) &= A_{i\leftarrow j}(t)^{-1} \\ A_{j\leftarrow k}(t) &= A_{j\leftarrow i}(t)A_{i\leftarrow k}(t). \end{aligned} \qquad (2.25)$$

$H_1(t)$ and $H_2(t)$ are finitely pulsed. Note however that in general $H_3(t)$ is *not* finitely pulsed and hence has not always a proper limit as $t \to +\infty$. This is due to the fact that $b(t)$ is constant for $t \geq \tau$ such that $c(t)$ grows linearly in $t \geq \tau$ whenever $b(\tau) \neq 0$, as is apparent from (2.21). Nevertheless the Kramers-Henneberger gauge is quite useful as we shall see below. These observations are related to the fact that $A_{2\leftarrow 1}(t)$ becomes stationary for large t but in general not $A_{2\leftarrow 3}(t)$ and $A_{1\leftarrow 3}(t)$. Note that by assumption on $E(t)$, $A_{j\leftarrow i}(-\infty) = \mathbf{1}$ and thus

$$H_{1,+\infty} = H_{1,-\infty} = H_{2,-\infty} = H_{3,-\infty} = H_0 + V. \qquad (2.26)$$

However, in general $H_{2,+\infty} \neq H_{2,-\infty}$.

The case $V \equiv 0$ is of special interest, since it corresponds to the situation in which the Schrödinger equation admits an exact solution, which is usually referred to as the Gordon-Volkov solution [6]. Call the resulting operators $H_{0,i}(t)(i = 1,2,3)$, such that in particular $H_{0,3}(t) = -\Delta/2$. The kernels of the resulting time evolution operators $U_{0,i}(t,t')$ can be calculated explicitly. Indeed, we start from the familiar relation for the free particle evolution operator (see e.g. [30])

$$\begin{aligned} U_{0,3}(\vec{x},t;\vec{x}',t') &= \langle \vec{x}|U_{0,3}(t,t')|\vec{x}'\rangle = \langle \vec{x}|\exp i(t-t')\frac{\Delta}{2}|\vec{x}'\rangle & (2.27) \\ &= \frac{1}{(2\pi i(t-t'))^{3/2}}\exp i\frac{(\vec{x}-\vec{x}')^2}{2(t-t')}. & (2.28) \end{aligned}$$

Obviously

$$\begin{aligned} \langle \vec{x}|\exp\ ib(t)z|\vec{x}'\rangle &= \exp\ ib(t)z\cdot\delta^3(\vec{x}-\vec{x}') \\ \langle \vec{x}|\exp\ ia(t)|\vec{x}'\rangle &= \exp\ ia(t)\cdot\delta^3(\vec{x}-\vec{x}') \end{aligned}$$

and

$$\langle \vec{x}|\exp ic(t)p_z|\vec{x}'\rangle = \langle \vec{x}-c(t)e_z,\vec{x}'\rangle = \delta^3(\vec{x}-\vec{x}'-c(t)e_z)$$

such that

$$\begin{aligned} \langle \vec{x}|T(t)|\vec{x}'\rangle &= \overline{\langle \vec{x}'|T(t)^{-1}|\vec{x}\rangle} \\ &= \exp -ia(t)\cdot\exp -ib(t)z\cdot\delta^3(\vec{x}-\vec{x}'-c(t)e_z). & (2.29) \end{aligned}$$

By (2.9) we immediately obtain the following (again well known) relations (see e.g. [26, 27, 6, 10]).

$$\begin{aligned} U_{0,1}(\vec{x},t;\vec{x}',t') &= \frac{1}{(2\pi i(t-t'))^{3/2}}\exp i(a(t')-a(t)) \\ &\quad\times\exp i(b(t')z-b(t)z')\exp i\frac{(\vec{x}-c(t)e_z-\vec{x}'+c(t')e_z)^2}{2(t-t')} \\ U_{0,2}(\vec{x},t;\vec{x}',t') &= \frac{1}{(2\pi i(t-t'))^{3/2}}\exp i(a(t')-a(t)) \\ &\quad\times\exp i\frac{(\vec{x}-c(t)e_z-\vec{x}'+c(t')e_z)^2}{2(t-t')}. \end{aligned}$$

The kernel of $U_{0,1}(t,t')$ is often called the Gordon-Volkov propagator (see e.g. [6, 26, 10]).

We finally give a discussion of time dependent perturbation theory and its gauge covariance. Returning to the general set-up, let $H(t)$ be a " perturbation" of $K(t)$. If $W(t,t')$ denotes the time evolution for $K(t)$, we have the generalized Du Hamel's formula (see e.g. [28]) in the form

$$\begin{aligned} U(t,t') &= W(t,t') - \int_{t'}^{t} \frac{d}{ds}\left[U(t,s)W(s,t')\right] ds \\ &= W(t,t') - i\int_{t'}^{t} U(t,s)\left[H(s)-K(s)\right]W(s,t')ds. \end{aligned} \tag{2.30}$$

We recall that in terms of the sometimes more familiar Green's function

$$G_{H(\cdot)}(t,t') = -iU(t,t')\theta(t-t') \ ,$$

which satisfies

$$(i\partial_t - H(t))G_{H(\cdot)}(t,t') = \delta(t-t')\mathbf{1} \ ,$$

relation (2.30) takes the form

$$G_{H(\cdot)}(t,t') = G_{K(\cdot)}(t,t') + \int_{-\infty}^{+\infty} G_{H(\cdot)}(t,s)\left[H(s)-K(s)\right]G_{K(\cdot)}(s,t')ds. \tag{2.31}$$

Similarly one derives the relation

$$U(t,t') = W(t,t') - i\int_{t'}^{t} W(t,s)\left[H(s)-K(s)\right]U(s,t')ds \tag{2.30$'$}$$

and hence

$$G_{H(\cdot)}(t,t') = G_{K(\cdot)}(t,t') + \int_{-\infty}^{+\infty} G_{K(\cdot)}(t,s)\left[H(s)-K(s)\right]G_{H(\cdot)}(s,t')ds. \tag{2.31$'$}$$

We recall that Du Hamel's formula in the form (2.31) and (2.31$'$) is the time dependent version of the Lippmann-Schwinger equation (see e.g.[29, 30]) in the case where both $H(s)$ and $K(s)$ are actually time-independent. Indeed, the Lippmann-Schwinger equation may be obtained from these relations by taking Laplace transforms.

The equations (2.30), (2.30'), (2.31) and (2.31') may be iterated by introducing the left hand side into the right hand side, resulting in a "power series expansion" of $U(t,t')$ in powers of $H(s)-K(s)$ and involving W of the form

$$U(t,t') = \sum_{n=0}^{\infty} U_n(t,t') \tag{2.32}$$

with $U_0(t,t') = W(t,t')$ and with an analogous expansion for $G_{H(\cdot)}$. Let us consider what happens under gauge transformations. Du Hamel's formula is compatible with gauge transformations in the sense that the relation

$$U'(t,t') = W'(t,t') - i\int_{t'}^{t} U'(t,s)\left[H'(s) - K'(s)\right] W'(s,t')ds \tag{2.33}$$

either follows directly for the pair $H'(t), K'(t)$ or by applying the gauge transformation to (2.30) and using (2.8) and (2.9). This implies in particular that

$$U'_n(t,t') = A(t)U_n(t,t')A(t')^{-1} \tag{2.34}$$

for all n. This applies in particular to the choices $H(t) = H_i(t)$ and $K(t) = H_{0,i}(t)$ given by (2.19) and the gauge transformations which relate them. However, some of the approximation methods used for high intensities, on which we shall comment more below, use the fact that one can decompose the Stark Hamiltonian in two different ways, that is either treating the potential or the term related to the electric field as a perturbation. Hence one obtains two versions for (2.30), which one may combine iteratively. The series generated in this manner in general does not respect gauge invariance order by order. A discussion of this problem and a remedy for restoring gauge invariance by including some terms of next order, thus leading to cancelations, may be found in [26, 31].

In the case of the interaction picture (see above) and with the choice $K(t) = H_0$ such that $K'(t) = 0$, the iteration of (2.33) in powers of $H'_I(t) = H'(t) - K'(t)$ is just the famous Dyson series of the S-Matrix $S(H(\cdot), H_0)$ in the limit $t \to +\infty, t' \to -\infty$. In the time independent context of the Lippmann-Schwinger equation (see above) this corresponds to iterating the Lippmann-Schwinger equation to obtain the Born series for the S-Matrix (see e.g. [30]). In the context we are presently interested in, such series expansions for the time evolution operator of finitely pulsed Hamiltonians lead to a series expansion for the ionization when inserted in (2.6). In the next section we will discuss the various approaches used so far for the Hamiltonian $H_1(t)$ and its gauge transforms $H_2(t)$ and $H_3(t)$ given in (2.19).

3 Ionization of atoms in strong, short electric fields

Using the notions of the previous section we start with a review and comparison of methods and results obtained by previous authors. Then we relate this in a first step to a new, rigorous upper bound on the ionization, valid for all small τ and small classical momentum transfer $b(\tau)$ and small displacement $c(\tau)$ (see the upper bound below). This result is also compared with another rigorous upper bound previously obtained by two of the authors (V.K., R.S.) [20] as well as with a result obtained using a time-energy uncertainty relation given by Pfeifer [32]. Secondly we prove a lower bound below, valid for all small τ and all large $b(\tau)$, which in particular proves the absence of stabilization.

Taking $H(t) = H_0 + V + z \cdot E(t) = H_1(t)$ and $K(t) = H_0 + V$ in (2.30), the resulting perturbation series in the time dependent interaction $H(t) - K(t) = zE(t)$ for $U_1(t,t')$, the time evolution operator for $H_1(t)$, has been used by Lambropolous [33]. Certainly for high intensities $E(t)$ this is very problematic, since one requires several terms in the expansion to achieve a reasonable result.

A more promising approach has been advocated by Perelomov, Popov and Terentev [4], who took $H(t) = H_0 + V + z \cdot E(t) = H_1(t)$ and $K(t) = H_0 + z \cdot E(t) = H_{0,1}(t)$. Then $H(t) - K(t) = V$, thus leading to a power series expansion in V. Whenever $|E| > |V|$ this seems to be very suggestive approximation and is based on the fact that the time evolution operator $U_{0,1}(t,t')$ for $H_{0,1}(t)$ is the Gordon-Volkov solution, which is known exactly (see equation after (2.29)). By the discussion of section 2, in the Kramers-Henneberger gauge this corresponds to a power series expansion in suitable translates of V.

A combination of these two methods has been proposed earlier in a seminal paper on the subject by Keldysh [3], who took the series for $U_1(t,t')$ with $z \cdot E(t)$ as a perturbation, but in the second iteration step inserted the time evolution operator $U_{0,1}$ instead of the time evolution operator for $H_0 + V$. In fact, it was demonstrated by Davidovich et al. [26] that to first order the Keldysh approximation and the one of Perelomov et al. precisely coincide. When carrying out the same steps in the velocity gauge, i.e. for $H(t) = H_2(t)$, one obtains the so-called Faisal-Reiss approximation [5].

We want to point out that all such series expansions are somewhat problematic since a proper convergence of the series has not yet been established (the only known case is the Born series in scattering theory at high energies, see e.g. [30] and the references given there. For the one dimensional situation convergence may be shown for integrable potentials [26]), nor is it straightforward to give precise quantum mechanical estimates of the first terms. Most statements seem to be based on crude semi-classical estimates [34] or in the belief that features which have been observed for relatively simple one dimensional models, which are anyway put in question [9], carry over in general [10, 8].

For realistic pulses for instance with smooth adiabatic turn on and off, the first terms will only give reasonable results when the full power is reached but will be poor, if not completely invalid near the turn on and turn off point.

We now give a new rigorous upper bound on the ionization. The proof of this bound bypasses the problem of summing the whole perturbation series by staying strictly with the Du Hamel formula. In what follows the potential V will be supposed to satisfy the conditions given in [20] which are tailored to ensure the existence of the time evolution operators for $H_i(t)(i = 1, 2, 3)$ given in (2.19). In particular such potentials V are Kato small (see e.g. [24, 28]), i.e. there are $a < 1$ and $b < \infty$ such that for all ψ in the domain $\mathcal{D}(H_0)$ of $H_0 = -\Delta/2$

$$\|V\psi\| \leq a\| - \Delta\psi\| + b\|\psi\|. \tag{3.1}$$

Also the domain $\mathcal{D}(H)$ of H coincides with $\mathcal{D}(H_0)$. We note that the potentials of atoms or molecules arising from Coulomb pair potentials belong to this wide class. Also Hamiltonians with Kato bounded potentials (3.1) are bounded below and if the (pair) potentials decay suitably at infinity, then there are no positive eigenvalues. Indeed, it has been shown [28, 35] for a large class of potentials including atomic and molecular ones that the eigenvalues are contained in $[inf\ \sigma(H), 0]$ where $\sigma(H)$ denotes the spectrum of H. With these specifications on V in mind we are now in the position to state, prove and comment on the first main result of this section.

Upper bound: *Let ψ be a normalized bound state of $H = H_0 + V$ with energy $E < 0$. Then for any pulse $E(t)$ with $1/2b(\tau)^2 < -E$, the ionization satisfies the upper bound*

$$I(\psi)^{\frac{1}{2}} \leq \int_0^\tau \|(V(\cdot - c(t)e_z) - V(\cdot))\psi\| dt + |c(\tau)| \|p_z\psi\| + \frac{|b(\tau)|}{-E - \frac{1}{2}b(\tau)^2}\|p_z\psi\|. \quad (3.2)$$

Note that the condition on $E(t)$ just says that the classical energy transfer of the pulse is less than the classical ionization energy.

The proof is based on a combination of arguments used in [19] and [20] and goes as follows. Since

$$\exp -itH\psi = \exp -itE\psi \ , \quad (3.3)$$

by using the Kramers-Henneberger transformation we have

$$\begin{aligned} I(\psi)^{\frac{1}{2}} &= \|(\mathbf{1} - P)U_1(\tau, 0)\psi\| = \|(\mathbf{1} - P)T(\tau)U_3(\tau, 0)\psi\| \\ &= \|(\mathbf{1} - P)\exp -ib(\tau)z \cdot \exp ic(\tau)p_z \cdot U_3(\tau, 0)\psi\| \\ &\leq \|(\mathbf{1} - P)\exp -ib(\tau)z \cdot \exp ic(\tau)p_z(U_3(\tau, 0) - \exp -i\tau H)\psi\| \\ &\quad + \|(\mathbf{1} - P)\exp -ib(\tau)z \cdot \exp ic(\tau)p_z \cdot \psi\|. \end{aligned} \quad (3.4)$$

We start with an estimate of the first term on the r.h.s of (3.4). Obviously it is bounded by

$$\|(U_3(\tau, 0) - \exp -i\tau H)\psi\|. \quad (3.5)$$

We now invoke Du Hamel's formula to rewrite (3.5) as

$$\left\| \int_0^\tau U_3(\tau, t)[V(\cdot - c(t)e_z) - V(\cdot)] \exp -i(\tau - t)H \cdot \psi \ dt \right\|. \quad (3.6)$$

Now we use the unitarity of $U_3(\tau,t)$ (besides the fact that we never iterate Du Hamel's formula, this is the crucial step in avoiding perturbation theory) and (3.3) to estimate (3.6) by

$$\int_0^\tau \|(V(\cdot - c(t)e_z) - V(\cdot))\psi\| dt \tag{3.7}$$

which is the first term on the r.h.s. of (3.2). To estimate the second term in (3.4), we use the triangle inequality to obtain

$$\begin{aligned} \|(\mathbf{1}-P)\exp -ib(\tau)z \cdot \exp ic(\tau)p_z \cdot \psi\| &\leq \|(\exp(ic(\tau)p_z) - \mathbf{1} \cdot \psi\| \\ &\quad +\|(\mathbf{1}-P)\exp -ib(\tau)z \cdot \psi\| \end{aligned} \tag{3.8}$$

Now we use the estimate

$$\|\exp iA - \mathbf{1}\psi\| \leq \|A\psi\|$$

valid for any selfadjoint operator to estimate the first term on the r.h.s. of (3.8) by

$$|c(\tau)| \; \|p_z\psi\| \tag{3.9}$$

which is the second term on the r.h.s. of (3.2). It remains to estimate the second term in (3.8). By assumption $(\mathbf{1}-P)H \geq 0$. Hence for any $\delta > 0$,$(\mathbf{1}-P)(H+\delta)^{-1}$ exists and is norm bounded by $1/\delta$. Therefore

$$\begin{aligned} & \|(\mathbf{1}-P)\exp -ib(\tau)z \cdot \psi\| \\ = \; & \|(\mathbf{1}-P)(H+\delta)^{-1}(H+\delta)\exp -ib(\tau)z \cdot \psi\| \\ = \; & \|(\mathbf{1}-P)(H+\delta)^{-1}\exp -ib(\tau)z \cdot \exp \; ib(\tau)z \cdot (H+\delta) \cdot \exp -ib(\tau)z \cdot \psi\| \\ \leq \; & \frac{1}{\delta}\|\exp \; ib(\tau)z \cdot (H+\delta) \cdot \exp -ib(\tau)z \cdot \psi\|. \end{aligned} \tag{3.10}$$

We now use the fact that

$$\begin{aligned} \exp \; ib(\tau)z \cdot H \cdot \exp -ib(\tau)z &= \frac{1}{2}(-i\nabla - b(\tau)e_z)^2 + V \\ &= H - b(\tau)p_z + \frac{1}{2}b(\tau)^2. \end{aligned} \tag{3.11}$$

Inserting this into (3.10) gives

$$\|(\mathbf{1}-P)\exp -ib(\tau)z\cdot\psi\| \leq \frac{1}{\delta}\|(E-b(\tau)p_z+\frac{1}{2}b(\tau)^2+\delta)\psi\| \tag{3.12}$$

Making the choice

$$\delta = -E-\frac{1}{2}b(\tau)^2 \tag{3.13}$$

which by assumption on $b(\tau)$ is >0 and inserting into (3.12) gives the third term in (3.2) concluding the proof of the upper bound.

We now comment on this result.

Inspection of the proof of the main result in [20] shows that one has the alternative upper bound

$$I(\psi)^{\frac{1}{2}} \leq \int_0^\tau \|(V(\cdot - c(t)e_z)-V(\cdot))\psi\|dt + |c(\tau)|\,\|p_z\psi\| + |b(\tau)|\,\|z\psi\| \tag{3.14}$$

which differs from (3.2) only in the last term. Typically near threshold, i.e. for small $|E|$, both $\|z\psi\|$ and $1/(-E-\frac{1}{2}b(\tau)^2)$ become large, whereas

$$\|p_z\psi\| = \langle\psi, p_z^2\psi\rangle^{\frac{1}{2}} \leq (2\langle\psi, H_0\psi\rangle)^{\frac{1}{2}}$$

stays finite. Thus for V being the Coulomb potential

$$\langle\psi, H_0\psi\rangle = -E \tag{3.15}$$

by the virial theorem (see e.g. [36]). For s-states ψ_{n00} one can improve slightly since $\|p_z\psi_{n00}\|^2 = \frac{2}{3}\langle\psi_{n00}, H_0\psi_{n00}\rangle = \frac{1}{3n^2}$. Similarly one has $\|z\psi\|^2 \leq \langle\psi, r^2\psi\rangle (= \frac{n^2}{2}[5n^2+1-3\ell(\ell+1)]$ if $\psi = \psi_{n\ell m}$, see e.g. [37]). Again for s-states $\|z\psi\|^2 = \frac{1}{3}\langle\psi, r^2\psi\rangle$, which is a slight improvement. Therefore (3.2) and (3.14) are essentially equivalent.

We now discuss the first two terms in (3.2) and (3.14). In general, for Kato bounded potentials $V(\cdot)(-\Delta+\mathbf{1})^{-1}$ is a bounded operator. Also since $-\Delta$ is translation invariant, we have

$$\|V(\cdot-\vec{y})(-\Delta+\mathbf{1})^{-1}\| = \|V(\cdot)(-\Delta+\mathbf{1})^{-1}\|. \tag{3.16}$$

In particular for the choice of the Coulomb potential, we prove in Appendix A that

$$\left\| \frac{1}{r}(-\Delta + \mathbf{1})^{-1} \right\| \leq 6.35. \tag{3.17}$$

Thus for the general potentials V considered, the first term in (3.2) and (3.14) is bounded by

$$\begin{aligned} \tau \|V(\cdot)(-\Delta+\mathbf{1})^{-1}\| \; \|(-\Delta+\mathbf{1})\psi\| \quad &+ \quad \tau\|V\psi\| = \tau\|V\psi\| \\ &+ \quad \tau\|V(\cdot)(-\Delta+\mathbf{1})^{-1}\| \; \|(2H_0+\mathbf{1})\psi\| \end{aligned} \tag{3.18}$$

which involves $E(t)$ only through its duration but not its strength. For the eigenfunctions $\psi_{n\ell m}$ of the hydrogen atom, one has (see e.g. [37])

$$\begin{aligned} \|(2H_0+\mathbf{1})\psi_{n\ell m}\|^2 &= \|((2H+\mathbf{1})-2V)\psi_{n\ell m}\|^2 \\ &= (2E_n+1)^2 + 2(2E_n+1)\langle\psi_{n\ell m}, \frac{1}{r}\psi_{n\ell m}\rangle + 4\langle\psi_{n\ell m}, \frac{1}{r^2}\psi_{n\ell m}\rangle \\ &= 1 - \frac{1}{n^4} + \frac{4}{n^3(\ell+\frac{1}{2})}. \end{aligned} \tag{3.19}$$

This quantity is ≤ 8 and behaves like $1+\mathcal{O}(\frac{1}{n^3})$ for n large uniformly in $0 \leq \ell \leq n-1$. Hence the right hand side of (3.18) for bound states of the hydrogen atom is bounded by 19.4τ uniformly in n and by 6.35τ for all large n.

For the ground state ψ_{100} of the hydrogen atom the first term in (3.2) and (3.14) has a much better estimate. As shown in Appendix B

$$\|(V(\cdot - \vec{y}) - V(\cdot))\psi_{100}\| \leq 2 \tag{3.20}$$

holds for all $\vec{y} \in \mathbb{R}^3$ such that the first term in (3.2) and (3.14) is now bounded by 2τ.

By a theorem of Pfeifer [32] for the survival probability $|\langle\psi, \psi_\tau\rangle|$ of a state with $\psi_\tau = U(\tau, 0)\psi$ for any time dependent Hamiltonian $H(t)$ one has for all small τ (see [32] for precise conditions)

$$|\langle\psi, \psi_\tau\rangle| \geq \cos(\int_0^\tau \Delta(t)dt) \tag{3.21}$$

where in the present context with $H(t) = H_1(t)$ (see (2.19))

$$\begin{aligned}\Delta(t) &= |E(t)| \cdot a_\psi \\ a_\psi &= (\|z\psi\|^2 - \langle \psi, z\psi \rangle^2)^{\frac{1}{2}} \leq \|z\psi\|. \end{aligned} \tag{3.22}$$

This gives for the ionization probability

$$\begin{aligned} I(\psi) &\leq 1 - |\langle \psi, \psi_\tau \rangle|^2 \leq a_\psi^2 \left(\int_0^\tau |E(t)| dt \right)^2 \\ &\leq \left(\int_0^\tau |E(t)| dt \right)^2 \|z\psi\|^2. \end{aligned} \tag{3.23}$$

This may be compared with the discussion above. (3.23) is weaker than (3.2) and (3.14) in the sense that it does not show independence of the field strength when $b(\tau)$ and $c(\tau)$ are small or even zero. Otherwise it is basically equivalent to (3.2) and (3.14) or even stronger whenever the last two terms there dominate. We note that the rigorous bound (3.23) may be compared with first order perturbation theory in $E(t)$ which gives

$$I^{(1)}_{\text{pert}} = \left(\int_0^\tau E(t) dt \right)^2 \|Pz\psi\|^2 \leq \left| \int_0^\tau E(t) dt \right|^2 \|z\psi\|^2 \ . \tag{3.24}$$

We now turn to a comparison with other approximation methods based on perturbative expansions used in this context. Since all these approximations resolve around the same principle, i.e. an expansion involving the Gordon-Volkov time evolution operator $U_{0,1}(t, t')$, we will mainly concentrate on a recent work by Geltman [10], who presented an explicit, partly analytical, partly numerical analysis of the full three-dimensional hydrogen atom. Also a nice discussion of other works may be found there. Geltman employs the approximation method of Perelomov et al. in order to compute the excitation and ionization rates for the $1s/2s/3s/2p/3p/3d$ states of the hydrogen atom struck by a linearly polarized monochromatic laser pulse of the form $E(t) = E_0 \cos \omega t$. The value for the electric field strength in atomic units is chosen to be $E_0 = 5, 10, 20$ and the frequency $\omega = 1.5$. Geltman obtained the following general features

a) At integer cycles, that is for $\tau = 2\pi n/\omega$ the rate of ionization becomes independent of the electric field strength. In particular for the non-s-states it goes to zero.

This is reflected qualitatively in our results in the following way. At integer cycles $b(\tau) = c(\tau) = 0$ such that the last two terms in (3.2) and (3.14) vanish. Also the first term is independent of the field strength, however, not zero and by the above discussion too large for the above choices of E_0 and ω.

b) The maxima of $b(\tau)$ and $c(\tau)$ are located at half-integer cycles, i.e. $\tau = 2\pi(n + \frac{1}{2})/\omega$.

For the applied pulse the bounds (3.2) and (3.14) also reproduce this feature qualitatively but again we emphasize that these bounds hold in more generality for all Kato potentials and all states.

c) Stabilization for strong, short electric fields has been a highly controversial issue with disagreeing results between numerous authors on one side, for a review see [17], as well as on the other side, see [16] for a review on these.

The following result shows absence of stabilization for sufficiently strong, short pulses, namely when $b(\tau)$ becomes large and τ small the ionization is close to 1.

Lower bound: *Let ψ be a normalized bound state of $H = H_0 + V$ with energy $E < 0$. Then for any pulse $E(t)$ with $1/2b(\tau)^2 > -E$ the ionization satisfies a lower bound of the form*

$$I(\psi) \geq 1 \quad - \quad \left\{ \int_0^\tau \|(V(\cdot - c(t)e_z) - V(\cdot))\psi\| dt \right. \qquad (3.25)$$
$$\left. + \frac{1}{E + \frac{1}{2}b(\tau)^2} \|(V(\cdot - c(\tau)e_z - V(\cdot))\psi\| + \frac{|b(\tau)|}{E + \frac{1}{2}b(\tau)^2} \|p_z\psi\| \right\}^2 .$$

Note that now the condition on $E(t)$ is that the classical energy transfer of the pulse is larger than the classical ionization energy. Recall that by our previous discussion the norms appearing in the first two terms in the bracket may be estimated independently of the field strength, such that (3.25) gives a bound which involves $E(t)$ only through τ and $b(\tau)$.

We turn to a proof. In order to obtain a lower bound on

$$I(\psi) = \|(\mathbf{1} - P)U_1(\tau, 0)\psi\|^2 = 1 - \|PU_1(\tau, 0)\psi\|^2$$

it suffices to obtain an upper bound on $\|PU_1(\tau, 0)\psi\|$. First we write

$$\begin{aligned}
\|PU_1(\tau, 0)\psi\| &= \|PT(\tau)U_3(\tau, 0)\psi\| \\
&= \|P\exp -ib(\tau)z \cdot \exp ic(\tau)p_z \cdot U_3(\tau, 0)\psi\| \\
&\leq \|P\exp -ib(\tau)z \cdot \exp ic(\tau)p_z \cdot (U_3(\tau, 0) - \exp -i\tau H)\psi\| \\
&\quad + \|P\exp -ib(\tau)z \cdot \exp ic(\tau)p_z \cdot \psi\|. \qquad (3.26)
\end{aligned}$$

The first term on the r.h.s. is estimated by (3.5) yielding the first term in the bracket in (3.25).

The second term in (3.26) is treated as follows. By assumption $PH \leq 0$. Let $\delta > 0$ be arbitrary. Then $P(H - \delta)^{-1}$ is a well defined operator with operator norm $\leq 1/\delta$. Hence

$$\begin{aligned}
& \|P\exp -ib(\tau)z \cdot \exp ic(\tau)p_z \cdot \psi\| \\
= & \|P(H - \delta)^{-1}(H - \delta)\exp -ib(\tau)z \cdot \exp ic(\tau)p_z \cdot \psi\| \\
\leq & \frac{1}{\delta}\|(H - \delta)\exp -ib(\tau)z \cdot \exp ic(\tau)p_z \cdot \psi\|. \qquad (3.27)
\end{aligned}$$

In analogy to (3.11) we now use the relation

$$\begin{aligned}
& \exp -ic(\tau)p_z \cdot \exp \ ib(\tau)z \cdot H \cdot \exp -ib(\tau)z \cdot \exp ic(\tau)p_z \\
& = \frac{1}{2}(-i\nabla - b(\tau)e_z)^2 + V(\cdot - c(\tau)e_z) \\
& = H - b(\tau)p_z + \frac{1}{2}b(\tau)^2 + V(\cdot - c(\tau)e_z) - V(\cdot). \qquad (3.28)
\end{aligned}$$

Inserting this into (3.27) we obtain

$$\begin{aligned}
\|P\exp -ib(\tau)z \cdot \exp ic(\tau)p_z\psi\| &\leq \frac{1}{\delta}\|(V(\cdot - c(\tau)e_z) - V(\cdot) \cdot \psi\| \\
&\quad + \frac{1}{\delta}\|(E - b(\tau)p_z + \frac{1}{2}b(\tau)^2 - \delta)\psi\| \quad (3.29)
\end{aligned}$$

We now make the choice

$$\delta = E + \frac{1}{2} b(\tau)^2 \tag{3.30}$$

which by assumption on $b(\tau)$ is > 0 and when inserted into (3.29) immediately yields the remaining two terms in the bracket of (3.25), thus concluding the proof of the lower limit.

We note that these two theorems are compatible with the result in [19] on the Stark kick, i.e. $E(t) = F_0 \delta(t)$. There it was shown that for fixed ionization probability of any bound state $\psi_{n\ell m}$ of the hydrogen atom F_0 scales like $1/n$, as predicted by Rheinhold et al. [38].

We now return to a comparison between our results and those obtained by employing approximation methods based on perturbative expansions and we will include the lower bound in the discussion. We stress once more the point that the lower bound definitely excludes the possibility of stabilization of the bound states for increasing electric field strength, when the applied pulse is short in duration, since (3.7), despite the dependence on $c(t)$, may be estimated by a constant, say C, independent of the electric field. Hence

$$\lim_{|E(t)| \to \infty} I(\psi) \;\geq\; 1 - \tau^2 \, C \quad . \tag{3.31}$$

This shows clearly that the electric field has no stabilizing effect and we are therefore in disagreement with [13, 12, 11]. The lower bound also reproduces the result obtained through an expansion around the Gordon-Volkov solution, namely for monochromatic linearly polarized laser pulses at integer cycles, i.e. $b(\tau) = c(\tau) = 0$, the ionization probability becomes independent of the electric field strength. We may qualitatively relate the term proportional to τ^2 to a term also observed in perturbative expansion methods and which is interpreted as the spreading of the wave.

To illustrate our results further, we consider now the concrete example of the hydrogen atom. For the ψ_{100} state we obtain as our best estimate

$$I(\psi_{100}) \;\leq\; \left(2\tau + \sqrt{\frac{7}{6}}|b(\tau)| + \frac{1}{\sqrt{3}}|c(\tau)|\right)^2 \tag{3.32}$$

$$I(\psi_{100}) \;\geq\; 1 - \left(2\tau + \frac{4}{b(\tau)^2 - 1} + \frac{2}{\sqrt{3}}\frac{b(\tau)}{b(\tau)^2 - 1}\right)^2 . \tag{3.33}$$

Taking the pulse to be of the form $E_z(t) = E_0 \cos \omega t$ for $0 \leq t \leq \tau$ and zero otherwise, we have

$$|b(\tau)| = \frac{E_0}{\omega}|\sin \omega\tau| \qquad \text{and} \qquad |c(\tau)| = \frac{2E_0}{\omega^2}\sin^2\left(\frac{\omega\tau}{2}\right) . \tag{3.34}$$

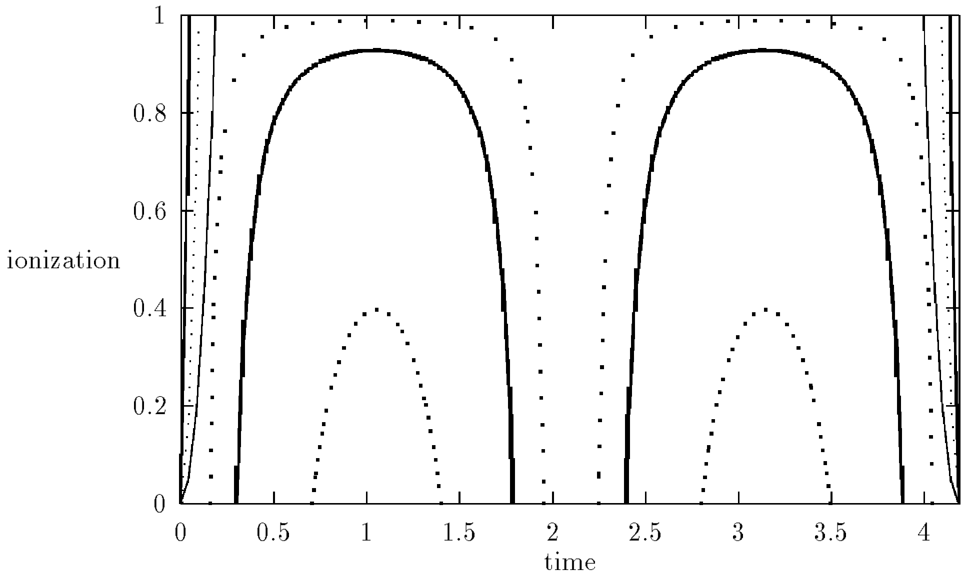

Figure 1: Upper and lower bounds for the ionization probability for the ψ_{100}-state of the hydrogen atom in the first cycle of an applied field $E_z = E_0 \cos(1.5t)$ for $E_0 = 5, 10, 20$, when neglecting the "spreading of the wave".

Figure 1 shows a plot of one cycle, when neglecting the term which is independent of the electric field strength, i.e. the term 2τ, in the upper and lower limit. At the far ends one observes the curves for the upper limits, which approach the vertical for increasing E_0. That is starting from the outside and going inwards the solid line $\equiv E_0 = 20$, dotted line $\equiv E_0 = 10$ and the next solid line $\equiv E_0 = 5$. Next we have the lower limits for $E_0 = 20 \equiv$ dotted line, $E_0 = 10 \equiv$ solid line and $E_0 = 5 \equiv$ dotted line. We have used the same values as in [10] and compare with figure 3 therein. Figure 1 clearly reproduces the features of Geltman's results, indicating the dips at the half cycles and producing an increasing ionization probability for increasing field strength. We do, however, not observe any crossing for different field intensities. When including the 2τ-term, this pattern will be moved above the trivial bound 1. This may be avoided when achieving a better estimate for the factor in front of τ, for instance when integrating explicitly (3.7) [39]. Figure 2 shows the upper bound for four cycles and reproduces the well known oscillatory behaviour superimposed by a spreading of the wave-packet of the Gordon-Volkov solution, the so-called over-the-barrier ionization.

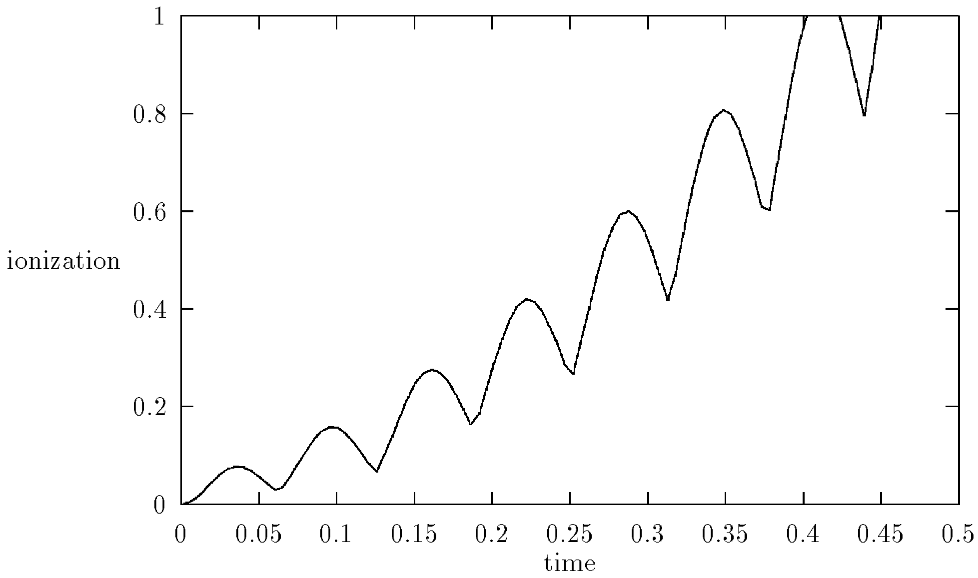

Figure 2: Upper bound for the ionization probability for the ψ_{100}-state of the hydrogen atom in the first four cycles of an applied field $E_z = 10\cos(50t)$.

4 Conclusions

In conclusion we can say that, according to our arguments atoms do not become resistant to ionization when exposed to short ultra-intense laser pulses. We therefore disagree with Eberly and Kulander [16] and do not share their opinion that one requires a new viewpoint on the physics in this regime. Their point of view is sustained through numerous numerical studies, which are based on perturbative methods. We have commented above on the problems of these methods and furthermore like to remark that those studies are in general very complex and subject to many possible errors which are difficult to check for second parties. We think that the virtue of our arguments is that they are analytic and transparent to the reader. An extension of our results to multi-particle systems (thus including atoms and molecules with several electrons and not necessarily electrically neutral) may be found in [21]. Needless to say, since our results are of a qualitative nature, in the sense that they merely provide bounds and that, since there are no explicit solutions for the Schrödinger equation available, for precise predictions of ionization rates one needs more numerical data.

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Appendix A

The aim of this appendix is to prove the following bound for the Coulomb potential $V = -\frac{1}{r}(r = |\vec{x}|)$ on $L^2(I\!R^3, d^3x) = L^2$.

Lemma A: *The following operator norm bound holds*

$$\left\| \frac{1}{r}(-\Delta + \mathbf{1})^{-1} \right\| \leq 6.35 \tag{A.1}$$

The proof optimizes well known a priori bounds which we take from [28]:

Let L^∞ be the space of all Lebesgue measurable functions φ on $I\!R^3$ such that $|\varphi(x)| \leq M < \infty$ almost everything. The smallest such M is denoted by $\|\varphi\|_\infty$. Then one has the a priori estimate ([28] p.56)

$$\|\varphi\|_\infty \leq a(\rho)\| - \Delta\varphi\|_2 + b(\rho)\|\varphi\|_2 \tag{A.2}$$

where $\| \quad \|_2$ denotes the L^2 -norm . Here

$$\begin{aligned} a(\rho) &= c\rho^{-1} \\ b(\rho) &= c\rho^3 \end{aligned} \tag{A.3}$$

with

$$c = \left(\int_{-\infty}^{+\infty} \frac{1}{(1+\lambda^2)^2} d\lambda\right)^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}} \tag{A.4}$$

and $\rho > 0$ may be chosen arbitrary.

Let $R > 0$ be arbitrary and write

$$\frac{1}{r} = V_1^R + V_2^R$$

with

$$\begin{aligned} V_1^R(r) &= \frac{\theta(r < R)}{r} \\ V_2^R(r) &= \frac{\theta(r \geq R)}{r}. \end{aligned}$$

Then one has the a priori bound (see [28] p.165)

$$\left\|\frac{1}{r}\varphi\right\|_2 \leq a(\rho)\ \|V_1^R\|_2\ \ \| - \Delta\varphi\|_2 + \left(b(\rho) + \|V_2^R\|_\infty\right) \|\varphi\|_2. \tag{A.5}$$

Since the estimates $\|\varphi\|_2 \leq \|(-\Delta + \mathbf{1})\varphi\|_2$, $\| - \Delta\varphi\|_2 \leq \|(-\Delta + \mathbf{1})\varphi\|_2$ are trivially valid, this gives

$$\left\|\frac{1}{r}(-\Delta + \mathbf{1})^{-1}\right\| \leq a(\rho)\|V_1^R\|_2 + b(\rho) + \|V_2^R\|_\infty. \tag{A.6}$$

Obviously

$$\|V_1^R\|_2 = (4\pi R)^{\frac{1}{2}},$$
$$\|V_2^R\|_\infty = \frac{1}{R}. \tag{A.7}$$

Inserting (A.3), (A.4) and (A.7) into (A.6) gives

$$\left\|\frac{1}{r}(-\Delta+\mathbf{1})^{-1}\right\| \le \pi\sqrt{2}\rho^{-1}R^{\frac{1}{2}} + \sqrt{\frac{\pi}{2}}\rho^3 + \frac{1}{R}. \tag{A.8}$$

for all $\rho > 0, R > 0$. The claim now follows by optimizing w.r.t. ρ and R. A short calculation finally gives

$$\left\|\frac{1}{r}(-\Delta+\mathbf{1})^{-1}\right\| \le 11 \cdot \frac{\pi^{\frac{7}{11}}}{2^{\frac{6}{11}}3^{\frac{9}{11}}} \tag{A.9}$$

which is (A1).

Appendix B

In this section we will study the quantity $\langle\psi|V(\cdot-\vec{y})^k|\psi\rangle$ for $k=1$ and 2 where $V(\vec{x}) = -\frac{1}{|\vec{x}|}(\vec{x}\in I\!R^3)$ is the Coulomb potential and where ψ is the normalized ground state wave function ψ_{100} for the hydrogen atom, which is rotationally invariant. Therefore this quantity depends on $|\vec{y}|$ only.

Lemma B: *Both* $-\langle\psi|V(\cdot-\vec{y})|\psi\rangle$ *and* $\langle\psi|V(\cdot-\vec{y})^2|\psi\rangle$ *are decreasing functions of* $|\vec{y}|$ *for* $\psi=\psi_{100}$.

Intuitively this result is clear: $\psi(\vec{x})$ has its maximum at $\vec{x}=0$ and $-V(\vec{x}-\vec{y})\ge 0$ has its singularity at $\vec{x}=\vec{y}$ so their overlap is maximal when $\vec{y}=0$.

Before we give a proof, we first establish an important consequence. Indeed we claim that for $\psi=\psi_{100}$

$$\|(V(\cdot-\vec{y})-V(\cdot))\psi\| \le 2 \tag{B.1}$$

for all $\vec{y}\in I\!R^3$. To see this we write

$$\begin{aligned}\|(V(\cdot-\vec{y})-V(\cdot))\psi\|^2 &= \langle\psi,V(\cdot-\vec{y})^2\psi\rangle - 2\langle\psi,V(\cdot-\vec{y})V(\cdot)\psi\rangle + \langle\psi,V(\cdot)^2\psi\rangle \\ &\le \langle\psi,V(\cdot-\vec{y})^2\psi\rangle + \langle\psi,V(\cdot)^2\psi\rangle \end{aligned} \tag{B.2}$$

since $V(\cdot - \vec{y})V(\cdot) \geq 0$ as an operator. By Lemma B, the right hand side of (B.2) takes its maximum at $\vec{y} = 0$ proving the claim since $\langle \psi | \frac{1}{|\vec{x}|^2} | \psi \rangle = 2$ (see e.g. [37]).

To prove the lemma, it suffices to consider $\vec{y}$ to be of the form $\vec{y} = ce_z$ with $c > 0$. Now we use the well known formula

$$\frac{1}{|\vec{x} - ce_z|} = \frac{1}{r_>} \sum_{\ell=0}^{\infty} \left(\frac{r_<}{r_>} \right)^{\ell} P_\ell(\cos \vartheta) \tag{B.3}$$

where $r_< = Max(r, c), r_< = Min(r, c)$ and where $(r = |\vec{x}|, \vartheta, \varphi)$ are the polar coordinates of $\vec{x}$. By the orthogonality relations of the Legendre polynomials and since ψ is the ground state we obtain

$$\begin{aligned} -\langle \psi | V(\cdot - ce_z) | \psi \rangle &= \langle \psi | \frac{1}{r_>} | \psi \rangle \\ &= 4 \int_0^\infty r^2 e^{-2r} \frac{1}{r_>} dr \\ &= \frac{4}{c} \int_0^c r^2 e^{-2r} dr + 4 \int_c^\infty r e^{-2r} dr. \end{aligned} \tag{B.4}$$

This function is differentiable in c for $c > 0$ and its derivative is easily seen to be ≤ 0, proving the first claim. Next we have (again by the orthogonality of the Legendre polynomials)

$$\begin{aligned} \langle \psi | V(\cdot - ce_z)^2 | \psi \rangle &= \langle \psi | \frac{1}{(r_>)^2} \sum_{\ell=0}^{\infty} \left(\frac{r_<}{r_>} \right)^{2\ell} \frac{2}{2\ell + 1} | \psi \rangle \\ &= \frac{4}{c^2} \int_0^c r^2 e^{-2r} \sum_{\ell=0}^{\infty} \left(\frac{r}{c} \right)^{2\ell} \frac{2}{2\ell + 1} dr \\ &\quad + 4 \int_c^\infty e^{-2r} \sum_{\ell=0}^{\infty} \left(\frac{c}{r} \right)^{2\ell} \frac{2}{2\ell + 1} dr. \end{aligned} \tag{B.5}$$

Now for $0 < x < 1$ we have

$$\sum_{\ell=0}^{\infty} x^{2\ell} \frac{1}{2\ell + 1} = \frac{1}{x} \sum_{\ell=0}^{\infty} \frac{x^{2\ell+1}}{2\ell + 1} = \frac{1}{2x} (\ln(1 + x) - \ln(1 - x)). \tag{B.6}$$

Inserting (B6) into (B5) gives

$$\begin{aligned} \langle \psi | V(\cdot - ce_z)^2 | \psi \rangle &= \frac{4}{c} \int_0^c r e^{-2r} \left(\ln \left(1 + \frac{r}{c} \right) - \ln \left(1 - \frac{r}{c} \right) \right) dr \\ &\quad + \frac{4}{c} \int_c^\infty r e^{-2r} \left(\ln \left(1 + \frac{c}{r} \right) - \ln \left(1 - \frac{c}{r} \right) \right) dr. \end{aligned} \tag{B.7}$$

Now the right hand side is not differentiable in c. To remedy this we regularize and consider the quantity $(0 < \varepsilon < 1)$

$$0 \le F(c,\varepsilon) = \frac{G(c,\varepsilon)}{c} = \frac{4}{c}\int_0^c re^{-2r}\left[\ln(1+\frac{r}{c}) - \ln(1-\frac{r}{c}(1-\varepsilon))\right] dr$$
$$+\frac{4}{c}\int_c^\infty re^{-2r}[\ln(1+\frac{c}{r}) - \ln(1-\frac{c}{r}(1-\varepsilon))]dr. \quad \text{(B.8)}$$

Since $\lim_{\varepsilon\to 0} F(c,\varepsilon) = \langle\psi|V(\cdot - ce_z)^2|\psi\rangle$ it suffices to show that

$$\frac{d}{dc}F(c,\varepsilon) = \frac{1}{c^2}(c\frac{dG}{dc}(c,\varepsilon) - G(c,\varepsilon)) \le 0 \quad \text{(B.9)}$$

for all $0 < \varepsilon < 1$, since then (B7) is also monotonically decreasing in $c > 0$. Now

$$\frac{d}{dc}G(c,\varepsilon) = \quad -4\int_0^c re^{-2r}\left[\frac{r}{c^2}\,\frac{1}{1+\frac{r}{c}} + \frac{r(1-\varepsilon)}{c^2}\,\frac{1}{1-\frac{r}{c}(1-\varepsilon)}\right] dr$$
$$+4\int_c^\infty re^{-2r}\left[\frac{1}{r}\,\frac{1}{1+\frac{c}{r}} + \frac{(1-\varepsilon)}{r}\,\frac{1}{1-\frac{c}{r}(1-\varepsilon)}\right] dr. \quad \text{(B.10)}$$

The first integral on the r.h.s. of (B10) is negative. By (B8), (B9) and (B10) it therefore suffices to show that for any $r > 0$

$$\frac{c}{r}\frac{1}{1+\frac{c}{r}} - \ln(1+\frac{c}{r}) \le 0. \quad \text{(B.11)}$$

and

$$\frac{c(1-\varepsilon)}{r}\frac{1}{1-\frac{c}{r}(1-\varepsilon)} + \ln\left(1-\frac{c}{r}(1-\varepsilon)\right) \le 0. \quad \text{(B.12)}$$

Now

$$\ln(1+\frac{c}{r}) = \frac{1}{r}\int_0^c \frac{1}{1+\frac{c'}{r}}dc' \ge \frac{1}{r}\frac{1}{1+\frac{c}{r}}\cdot c$$

which is (B11). (B12) is proved in the same fashion. This concludes the proof of Lemma B.

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