On The Even CAR Algebra

Carsten Binnenhei *
Institut für Theoretische Physik
Freie Universität Berlin, Germany
E–mail: binnenhe@omega.physik.fu-berlin.de

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Abstract. We exhibit a family of *-isomorphisms mapping the CAR algebra onto its even subalgebra.

In 1970 E. Størmer proved that the even subalgebra of the CAR algebra over an infinite dimensional separable Hilbert space is UHF of type 2^∞, hence *-isomorphic to the CAR algebra itself [1]. But it seems to be unknown that such isomorphisms have a nice and simple realization in terms of Bogoliubov endomorphisms with "statistical dimension" $\sqrt{2}$.

Bogoliubov endomorphisms are conveniently described using Araki's "selfdual" CAR algebra formalism [2]. Let $\mathcal{K}$ be an infinite dimensional separable complex Hilbert space, equipped with a complex conjugation $k \mapsto k^*$, and let $\mathcal{C}(\mathcal{K})$ denote the unique (simple) C*-algebra generated by $\mathbf{1}$ and the elements of $\mathcal{K}$, subject to the anticommutation relation

$$k^* k' + k' k^* = \langle k, k' \rangle \mathbf{1}, \quad k, k' \in \mathcal{K}.$$

$\mathcal{K}$ will henceforth be viewed as a subspace of $\mathcal{C}(\mathcal{K})$ (but we caution the reader that the Hilbert space norm and the C*-norm differ on $\mathcal{K}$). Each isometry V on $\mathcal{K}$ that commutes with complex conjugation extends to a unique unital *-endomorphism ϱ_V of $\mathcal{C}(\mathcal{K})$. Such V has a well-defined Fredholm index $\operatorname{ind} V = -\dim\ker V^*$, and the statistical dimension of ϱ_V (i.e. the square root of the index of the inclusion $\varrho_V(\mathcal{C}(\mathcal{K})) \subset \mathcal{C}(\mathcal{K})$) was shown to equal $2^{-\frac{1}{2}\operatorname{ind} V}$ in [3]. The semigroup consisting of these isometries V will be denoted by $\mathcal{I}(\mathcal{K})$, and the distinguished automorphism corresponding to $-\mathbf{1} \in \mathcal{I}(\mathcal{K})$ by γ. This automorphism turns $\mathcal{C}(\mathcal{K})$ into a $\mathbb{Z}_2$-graded algebra: $\mathcal{C}(\mathcal{K}) = \mathcal{C}(\mathcal{K})_0 \oplus \mathcal{C}(\mathcal{K})_1$ and $\mathcal{C}(\mathcal{K})_g \cdot \mathcal{C}(\mathcal{K})_{g'} \subset \mathcal{C}(\mathcal{K})_{g+g'}$ where $\mathcal{C}(\mathcal{K})_g := \{a \mid \gamma(a) = (-1)^g a\}$, $g \in \mathbb{Z}_2 = \{0,1\}$.

PROPOSITION 1. *Let* $V \in \mathcal{I}(\mathcal{K})$ *with* $\operatorname{ind} V = -1$, *and let* $k_V \in \ker V^*$ *be unitary and skew-adjoint. Then* $u_V := \frac{1}{\sqrt{2}}(\mathbf{1} + k_V) \in \mathcal{C}(\mathcal{K})$ *is unitary, and*

$$\sigma_V(a) := u_V \varrho_V(a) u_V^*, \quad a \in \mathcal{C}(\mathcal{K}) \tag{1}$$

*defines a unital *-isomorphism* σ_V *from* $\mathcal{C}(\mathcal{K})$ *onto its even subalgebra* $\mathcal{C}(\mathcal{K})_0$.

Proof. (1) clearly defines an injective *-endomorphism σ_V of $\mathcal{C}(\mathcal{K})$. The direct sum decomposition $\mathcal{K} = \ker V^* \oplus \operatorname{ran} V$ induces a $\mathbb{Z}_2$-graded tensor product decomposition $\mathcal{C}(\mathcal{K}) = \mathcal{C}(\ker V^*) \cdot \mathcal{C}(\operatorname{ran} V)$, where the factors on the right are the subalgebras generated by $\ker V^*$ and $\operatorname{ran} V$, respectively. From $\mathcal{C}(\ker V^*)_0 = \mathbb{C}\mathbf{1}$ and $\mathcal{C}(\ker V^*)_1 = \mathbb{C}k_V$ it follows that $\mathcal{C}(\mathcal{K})_0 = \mathcal{C}(\operatorname{ran} V)_0 + \mathbb{C}k_V \cdot \mathcal{C}(\operatorname{ran} V)_1$. Since $u_V a u_V^* = a$ if

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$a \in \mathcal{C}(\operatorname{ran} V)_0$ and $u_V b u_V^* = u_V^2 b = k_V b$ if $b \in \mathcal{C}(\operatorname{ran} V)_1$, $\operatorname{ran} \sigma_V = u_V(\operatorname{ran} \varrho_V) u_V^* = u_V \mathcal{C}(\operatorname{ran} V)_0 u_V^* + u_V \mathcal{C}(\operatorname{ran} V)_1 u_V^* = \mathcal{C}(\operatorname{ran} V)_0 + \mathbb{C} k_V \cdot \mathcal{C}(\operatorname{ran} V)_1 = \mathcal{C}(\mathcal{K})_0$.

Remarks. 1. V and k_V as above always exist since $\dim \mathcal{K} = \infty$ and since $\ker V^*$ is invariant under complex conjugation. k_V is determined by V up to a sign. The group of Bogoliubov automorphisms $\{\varrho_U \mid U \in \mathcal{I}(\mathcal{K}),\ \operatorname{ind} U = 0\}$ acts transitively on the set of isomorphisms $\{\sigma_V\}$ by left multiplication.

2. The identification of $\mathcal{C}(\mathcal{K})$ with a 'conventional' CAR algebra $\mathcal{A}(K)$, generated by elements $a(f)$ and $a(f)^*$, $f \in K$ as in [1][1], depends on the choice of a closed subspace $K \subset \mathcal{K}$ such that $\mathcal{K} = K \oplus K^*$. Given K, one gets $\mathcal{C}(\mathcal{K}) \cong \mathcal{A}(K)$ by identifying $f \in K$ with $a(f)$.

3. In [3] we constructed left inverses Φ_V for Bogoliubov endomorphisms ϱ_V. We get a left inverse φ_V for σ_V by setting $\varphi_V(a) := \Phi_V(u_V^* a u_V)$. Each $a \in \mathcal{C}(\mathcal{K})$ has a unique decomposition $a = a_0 + k_V a_1 + b_1$ where $a_g \in \mathcal{C}(\operatorname{ran} V)_g$, $b_1 \in \mathcal{C}(\mathcal{K})_1$, and then $\varphi_V(a) = \varrho_V^{-1}(a_0 + a_1)$. The conditional expectation $\sigma_V \circ \varphi_V$ from $\mathcal{C}(\mathcal{K})$ onto $\mathcal{C}(\mathcal{K})_0$ coincides with the mean over $\mathbb{Z}_2$: $\sigma_V(\varphi_V(a)) = \frac{1}{2}(a + \gamma(a))$.

4. Since the representation theory of the CAR algebra is well developed, one may use the isomorphisms from the Proposition to study representations of the even CAR algebra. For example, in the algebraic approach to the chiral Ising model [4], one has to study representations of the form $\pi \circ \varrho_W$ with π a Fock representation of $\mathcal{C}(\mathcal{K})$ and ϱ_W a Bogoliubov endomorphism. It was observed [5, 6], but regarded as a curiosity, that the restriction of $\pi \circ \varrho_W$ to $\mathcal{C}(\mathcal{K})_0$ behaves like a representation $\pi \circ \varrho_{W'}$ of $\mathcal{C}(\mathcal{K})$ for some $W' \in \mathcal{I}(\mathcal{K})$ with $\operatorname{ind} W' = \operatorname{ind} W - 1$. This phenomenon is an immediate consequence of the Proposition since $\pi \circ \varrho_W(\mathcal{C}(\mathcal{K})_0) = \pi \circ \varrho_W \circ \sigma_V(\mathcal{C}(\mathcal{K})) \simeq \pi \circ \varrho_{WV}(\mathcal{C}(\mathcal{K}))$ and $\operatorname{ind} WV = \operatorname{ind} W - 1$. In combination with results on representations of $\mathcal{C}(\mathcal{K})$ [6, 3], one obtains the decomposition of the restriction of $\pi \circ \varrho_W$ to $\mathcal{C}(\mathcal{K})_0$ into irreducible subrepresentations as discussed in [6].

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[1] $a(f)$ depends linearly on f in [1].