$G^*_{2(2)}$-Structures on pseudo-Riemannian manifolds

I. Kath

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Abstract

We will give the definition and basic properties of nearly-parallel $G^*_{2(2)}$-structures on pseudo-Riemannian manifolds of signature (4,3). In particular we explaine the equivalence of their existence with that of Killing spinor fields. Furthermore, we will give first examples of pseudo-Riemannian manifolds of signature (4,3) with Killing spinors.

Contents

1 Introduction

This article relates to the paper [4] of Th. Friedrich and others on nearly parallel G_2-structures. G_2-structures are topological reductions of the frame bundle of a 7-dimensional manifold to the exceptional group G_2. They can be described by 3-forms of special algebraic type on the manifold. Since $G_2 \subset SO(7)$ such a structure induces a Riemannian metric and in particular a Levi-Civita connection ∇ on the manifold. It is called nearly parallel if the associated 3-form ω^3 satisfies $\nabla_Z \omega^3 = -2\lambda(Z \lrcorner * \omega^3)$. The existence of such a 3-form is equivalent to the existence of a spin structure with a Killing spinor field.

Now we are interested in similar structures on pseudo-Riemannian manifolds, more exactly, on manifolds admitting a metric of signature (4,3). There are two real connected non-compact groups of type G_2. One of them denoted by $G^*_{2(2)}$ is a subgroup of $SO(4,3)$. $G^*_{2(2)}$ is one of the possible "exceptional" holonomy groups of non-symmetric irreducible pseudo-Riemannian manifolds [2].

The $Spin(4,3)$-representation $\Delta_{4,3}$ has some algebraic properties similar to those of the $Spin(7)$-representation Δ_7. In particular, both are real. Furthermore, while $Spin(7)$ acts transitively on the sphere S^7 with isotropy group G_2 the action of the connected component $Spin^+(4,3)$ of $Spin(4,3)$ on the pseudo-sphere in $\Delta_{4,3}$ is transitive with isotropy group $G^*_{2(2)}$. For a fixed spinor $\psi \neq 0$ in Δ_7 the Clifford multiplication $X \longmapsto X \cdot \psi$ is an isomorphism from $\mathbb{R}^7$ to the orthogonal complement of ψ. The same is true in $\Delta_{4,3}$ for any non-isotropic spinor ψ.

These properties will allow us to translate several results from the Riemannian case to signature (4,3). We will give the definition and basic properties of nearly-parallel $G^*_{2(2)}$-structures. In particular we explaine the equivalence of their existence with that of Killing spinor fields. Furthermore, we will give first examples of pseudo-Riemannian manifolds of signature (4,3) with Killing spinors.

2 The exceptional non-compact group $G^*_{2(2)}$

Let $g_{4,3}$ be the symmetric bilinear form on $\mathbb{R}^7$ which is given by $g_{4,3} = \mathrm{diag}$ $(-1,-1,-1,-1,1,1,1)$ with respect to the standard basis $e_1, e_2, ..., e_7$ of $\mathbb{R}^7$. Define ε_i by $\varepsilon_i = g_{4,3}(e_i, e_i)$. The real Clifford algebra $\mathcal{C}_{4,3} = \mathrm{Cliff}(\mathbb{R}^7, -g_{4,3})$ is the algebra generated by $e_1, e_2, ..., e_7$ with the relations $e_i^2 = -\varepsilon_i$, $e_ie_j + e_je_i = 0$ if $i \neq j$. It is isomorphic to the direct sum $\mathbb{R}(8) \oplus \mathbb{R}(8)$ of algebras of matrices. We will use the isomorphism which is defined by

$$\begin{aligned}
e_1 &= (\varepsilon \otimes \varepsilon \otimes \sigma, \varepsilon \otimes \varepsilon \otimes \sigma) \\
e_2 &= (-\sigma \otimes \sigma \otimes \tau, -\sigma \otimes \sigma \otimes \tau) \\
e_3 &= (-\sigma \otimes I \otimes \sigma, -\sigma \otimes I \otimes \sigma) \\
e_4 &= (\sigma \otimes \tau \otimes \tau, \sigma \otimes \tau \otimes \tau) \qquad (1) \\
e_5 &= (-I \otimes \varepsilon \otimes \tau, -I \otimes \varepsilon \otimes \tau)
\end{aligned}$$

$$
\begin{aligned}
e_6 = &\quad (-\tau \otimes \varepsilon \otimes \sigma, -\tau \otimes \varepsilon \otimes \sigma)\\
e_7 = &\quad (I \otimes I \otimes \varepsilon, -I \otimes I \otimes \varepsilon),
\end{aligned}
$$

where

$$
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
$$

The projection of this isomorphism onto the first component restricted to $Spin(4,3) \subset \mathcal{C}_{4,3}$ yields the $Spin(4,3)$-representation on $\mathbb{R}^8 =: \Delta_{4,3}$. Furthermore this projection defines the Clifford multiplication of a vector $X \in \mathbb{R}^7 \subset \mathcal{C}_{4,3}$ with a spinor $\psi \in \Delta_{4,3}$ which we will denote by $X \cdot \psi$. Denote by $\psi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} =^t (1,0,...,0)$, $\psi_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} =^t (0,1,...,0)$, ..., $\psi_8 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} =^t (0,...,0,1)$ the standard basis of $\mathbb{R}^8$. We identify the Lie algebra of $Spin(4,3)$ with $\mathfrak{spin}(4,3) = \{\omega = \sum_{i<j} \omega_{ij} e_i e_j \mid \omega_{ij} \in \mathbb{R}\} \subset \mathcal{C}_{4,3}$. We will need the following formulae for the action of $\mathfrak{spin}(4,3)$ on $\Delta_{4,3}$ which follow from (1). Let D_{ij} be the 8×8-matrix whose (i,j)-entry is 1 and all of whose other entries are 0. We set $E_{ij} = -D_{ij} + D_{ji}$ and $A_{ij} = D_{ij} + D_{ji}$. Then we have

$$
\begin{aligned}
e_1e_2 = &\quad -E_{12} + E_{34} + E_{56} - E_{78}\\
e_1e_3 = &\quad E_{13} + E_{24} - E_{57} - E_{68}\\
e_1e_4 = &\quad -E_{14} + E_{23} + E_{58} - E_{67}\\
e_1e_5 = &\quad A_{16} - A_{25} + A_{38} - A_{47}\\
e_1e_6 = &\quad A_{15} + A_{26} + A_{37} + A_{48}\\
e_1e_7 = &\quad A_{17} - A_{28} - A_{35} + A_{46}\\
e_2e_3 = &\quad -E_{14} + E_{23} - E_{58} + E_{67}\\
e_2e_4 = &\quad -E_{13} - E_{24} - E_{57} - E_{68}\\
e_2e_5 = &\quad A_{15} + A_{26} - A_{37} - A_{48}\\
e_2e_6 = &\quad -A_{16} + A_{25} + A_{38} - A_{47}\\
e_2e_7 = &\quad A_{18} + A_{27} + A_{36} + A_{45} \qquad (2)\\
e_3e_4 = &\quad -E_{12} + E_{34} - E_{56} + E_{78}\\
e_3e_5 = &\quad A_{18} - A_{27} - A_{36} + A_{45}\\
e_3e_6 = &\quad A_{17} + A_{28} - A_{35} - A_{46}
\end{aligned}
$$

$$
\begin{aligned}
e_3e_7 &= \ -A_{15}+A_{26}-A_{37}+A_{48}\\
e_4e_5 &= \ \ A_{17}+A_{28}+A_{35}+A_{46}\\
e_4e_6 &= \ -A_{18}+A_{27}-A_{36}+A_{45}\\
e_4e_7 &= \ -A_{16}-A_{25}+A_{38}+A_{47}\\
e_5e_6 &= \ \ E_{12}+E_{34}-E_{56}-E_{78}\\
e_5e_7 &= \ \ E_{14}+E_{23}+E_{58}+E_{67}\\
e_6e_7 &= \ -E_{13}+E_{24}+E_{57}-E_{68}
\end{aligned}
$$

The following two bilinear forms on $\Delta_{4,3}$ are related to the $Spin(4,3)$-representation. On one hand we have the standard inner product of $\mathbb{R}^8$ which we denote by $(\ ,\ )$. It is invariant with respect to the maximal compact subgroup $((Pin(4)\times Pin(3))/\mathbb{Z}_2)\cap Spin(4,3)$ of $Spin(4,3)$ and admits the property $(X\cdot\varphi,\psi)+(\varphi,\theta(X)\cdot\psi)=0$ for all $X\in\mathbb{R}^7$and $\varphi,\psi\in\Delta_{4,3}$, where $\theta:\mathbb{R}^7\longrightarrow\mathbb{R}^7$ denotes the reflection with respect to $span\{e_5,e_6,e_7\}$. On the other hand we consider the product $\langle\ ,\ \rangle_\Delta$ of signature (4,4) defined by $\langle\varphi,\psi\rangle_\Delta := (e_1e_2e_3e_4\varphi,\psi)$. It is invariant with respect to the connected component $Spin^+(4,3)$ of $1\in Spin(4,3)$ and the equation $\langle X\cdot\varphi,\psi\rangle_\Delta+\langle\varphi,X\cdot\psi\rangle_\Delta=0$ holds for all $X\in\mathbb{R}^7$ and $\varphi,\psi\in\Delta_{4,3}$. The matrix of $\langle\ ,\ \rangle_\Delta$ with respect to the standard basis $\psi_1,...,\psi_8$ equals $diag(-1,-1,-1,-1,1,1,1,1)$. In particular, we obtain an imbedding $Spin(4,3)\subset SO(4,4)$.

Because of the $Spin^+(4,3)$-invariance of $\langle\ ,\ \rangle_\Delta$ the group $Spin^+(4,3)$ acts on $\mathcal{M}_c=\{\psi\in\Delta_{4,3}\mid\langle\psi,\psi\rangle_\Delta=c\}$, $c\in\mathbb{R}$. We will prove that this action is transitive for $c\neq 0$ and has two orbits for $c=0$.

Proposition 2.1 *The action of $Spin^+(4,3)$ on*

$$S^{4,3}:=\{\psi\in\Delta_{4,3}\mid\langle\psi,\psi\rangle_\Delta=1\}$$

is transitive. The same is valid for

$$H^{3,4}:=\{\psi\in\Delta_{4,3}\mid\langle\psi,\psi\rangle_\Delta=-1\}$$

The orbits of the $Spin(4,3)^+$-action on

$$\mathcal{C}:=\{\psi\in\Delta_{4,3}\mid\langle\psi,\psi\rangle_\Delta=0\}$$

are $\{0\}$ and $\mathcal{C}\setminus\{0\}$.

Proof. We consider the subalgebra $\mathfrak{spin}(4,0) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ of $\mathfrak{spin}(4,3)$. It is spanned by

$$\begin{aligned} e_1e_2 + e_3e_4 &= 2(-E_{12} + E_{34}) & e_1e_2 - e_3e_4 &= 2(E_{56} - E_{78}) \\ e_1e_3 - e_2e_4 &= 2(E_{13} + E_{24}) & e_1e_3 + e_2e_4 &= -2(E_{57} + E_{68}) \\ e_1e_4 + e_2e_3 &= 2(-E_{14} + E_{23}) & e_1e_4 - e_2e_3 &= 2(E_{58} - E_{67}) \end{aligned} \tag{3}$$

(see (2). Thus, $\mathfrak{spin}(4,0) \hookrightarrow \mathfrak{so}(\Delta_{4,3}) = \mathfrak{so}(4,4)$ equals the standard imbedding of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ into $\mathfrak{so}(4,4)$ Consequently, given an element $\psi \in \Delta_{4,3}$ we find an element g of $Spin(4,0) \subset Spin^+(4,3)$ such that $g\psi = x_1\psi_1 + x_5\psi_5$. Let now $\psi = x_1\psi_1 + x_5\psi_5$ be an element of $S^{4,3}$, i.e. $-x_1^2 + x_5^2 = 1$. Then $(x_1e_1 - x_5e_6)e_6 \in Spin^+(4,3)$ and $(x_1e_1 - x_5e_6)e_6\psi_5 = x_1\psi_1 + x_5\psi_5$. Analogously, given a $\psi = x_1\psi_1 + x_5\psi_5 \in H^{3,4}$, i.e. $-x_1^2 + x_5^2 = -1$ we have $e_1(x_1e_1 + x_5e_6) \in Spin^+(4,3)$ and $e_1(x_1e_1 + x_5e_6)\psi_1 = x_1\psi_1 + x_5\psi_5$. This proves the transitivity of the $Spin^+(4,3)$-action on $S^{4,3}$ and $H^{3,4}$, respectively. Now we consider the case $\psi = x_1\psi_1 + x_5\psi_5 \in \mathcal{C} \setminus \{0\}$. First let ψ be equal to $x\psi_1 + x\psi_5$ $(x \neq 0)$. Then

$$e_1(\frac{x^2+1}{2x}e_1 + \frac{x^2-1}{2x}e_6) \in Spin^+(4,3)$$

and

$$e_1(\frac{x^2+1}{2x}e_1 + \frac{x^2-1}{2x}e_6)(\psi_1 + \psi_5) = x\psi_1 + x\psi_5 = \psi$$

For $\psi = x_1\psi_1 - x_5\psi_5$ the assertion follows from $-e_1e_2e_3e_4(\psi_1 + \psi_5) = \psi_1 - \psi_5$.

Corollary 2.2 *1. The isotropy group $H(\psi) = \{h \in Spin^+(4,3) \mid h\psi = \psi\}$ of a non-isotropic spinor $\psi \in \Delta_{4,3}$ (i.e. $\langle\psi,\psi\rangle_\Delta \neq 0$) with respect to the $Spin^+(4,3)$-action is a connected non-compact group of type G_2 with fundamental group $\mathbb{Z}_2$.*

2. The Lie algebra of the isotropy group of an isotropic spinor is the semidirect sum of a 6-dimensional nilpotent algebra and $\mathfrak{sl}(3,\mathbb{R})$.

Proof. ad 1. Because of the transitivity of the $Spin^+(4,3)$-action it suffices to prove that $H(\psi_1)$ has the required properties. We first consider the Lie algebra $\mathfrak{h}(\psi_1)$ of this group. Because of (2) it equals

$$\begin{aligned} \mathfrak{h}(\psi_1) = \{ \sum_{i<j} \omega_{ij}e_ie_j \mid \quad & -\omega_{12} - \omega_{34} + \omega_{56} = 0 \ , \\ & \omega_{13} - \omega_{24} - \omega_{67} = 0 \ , \ -\omega_{14} - \omega_{23} + \omega_{57} = 0 \ , \\ & \omega_{16} + \omega_{25} - \omega_{37} = 0 \ , \ \omega_{15} - \omega_{26} - \omega_{47} = 0 \ , \\ & \omega_{17} + \omega_{36} + \omega_{45} = 0 \ , \ \omega_{27} + \omega_{35} - \omega_{46} = 0 \quad \}. \end{aligned} \tag{4}$$

It is spanned by $X_1 = e_1e_2 - e_3e_4$, $Y_1 = e_3e_4 + e_5e_6$, $X_2 = e_1e_3 + e_2e_4$, $Y_2 = e_2e_4 - e_6e_7$, $X_3 = e_1e_4 - e_2e_3$, $Y_3 = e_2e_3 + e_5e_7$, $X_4 = e_1e_6 - e_2e_5$, $Y_4 = e_1e_6 + e_3e_7$, $X_5 = e_2e_6 + e_1e_5$, $Y_5 = e_2e_6 - e_4e_7$, $X_6 = e_1e_7 - e_3e_6$, $Y_6 = e_1e_7 - e_4e_5$, $X_7 = e_2e_7 + e_4e_6$ and $Y_7 = e_2e_7 - e_3e_5$. Using the isomorphism of $\mathfrak{spin}(4,3)$ and $\mathfrak{so}(4,3)$, we see that the Killing form on $\mathfrak{h}(\psi_1)$ is non-degenerate and has index 6. Therefore, $\mathfrak{h}(\psi_1)$ is a non-compact real form of the semisimple Lie algebra $\mathfrak{h}(\psi_1)^{\mathbb{C}}$. Furthermore one reads from the relations

$$
\begin{array}{lll}
 & [X_1, Y_1] = 0 & \\
[X_1, X_2] = 4X_3 & [X_1, Y_2] = 2X_3 & \\
[X_1, X_3] = -4X_2 & [X_1, Y_3] = 2X_2 & \\
[X_1, X_i] = -2X_{i+1} & [X_1, Y_i] = -2Y_{i+1} & (i = 4, 6) \\
[X_1, X_j] = 2X_{j-1} & [X_1, Y_j] = 2Y_{j-1} & (j = 5, 7) \\
[Y_1, X_2] = -2X_3 & [Y_1, Y_2] = 4Y_3 & \\
[Y_1, X_3] = 2X_2 & [Y_1, Y_3] = -4Y_2 &
\end{array}
$$

that X_1 and Y_1 commute, but no element out of $span\{X_1, Y_1\}$ commutes with both X_1 and Y_1, i.e $\mathfrak{h}(\psi_1)^{\mathbb{C}}$ has rank 2 and thus it must be simple. Since its dimension is 14 it is of type G_2. There is only one non-compact real form of the complex Lie algebra of type G_2 (see e.g. [8]). Now we determine $H(\psi_1)$. Recall that there are two non-compact connected groups of type G_2 (see [8]). The simply connected one has centre $\mathbb{Z}_2$. Because of the transitivity of the $Spin^+(4,3)$-action $H^{3,4}$ is diffeomorphic to the homogeneous space $Spin^+(4,3)/H(\psi_1)$. Using the exact homotopy sequence of this fibration we conclude from $\pi_2(H^{3,4}) = \pi_1(H^{3,4}) = \pi_0(H^{3,4}) = 0$ and from $\pi_1(Spin^+(4,3)) = \mathbb{Z}_2$, $\pi_0(Spin^+(4,3)) = 0$ that $H(\psi_1)$ is connected and admits fundamental group $\pi_1(H(\psi_1)) = \mathbb{Z}_2$.

ad 2. We calculate the Lie algebra $\mathfrak{h}(\psi_1 + \psi_5)$ of the isotropy group of $\psi_1 + \psi_5$ and obtain using (2)
$\mathfrak{h}(\psi_1 + \psi_5) =$

$$
\begin{aligned}
= \{ \sum_{i<j} \omega_{ij} e_i e_j \mid \quad & \omega_{16} + \omega_{25} - \omega_{37} = 0 \ , \\
& \omega_{15} - \omega_{26} - \omega_{12} + \omega_{56} = 0 \ , \ \omega_{34} - \omega_{47} = 0 \ , \\
& \omega_{27} + \omega_{23} + \omega_{35} - \omega_{57} = 0 \ , \ \omega_{14} + \omega_{46} = 0 \ , \\
& \omega_{13} + \omega_{17} + \omega_{36} - \omega_{67} = 0 \ , \ \omega_{24} + \omega_{45} = 0 \quad \}.
\end{aligned} \tag{5}
$$

Hence, $\mathfrak{h}(\psi_1 + \psi_5)$ is the semidirect sum of the null space $\mathfrak{n}$ of its Killing form spanned by $e_3e_4 + e_4e_7$, $e_2e_4 - e_4e_5$, $e_1e_4 - e_4e_6$, $e_6e_7 - e_1e_3 + e_1e_7 +$

e_3e_6 , $e_1e_2 - e_5e_6 + e_1e_5 - e_2e_6$, $e_2e_3 - e_5e_7 - e_2e_7 - e_3e_5$ and the 8-dimensional subalgebra $\mathfrak{p}$ spanned by $e_1e_6 + e_3e_7$, $e_1e_6 - e_2e_5$, $e_1e_2 + e_5e_6$, $e_1e_5 + e_2e_6$, $e_1e_3 + e_6e_7$, $e_1e_7 - e_3e_6$, $e_2e_3 + e_5e_7$, $e_2e_7 - e_3e_5$. Obviously, $\mathfrak{n}$ is nilpotent. The Killing form restricted to $\mathfrak{p}$ is non-degenerate and has index 3. Consequently, $\mathfrak{p}$ equals $\mathfrak{sl}(3, \mathbb{R})$.

Definition 2.3 $G^*_{2(2)} := H(\psi_1)$

Remark 2.4 *In this notation $H^{3,4}$ is diffeomorphic to $Spin^+(4,3)/G^*_{2(2)}$.*

Corollary 2.5 *For a fixed spinor $\psi \in \Delta_{4,3}$ the kernel of the homomorphism*

$$\begin{array}{ccc} \mathbb{R}^7 & \longrightarrow & \{\psi\}^\perp \subset \Delta_{4,3} \\ X & \longmapsto & X \cdot \psi \end{array}$$

(i) is trivial iff $\psi \neq 0$ is non-isotropic

(ii) has dimension 3 iff $\psi \neq 0$ is isotropic.

Proof. Using (1) the assertions (i) and (ii) can be easily verified for $\psi = \psi_1$ and $\psi = \psi_1 + \psi_5$, respectively. Hence, they hold for any $\psi \neq 0$.

Now we consider the universal covering $\lambda : Spin(4,3) \longrightarrow SO(4,3)$. Because of $-1 \notin G^*_{2(2)}$ there is an isomorphism from $G^*_{2(2)}$ onto a subgroup of $SO(4,3)$, which we also denote by $G^*_{2(2)}$. We now describe this group using 3-forms on R^7. The key point is a special relation between non-isotropic spinors in $\Delta_{4,3}$ and generic 3-forms in $\Lambda^3(\mathbb{R}^7)$.

Let $\psi \in \Delta_{4,3}$ be a fixed non isotropic spinor. Then the map

$$R^7 \ni X \longmapsto X\psi \in \Delta_{4,3}$$

is an isomorphism because of the orthogonality of $X\psi$ and ψ and since $dim_R \Delta_{4,3} = 8$. We observe now that for $X, Y \in R^7$ the spinors ψ and $YX\psi + g_{4,3}(X,Y)\psi$ are orthogonal to each other. Therefore we can define a (2,1)-tensor A_ψ by

$$YX\psi + g_{4,3}(X,Y)\psi = A_\psi(Y,X)\psi. \tag{6}$$

A_ψ has the following properties

1. $A_\psi(X,Y) = -A_\psi(Y,X)$

2. $g_{4,3}(Y, A_\psi(Y,X)) = 0$

3. $A_\psi(Y, A_\psi(Y,X)) = -\|Y\|^2_{4,3}X + g_{4,3}(X,Y)Y$.

It defines a 3-form ω^3_ψ by $\omega^3_\psi(X,Y,Z) = g_{4,3}(X, A_\psi(Y,Z))$.

Clearly,

$$\omega^3_{\alpha\psi} = \omega^3_\psi \qquad \alpha \in \mathbb{R},\, \alpha \neq 0\,. \tag{7}$$

In particular, if $\psi = \psi_1$ then a direct calculation yields $\omega^3_{\psi_1} = \omega^3_0$, where ω^3_0 is given by

$$\begin{aligned} \omega^3_0 &= -e_1\wedge e_2\wedge e_7 - e_1\wedge e_3\wedge e_5 + e_1\wedge e_4\wedge e_6 \\ &\quad + e_2\wedge e_3\wedge e_6 + e_2\wedge e_4\wedge e_5 - e_3\wedge e_4\wedge e_7 + e_5\wedge e_6\wedge e_7. \end{aligned} \tag{8}$$

Definition 2.6 *Let ω^3 be a 3-form on $\mathbb{R}^7$. Furthermore let $X_1, ..., X_7$ be an arbitrary pseudo-orthonormal basis of $(\mathbb{R}^7, g_{4,3})$. We define a 4-form σ^4 by $\sigma^4 = -\sum_{i=1}^4 (X_i \lrcorner \omega^3)\wedge(X_i \lrcorner \omega^3) + \sum_{i=5}^7 (X_i \lrcorner \omega^3)\wedge(X_i \lrcorner \omega^3)$ which does not depend on the choosen basis. We will say that ω^3 defines the orientation of $\mathbb{R}^7$ if $\omega^3\wedge\sigma^4$ is a positive multiple of the volume form of $\mathbb{R}^7$. Furthermore we will say that ω^3 defines the space and time orientation of $(\mathbb{R}^7, g_{4,3})$ if it defines the orientation of $\mathbb{R}^7$ and if $\omega^3(X_5, X_6, X_7) > 0$ for any positively space and time oriented pseudo-orthonormal basis $X_1, ..., X_7$.*

Now let $\psi \neq 0$ be a fixed non-isotropic spinor and ω^3_ψ the associated 3-form. With the same notation as above we obtain $\omega^3_\psi \wedge \sigma^4_\psi = 42 e_1\wedge\ldots\wedge e_7$. Hence, ω^3_ψ defines the orientation of $\mathbb{R}^7$.

Now fix a spinor ψ with $\langle\psi,\psi\rangle_\Delta = -1$ and let $X_1, ..., X_7$ be a positively space and time oriented pseudo-orthonormal basis. From the definition of A_ψ we know that $g_{4,3}(A_\psi(X_5,X_6), A_\psi(X_5,X_6)) = 1$ and therefore $A_\psi(X_5,X_6) \notin \{X_5,X_6,X_7\}^\perp$. Since, however, $A_\psi(X_5,X_6) \perp X_5, X_6$ the vectors $A_\psi(X_5,X_6)$ and X_7 can not be orthogonal. Hence, $\omega^3_\psi(X_5,X_6,X_7) \neq 0$. Since on the other hand $\omega^3_{\psi_1}(e_5,e_6,e_7) = 1$ we obtain $\omega^3_\psi(X_5,X_6,X_7) > 0$. Hence ω^3_ψ defines the space and time orientation of $(\mathbb{R}^7, g_{4,3})$.

Vice versa, let A be a (2,1)-tensor on $\mathbb{R}^7$ which has the properties 1., 2., 3. Then A defines a 3-form $\omega^3 = g_{4,3}(\cdot, A(\cdot,\cdot))$. We can define σ^4 in the same way as above. From the properties 1., 2., 3. we conclude $\omega^3\wedge\sigma^4 \neq 0$. Suppose that ω^3 defines the orientation of $\mathbb{R}^7$. Furthermore from the properties 1., 2., 3 we deduce as above that $\omega^3(X_5,X_6,X_7) \neq 0$ for any oriented pseudo-orthonormal basis $X_1, ..., X_7$. Suppose that ω^3 defines the space and time orientation of $(\mathbb{R}^7, g_{4,3})$. Consider now the subspace

$$E = \{\psi \in \Delta_{4,3} \mid XY\psi = -g_{4,3}(X,Y)\psi + A(X,Y)\psi\}.$$

Then E is 1-dimensional and spanned by a spinor ψ_0 with $\langle\psi_0, \psi_0\rangle_\Delta = -1$. In particular, $\omega^3 = \omega_{\psi_0}$. We obtain

Theorem 2.7 *There is a 1 -1 correspondence between $S^{4,3}/\{1, -1\}$ and those $\omega^3 \in \Lambda^3(\mathbb{R}^7)$ which define the space and time orientation of $(\mathbb{R}^7, g_{4,3})$ and for which the bilinear map A defined by $\omega^3(X, Y, Z) = g_{4,3}(X, A(Y, Z))$ admits the properties 1., 2., 3. .*

Analogously, there is a 1 -1 correspondence between $H^{4,3}/\{1, -1\}$ and those $\omega^3 \in \Lambda^3(\mathbb{R}^7)$ which define the inverse space and time orientation of $(\mathbb{R}^7, g_{4,3})$ and for which the bilinear map A defined by $\omega^3(X, Y, Z) = g_{4,3}(X, A(Y, Z))$ admits the properties 1., 2., 3. .

In particular, since we have for $g \in Spin^+(4, 3)$

$$\omega^3_{g\psi} = (\lambda(g^{-1}))^* \omega_\psi,$$

we conclude

Corollary 2.8 *The image of $G^*_{2(2)}$ with respect to $\lambda : Spin(4, 3) \longmapsto SO(4, 3)$ equals*

$$G^*_{2(2)} = \{A \in SO^+(4, 3) \mid A^*\omega_0 = \omega_0\}.$$

Note that $A \in SO(4, 3)$ and $A^*\omega_0 = \omega_0$ imply $A \in SO^+(4, 3)$ since ω_0 defines a space and time orientation.

On the other hand the equation $A^*\omega_0^3 = \omega_0^3$ for $A \in GL(7)$ implies $A \in SO(4, 3)$. See for a proof in [2]. Consequently, we obtain

$$G^*_{2(2)} = \{A \in GL(7) \mid A^*\omega^3_0 = \omega^3_0\}.$$

Next we investigate in the same way as above the action of $Spin^+(4, 3)$ on some of the Stiefel manifolds

$$\begin{aligned} V(\varepsilon_1, ..., \varepsilon_l) = \{ \ (\varphi_1, ..., \varphi_l) \quad | \quad & \varphi_i \in \Delta_{4,3} \quad (i = 1, ..., l), \\ & \langle\varphi_i, \varphi_i\rangle_\Delta = \varepsilon_i \quad (i = 1, ..., l), \\ & \langle\varphi_i, \varphi_j\rangle_\Delta = 0 \text{ if } i \neq j \quad (i, j = 1, ..., l)\}, \end{aligned}$$

where $\varepsilon_i = -1$ for $i = 1, ..., k$ $(k \leq l)$ and $\varepsilon_i = 1$ for $i = k + 1, ..., l$.

Proposition 2.9 *The action of $Spin^+(4, 3)$ on $V(-1, -1), V(-1, 1)$ and $V(1, 1)$ is transitive.*

Proof. Since $e_1e_5 \in Spin(4,3)$ maps $S^{4,3}$ one-to-one onto $H^{3,4}$ and

$$(e_1e_5)Spin^+(4,3)(e_1e_5)^{-1} = (e_1e_5)Spin^+(4,3)(-e_5e_1) = Spin^+(4,3) \quad (9)$$

the situation on $V(-1,-1)$ and $V(1,1)$ is essentially the same.

We calculate the dimension of the isotropy group $H(\varphi_1,\varphi_2)$ of an arbitrary pair (φ_1,φ_2) with $\langle\varphi_1,\varphi_1\rangle_\Delta = -1$, $\langle\varphi_1,\varphi_2\rangle_\Delta = 0$ and $\varphi_2 \neq 0$. Clearly (see Proposition 2.1), we may assume $\varphi_1 = \psi_1$. Next we shall explain, why we can assume furthermore $\varphi_2 = x_2\psi_2 + x_5\psi_5$. The isotropy group $G^*_{2(2)}$ of ψ_1 contains $SO(3)$ and $SU(2)$ as subgroups. The Lie algebra $\mathfrak{so}(3) \subset \mathfrak{so}(4,4)$ is spanned by $e_3e_4 + e_5e_6 = 2(E_{34} - E_{56})$, $e_2e_4 - e_6e_7 = 2(-E_{24} - E_{57})$, $e_2e_3 + e_5e_7 = 2(E_{23} + E_{67})$ and $\mathfrak{su}(2) \subset \mathfrak{so}(4,4)$ by $e_1e_2 - e_3e_4 = 2(E_{56}-E_{78})$, $e_1e_3+e_2e_4 = -2(E_{57}+E_{68})$, $e_1e_4-e_2e_3 = 2(E_{58}-E_{67})$. Therefore we can first achieve that $\varphi_2 = x_2\psi_2 + x_5\psi_5 + x_6\psi_6 + x_7\psi_7 + x_8\psi_8$ using the action of $SO(3) \subset G^*_{2(2)}$ and after that $\varphi_2 = x_2\psi_2 + x_5\psi_5$ using $SU(2)$.

Thus, let φ_2 be $x_2\psi_2+x_5\psi_5$. The equations (2) imply that the Lie algebra $\mathfrak{h}(\psi_1, x_2\psi_2 + x_5\psi_5)$ of the isotropy group of $(\psi_1, x_2\psi_2 + x_5\psi_5)$ consists of all $\omega = \sum_{i<j}\omega_{ij}e_ie_j \in \mathfrak{spin}(4,3)$ satisfying

$$\begin{array}{lll}
-\omega_{12} - \omega_{34} + \omega_{56} = 0 & \omega_{13} - \omega_{24} - \omega_{67} = 0 & \omega_{14} + \omega_{23} - \omega_{57} = 0 \\
-\omega_{16} - \omega_{25} + \omega_{37} = 0 & \omega_{15} - \omega_{26} - \omega_{47} = 0 & \omega_{17} + \omega_{36} + \omega_{45} = 0 \\
 & \omega_{27} + \omega_{35} - \omega_{46} = 0 &
\end{array}$$

$$\begin{array}{lcll}
x_2(\omega_{12} + \omega_{34} - \omega_{56}) & + & x_5(-\omega_{16} - \omega_{25} + \omega_{37}) & = 0 \\
 & & x_5(\omega_{15} - \omega_{26} + \omega_{47}) & = 0 \\
x_2(\omega_{14} + \omega_{23} + \omega_{57}) & + & x_5(\omega_{17} + \omega_{36} - \omega_{45}) & = 0 \\
x_2(\omega_{13} - \omega_{24} + \omega_{67}) & + & x_5(-\omega_{27} - \omega_{35} - \omega_{46}) & = 0 \\
x_2(-\omega_{15} + \omega_{26} - \omega_{47}) & & & = 0 \\
x_5(-\omega_{12} + \omega_{34} + \omega_{56}) & + & x_2(\omega_{16} + \omega_{25} + \omega_{37}) & = 0 \\
x_5(\omega_{13} + \omega_{24} - \omega_{67}) & + & x_2(\omega_{27} - \omega_{35} + \omega_{46}) & = 0 \\
x_5(-\omega_{14} + \omega_{23} - \omega_{57}) & + & x_2(-\omega_{17} + \omega_{36} + \omega_{45}) & = 0.
\end{array}$$

By simplifying these equations we obtain

$$\mathfrak{h}(\psi_1, x_2\psi_2 + x_5\psi_5) =$$

$$\begin{array}{ll}
= \{ \sum_{i<j}\omega_{ij}e_ie_j \mid & -\omega_{12} - \omega_{34} + \omega_{56} = 0 \ , \\
 & \omega_{13} - \omega_{24} - \omega_{67} = 0 \ , \ \omega_{14} + \omega_{23} - \omega_{57} = 0 \ , \\
 & -\omega_{16} - \omega_{25} + \omega_{37} = 0 \ , \ \omega_{15} - \omega_{26} - \omega_{47} = 0 \ ,
\end{array} \quad (10)$$

$$\begin{array}{c} \omega_{17}+\omega_{36}+\omega_{45}=0\ ,\ \ \omega_{27}+\omega_{35}-\omega_{46}=0\ , \\ x_5\omega_{47}=0\ ,\ \ x_2\omega_{47}=0\ , \\ x_2\omega_{57}-x_5\omega_{45}=0\ ,\ \ x_2\omega_{67}-x_5\omega_{46}=0\ , \\ x_5\omega_{34}+x_2\omega_{37}=0\ ,\ \ x_5\omega_{24}+x_2\omega_{27}=0\ , \\ x_5\omega_{14}+x_2\omega_{17}=0 \qquad\qquad\qquad \}. \end{array}$$

Since not $x_2 = x_5 = 0$ the dimension of the Lie algebra $\mathfrak{h}(\varphi_1, \varphi_2)$ of $H(\varphi_1, \varphi_2)$ equals 8 and the one of the orbit of (φ_1, φ_2) equals 13. Hence, all orbits are open sets and the action of $Spin^+(4,3)$ is transitive.

Corollary 2.10 *The isotropy group of a pair (φ_1, φ_2) of pseudo-orthonormal spinors with respect to the $Spin^+(4,3)$-action equals*

1. *$SU(1,2)$ if $(\varphi_1, \varphi_2) \in V(-1,-1)$ or $V(1,1)$*
2. *$SL(3,\mathbb{R})$ if $(\varphi_1, \varphi_2) \in V(-1,1)$.*

Proof. The Lie algebra of $H(\psi_1, \psi_2)$ equals

$$\mathfrak{h}(\psi_1,\psi_2) = \{\ \sum_{i<j}\omega_{ij}e_ie_j\ |\ \begin{array}{c} -\omega_{12}-\omega_{34}+\omega_{56}=0\ , \\ \omega_{13}-\omega_{24}=0\ ,\ \ \omega_{14}+\omega_{23}=0\ , \\ \omega_{16}+\omega_{25}=0\ ,\ \ \omega_{15}-\omega_{26}=0\ , \\ \omega_{36}+\omega_{45}=0\ ,\ \ \omega_{35}-\omega_{46}=0\ , \\ \omega_{i7}=0\ ,\ \ i=1,...,6 \qquad \}. \end{array} \tag{11}$$

As a subalgebra of $\mathfrak{so}(4,4)$ it is spanned by $E_{34}-E_{78}$, $E_{56}-E_{78}$, $E_{57}+E_{68}$, $E_{58}-E_{67}$, $A_{37}+A_{48}$, $A_{38}-A_{47}$, $A_{35}+A_{46}$, $A_{36}-A_{45}$ and equals therefore $\mathfrak{su}(1,2)$ where $SU(1,2) \subset SU(2,2) \subset SO(4,4)$ is imbedded in the usual way. We conclude that the connected component of $H(\psi_1, \psi_2)$ must be $SU(1,2)$. On the other hand $V(-1,-1)$ is simply connected. This follows from the exact homotopy sequence of the fibration $SO^+(2,4) \xrightarrow{i} SO^+(4,4) \longrightarrow V(-1,-1)$ since

$$\begin{aligned} ... \longrightarrow \pi_1(SO^+(2,4)) \xrightarrow{i_\#} \pi_1(SO^+(4,4)) \longrightarrow \pi_1(V(-1,-1)) \longrightarrow \\ \longrightarrow \pi_0(SO^+(2,4), I) \longrightarrow ... \end{aligned}$$

is the sequence of the following groups

$$... \longrightarrow \mathbb{Z}\oplus\mathbb{Z}_2 \xrightarrow{i_\#} \mathbb{Z}_2\oplus\mathbb{Z}_2 \longrightarrow \pi_1(V(-1,-1)) \longrightarrow 0 \longrightarrow ...$$

and $i_\#$ sends $(1,1) \in \mathbb{Z} \oplus \mathbb{Z}_2$ to $(1,1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and is therefore surjective. Using now the exact homotopy sequence of $H(\psi_1,\psi_2) \longrightarrow Spin^+(4,3) \longrightarrow V(-1,-1)$

$$... \longrightarrow \pi_1(V(-1,-1)) \longrightarrow \pi_0(H(\psi_1,\psi_2),1) \longrightarrow \pi_0(Spin^+(4,3),1) = 0$$

we deduce from $\pi_1(V(-1,-1)) = 0$ that $H(\psi_1,\psi_2)$ is connected. Thus $H(\psi_1,\psi_2) = SU(1,2)$.

Now we turn to the Lie algebra of $H(\psi_1,\psi_5)$. It is equal to

$$\mathfrak{h}(\psi_1,\psi_5) = \{ \sum_{i<j} \omega_{ij} e_i e_j \mid \begin{array}{c} -\omega_{16} - \omega_{25} + \omega_{37} = 0 \ , \\ \omega_{12} - \omega_{56} = 0 \ , \ \omega_{13} - \omega_{67} = 0 \ , \\ \omega_{23} - \omega_{57} = 0 \ , \ \omega_{15} - \omega_{26} = 0 \ , \\ \omega_{17} + \omega_{36} = 0 \ , \ \omega_{27} + \omega_{35} = 0 \ , \\ \omega_{i4} = 0 \ , \ i = 1,2,3 \\ \omega_{4j} = 0 \ , \ i = 5,6,7 \end{array} \}. \tag{12}$$

Using the isomorphism of $\mathfrak{spin}(4,3)$ and $\mathfrak{so}(4,3)$, we see that the Killing form on $\mathfrak{h}(\psi_1,\psi_5)$ is non-degenerate and has index 3. Therefore, $\mathfrak{h}(\psi_1,\psi_5)$ is a non-compact real form of the semisimple Lie algebra $\mathfrak{h}(\psi_1,\psi_5)^{\mathbb{C}}$. Since, furthermore, $\mathfrak{h}(\psi_1,\psi_5)^{\mathbb{C}}$ has dimension 8 it must be simple and therefore equal to $\mathfrak{sl}(3,\mathbb{C})$. The index of the Killingform distinguishes the various real forms of $\mathfrak{sl}(3,\mathbb{C})$. We conclude that $\mathfrak{h}(\psi_1,\psi_5)$ equals $\mathfrak{sl}(3,\mathbb{R})$. Next we prove that $H(\psi_1,\psi_5)$ is connected and has fundamental group $\mathbb{Z}_2$ what implies immediately $H(\psi_1,\psi_5) = SL(3,\mathbb{R})$ since the center of the universal covering of $SL(3,\mathbb{R})$ equals $\mathbb{Z}_2$. To begin with, we compute the first and second homotopy group of $V(-1,1)$ using the exact homotopy sequence of the fibration $SO^+(3,3) \xrightarrow{i} SO^+(4,4) \longrightarrow V(-1,1)$. This sequence equals

$$\begin{aligned} ... \longrightarrow \pi_2(SO^+(4,4)) \longrightarrow \pi_2(V(-1,1)) \longrightarrow \pi_1(SO^+(3,3)) \xrightarrow{i_\#} \\ \xrightarrow{i_\#} \pi_1(SO^+(4,4)) \longrightarrow \pi_1(V(-1,1)) \longrightarrow \pi_0(SO^+(3,3),I) \longrightarrow ... \end{aligned}$$

and consists of the groups

$$\begin{aligned} ... \longrightarrow 0 \longrightarrow \pi_2(V(-1,1)) \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \xrightarrow{i_\#} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \\ \longrightarrow \pi_1(V(-1,1)) \longrightarrow 0 \longrightarrow ... \end{aligned}$$

Since $i_\#$ is an isomorphism we see that $\pi_2(V(-1,1)) = \pi_1(V(-1,1)) = 0$. A look at the exact homotopy sequence of the fibration $H(\psi_1,\psi_5) \longrightarrow$

$Spin^+(4,3) \longrightarrow V(-1,1)$ now shows that $\pi_1(H(\psi_1,\psi_5)) = \pi_1(Spin^+(4,3))$ $= \mathbb{Z}_2$ and $\pi_0(H(\psi_1,\psi_5),1) = 0$.

Proposition 2.11 *The action of $Spin^+(4,3)$ on the Stiefel manifolds $V(-1,-1,-1), V(-1,-1,1), V(-1,1,1)$ and $V(1,1,1)$ is transitive.*

Proof. As in the proof of Proposition 2.9 it suffices to consider $V(-1,-1,-1)$ and $V(-1,-1,1)$. Again we calculate the Lie algebras of the corresponding isotropy groups. Let φ_1,φ_2 and φ_3 pseudo-orthonormal spinors with $\langle\varphi_1,\varphi_1\rangle_\Delta = -1$ and $\langle\varphi_2,\varphi_2\rangle_\Delta = -1$. Because of Proposition 2.9 we may assume $\varphi_1 = \psi_1$ and $\varphi_2 = \psi_2$. Again the isotropy group of (ψ_1,ψ_2) contains the same subgroup isomorphically to $SU(2)$ as mentioned in the proof of Proposition 2.9 and the group $SO(2) \subset SO(3)$ acting on $span\{\psi_3,\psi_4\}$. Therefore we may set $\varphi_3 = x_3\psi_3 + x_5\psi_5$. Then the isotropy group of $(\varphi_1,\varphi_2,\varphi_3)$ has the Lie algebra

$$\begin{aligned}
&\mathfrak{h}(\psi_1,\psi_2,x_3\psi_3+x_5\psi_5) = \\
&= \{ \sum_{i<j}\omega_{ij}e_ie_j \mid \quad -\omega_{12}-\omega_{34}+\omega_{56} = 0 \ , \\
&\qquad \omega_{13}-\omega_{24} = 0 \ , \ \omega_{14}+\omega_{23} = 0 \ , \\
&\qquad \omega_{16}+\omega_{25} = 0 \ , \ \omega_{15}-\omega_{26} = 0 \ , \\
&\qquad \omega_{36}+\omega_{45} = 0 \ , \ \omega_{35}-\omega_{46} = 0 \ , \\
&\qquad x_4\omega_{56}-x_5\omega_{45} = 0 \ , \ x_5\omega_{34}+x_4\omega_{36} = 0 \ , \\
&\qquad x_5\omega_{24}+x_4\omega_{26} = 0 \ , \ x_5\omega_{14}+x_4\omega_{16} = 0 \\
&\qquad x_5\omega_{46} = 0 \ , \ x_4\omega_{46} = 0 \\
&\qquad \omega_{i7} = 0 \ , \ i = 1,...,6 \qquad \}.
\end{aligned} \tag{13}$$

Since not $x_3 = x_5 = 0$, the dimension of $\mathfrak{h}(\varphi_1,\varphi_2,\varphi_3)$ equals 3 and the action is transitive.

Corollary 2.12 *The isotropy group of a triple $(\varphi_1,\varphi_2,\varphi_3)$ of pseudo-orthonormal spinors with respect to the $Spin^+(4,3)$-action equals*

1. *$SU(2)$ if $(\varphi_1,\varphi_2,\varphi_3) \in V(-1,-1,-1)$ or $V(1,1,1)$*
2. *$SL(2,\mathbb{R})$ if $(\varphi_1,\varphi_2,\varphi_3) \in V(-1,-1,1)$ or $V(-1,1,1)$.*

Proof. The Lie algebra of the isotropy group $H(\psi_1,\psi_2,\psi_3)$ of (ψ_1,ψ_2,ψ_3) equals

$$\begin{aligned}
\mathfrak{h}(\psi_1,\psi_2,\psi_3) = \{ \sum_{i<j}\omega_{ij}e_ie_j \mid \quad & \omega_{12}+\omega_{34} = 0 \ , \\
& \omega_{13}-\omega_{24} = 0 \ , \ \omega_{14}+\omega_{23} = 0 \ , \\
& \omega_{i5} = \omega_{i6} = \omega_{i7} = 0 \qquad \}.
\end{aligned} \tag{14}$$

As a subalgebra of $\mathfrak{so}(4,4)$ it is spanned by $E_{56} - E_{78}$, $E_{57} + E_{68}$ and $E_{58} - E_{67}$ and equals therefore $\mathfrak{su}(2)$ where $SU(2) \subset SU(2,2) \subset SO(4,4)$ is imbedded in the usual way. In particular, the connected component of the unity of $H(\psi_1, \psi_2, \psi_3)$ is isomorphic to $SU(2)$. It remains to prove that $H(\psi_1, \psi_2, \psi_3)$ is connected. A look at the exact homotopy sequence of the fibration $H(\psi_1, \psi_2, \psi_3) \longrightarrow Spin^+(4,3) \longrightarrow V(-1,-1,-1)$ shows that

$$\begin{aligned} ... \longrightarrow \pi_1(H(\psi_1, \psi_2, \psi_3), 1) \longrightarrow \mathbb{Z}_2 \longrightarrow \pi_1(V(-1,-1,-1)) \longrightarrow \\ \longrightarrow \pi_0(H(\psi_1, \psi_2, \psi_3)) \longrightarrow 0 \longrightarrow ... \end{aligned}$$

is exact. Since $\pi_1((H(\psi_1, \psi_2, \psi_3), 1) = \pi_1(SU(2)) = 0$ it suffices to prove that $\pi_1(V(-1,-1,-1))$ equals $\mathbb{Z}_2$. But this is clear from the homotopy sequence of $SO^+(1,4) \xrightarrow{i} SO^+(4,4) \longrightarrow V(-1,-1,-1)$. Indeed,

$$\begin{aligned} ... \longrightarrow \pi_1(SO^+(1,4)) \xrightarrow{i_\#} \pi_1(SO^+(4,4)) \longrightarrow \pi_1(V(-1,-1,-1)) \longrightarrow \\ \longrightarrow \pi_0(SO^+(1,4), I) \longrightarrow ... \end{aligned}$$

is the sequence of the following groups

$$... \longrightarrow \mathbb{Z}_2 \xrightarrow{i_\#} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \pi_1(V(-1,-1,-1)) \longrightarrow 0 \longrightarrow ...$$

where $i_\#$ sends $1 \in \mathbb{Z}_2$ to $(0,1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

We now prove the second assertion in the same way. The Lie algebra of the isotropy group $H(\psi_1, \psi_5, \psi_6)$ is

$$\mathfrak{h}(\psi_1, \psi_5, \psi_6) = \{ \sum_{i<j} \omega_{ij} e_i e_j \mid \begin{array}{c} \omega_{12} - \omega_{56} = 0 \ , \\ \omega_{16} + \omega_{25} = 0 \ , \ \omega_{15} - \omega_{26} = 0 \ , \\ \omega_{i3} = \omega_{i4} = \omega_{i7} = 0 \end{array} \}. \tag{15}$$

As a subalgebra of $\mathfrak{so}(4,4)$ it is spanned by $E_{34} - E_{78}$, $A_{37} + A_{48}$ and $A_{38} - A_{47}$ and equals therefore $\mathfrak{su}(1,1)$ where $SU(1,1) \subset SU(2,2) \subset SO(4,4)$ is imbedded in the usual way. In particular, the connected component of $H(\psi_1, \psi_2, \psi_5)$ is isomorphic to $SU(2)$ which is on the other hand isomorphic to $SL(2;\mathbb{R})$. To show that $H(\psi_1, \psi_5, \psi_6)$ is connected it suffices to verify that the Stiefel manifold is simply connected. But this follows again from the exact homotopy sequence of the corresponding fibration $SO^+(3,2) \xrightarrow{i} SO^+(4,4) \longrightarrow V(-1,1,1)$. Indeed,

$$\begin{aligned} ... \longrightarrow \pi_1(SO^+(3,2)) \xrightarrow{i_\#} \pi_1(SO^+(4,4)) \longrightarrow \pi_1(V(-1,1,1)) \longrightarrow \\ \longrightarrow \pi_0(SO^+(3,2), I) \longrightarrow ... \end{aligned}$$

is the sequence of the following groups

$$... \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z} \xrightarrow{i_\#} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \pi_1(V(-1,1,1)) \longrightarrow 0 \longrightarrow ...$$

where $i_\#$ sends $(1,1) \in \mathbb{Z}_2 \oplus \mathbb{Z}$ to $(1,1) \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and is therefore surjective.

The rest of this section is devoted to real representations of $G^*_{2(2)}$. Recall that all complex representations of $\mathfrak{g}_{2(2)}$ are of real type [8]. Therefore, the real irreducible representations of the universal covering $\widetilde{G_{2(2)}}$ of $G^*_{2(2)}$ correspond to the real forms of the complex irreducible representations of $\mathfrak{g}_{2(2)}$. On the other hand the fundamental representations of $\widetilde{G_{2(2)}}$, i.e. the standard representation on $\mathbb{R}^7$ and the adjoint representation are in fact representations of $G^*_{2(2)}$. Thus all representations of $\widetilde{G_{2(2)}}$ are representations of $G^*_{2(2)}$. We conclude that the real irreducible representations of $G^*_{2(2)}$ correspond exactly to the complex irreducible representations of $\mathfrak{g}_{2(2)}$. In particular, the dimensions of the irreducible real representations are 1, 7, 14, 27, ... Furthermore the decomposition of $\Lambda^p(R^7)$ into irreducible components of the $G^*_{2(2)}$-action is similar to that with respect to the action of the compact group G_2. Denote by $*$ the Hodgeoperator of the pseudo-Euclidean space $(\mathbb{R}^7, g_{4,3})$ and let ω^3_0 be the 3-form defined by (8). Then we have

Proposition 2.13 *1. $R^7 = \Lambda^1(R^7) =: \Lambda^1_7$ is irreducible.*

2. $\Lambda^2(R^7) = \Lambda^2_7 \oplus \Lambda^2_{14}$, where

$$\begin{aligned} \Lambda^2_7 &= \{\alpha^2 \in \Lambda^2 \mid *(\omega^3_0 \wedge \alpha^2) = 2\alpha^2\} = \{X \lrcorner \omega^3_0 \mid X \in R^7\} \\ \Lambda^2_{14} &= \{\alpha^2 \in \Lambda^2 \mid *(\omega^3_0 \wedge \alpha^2) = -\alpha^2\} = \mathfrak{g}_{2(2)} \end{aligned}$$

3. $\Lambda^3(R^7) = \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$, where

$$\begin{aligned} \Lambda^3_1 &= \{t\omega^3_0 \mid t \in R^1\} \\ \Lambda^3_7 &= \{*(\omega^3_0 \wedge \alpha^1) \mid \alpha^1 \in \Lambda^1_7\} \\ \Lambda^3_{27} &= \{\alpha^3 \in \Lambda^3 \mid \alpha^3 \wedge \omega^3_0 = 0, \alpha^3 \wedge *\omega^3_0 = 0\} \end{aligned}$$

3 Killing spinors

Now let $(M^{4,3}, g_{4,3})$ be a 7-dimensional pseudo-Riemannian spin manifold of signature (4,3) which is space and time oriented. Assume that $M^{4,3}$ admits a spin structure $Q(M^{4,3})$. This is a $Spin^+(4,3)$-reduction of the bundle

$R(M^{4,3})$ of all space and time oriented pseudo-orthonormal frames. Then the spinor bundle S of $M^{4,3}$ is the associated bundle $Q(M^{4,3}) \times_{Spin^+(4,3)} \Delta_{4,3}$. Furthermore ∇ denotes the Levi-Civita-connection on the tangent bundle $TM^{4,3}$ as well as the induced covariant derivative on S. The pseudo-Euclidean product $\langle\ ,\ \rangle_\Delta$ on $\Delta_{4,3}$ induces a product of signature (4,4) on S.

Definition 3.1 *A section $\psi \in \Gamma(S)$ is called Killing spinor if there is a real number $\lambda \neq 0$ such that the differential equation*

$$\nabla_X \psi = \lambda X \cdot \psi$$

is satisfied for all vector fields $X \in \mathfrak{X}(M^{4,3})$. We call λ the Killing number of ψ.

The following properties of Killing spinors are well known [1]. Let $\psi \in \Gamma(S)$ be a Killing spinor on $M^{4,3}$ with Killing number λ. Then $\langle\psi, \psi\rangle_\Delta$ is constant on $M^{4,3}$. For the Ricci map $Ric : TM^{4,3} \longrightarrow TM^{4,3}$ of the tangent bundle the equation $Ric(X)\psi = 24\lambda^2 X \cdot \psi$ holds. If ψ is non-isotropic this means that $M^{4,3}$ is an Einstein manifold of scalar curvature $\tau = 168\lambda^2$. Now let W be the Weyl tensor of $M^{4,3}$. Then $W(X,Y) \cdot \psi = 0$ for all $X, Y \in \mathfrak{X}(M^{4,3})$, where this product is defined in the following way. Let $s_1, s_2, ..., s_7$ be a local pseudo-orthonormal frame, $\varepsilon_i = g(s_i, s_i)$ and $W_{ijkl} = W(s_i, s_j, s_k, s_l)$ Then set

$$W(s_i, s_j) \cdot \psi = \sum_{k<l} \varepsilon_k \varepsilon_l W_{ijkl} s_k \cdot s_l \cdot \psi \quad .$$

Of course, parallel spinors have the same properties. We now turn to the question how many Killing spinors can exist on $(M^{4,3}, g_{4,3})$.

Theorem 3.2 *If there exist four orthogonal non-isotropic Killing spinors with the same Killing number on $(M^{4,3}, g_{4,3})$ such that at least three of them have the same length then $M^{4,3}$ is conformally flat.*

Proof. Let $\varphi_1, ..., \varphi_4$ be four such Killing spinors. Let $\langle\varphi_\alpha, \varphi_\alpha\rangle_\Delta = -1$ for $\alpha = 1, 2, 3$. Because of the transitivity of the $Spin^+(4,3)$-action on $V(-1,-1,-1)$ we may assume that for some local time and space oriented pseudo-orthonormal frame $s_1, ..., s_7$ the spinor φ_α equals ψ_α for $\alpha = 1, 2, 3$. Moreover, since the isotropy group of (ψ_1, ψ_2, ψ_3) equals $SU(2)$ acting on $span\{\psi_5, \psi_6, \psi_7, \psi_8\}$ we can assume $\varphi_4 = x_4\psi_4 + x_5\psi_5$ where x_4 and x_5 are real functions. The condition $W(s_i, s_j) \cdot \varphi_\alpha = 0\,(\alpha = 1, 2, 3)$ implies

$$W_{ij12} + W_{ij34} = 0\,,\ W_{ij13} - W_{ij24} = 0\,,\ W_{ij14} + W_{ij23} = 0$$

and $W_{ijkl} = 0$ for any other k, l. Furthermore, we have

$$\begin{aligned} 0 &= W(s_i, s_j) \cdot (x_4\psi_4 + x_5\psi_5) = \sum_{k<l} \varepsilon_k\varepsilon_l W_{ijkl} s_k \cdot s_l \cdot (x_4\psi_4 + x_5\psi_5) \\ &= x_5\{(-W_{ij12} + W_{ij34})\psi_6 + (W_{ij13} + W_{ij24})\psi_7 + (W_{ij14} + W_{ij23})\psi_8\}. \end{aligned}$$

Consequently, in case $x_5 \neq 0$ the Weyl tensor must vanish and we are done. Consider now the case $x_5(m) = 0$ for $m \in M^{4,3}$. If there is any sequence $m_n \in M^{4,3}$ which converges to m and such that $x_5(m_n) \neq 0$ then by continuity of the Weyl tensor we have again $W(m) = 0$. Assume now that $x_5(m) = 0$ on an open set containing m., i.e. $\varphi_4 = \psi_4$. Since $\varphi_1, ..., \varphi_4$ are Killing spinors we have $\nabla_{s_1}\psi_\alpha = \lambda s_1 \cdot \psi_\alpha$ $(\alpha = 1, ..., 4)$. We can calculate the covariant derivative using the local connection forms $\omega_{ij} = \varepsilon_i\varepsilon_j g_{4,3}(\nabla s_i, s_j)$ and obtain

$$\nabla_{s_1}\psi_\alpha = \frac{1}{2}\sum_{i<j} \varepsilon_i\varepsilon_j\omega_{ij}(s_1) s_i \cdot s_j \cdot \psi_\alpha = \lambda s_1 \cdot \psi_\alpha \ \ (\alpha = 1, ..., 4)$$

In particular,

$$\begin{aligned} -\omega_{27}(s_1) - \omega_{35}(s_1) + \omega_{46}(s_1) &= 2\lambda \\ -\omega_{27}(s_1) + \omega_{35}(s_1) - \omega_{46}(s_1) &= 2\lambda \\ -\omega_{27}(s_1) + \omega_{35}(s_1) + \omega_{46}(s_1) &= -2\lambda \\ -\omega_{27}(s_1) - \omega_{35}(s_1) - \omega_{46}(s_1) &= -2\lambda \end{aligned}$$

which is impossible if $\lambda \neq 0$. The assertion can be proved similarly if $\langle\varphi_\alpha, \varphi_\alpha\rangle_\Delta = 1$ for $\alpha = 1, 2, 3$.

3.1 Geometrical and nearly parallel $G^*_{2(2)}$-structures

Let M^7 be a 7-dimensional manifold and $R(M^7)$ the frame bundle of M^7. We define the bundle $\Lambda^3_*(M^7)$ by

$$\Lambda^3_*(M^7) := R(M^7) \times_{GL(7)} \Lambda^3_*(R^7) \subset R(M^7) \times_{GL(7)} \Lambda^3(R^7) = \Lambda^3(M^7).$$

where $\Lambda^3_*(R^7)$ is the open subset $\{A^*\omega^3_0 \mid A \in GL(7)\}$ of $\Lambda^3(R^7)$.

Definition 3.3 *A topological G_2-structure ($Spin^+(4,3)$-structure) on M^7 is a G_2- reduction ($Spin^+(4,3)$-reduction) of the frame bundle $R(M^7)$.*

The fact that $G^*_{2(2)}$ is a subset of $SO^+(4,3)$ and of $Spin^+(4,3)$ implies that a $G^*_{2(2)}$- structure $P \subset R(M^7)$ on M^7 induces an orientation of M^7 (i.e. $\omega_1 = 0$) , a pseudo-Riemannian metric $g_{4,3}$ of index 4 on M^7 together with a space and time orientation such that the corresponding $SO^+(4,3)$-bundle equals $P \times_{G^*_{2(2)}} SO^+(4,3)$ and a spin structure $P \times_{G_2} Spin^+(4,3)$. Furthermore it defines the following spinor $\psi \in \Gamma(S)$ of length -1 in the real spinor bundle $S = P \times_{G^*_{2(2)}} \Delta_{4,3}$ of M^7. Since $G^*_{2(2)} \subset Spin^+(4,3)$ is the isotropy group of $\psi_1 \in \Delta_{4,3}$ the map $\psi : P \longrightarrow \Delta_{4,3};\ \psi(p) = \psi_1$ has the property $\psi(pg) = g^{-1}\psi$ for all $g \in G^*_{2(2)}$ and is therefore a section in S. Because of the $G^*_{2(2)}$ - invariance of ω_0 the $G^*_{2(2)}$-structure defines in the same way a section ω^3 in $\Lambda^3_*(M^7) = R(M^7) \times_{GL(7)} \Lambda^3_*(R^7) = P_{G_2} \times_{G^*_{2(2)}} \Lambda^3_*(R^7)$ by $\omega^3 : P \longrightarrow \Lambda^3_*(R^7);\ \omega^3(p) = \omega^3_0$. On the other hand the spinor ψ defines a (2,1)-tensor field $A = A_\psi$ (see equation (6)) on M^7 and $\omega^3 = g_{4,3}(\cdot, A(\cdot,\cdot))$ holds.

Vice versa, suppose we are given a 3-form ω^3 in $\Lambda^3_*(M^7)$ then M^7 admits a $G^*_{2(2)}$- structure P consisting of all frames relative to those ω^3 equals ω^3_0. Secondly, given a pseudo-Riemannian metric $g_{4,3}$, a space and time orientation, a $Spin^+(4,3)$-structure and a spinor ψ of length -1 on M^7 then M^7 admits a $G^*_{2(2)}$-structure P consisting of all frames relative to those ψ equals ψ_0.

Now we turn to geometrical $G^*_{2(2)}$ - stuctures.

Definition 3.4 *Let $P \subset R(M^7)$ be a topological $G^*_{2(2)}$-structure on M^7 and $g_{4,3}$ the associated Riemannian metric with Hodge operator $*$. P is said to be geometrical if one of the following equivalent conditions is satisfied.*

(i) ∇ *reduces to P.*

(ii) The holonomy group Hol(M^7, g) of M^7 is contained in $G^*_{2(2)}$.

(iii) The associated 3-form ω^3 is parallel, i.e. $\nabla\omega^3 = 0$.

(iv) $d\omega^3 = 0, \quad d * \omega^3 = 0.$

(v) The associated spinor field ψ is parallel, i.e. $\nabla\psi = 0$.

For a proof of $(iii) \Longleftrightarrow (iv)$ see [3], [5], [6].

Now we can generalize the condition $\nabla\psi = 0$ and obtain the notion of a nearly parallel $G^*_{2(2)}$-structure.

Definition 3.5 *Let $P \subset R(M^7)$ be a topological $G^*_{2(2)}$-structure on M^7 and $g_{4,3}$ the associated Riemannian metric with Hodge operator $*$. P is said to be nearly parallel if one of the following equivalent conditions is satisfied.*

(i) The associated spinor ψ is a Killing spinor with Killing number λ.

(ii) The associated tensor A satisfies

$$(\nabla_Z A)(Y, X) = 2\lambda\{g_{4,3}(Y, Z)X - g_{4,3}(X, Z)Y + A(Z, A(Y, X))\}.$$

(iii) The associated 3-form ω^3 satisfies

$$\nabla_Z \omega^3 = -2\lambda(Z \lrcorner * \omega^3).$$

(iv) The associated 3-form ω^3 satisfies

$$d * \omega^3 = 0, \quad d\omega^3 = -8\lambda * \omega^3.$$

For a proof of $(iii) \Longleftrightarrow (iv)$ see [4].

3.2 Examples of homogeneous spaces with Killing spinors

This section is devoted to the construction of first examples of pseudo-Riemannian manifolds of signature (4,3) with Killing spinors. We use the twistor spaces $Z^-(\mathbb{R}P^{4,0})$ and $Z^-(\mathbb{C}P^{2,0})$ of the negativ definit elliptic spaces $\mathbb{R}P^{4,0}$ and $\mathbb{C}P^{2,0}$. One obtains $\mathbb{R}P^{4,0}$ and $\mathbb{C}P^{2,0}$ from the real and the complex hyperbolic space on replacing the metric by its negative. $Z^-(\mathbb{R}P^{4,0})$ is the homogeneous space $Sp(1,1)/U(1) \times Sp(1) = SO^+(4,1)/U(2)$ and $Z^-(\mathbb{C}P^{2,0})$ equals $U(2,1)/U(1) \times U(1) \times U(1)$. There exist two homogeneous Einstein metrics of signature $(4,2)$ on each of these spaces one of both being Kählerian. These Kählerian metrics are used to construct S^1-bundles over $Z^-(\mathbb{R}P^{4,0})$ and $Z^-(\mathbb{C}P^{2,0})$ admitting three linearly independent Killing spinors. One can obtain a further Einstein metric with one Killing spinor on each of these bundles by squashing the metric on the S^3-fibres over $\mathbb{R}P^{4,0}$ and $\mathbb{C}P^{2,0}$, respectively. The other examples are warped products with six-dimensional pseudo-Riemannian manifolds of signature (4,2) with Killing spinors, in particular with the second Einstein metric on the twistor spaces.

3.2.1 The squashed 7-sphere

Consider the sphere $S^{4,3} = Sp(1,1)/Sp(1)$ where

$$Sp(1,1) = \{A \in \mathbb{H}(2) \mid \ {}^tA \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \bar{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\}$$

and the imbedding $Sp(1) \hookrightarrow Sp(1,1)$ is given by

$$Sp(1) \ni C \longmapsto \begin{pmatrix} 1 & 0 \\ 0 & C \end{pmatrix} \in Sp(1,1) \ .$$

Let B be the $Ad(Sp(1))$-invariant bilinear form on $\mathfrak{sp}(1,1)$ defined by $B(X,Y) = -\mathrm{Re}\ \mathrm{tr}XY$ for $X, Y \in \mathfrak{sp}(1,1)$. We decompose

$$\mathfrak{sp}(1,1) = \mathfrak{sp}(1) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$$

where

$$\mathfrak{m}_1 = \mathrm{span} \quad \{ \quad \bar{e}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \bar{e}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \bar{e}_3 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \ \bar{e}_4 = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} \ \}$$

$$\mathfrak{m}_2 = \mathrm{span} \quad \{ \quad \bar{e}_5 = \begin{pmatrix} j & 0 \\ 0 & 0 \end{pmatrix}, \ \bar{e}_6 = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}, \ \bar{e}_7 = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} \ \} \ .$$

Note that $\mathfrak{m}_1 = \mathbb{H}, \mathfrak{m}_2 = \mathrm{im}\mathbb{H}$. We have $[\mathfrak{sp}(1), \mathfrak{m}_1] = \mathfrak{m}_1$ and $[\mathfrak{sp}(1), \mathfrak{m}_2] = 0$. Now we rescale B in direction $\mathfrak{m}_2$ and get a bilinear form

$$B_t = B\mid_{\mathfrak{m}_1 \times \mathfrak{m}_1} \oplus 2tB\mid_{\mathfrak{m}_2 \times \mathfrak{m}_2}$$

on $\mathfrak{m}_1 \oplus \mathfrak{m}_2$. Then

$$e_i = \frac{1}{\sqrt{2}}\bar{e}_i (i = 1, ..., 4), \quad e_j = \frac{1}{\sqrt{2t}}\bar{e}_j (j = 5, 6, 7)$$

is a pseudo-orthonormal basis of $\mathfrak{m}_1 \oplus \mathfrak{m}_2$. Identifying $\mathfrak{m}_1 \oplus \mathfrak{m}_2$ with the tangent space of $Sp(1,1)/Sp(1)$ at $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot Sp(1,1)$ we obtain a left invariant metric g_t on $S^{4,3}$. The isotropy representation $\alpha : Sp(1) \longrightarrow SO(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$

assigns to an element $q \in Sp(1)$ that linear map which is the multiplication by q on $\mathfrak{m}_1$ and the identity on $\mathfrak{m}_2$. Therefore α is the standard imbedding

$$\alpha : Sp(1) \hookrightarrow SO(4) \subset SO(4,3) \ .$$

Since $Spin(4) = Sp(1) \times Sp(1)$,

$$\begin{array}{rcl} \tilde{\alpha} : Sp(1) & \hookrightarrow & Sp(1) \times Sp(1) \subset Spin^+(4,3) \\ q & \longmapsto & (q,1) \end{array}$$

defines a lift of α. From equation (4) we conclude that $\tilde{\alpha}(q)$ acts trivially on $\mathrm{span}\{\psi_5, \psi_6, \psi_7, \psi_8\}$ for any $q \in Sp(1)$. $\tilde{\alpha}$ defines a spin structure

$$Q = Sp(2,1) \times_{\tilde{\alpha}} Spin(4,3)$$

and the associated spinor bundle

$$S = Sp(2,1) \times_{\tilde{\alpha}} \Delta_{4,3} \ .$$

Sections of S are identified with functions $\psi : Sp(2,1) \longrightarrow \Delta_{4,3}$ satisfying the equation $\psi(gh) = h^{-1}\psi(g)$ for all $g \in Sp(2,1)$ and $h \in Sp(1)$. In particular constant functions $\psi(g) = \psi_0 \in \mathrm{span}\{\psi_5, \psi_6, \psi_7, \psi_8\}\,, g \in Sp(2,1)$ define spinors on $S^{4,3}$.

Using Wang's theorem we describe the connection on S induced by the Levi-Civita connection of $(S^{4,3}, g_t)$. It is given by

$$\begin{array}{rcl} \tilde{\Lambda} : \ \mathfrak{m}_1 \oplus \mathfrak{m}_2 & \longmapsto & \mathfrak{spin}(4,3) \\ \tilde{\Lambda}(e_1) & = & \dfrac{\sqrt{2t}}{4}(-e_2e_7 - e_3e_5 - e_4e_6) \\ \tilde{\Lambda}(e_2) & = & \dfrac{\sqrt{2t}}{4}(e_1e_7 + e_3e_6 - e_4e_5) \\ \tilde{\Lambda}(e_3) & = & \dfrac{\sqrt{2t}}{4}(e_1e_5 - e_2e_6 + e_4e_7) \\ \tilde{\Lambda}(e_4) & = & \dfrac{\sqrt{2t}}{4}(e_1e_6 + e_2e_5 - e_3e_7) \\ \tilde{\Lambda}(e_5) & = & \dfrac{t-1}{2\sqrt{2t}}(-e_1e_3 - e_2e_4) + \dfrac{1}{2\sqrt{2t}}e_6e_7 \\ \tilde{\Lambda}(e_6) & = & \dfrac{t-1}{2\sqrt{2t}}(-e_1e_4 + e_2e_3) - \dfrac{1}{2\sqrt{2t}}e_5e_7 \\ \tilde{\Lambda}(e_7) & = & \dfrac{t-1}{2\sqrt{2t}}(-e_1e_2 + e_3e_4) + \dfrac{1}{2\sqrt{2t}}e_5e_6 \end{array}$$

We have now to check for which choice of t and λ do exist common solutions $\psi_0 \in \mathrm{span}\{\psi_5, \psi_6, \psi_7, \psi_8\}$ of

$$\tilde{\Lambda}(e_i)\psi_0 - \lambda e_i \psi_0 = 0, \quad i = 1, ..., 7 .$$

There are exactly two possibilities. In case $t = 1$, $\lambda = -\frac{1}{2\sqrt{2}}$ we obtain three linear independent Killing spinors (ψ_6, ψ_7, ψ_8) on the standard sphere. In case $t = \frac{1}{5}$, $\lambda = \frac{3}{2\sqrt{10}}$ we get a further Einstein metric on $S^{4,3}$ together with one Killing spinor (ψ_5).

3.2.2 The space $\tilde{N}(1,1)$

Consider now the homogeneous space $\tilde{N}(1,1) = SU(2,1)/S^1$ where the imbedding of S^1 into $SU(2,1)$ is given by

$$\begin{array}{ccc} S^1 & \hookrightarrow & SU(2,1) \\ \theta & \longmapsto & \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta} & 0 \\ 0 & 0 & e^{-2i\theta} \end{pmatrix} \end{array}$$

Denote by $\mathfrak{k}$ the Lie algebra of S^1 contained in $\mathfrak{su}(2,1)$. Let B be the $Ad(S^1)$-invariant bilinear form on $\mathfrak{su}(2,1)$ defined by $B(X,Y) = -\mathrm{Re}\ \mathrm{tr} XY$ for $X, Y \in \mathfrak{su}(2,1)$. Now we decompose $\mathfrak{su}(2,1)$ into

$$\mathfrak{su}(2,1) = \mathfrak{k} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$$

where

$$\mathfrak{m}_1 = \mathrm{span} \quad \{ \quad \bar{e}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ \bar{e}_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$
$$\bar{e}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \bar{e}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \}$$

$$\mathfrak{m}_2 = \mathrm{span} \quad \{ \quad \bar{e}_5 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \bar{e}_6 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\bar{e}_7 = \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \} .$$

We have $[\mathfrak{k}, \mathfrak{m}_1] = \mathfrak{m}_1$ and $[\mathfrak{k}, \mathfrak{m}_2] = 0$. Now we continue as in the case of the sphere. We rescale B and obtain

$$B_t = B\,|_{\mathfrak{m}_1 \times \mathfrak{m}_1} \oplus 2tB\,|_{\mathfrak{m}_2 \times \mathfrak{m}_2}$$

on $\mathfrak{m}_1 \oplus \mathfrak{m}_2$, which yields a left invariant metric g_t on $\tilde{N}(1,1)$. Then

$$e_i = \frac{1}{\sqrt{2}}\bar{e}_i (i = 1, ..., 4), \quad e_j = \frac{1}{2\sqrt{t}}\bar{e}_j (j = 5, 6, 7)$$

is a pseudo-orthonormal basis of $\mathfrak{m}_1 \oplus \mathfrak{m}_2$. With respect to this basis the isotropy representation $\alpha : S^1 \longrightarrow SO(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$ is given by

$$\alpha(\theta) = \begin{pmatrix} \cos 3\theta & \sin 3\theta & 0 & 0 \\ -\sin 3\theta & \cos 3\theta & 0 & 0 \\ 0 & 0 & \cos 3\theta & \sin 3\theta \\ 0 & 0 & -\sin 3\theta & \cos 3\theta \end{pmatrix} \oplus id\,|_{\mathfrak{m}_2} \quad .$$

We can lift α into $Spin^+(4,3)$ and obtain

$$\begin{array}{rcl} \tilde{\alpha} : S^1 & \hookrightarrow & Spin^+(4,3) \\ \theta & \longmapsto & (\cos\frac{3}{2}\theta + \sin\frac{3}{2}\theta\, e_1 e_2)(\cos\frac{3}{2}\theta + \sin\frac{3}{2}\theta\, e_3 e_4) \quad . \end{array}$$

Therefore $\tilde{N}(1,1)$ admits a spin structure. In particular constant functions $\psi(g) = \psi_0 \in \mathrm{span}\{\psi_5, \psi_6, \psi_7, \psi_8\}$, $g \in SU(2,1)$ define sections in the spinor bundle S of $\tilde{N}(1,1)$. The connection on S with respect to g_t is given by

$$\begin{array}{rcl} \tilde{\Lambda} : \mathfrak{m}_1 \oplus \mathfrak{m}_2 & \longmapsto & \mathfrak{spin}(4,3) \\ \tilde{\Lambda}(e_1) & = & \frac{\sqrt{t}}{4}(-e_2e_7 + e_3e_5 - e_4e_6) \\ \tilde{\Lambda}(e_2) & = & \frac{\sqrt{t}}{4}(e_1e_7 + e_3e_6 + e_4e_5) \\ \tilde{\Lambda}(e_3) & = & \frac{\sqrt{t}}{4}(-e_1e_5 - e_2e_6 + e_4e_7) \\ \tilde{\Lambda}(e_4) & = & \frac{\sqrt{t}}{4}(e_1e_6 - e_2e_5 - e_3e_7) \\ \tilde{\Lambda}(e_5) & = & \frac{t-1}{4\sqrt{t}}(e_1e_3 + e_2e_4) - \frac{1}{4\sqrt{t}}e_6e_7 \\ \tilde{\Lambda}(e_6) & = & \frac{t-1}{4\sqrt{t}}(-e_1e_4 + e_2e_3) + \frac{1}{4\sqrt{t}}e_5e_7 \\ \tilde{\Lambda}(e_7) & = & \frac{t-1}{4\sqrt{t}}(-e_1e_2 + e_3e_4) - \frac{1}{4\sqrt{t}}e_5e_6 \quad . \end{array}$$

Again we have to check whether there are common solutions $\psi_0 \in \mathrm{span}\{\psi_5, \psi_6, \psi_7, \psi_8\}$ of

$$\tilde{\Lambda}(e_i)\psi_0 - \lambda e_i \psi_0 = 0, \quad i = 1, ..., 7 .$$

Again there are two possible choices of t . In case $t = 1$ we obtain the standard metric together with three linear independent Killing spinors (ψ_5, ψ_6, ψ_8) with Killing number $\lambda = \frac{1}{4}$. For $t = \frac{1}{5}$ we obtain a further Einstein metric on $\tilde{N}(1,1)$ together with one Killing spinor (ψ_7) with Killing number $\lambda = -\frac{3}{4\sqrt{5}}$.

3.3 Warped products with Killing spinors

3.3.1 pseudo-Riemannian manifolds of signature (4,2)

Consider first $\mathbb{R}^6 = span\{e_1, ..., e_6\} \subset \mathbb{R}^7$ with pseudo-Euclidean product $g_{4,2} = g_{4,3}|_{\mathbb{R}^7}$. We may restrict the real $Spin(4,3)$-representation to $Spin(4,2)$ and obtain the unique irreducible real representation $\Delta_{4,2}$ of $Spin(4,2)$. The connected component $Spin^+(4,2)$ of $1 \in Spin(4,2)$ acts transitively on $S^{4,3}$ and $H^{3,4}$. Actually, the proof of Proposition 2.1 remains valid.

The multiplication of spinors by the volume form of $(\mathbb{R}^6, g_{4,2})$ yields a complex structure on $\Delta_{4,2}$. In fact, let $X_1, ..., X_6$ be a positively oriented pseudo-orthonormal basis of $(\mathbb{R}^6, g_{4,2})$. Then we define J^Δ by $J^\Delta(\psi) = X_1 \cdot ... \cdot X_6 \psi$. J^Δ does not depend on the choice of the pseudo-orthonormal basis. We have $J^\Delta = -I \otimes I \otimes \varepsilon$ with respect to the standard basis $\psi_1, ..., \psi_8$. Furthermore J^Δ has the following properties.

1. $(J^\Delta)^2 = -1$
2. $X \cdot J^\Delta(\psi) = -J^\Delta(X \cdot \psi)$ for any $X \in \mathbb{R}^6$
3. Besides $\langle X \cdot \psi, \psi \rangle_\Delta = 0$ we also have $\langle X \cdot \psi, J^\Delta(\psi) \rangle_\Delta = 0$.

Therefore the map

$$\begin{array}{ccc} \mathbb{R}^6 & \longrightarrow & \{\psi \, , J^\Delta(\psi)\}^\perp \subset \mathbb{R}^8 \\ X & \longmapsto & X \cdot \psi \end{array}$$

is an isomorphism for any spinor $\psi \in \Delta_{4,2}$ with $\langle \psi, \psi \rangle_\Delta \neq 0$. In particular, we obtain a complex structure J_ψ of $\mathbb{R}^6$ defined by

$$J_\psi(X) \cdot \psi := J^\Delta(X \cdot \psi) \text{ for any } X \in \mathbb{R}^6.$$

For instance,

$$J_{\psi_1} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

with respect to $e_1, ..., e_6$.

Now let $(F^{4,2}, h)$ be a pseudo-Riemannian manifold of signature (4,2). J^{Δ} defines a complexe structure J^S on the spinorbundle S^F of $F^{4,2}$. We have $\nabla J^S = 0$. Furthermore any nowhere vanishing nor isotropic section $\psi \in \Gamma(S^F)$ defines a complex structure J_ψ on $F^{4,2}$. Assume now that $F^{4,2}$ admits a Killing spinor $\varphi \neq 0$ with Killing number λ. Then $J^S(\varphi)$ is a Killingspinor with Killingnumber $-\lambda$, since

$$\nabla_X(J^S(\varphi)) = J^S(\nabla_X\varphi) = J^S(\lambda X \cdot \varphi) = -\lambda X \cdot J^S(\varphi).$$

If φ is non-isotropic then we can define J_φ which is nearly-Kaehlerian.

Next we discuss two examples of such manifolds with Killing spinors.

3.3.2 $U(2,1)/U(1) \times U(1) \times U(1)$

Consider the homogeneous space $U(2,1)/U(1) \times U(1) \times U(1)$ where the imbedding of $U(1) \times U(1) \times U(1)$ into $U(2,1)$ is given by

$$\begin{array}{rcl} U(1) \times U(1) \times U(1) & \hookrightarrow & U(2,1) \\ (e^{it}, e^{is}, e^{ir}) & \longmapsto & diag(e^{it}, e^{is}, e^{ir}) \end{array}$$

Denote by $\mathfrak{k}$ the Lie algebra of $U(1) \times U(1) \times U(1)$ contained in $\mathfrak{u}(2,1)$. Let B be the $Ad(U(1) \times U(1) \times U(1))$-invariant bilinear form on $\mathfrak{u}(2,1)$ defined by $B(X,Y) = -\mathrm{Re}\ \mathrm{tr}XY$ for $X, Y \in \mathfrak{su}(2,1)$. We decompose $\mathfrak{u}(2,1)$ into

$$\mathfrak{u}(2,1) = \mathfrak{k} \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$$

where

$$\mathfrak{m}_1 = \mathrm{span} \quad \{ \quad \bar{e}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; \bar{e}_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix},$$

$$\bar{e}_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; \bar{e}_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \}$$

$$\mathfrak{m}_2 = \mathrm{span} \quad \{ \quad \bar{e}_5 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \bar{e}_6 = \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \} \; .$$

We rescale B and obtain

$$B_t = B \mid_{\mathfrak{m}_1 \times \mathfrak{m}_1} \oplus 2tB \mid_{\mathfrak{m}_2 \times \mathfrak{m}_2}$$

on $\mathfrak{m}_1 \oplus \mathfrak{m}_2$, which yields a left invariant metric g_t on $U(2,1)/U(1) \times U(1) \times U(1)$. Then

$$e_i = \frac{1}{\sqrt{2}} \bar{e}_i (i = 1, ..., 4), \quad e_j = \frac{1}{2\sqrt{t}} \bar{e}_j (j = 5, 6)$$

is a pseudo-orthonormal basis of $\mathfrak{m}_1 \oplus \mathfrak{m}_2$. With respect to this basis the isotropy representation $\alpha : U(1) \times U(1) \times U(1) \longrightarrow SO(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$ is given by

$$\alpha(e^{it}, e^{is}, e^{ir}) = \begin{pmatrix} D_{r-t} & 0 & 0 \\ 0 & D_{r-s} & 0 \\ 0 & 0 & D_{s-t} \end{pmatrix}$$

where D_θ denotes the rotation of the plane about θ. We can lift α into $Spin(4,2)^+$ and obtain

$$\begin{aligned} \tilde{\alpha} : U(1) \times U(1) \times U(1) \hookrightarrow Spin^+(4,2) \\ \tilde{\alpha}(e^{it}, e^{is}, e^{ir}) = (\cos \tfrac{r-t}{2} - \sin \tfrac{r-t}{2}\, e_1 e_2)(\cos \tfrac{r-s}{2} - \sin \tfrac{r-s}{2}\, e_3 e_4) \cdot \\ \cdot (\cos \tfrac{s-t}{2} + \sin \tfrac{s-t}{2}\, e_5 e_6) \quad . \end{aligned}$$

Therefore $U(2,1)/U(1) \times U(1) \times U(1)$ admits a spin structure. In particular constant functions $\psi(g) = \psi_0 \in \mathrm{span}\{\psi_7, \psi_8\}$, $g \in U(2,1)$ define sections in the spinor bundle S of $U(2,1)/U(1) \times U(1) \times U(1)$. The connection on S with respect to g_t is given by

$$\begin{aligned} \tilde{\Lambda} : \; \mathfrak{m}_1 \oplus \mathfrak{m}_2 \; &\longmapsto \; \mathfrak{spin}(4,3) \\ \tilde{\Lambda}(e_1) \; &= \; \frac{\sqrt{t}}{4}(e_3 e_5 - e_4 e_6) \\ \tilde{\Lambda}(e_2) \; &= \; \frac{\sqrt{t}}{4}(+e_3 e_6 + e_4 e_5) \\ \tilde{\Lambda}(e_3) \; &= \; \frac{\sqrt{t}}{4}(-e_1 e_5 - e_2 e_6) \end{aligned}$$

$$\begin{aligned}
\tilde{\Lambda}(e_4) &= \frac{\sqrt{t}}{4}(e_1e_6 - e_2e_5) \\
\tilde{\Lambda}(e_5) &= \frac{t-1}{4\sqrt{t}}(e_1e_3 + e_2e_4) \\
\tilde{\Lambda}(e_6) &= \frac{t-1}{4\sqrt{t}}(-e_1e_4 + e_2e_3) \ .
\end{aligned}$$

If $t = \frac{1}{2}$ we obtain an Einstein metric with one Killingspinor for each of the values $\lambda = \pm\frac{1}{2\sqrt{2}}$ (ψ_8 and ψ_7, respectively).

3.3.3 $SO^+(4,1)/U(2)$

with canonical imbedding

$$U(2) \hookrightarrow SO(4) \hookrightarrow \begin{pmatrix} SO(4) & 0 \\ 0 & 1 \end{pmatrix} \hookrightarrow SO^+(4,1)\,.$$

Denote by D_{ij} the 5×5-matrix consisting of a single 1 in the i-th row and j-th column, and zeros elsewhere. We set $E_{ij} = -D_{ij} + D_{ji}$ and $A_{ij} = D_{ij} + D_{ji}$. Then we have

$$\mathfrak{u}(2) = span\{E_{12}, E_{34}, E_{13} + E_{24}, E_{14} - E_{23}\}.$$

Let B be the $Ad(U(2))$-invariant bilinear form on $\mathfrak{so}(4,1)$ defined by $B(X,Y) = --\frac{1}{2}\mathrm{Re}\ \mathrm{tr}XY$ for $X,Y \in \mathfrak{so}(4,1)$. We decompose $\mathfrak{so}(4,1)$ into

$$\mathfrak{so}(4,1) = \mathfrak{u}(2) \oplus \mathfrak{m}_1 \oplus \mathfrak{m}_2$$

where

$$\mathfrak{m}_1 = span\{A_{15}, A_{25}, A_{35}, A_{45}\}\,,\ \mathfrak{m}_2 = span\{-E_{13} + E_{24}, -E_{14} - E_{23}\}.$$

We rescale B and obtain

$$B_t = B\mid_{\mathfrak{m}_1\times\mathfrak{m}_1} \oplus tB\mid_{\mathfrak{m}_2\times\mathfrak{m}_2}$$

on $\mathfrak{m}_1 \oplus \mathfrak{m}_2$, which yields a left invariant metric g_t on $SO^+(4,1)/U(2)$. Then

$$\begin{gathered}
e_1 = A_{15}, e_2 = A_{25}, e_3 = A_{35}, e_4 = A_{45} \\
e_5 = \tfrac{1}{\sqrt{2t}}(E_{13} + E_{24}), e_6 = \tfrac{1}{\sqrt{2t}}(-E_{14} - E_{23})
\end{gathered}$$

is a pseudo-orthonormal basis of $\mathfrak{m}_1 \oplus \mathfrak{m}_2$. With respect to this basis the isotropy representation $\alpha : U(2) \longrightarrow SO(\mathfrak{m}_1 \oplus \mathfrak{m}_2)$ is given by

$$\begin{array}{rcl} \alpha : U(2) & \longrightarrow & SO(4,2) \\ h & \longmapsto & (h, \det h) \ , \text{ where } \det h \in S^1 = SO(2). \end{array}$$

Considering this map at the level of homotopy groups we see that we can lift α into $Spin^+(4,2)$. Let $\tilde{\alpha}$ denote this lift. One calculates

$$\begin{aligned} \tilde{\alpha}_*(E_{12}) &= \frac{1}{2}(-e_1e_2 + e_5e_6) \\ \tilde{\alpha}_*(E_{34}) &= \frac{1}{2}(-e_1e_2 + e_5e_6) \\ \tilde{\alpha}_*(E_{13} + E_{24}) &= \frac{1}{2}(-e_1e_3 - e_2e_4) \\ \tilde{\alpha}_*(E_{14} - E_{23}) &= \frac{1}{2}(-e_1e_4 + e_2e_3) \,. \end{aligned}$$

Consequently, constant functions $\psi(g) = \psi_0 \in \mathrm{span}\{\psi_3, \psi_4\}\,, g \in SO^+(4,1)$ define sections in the spinor bundle S of $SO^+(4,1)/U(2)$. The connection on S with respect to g_t is given by

$$\begin{array}{rcl} \tilde{\Lambda} : \ \mathfrak{m}_1 \oplus \mathfrak{m}_2 & \longmapsto & \mathfrak{spin}(4,3) \\ \tilde{\Lambda}(e_1) & = & \frac{1}{4}\sqrt{\frac{t}{2}}(-e_3e_5 - e_4e_6) \\ \tilde{\Lambda}(e_2) & = & \frac{1}{4}\sqrt{\frac{t}{2}}(-e_3e_6 + e_4e_5) \\ \tilde{\Lambda}(e_3) & = & \frac{1}{4}\sqrt{\frac{t}{2}}(e_1e_5 + e_2e_6) \\ \tilde{\Lambda}(e_4) & = & \frac{1}{4}\sqrt{\frac{t}{2}}(e_1e_6 - e_2e_5) \\ \tilde{\Lambda}(e_5) & = & \frac{t-2}{2\sqrt{2t}}(-e_1e_3 + e_2e_4) \\ \tilde{\Lambda}(e_6) & = & \frac{t-2}{2\sqrt{2t}}(-e_1e_4 - e_2e_3) \ . \end{array}$$

If $t = 1$ we obtain an Einstein metric with one Killing spinor for each of the values $\lambda = \pm\frac{1}{2\sqrt{2}}$ (ψ_4 and ψ_3, respectively).

3.3.4 Construction of warped products with Killing spinors

Let $(F^{4,2}, h)$ be a pseudo-Riemannian spin manifold of signature (4,2) with spin structure Q_F and spinor bundle S_F. Furthermore let $I = (a, b) \subseteq \mathbb{R}$ be an open interval and $\sigma \in C^\infty(I, (0, \infty))$ be a smooth positive function. We consider the warped product

$$(M^{4,3}, g) := F^{4,2} \times_\sigma I := (F^{4,2} \times I, \sigma(t)h \oplus dt^2).$$

Denote by $\pi : F^{4,2} \times I \longrightarrow F^{4,2}$ the projection. Let $\tilde{Q}$ be that spin structure of $(M^{4,3}, g)$ whose $Spin(n-1)$-reduction with respect to $\xi = \frac{\partial}{\partial t}$ restricted to any fibre $F^{4,2} \times \{t\}$ yields that spin structure of $(F^{4,2}, \sigma(t)h)$ which is conformally equivalent to the spin structure Q_F of $(F^{4,2}, h)$. The spinor bundle S of $(M^{4,3}, g)$ can be identified with the bundle $\pi^* S_F$ by

$$\begin{array}{rcl} \pi^* S_F & \xrightarrow{\sim} & S = \tilde{Q} \times_{Spin(4,3)} \Delta_{4,3} \\ \psi = [q, u(x,t)] & \longmapsto & \widetilde{\psi} = [\tilde{q}, u(x,t)] \end{array}$$

where $\tilde{q}$ denotes that element of $\tilde{Q}_{(x,t)}$ which corresponds to $q \in (Q_F)_x$ relative to the conformal equivalence of Q_F and $\tilde{Q}|_{F^{4,2}\times\{t\}}$. For a section $\psi \in \Gamma(\pi^* S_F)$ we denote by $\psi_t \in \Gamma(S_F)$ the spinor field $\psi_t(x) := \psi(x,t)$. Furthermore, for a vector field X on $F^{4,2}$ let $\tilde{X}$ be the vector field $\tilde{X}(x,t) := \sigma(t)^{-\frac{1}{2}} X(x)$ on $M^{4,3}$. Then the following formulae for the Clifford multiplication and the spinor derivative hold.

$$\tilde{X}(x,t) \cdot \widetilde{\psi}(x,t) = \widetilde{X(x) \cdot \psi_t(x)} \tag{16}$$

$$\xi \cdot \widetilde{\psi} = -\widetilde{J^S \psi} \tag{17}$$

$$\nabla_{\tilde{X}} \widetilde{\psi} = \sigma(t)^{-\frac{1}{2}} \widetilde{\nabla_X \psi_t} - \tfrac{1}{4} \sigma^{-1} \sigma' \tilde{X} \cdot \xi \cdot \widetilde{\psi} \tag{18}$$

$$\nabla_\xi \widetilde{\psi} = \widetilde{\tfrac{\partial}{\partial t} \psi} \tag{19}$$

Theorem 3.6 *Let now φ^+ and $\varphi^- := J^S(\varphi^+)$ be Killing spinors on $F^{4,2}$ with Killing numbers λ and $-\lambda$, respectively. We may assume $\lambda > 0$. Denote by ψ^+ and ψ^- the sections $\psi^+(x,t) = \cos(\lambda t)\varphi^+(x) - \sin(\lambda t)\varphi^-(x)$ and $\psi^-(x,t) = \sin(\lambda t)\varphi^+(x) - \cos(\lambda t)\varphi^-(x)$ of $\pi^* S_F$. Then $\widetilde{\psi^+}$ and $\widetilde{\psi^-}$ are Killing spinors on $F^{4,2} \times_{\cos^2(2\lambda t)} (-\frac{\pi}{4\lambda}, \frac{\pi}{4\lambda})$ with Killing numbers λ and $-\lambda$, respectively.*

Proof. Direct calculations using (16) – (19).

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