Theta functions for infinite period matrices

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Abstract

For a certain class of infinite period matrices we construct ϑ-functions. This construction involves a certain renormalization procedure and extends discussions given by Tahara and also by McKean and Trubowitz.

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1 Introduction

Theta functions of a single variable were first introduced by Jacobi in his work on elliptic functions. The study of abelian functions led Riemann and Weierstraß to the extension of Jacobi's theory to the case of several complex variables. Theta functions are now an important tool in the function theory on Riemann surfaces and abelian varieties. For a comprehensive exposition see e.g. [11].

More recently, it was discovered that theta functions also give raise to finite gap solutions of Korteweg-de Vries equations and other integrable non-linear differential equations (see [1] and references cited therein). Using theta functions associated with certain hyperelliptic Riemann surfaces of infinite genus, McKean and Trubowitz [8], [9] produced non-finite gap solutions. This approach has been continued by Feldmann, Knörrer and Trubowitz [3] as well as by one of the authors [12]. The construction of such theta functions as used in [8] is originally due to Tahara [14], [15]. Roughly speaking, the period matrices Ω involved are such that they are "close" to $\sqrt{-1}\ T = \sqrt{-1}\ \mathrm{diag}(\tau_1, ..., \tau_n, ...)$ where the $\tau_n > 0$ are such that

$$\prod_{j=1}^{\infty}\left(\sum_{n\in\mathbb{Z}}\exp\left(-\pi\tau_j n^2\right)\right) < \infty. \tag{1.1}$$

The purpose of this paper is to construct theta functions with period matrices Ω which are close to $\sqrt{-1}\ T$ (in a sense made precise below) where the $\tau_j > 0$ are arbitrary. These functions are renormalized in the sense that they formally are given as a quotient

$$\vartheta_T(z|\Omega) = \frac{\vartheta(z|\Omega)}{\vartheta(0|\sqrt{-1}\ T)} \tag{1.2}$$

with z being an infinite sequence $z = (z_1, ..., z_n, ...)$, $z_n \in \mathbb{C}$, lying in a suitable linear space depending on T.

More precisely, the construction is such that one first considers a finite dimensional approximation to Ω and z, then forms the corresponding quotient on the r.h.s. of (1.2) and finally takes the limit. In the cases considered by Tahara, McKean and Trubowitz, Feldmann, Knörrer and Trubowitz, $\vartheta(z|\Omega)$ and $\vartheta(0|\sqrt{-1}\ T)$ are separately well defined. In fact, $\vartheta(0|\sqrt{-1}\ T)$ is then just the expression in (1.1). In that sense our construction is a generalization of their results and is motivated by the theory of Hilbert Schmidt perturbations of Gaussian processes, well known in quantum field theory (see e.g. [5] and references therein).

In a subsequent article we will construct (connected) hyperelliptic Riemann surfaces of infinite genus whose period matrices are "close" to an arbitrary prescribed T of the above form. This will lead to new non-periodic solutions of the KdV equations in terms of these theta functions [10].

The paper is organized as follows. In section 2, we associate to a given T a suitable infinite dimensional space H_T^p which, in analogy with the finite-dimensional case, we call Siegel upper half-space. This is the appropriate space of period matrices Ω we shall work with. Section 3 contains the construction of the corresponding theta functions. In section 4 we derive some relations of these theta functions which are analogous to the finite dimensional case. Finally, in section 5 we prove the functional equation satisfied by our theta functions.

2 Infinite dimensional Siegel upper half–planes

First, we shall introduce some notation. Let ℓ^2 be the Hilbert space of all infinite sequences $z = (z_1, \dots, z_n, \dots)$, $z_i \in \mathbb{C}$, with $\sum_{i=1}^{\infty} |z_i|^2 < \infty$. We denote by $\langle \cdot, \cdot \rangle$ the standard inner product in ℓ^2.

Let $\ell^2_{\rm fn}$ be the dense linear–subspace of ℓ^2 consisting of all $z \in \ell^2$ with $z_i = 0$ for all large i.

Let $M(\infty, \mathbb{C})$ be the linear space of all infinite matrices $A = \{A_{ij}\}_{1 \leq i,j \leq \infty}$ with complex entries. We observe that each $A \in M(\infty, \mathbb{C})$ defines a quadratic form on ℓ^2, namely $z \longmapsto \langle Az, z \rangle$, with domain containing $\ell^2_{\rm fn}$.

For each $p \geq 0$, we introduce the following p–norm in $M(\infty, \mathbb{C})$:

$$||A||_p = \sup_{i,j \geq 1} \{|A_{ij}| i^p j^p\}. \tag{2.1}$$

Let $M_{\rm fn}(\infty, \mathbb{C})$ be the linear subspace of $M(\infty, \mathbb{C})$ consisting of all A with $A_{ij} = 0$ for sufficiently large i and j. We define the Banach space $\mathcal{B}^p$ to be the completion of $M_{\rm fn}(\infty, \mathbb{C})$ with respect to this p–norm. Let $\{e_i\}_{i \in \mathbb{N}}$ be the standard orthonormal basis of ℓ^2, i.e. the j–th component of e_i is equal to δ_{ij}. Suppose that $p > 1/2$ and let $A \in \mathcal{B}^p$. Using (2.1), we get

$$\sum_{i=1}^{\infty} ||Ae_i||^2 = \sum_{i,j=1}^{\infty} |A_{ij}|^2 \leq C_{2p}^2 ||A||_p^2, \tag{2.2}$$

with

$$C_p = \sum_{i=1}^{\infty} i^{-p}. \tag{2.3}$$

Thus, each $A \in \mathcal{B}^p$ defines a Hilbert–Schmidt operator in ℓ^2. In fact, a stronger result holds.

Lemma 2.1 *Let $p > 1$. Then each $A \in \mathcal{B}^p$ defines a trace class operator in ℓ^2.*

Proof. Let $f_i = (f_{i1}, f_{i2}, \dots, f_{in}, \dots)$ be any orthonormal basis in ℓ^2. Then we have

$$\begin{aligned} \sum_{i \geq 1} |\langle Af_i, f_i \rangle| &= \sum_i |\sum_{k,l} A_{kl} f_{ik} \overline{f}_{il}| \leq ||A||_p \sum_{i,k,l} |f_{ik}|\,|f_{il}|\, k^{-p} l^{-p} \\ &\leq ||A||_p \sum_{k,l} k^{-p} l^{-p} \left(\sum_i |f_{ik}|^2 \right)^{1/2} \left(\sum_i |f_{il}|^2 \right)^{1/2} = C_p^2 ||A||_p < \infty. \end{aligned}$$

By Theorem 8.1, Chap.III, §8, of [6], this implies that A is trace class.□ For $p > 1$, let H^p be the convex set consisting of all $\Omega \in M(\infty, \mathbb{C})$ with the properties

(i) $\Omega - \sqrt{-1}\,\mathbf{1} \in \mathcal{B}^p$

(ii) Ω is symmetric, i.e., $\Omega_{ij} = \Omega_{ji}$ for all i, j.

(iii) $\mathrm{Im}(\Omega) > 0$.

We equip H^p with the topology induced by $\mathcal{B}^p$, which is given in terms of the distance function $d_p(\Omega, \Omega') = ||\Omega - \Omega'||_p$.

Note that by Lemma 2.1, each $\Omega \in H^p$, regarded as operator in ℓ^2, is a trace class perturbation of $\sqrt{-1}\,\mathbf{1}$. In particular, each $\Omega \in H^p$ is a bounded operator in ℓ^2 so that (iii) means

$$\langle \mathrm{Im}(\Omega)e, e\rangle > 0 \quad \text{for all} \quad e \in \ell^2.$$

In fact, an even stronger condition holds. Let ϵ_n be the real eigenvalues of $\mathrm{Im}(\Omega) - \mathbf{1}$. By (i) and Lemma 2.1 it follows that $\sum_{n=1}^{\infty} \epsilon_n^2 < \infty$, and by (iii), we have $\epsilon_n > -1$, $n \in \mathbb{N}$. This implies

Lemma 2.2 *Let $\Omega \in H^p$. There exists $\delta = \delta(\Omega) > 0$ such that*

$$\mathrm{Im}(\Omega) \geq \delta.$$

We may phrase this differently. For given $\delta > 0$, let $H^{p,\delta}$ denote the subset of H^p consisting of all Ω such that $\mathrm{Im}(\Omega) \geq \delta$. Then $H^p = \cup_{\delta>0} H^{p,\delta}$.

We set

$$H_{\mathrm{fn}} = \{\Omega \mid \Omega - \sqrt{-1}\,\mathbf{1} \in M_{\mathrm{fn}}(\infty, \mathbb{C}),\ \Omega = {}^t\Omega,\ \mathrm{Im}(\Omega) > 0\}.$$

Then it is easy to see that H_{fn} is dense in each H^p. Moreover each element $\Omega \in H^{p,\delta}$ may be approximated by elements in $H_{\mathrm{fn}} \cap H^{p,\delta'}$ with $0 < \delta' < \min(1, \delta)$.

Let now $T = \mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_n, \ldots)$ be an arbitrary infinite diagonal matrix with strictly positive diagonal entries, i.e. $\tau_n > 0$ for all n. Then $T^{-1} = \mathrm{diag}(\tau_1^{-1}, \tau_2^{-1}, \ldots, \tau_n^{-1}, \ldots)$ and $T^{1/2} = \mathrm{diag}(\tau_1^{1/2}, \tau_2^{1/2}, \ldots, \tau_n^{1/2}, \ldots)$ are well–defined. For $1 < p$ we put

$$H_T^p = T^{1/2} H^p T^{1/2} = \{\Omega \mid \Omega = T^{1/2}\tilde{\Omega}T^{1/2}, \quad \tilde{\Omega} \in H^p\}. \tag{2.4}$$

We equip H_T^p with the topology which makes the map $\Omega \longmapsto T^{1/2}\Omega T^{1/2}$ of H^p onto H_T^p a homeomorphism.

Let $\mathcal{H}^n$ denote the n–dimensional Siegel upper–half space, i.e., $\mathcal{H}^n$ consists of all complex, symmetric $n \times n$ matrices whose imaginary part is positive definite. Each $\Omega \in \mathcal{H}^n$ is invertible and the map $\Omega \longmapsto -\Omega^{-1}$ is an analytic homeomorphism of $\mathcal{H}^n$.

Let $H_{\mathrm{fn},T} = T^{1/2} H_{\mathrm{fn}} T^{1/2}$. Then $\Omega \longmapsto -\Omega^{-1}$ is also well–defined as a map from $H_{\mathrm{fn},T}$ onto $H_{\mathrm{fn},T^{-1}}$. Note that for any $\Omega \in H_{\mathrm{fn},T}$, $\Omega_{ij} = \sqrt{-1}\tau_j\delta_{ij}$ for sufficiently large i and j. For $\Omega \in M(\infty, \mathbb{C})$, let $r_n(\Omega) \in M(n, \mathbb{C})$ be defined by

$$r_n(\Omega) = \{\Omega_{ij}\}_{1 \leq i,j \leq n}. \tag{2.5}$$

It is clear that for each $p > 1$, r_n is a continuous map of H_T^p onto $\mathcal{H}^n$.

Given $\Omega \in \mathcal{H}_T^p$, we have $r_n(T^{-1/2}\Omega T^{-1/2}) \in \mathcal{H}^n$ and it follows from [4], p.19, that the maps indexed by n

$$\Omega \longmapsto \det\left(\frac{r_n(T^{-1/2}\Omega T^{-1/2})}{\sqrt{-1}}\right)^{1/2} \in \mathbb{C} \tag{2.6}$$

are well defined on $H_{\mathrm{fn},T}$ and stationary for all large n and fixed $\Omega \in H_{\mathrm{fn},T}$. Hence the limit $n \to \infty$ exists trivially on $H_{\mathrm{fn},T}$ and is denoted by

$$\Omega \longmapsto \det \left(\frac{T^{-1/2}\Omega T^{-1/2}}{\sqrt{-1}} \right)^{1/2}. \tag{2.7}$$

Lemma 2.3 *The map $\Omega \longmapsto -\Omega^{-1}$ from $H_{fn,T}$ onto $H_{fn,T^{-1}}$ can be extended to a continuous map from $H_T^{2p+\epsilon}$ to $H_{T^{-1}}^p$ for all $p > 1$ and $\epsilon > 0$. Furthermore, the map (2.7) extends to a continuous function on H_T^p, again denoted by (2.7).*

Proof. By definition of H_T^p, it obviously suffices to consider the case $T = \mathbf{1}$. The second part of the lemma is then well known (see e.g. [13]), since $\frac{\Omega}{\sqrt{-1}} - \mathbf{1}$ is of trace class and since $\mathrm{Im}(\Omega) > 0$.

To prove the first part of the lemma, we note that for $\Omega \in H^p$, $-\Omega^{-1}$ exists as a bounded linear operator in ℓ^2. Indeed Ω is a Fredholm operator and using $\mathrm{Im}(\Omega) > 0$ it is easy to see that $\ker(\Omega) = \mathrm{coker}(\Omega) = 0$. Hence $-\Omega^{-1}$ exists as a linear operator in ℓ^2. By the closed graph theorem (see e.g. [2] page 57), $-\Omega^{-1}$ is continuous. Also an elementary calculation shows that

$$\mathrm{Im}(-\Omega^{-1}) \geq ||\Omega||^{-2}\, \mathrm{Im}(\Omega) > 0. \tag{2.8}$$

For $\Omega \in H^{2p+\epsilon}$ we will write $-\Omega^{-1}$ as a Neumann series which converges in H^p. Now for each n, let i_n denote the map from $\mathcal{H}^n$ into H_{fn} given as

$$i_n(\Omega)_{ij} = \begin{cases} \Omega_{ij}, & 1 \leq i,j \leq n \\ \sqrt{-1}\delta_{ij}, & \text{otherwise.} \end{cases}$$

Then we have $r_n \circ i_n = \mathrm{Id}_{\mathcal{H}^n}$. Regarded as map from $\mathcal{H}^n$ into H^p, i_n is continuous. Now let $\Omega \in H^{2p+\epsilon}$ and write

$$\Omega = i_n \circ r_n(\Omega) + (\Omega - i_n \circ r_n(\Omega)) = U_n + V_n.$$

Since $r_n(\Omega) \in \mathcal{H}^n$, we have $-r_n(\Omega)^{-1} \in \mathcal{H}^n$ and hence $-U_n^{-1} = -i_n(r_n(\Omega)^{-1}) \in H_{\mathrm{fn}}$. Formally we have

$$-\Omega^{-1} = -U_n^{-1} - U_n^{-1} \sum_{k=1}^{\infty} \left(V_n U_n^{-1} \right)^k$$

such that

$$\begin{aligned} -\Omega^{-1} - \sqrt{-1}\,\mathbf{1} = -U_n^{-1} - \sqrt{-1}\,\mathbf{1} + \left(-U_n^{-1} - \sqrt{-1}\,\mathbf{1} \right) \sum_{k=1}^{\infty} \left(V_n U_n^{-1} \right)^k \\ + \sqrt{-1} \sum_{k=1}^{\infty} \left(V_n U_n^{-1} \right)^k. \end{aligned} \tag{2.9}$$

By iteration we will show that for $p > 1$, the two series on the r.h.s. are both convergent in H^p for all $n \geq n(p,\epsilon)$ and $|| - \Omega^{-1} + U_n^{-1}||_p \to 0$ as $n \to \infty$. In the first step we estimate

$||-U_n^{-1}-\sqrt{-1}\,\mathbf{1}||_p$ as follows. By Lemma 2.2 there exists $\delta>0$ such that $\mathrm{Im}(\Omega)\geq\delta$. Put $\delta'=\max(1,\delta^{-1})$. We claim that

$$|(U_n^{-1})_{ij}|\leq\delta' \tag{2.10}$$

for all n,i,j. To prove (2.10), we observe that $U_n^{-1}=(i_n\circ r_n(\Omega))^{-1}=i_n(r_n(\Omega)^{-1})$. By assumption, we know that $\mathrm{Im}(r_n(\Omega))=r_n(\mathrm{Im}(\Omega))\geq\delta$. Now write $r_n(\Omega)=A+iB$ where A,B are real symmetric matrices. Then we have $r_n(\Omega)=iB(\mathbf{1}-iB^{-1}A)$. Since $B\geq\delta$, it follows that $||r_n(\Omega)||\leq\delta^{-1}$ which implies that $||i_n(r_n(\Omega)^{-1})||\leq\max(1,\delta^{-1})=\delta'$, where the l.h.s. is the operator norm in ℓ^2. In particular, all matrix elements of U_n^{-1} are bounded by δ' which proves (2.10). Using the block form of $U_n^{-1}=i_n\circ r_n(\Omega)^{-1}$, it follows from (2.10) that

$$||-U_n^{-1}-\sqrt{-1}\,\mathbf{1}||_p\leq(1+\delta')n^{2p},\quad n\in\mathbb{N}. \tag{2.11}$$

Using the trivial estimate

$$||A\cdot B||_p\leq C_{2p}||A||_p||B||_p,$$

we obtain the a priori estimate

$$\begin{aligned}||-\Omega^{-1}-\sqrt{-1}\,\mathbf{1}||_p&\leq||-U_n^{-1}-\sqrt{-1}\,\mathbf{1}||_p\\&\quad+(C_{2p}||-U_n^{-1}-\sqrt{-1}\,\mathbf{1}||_p+1)\sum_{k=1}^{\infty}C_{2p}^{k-1}||V_nU_n^{-1}||_p^k\end{aligned} \tag{2.12}$$

and

$$||-\Omega^{-1}+U_n^{-1}||_p\leq(C_{2p}||-U_n^{-1}-\sqrt{-1}\,\mathbf{1}||_p+1)\cdot\sum_{k=1}^{\infty}C_{2p}^{k-1}||V_nU_n^{-1}||_p^k. \tag{2.13}$$

Next we shall estimate $||V_nU_n^{-1}||_p$. Write

$$(V_nU_n^{-1})_{ij}=\sum_k(V_n)_{ik}(U_n^{-1})_{kj}$$

and split the sum into two parts according to whether $k\leq n$ or $k>n$. Let $\Theta(x)$ denote the Heaviside step function, i.e., $\Theta(x)=1$ if $x\geq0$ and zero otherwise. Recall that $(V_n)_{ij}=0$ unless $i>n$ or $j>n$. Let $k\leq n$. Using (2.10), it follows that

$$\left|\sum_{k\leq n}(V_n)_{ik}(U_n^{-1})_{kj}\right|\leq\sum_{k\leq n}|(V_n)_{ik}|\,\delta'\Theta(n-j)\leq C_p\delta' i^{-p}||V_n||_p\Theta(n-j). \tag{2.14}$$

Now suppose that $k>n$. Since $(U_n^{-1})_{kj}=-\sqrt{-1}\delta_{kj}$ for $k>n$, or $j>n$, we obtain

$$\left|\sum_{k>n}(V_n)_{ik}(U_n^{-1})_{kj}\right|\leq|(V_n)_{ij}|. \tag{2.15}$$

Combining the estimates (2.14) and (2.15) gives

$$||V_nU_n^{-1}||_p\leq(C_p\delta' n^p+1)||V_n||_p.$$

Furthermore, by the form of V_n we have

$$||V_n||_p \leq ||\Omega - \sqrt{-1}\,\mathbf{1}||_{2p+\epsilon}\, n^{-p-\epsilon}.$$

Hence, we get

$$||V_n U_n^{-1}||_p \leq (C_p\delta' n^p + 1)||\Omega - \sqrt{-1}\,\mathbf{1}||_{2p+\epsilon}\, n^{-p-\epsilon} \leq (C_p\delta' + 1)||\Omega - \sqrt{-1}\,\mathbf{1}||_{2p+\epsilon}\, n^{-\epsilon},$$

and therefore, for given p and ϵ, there exists $n(p,\epsilon)$ such that

$$||V_n U_n^{-1}||_p \leq \frac{1}{2} C_{2p}^{-1} n^{-\epsilon/2}$$

for all $n \geq n(p,\epsilon)$. Inserting these estimates into (2.12) and (2.13) shows that

$$||-\Omega^{-1} - \sqrt{-1}\,\mathbf{1}||_p < c \tag{2.16}$$

and

$$||-\Omega^{-1} + U_n^{-1}||_p < c\, n^{\max(0,2p-\epsilon/2)} \tag{2.17}$$

for some $c < \infty$ and all $n \geq n(p,\epsilon)$. This concludes the first step in our iterative procedure. Now we notice that

$$\begin{aligned} ||-U_n^{-1} - \sqrt{-1}\,\mathbf{1}||_p &\leq ||-\Omega^{-1} + U_n^{-1}||_p + ||-\Omega^{-1} - \sqrt{-1}\,\mathbf{1}||_p \\ &\leq 2c\, n^{\max(0,2p-\epsilon/2)}. \end{aligned} \tag{2.18}$$

In the second step we proceed as in the first step but replace (2.11) by the estimate (2.18). This gives

$$||-\Omega^{-1} + U_n^{-1}||_p < c_1 n^{\max(0,2p-\epsilon)}$$

Iterating this k times with $\epsilon k > 4p$ we arrive at the estimate

$$||-U_n^{-1} - \sqrt{-1}\,\mathbf{1}||_p \leq c_2 < \infty$$

for some $c_2 < \infty$. Inserting this into (2.13) gives

$$||-\Omega^{-1} + U_n^{-1}||_p < c_3 n^{-\epsilon/2}.$$

Using $-U_n^{-1} \in H_{\mathrm{fn}}$ and (2.8), the last estimate implies that $-\Omega^{-1} \in H^p$. The continuity of the map $\Omega \longmapsto -\Omega^{-1}$ from $H^{2p+\epsilon}$ to H^p is proved by similar arguments. $\square$

3 The construction of ϑ functions

We start recalling the standard definition of ϑ-functions. For a given matrix M, we shall denote the transpose of M by ${}^tM$. Let $\Omega \in \mathcal{H}^n$, $z \in \mathbb{C}^n$. Then the theta function associated to Ω and z is defined by

$$\vartheta_n(z|\Omega) = \sum_{g\in\mathbb{Z}^n} \exp(\sqrt{-1}\pi\ {}^tg\cdot\Omega g + 2\pi\sqrt{-1}\ {}^tg\cdot z). \tag{3.1}$$

Here g and z are taken to be column vectors such that ${}^tg$ is a row vector and ${}^tg\cdot z = \sum_{i=1}^n g_iz_i$.

By a slight modification (a "renormalization") we will extend this definition to the infinite dimensional case. For this purpose let $\ell_p \subset \ell^2$, $p \geq 1$, be the set of infinite sequences $z = (z_1, z_2, \cdots, z_i, \cdots)$ such that

$$\|z\|_p = \left(\sum_{i\geq 1} |z_i|^2 i^p\right)^{1/2} < \infty.$$

Note that $\ell^2_{\mathrm{fn}} \subset \ell_p$. Let $\ell_{p,T}$ be the space

$$\ell_{p,T} = \{z \in \mathbb{C}^\infty |\ ||z||^2_{p,T} = \sum_{i\geq 1} |z_i|^2\tau_i^{-1} i^p < \infty\}.$$

For $\Omega \in H_{\mathrm{fn},T}$ and $z \in \ell^2_{\mathrm{fn}}$ we set

$$\vartheta_T(z|\Omega) = \lim_{n\to\infty} \frac{\vartheta_n(r_n(z)|r_n(\Omega))}{\vartheta_n(0|r_n(\sqrt{-1}T))} \tag{3.2}$$

with $r_n(z) \in \mathbb{C}^n$ given by $r_n(z)_i = z_i$ for $1 \leq i \leq n$. By construction the quotient on the r.h.s. of (3.2) becomes stationary for all large n, such that the limit trivially exists. Our main result is contained in

Theorem 3.1 *For every $p \geq 1$, the map*

$$(z, \Omega) \mapsto \vartheta_T(z|\Omega)$$

from $\ell^2_{fn} \times H_{fn,T}$ into $\mathbb{C}$ extends to a continuous map from $\ell_{p,T} \times H^p_T$ into $\mathbb{C}$.

Proof: To simplify notation, we assume that $T = \mathbf{1}$. The proof for a general T is similar. Pick $\delta > 0$ and consider $\Omega_0, \Omega_1 \in H_{\mathrm{fn}} \cap H^{p,\delta}$. Also let $z_0, z_1 \in \ell^2_{\mathrm{fn}}$ and for $0 \leq s \leq 1$, put

$$\Omega(s) = s\Omega_1 + (1-s)\Omega_0 \quad \in H_{\mathrm{fn}} \cap H^{p,\delta} \tag{3.3}$$

$$z(s) = sz_1 + (1-s)z_0 \quad \in \ell^2_{\mathrm{fn}}. \tag{3.4}$$

Then we have

$$|\vartheta_{\mathbf{1}}(z_1|\Omega_1) - \vartheta_{\mathbf{1}}(z_0|\Omega_0)| \leq \int_0^1 |\frac{d}{ds}\vartheta_{\mathbf{1}}(z(s)|\Omega(s))|ds. \tag{3.5}$$

Let n be such that

$$(\Omega_0)_{ij} = (\Omega_1)_{ij} = \sqrt{-1}\delta_{ij} \text{ for } i > n \text{ or } j > n,$$

and $(z_0)_i = (z_1)_i = 0$ for $i > n$.

We now introduce a probability measure $d\mu_n$ on $\mathbb{Z}^n$ by

$$\int f(g)d\mu_n(g) = \frac{1}{\sum_{g'\in\mathbb{Z}^n} e^{-\pi\ {}^tg'\cdot g'}} \sum_{g\in\mathbb{Z}^n} f(g)e^{-\pi\ {}^tg\cdot g}. \tag{3.6}$$

With $\Omega(s) = \sqrt{-1}\ \mathbf{1} + V(s)$ we have

$$\begin{aligned} \frac{d}{ds}\vartheta_{\mathbf{1}}(z(s)|\Omega(s)) \quad &= \quad \int \Big[2\pi\sqrt{-1}\ {}^tg\cdot(z_1 - z_0) + \pi\sqrt{-1}\ {}^tg\cdot r_n(\Omega_1 - \Omega_0)g\Big] \\ &\cdot \quad \exp\Big[2\pi\sqrt{-1}\ {}^tg\cdot z(s) + \pi\sqrt{-1}\ {}^tg\cdot V(s)g\Big]\ d\mu_n(g). \end{aligned} \tag{3.7}$$

To estimate this expression, let $\|\cdot\|$ denote the norm on ℓ^2 and let B_p be the diagonal matrix $B_{ij} = i^{-\frac{p}{2}}\delta_{ij}$ such that $\|B_p^{-1}z\| = \|z\|_p$. Then by Schwarz inequality

$$\begin{aligned} |{}^tg\cdot(z_1 - z_0)| &\le \|B_pg\|\ \|z_1 - z_0\|_p \\ &\le \frac{1}{2}(1 + \|B_pg\|^2)\|z_1 - z_0\|_p \\ &\le \frac{1}{2}(1 + \frac{4}{\pi\delta})\exp\left(\frac{\pi}{4}\delta\|B_pg\|^2\right)\cdot\|z_1 - z_0\|_p. \end{aligned} \tag{3.8}$$

Similarly we obtain

$$\begin{aligned} |{}^tg\cdot(\Omega_1 - \Omega_0)g| &\le C_p\|B_pg\|^2\ \|\Omega_1 - \Omega_0\|_p \\ &\le C_p\frac{4}{\pi\delta}\exp\left(\frac{\pi}{4}\delta\|B_pg\|^2\|\right)\|\Omega_1 - \Omega_0\|_p. \end{aligned} \tag{3.9}$$

Also we have

$$\begin{aligned} |\exp 2\pi\sqrt{-1}\ {}^tg\cdot z(s)| &\le \exp\Big(2\pi\|B_pg\|\mathrm{Max}(\|z_0\|_p, \|z_1\|_p)\Big) \\ &\le \exp\left(\frac{\pi\delta}{4}\|B_pg\|^2 + \frac{4\pi}{\delta}\mathrm{Max}(\|z_0\|_p, \|z_1\|_p)^2\right). \end{aligned} \tag{3.10}$$

Finally

$$|\exp\pi\sqrt{-1}\ {}^tg\cdot V(s)g| \le \exp\Big(-\pi\ {}^tg\cdot\mathrm{Im}(V(s))g\Big). \tag{3.11}$$

By the discussion in section 2, $\mathrm{Im}\Big(V(s)\Big)$ is self-adjoint and of trace class on ℓ^2. Let $P(s)$ be the orthogonal projection onto the space of negative eigenvalues. Using (3.11), we obtain

$$|\exp\pi\sqrt{-1}\ {}^tg\cdot V(s)g| \le \exp\Big(\pi\ {}^tg\cdot P(s)V(s)g\Big). \tag{3.12}$$

We now define

$$W(s) = P(s)\,\mathrm{Im}\,V(s) - \frac{\delta}{2}B_p^2 \le 0. \tag{3.13}$$

Since $\mathrm{Im}(\Omega_1) > \delta, \mathrm{Im}\ \Omega_2 > \delta$, we also have $\mathbf{1} + \mathrm{Im}\ V(s) > \delta$ and in particular $\sqrt{-1}(\mathbf{1} + W(s)) \in H_{\mathrm{fn}}$ with

$$\mathbf{1} + W(s) > \frac{\delta}{2}. \tag{3.14}$$

Inserting these estimates into (3.5) we obtain

$$\begin{aligned} |\vartheta_{\mathbf{1}}(z_1|\Omega_1) - \vartheta_{\mathbf{1}}(z_0|\Omega_0)| \leq \Big(\frac{1}{2}(1 + \frac{4}{\pi\delta})\|z_1 - z_0\|_p + C_p \frac{4}{\pi\delta}\|\Omega_1 - \Omega_0\|_p\Big) \cdot \\ \exp\Big(\frac{4\pi}{\delta}\mathrm{Max}(\|z_0\|_p, \|z_1\|_p)^2\Big) \cdot \int_0^1 \vartheta_{\mathbf{1}}\Big(0|\sqrt{-1}(\mathbf{1} + W(s)\Big)ds. \end{aligned} \tag{3.15}$$

Now it follows easily from the corresponding property in the finite dimensional case that

$$\vartheta_{\mathbf{1}}\Big(0|\sqrt{-1}(\mathbf{1} + W(s))\Big) = \det(\mathbf{1} + W(s))^{-\frac{1}{2}}\ \vartheta_{\mathbf{1}}(0|\sqrt{-1}(1 + W(s))^{-1}). \tag{3.16}$$

However, since $\mathbf{1} + W(s) \leq \mathbf{1}$ we have $(\mathbf{1} + W(s))^{-1} \geq \mathbf{1}$ such that

$$\vartheta_{\mathbf{1}}(0|\sqrt{-1}(\mathbf{1} + W(s))^{-1}) \leq 1 \tag{3.17}$$

holds by definition of $\vartheta_{\mathbf{1}}$.

By (3.15) and (3.16) it therefore remains to estimate $\det(1 + W(s))^{-\frac{1}{2}}$. Let Tr denote the trace functional. Then we have

$$\det(\mathbf{1} + W(s))^{-\frac{1}{2}} = \exp\Big\{-\frac{1}{2}\,\mathrm{Tr}\Big[\ln(\mathbf{1} + W(s)) - W(s)\Big] - \frac{1}{2}\,\mathrm{Tr}\Big(W(s)\Big)\Big\} \tag{3.18}$$

and we start with the estimate

$$\begin{aligned} \exp\Big(-\frac{1}{2}\,\mathrm{Tr}\Big(W(s)\Big)\Big) &\leq \exp\Big(\frac{1}{2}\sum_i |W(s)_{ii}|\Big) \\ &\leq \exp\Big(\frac{1}{2}\sum_{i,j} |P(s)_{ij}\ \mathrm{Im}\ V(s)_{ji}| + \frac{\delta}{4}C_p\Big) \\ &\leq \exp\Big(\frac{1}{2}\sum_i (\sum_j |P(s)_{ij}|^2)^{\frac{1}{2}} (\sum_j |V(s)_{ji}|^2)^{\frac{1}{2}} + \frac{\delta}{4}C_p\Big) \\ &\leq \exp\Big(\frac{1}{2}\sum_i (\sum_j |V(s)_{ji}|^2)^{\frac{1}{2}} + \frac{\delta}{4}C_p\Big). \end{aligned} \tag{3.19}$$

Here we have used the fact that $P(s)$ is a bounded operator with norm ≤ 1 such that, in particular, $\sum_j |P(s)_{ij}|^2 \leq 1$ for all i. The r.h.s. of (3.19) may easily be estimated to give

$$\begin{aligned} &\exp\Big(-\frac{1}{2}\,\mathrm{Tr}\,\Big(W(s)\Big)\Big) \\ &\quad\leq \exp\Big\{\frac{1}{2}\ C_p C_{2p}^{1/2}\mathrm{Max}(\|\Omega_1 - \sqrt{-1}\mathbf{1}\|_p, \|\Omega_0 - \sqrt{-1}\mathbf{1}\|_p) + \frac{\delta}{4}\ C_p\Big\}. \end{aligned} \tag{3.20}$$

To complete the estimate for (3.18) we observe that

$$\begin{aligned}\left|\frac{1}{2}\operatorname{Tr}\left\{\ln(\mathbf{1}+W(s))-W(s)\right\}\right| &= \left|\frac{1}{2}\ \operatorname{Tr}\int_0^1 \Big(W(s)(1+tW(s))^{-1}-W(s)\Big)\,dt\right| \\ &= \left|\frac{1}{2}\operatorname{Tr}\int_0^1 W(s)^2 t(1+tW(s))^{-1}dt\right| \\ &\leq \frac{1}{4}\sup_{0\leq t\leq 1}\|(1+tW(s))^{-1}\|\sum_{i,j}|W(s)_{ij}|^2.\end{aligned} \tag{3.21}$$

By (3.14) we have

$$\sup_{0\leq t\leq 1}\|(1+tW(s))^{-1}\| \leq \sup_{0\leq t\leq 1}\Big[\operatorname{infspec}\Big(1+tW(s)\Big)\Big]^{-1} \leq \frac{2}{\delta}. \tag{3.22}$$

Also

$$\begin{aligned}\sum_{i,j}|W(s)_{ij}|^2 &\leq 2\sum_{i,j}\left|\Big(P(s)\operatorname{Im}\ V(s)\Big)_{ij}\right|^2+\frac{\delta^2}{2}C_{2p} \\ &\leq 2\sum_{i,j}|V(s)_{ij}|^2+\frac{\delta^2}{2}C_{2p} \\ &\leq 4C_{2p}^2\operatorname{Max}(\|\Omega_1-\sqrt{-1}\mathbf{1}\|_p^2,\|\Omega_0-\sqrt{-1}\mathbf{1}\|_p^2)+\frac{\delta^2}{2}C_{2p}.\end{aligned} \tag{3.23}$$

Inserting these estimates into (3.18) gives

$$\det(\mathbf{1}+W(s))^{-\frac{1}{2}} \leq \exp A \tag{3.24}$$

with

$$\begin{aligned}A = &\frac{1}{2}C_p C_{2p}^{1/2}\operatorname{Max}(\|\Omega_1-\sqrt{-1}\mathbf{1}\|_p,\|\Omega_0-\sqrt{-1}\|_p)+\frac{\delta^2}{4}\ C_p \\ &+\frac{1}{2\delta}\Big\{4C_{2p}^2\operatorname{Max}(\|\Omega_1-\sqrt{-1}\mathbf{1}\|_p^2,\|\Omega_0-\sqrt{-1}\mathbf{1}\|_p^2)+\frac{\delta}{2}\ C_{2p}\Big\}.\end{aligned} \tag{3.25}$$

Inserting (3.16) (3.17) and (3.25) into (3.15), we get the desired estimate. In the same way one can prove that

$$\begin{aligned}|\vartheta_T(z_1|\Omega_1)-\vartheta_T(z_0|\Omega_0)| \leq &\left(\frac{1}{2}(1+\frac{4}{\pi\delta})\|z_1-z_0\|_{p,T}+C_p\frac{4}{\pi\delta}\|\Omega_1-\Omega_0\|_{p,T}\right)\cdot \\ &\exp\left(A+\frac{4\pi}{\delta}\operatorname{Max}(\|z_0\|_{p,T},\|z_1\|_{p,T})^2\right).\end{aligned} \tag{3.26}$$

This concludes the proof of the theorem, since elements in $H_T^{p,\delta}$ may be approximated by elements in $H_{\mathrm{fn},T}\cap H_T^{p,\delta'}$ with $\delta' < \operatorname{Min}(1,\delta)$.□

Let T and T' be such that they differ at finitely many entries only. Then the spaces $\ell_{p,T}$ and $\ell_{p,T'}$ as well as H_T^p and $H_{T'}^p$ are equal. By construction we therefore have the

Corollary 3.2 *Let T and T' differ at finitely many entries. Then the relation*

$$\vartheta_T(z|\Omega) = \vartheta_T(0|\sqrt{-1}T')\vartheta_{T'}(z|\Omega) \tag{3.27}$$

holds.

Remark 3.3: For each T, the function $\vartheta_T(z|\Omega)$ is a universal ϑ-function in the following sense. Let $z \in \mathbb{C}^n$ and $\Omega \in \mathcal{H}^n$ and set

$$i_{n,T}(\Omega)_{ij} = \begin{cases} \Omega_{ij}, & 1 \leq i,j \leq n, \\ \sqrt{-1}\tau_j\delta_{ij}, & \text{otherwise.} \end{cases} \tag{3.28}$$

Then the relation

$$\vartheta_n(z|\Omega) = \vartheta_T(i_n(z)|i_{n,T}(\Omega)) \cdot \vartheta_n(0|\sqrt{-1}r_n(T)). \tag{3.29}$$

holds.

Remark 3.4: In finite dimensions, there exist slight generalizations of theta functions known as theta functions $\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z|\Omega)$ with rational characteristics $a, b \in \mathbb{Q}^n$. They are defined as

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z|\Omega) = \sum_{g\in\mathbb{Z}^n} \exp\Big(\sqrt{-1}\pi\ {}^t(g+a)\cdot\Omega(g+a) + 2\sqrt{-1}\ {}^t(g+a)(z+b)\Big). \tag{3.30}$$

Thus, the theta functions with characteristics are just translates of the ordinary theta functions multiplied by an exponential factor:

$$\vartheta\begin{bmatrix}a\\b\end{bmatrix}(z|\Omega) = \exp\Big(\sqrt{-1}\pi\ {}^ta\cdot\Omega a + 2\sqrt{-1}\pi\ {}^ta\cdot(z+b)\Big)\ \vartheta(z+\Omega a+b|\Omega). \tag{3.31}$$

This can be easily extended to our setting. Let $\ell_{p,T}(\mathbb{Q})$ be the subspace of $\ell_{p,T}$ consisting of all sequences in $\ell_{p,T}$ with rational components. Let $a \in \ell_{p,T^{-1}}(\mathbb{Q})$ and $b \in \ell_{p,T}(\mathbb{Q})$. Then, for $\Omega \in H_T^p$, $z \in \ell_{p,T}$, we may define $\vartheta_T\begin{bmatrix}a\\b\end{bmatrix}(z|\Omega)$ by a formula analogous to (3.2), replacing the numerator on the r.h.s. by the corresponding theta function with rational characteristics $r_n(a), r_n(b)$. By 3.31, it is clear that $\vartheta_T\begin{bmatrix}a\\b\end{bmatrix}(z|\Omega)$ is again obtained from $\vartheta_T(z|\Omega)$ by translating z and multiplying by an exponential factor, i.e., we have

$$\vartheta_T\begin{bmatrix}a\\b\end{bmatrix}(z|\Omega) = \exp\Big(\sqrt{-1}\pi\ {}^ta\cdot\Omega a + 2\sqrt{-1}\pi\ {}^ta\cdot(z+b)\Big)\ \vartheta_T(z+\Omega a+b|\Omega). \tag{3.32}$$

The choice of a and b guarantees that all terms on the r.h.s. exist.

4 Some basic properties of the ϑ-functions

In this section we prove some transformation properties and a non-vanishing theorem for ϑ-functions.

Corollary 4.1 *Let $n \in \ell^2_{fn}$ have integer coefficients, i.e. $n = (n_1, n_2, \cdots n_i, \cdots)$ with $n_i \in \mathbb{Z}$ and $n_i = 0$ for all large i. Then the relations*

$$\vartheta_T(z + n|\Omega) = \vartheta_T(z|\Omega), \tag{4.1}$$

$$\vartheta_T(z + \Omega n|\Omega) = \exp\Big(- \pi\sqrt{-1}\ {}^t n \cdot \Omega n - 2\pi\sqrt{-1}\ {}^t n \cdot z\Big)\ \vartheta_T(z|\Omega) \tag{4.2}$$

hold for all $(z, \Omega) \in \ell_{p,T} \times H^p_T$.

This follows easily from Theorem 3.1 and the familiar transformation properties in the finite dimensional case (see e.g. [11]).

The proof of Theorem 3.1 also gives

Corollary 4.2 *For any $z, z', z'' \in \ell_{p,T}$ and $\Omega \in H^p_T$ the function*

$$t \mapsto \vartheta_T(z + sz' + tz''|\Omega) \tag{4.3}$$

is analytic in $(s, t) \in \mathbb{C}^2$ and non-constant when $z' \neq 0 \neq z''$. The derivatives are jointly continuous in s, t, z, z', z'' and Ω.

Corollary 4.3 *For $(z, \Omega) \in \ell_{p,T} \times H^p_T$ with $p > 2$ one has*

$$\vartheta_{T^{-1}}(\Omega^{-1}z| - \Omega^{-1}) = \exp\Big(\pi\sqrt{-1}\ {}^t z \cdot \Omega^{-1}z\Big) \cdot \det\left(\frac{T^{-\frac{1}{2}}\Omega T^{-\frac{1}{2}}}{\sqrt{-1}}\right)^{\frac{1}{2}} \vartheta_T(z|\Omega). \tag{4.4}$$

Proof: By lemma 2.1, $-\Omega^{-1} \in H^{p''}_{T^{-1}}$ for any p'' with $1 < p'' < p/2$. In particular ${}^t z \cdot \Omega^{-1}z$ is well defined. We claim that $\Omega^{-1}z \in \ell_{p',T^{-1}}$ for any $1 < p'$ with $2p'' - p' > 1$ so, in particular, for $p' = p''$. Combined with Lemma 2.1, Theorem 3.1 and the analogous transformation properties in the finite dimensional case (see [11] p.195) this proves the corollary.

To prove the claim, we note that

$$\begin{aligned} \|\Omega^{-1}z\|_{p',T^{-1}} \leq & \|(-\Omega^{-1} - \sqrt{-1}T^{-1})z\|_{p',T^{-1}} \\ & + \|\sqrt{-1}T^{-1}z\|_{p',T^{-1}}. \end{aligned} \tag{4.5}$$

The second term on the r.h.s of (4.5) is easy to control since

$$\|\sqrt{-1}T^{-1}z\|_{p',T^{-1}} = \|z\|_{p',T} \leq \|z\|_{p,T}. \tag{4.6}$$

Since $-\Omega^{-1} \in H^{p''}_{T^{-1}}$, we have $-\Omega^{-1} = T^{-\frac{1}{2}} V T^{-\frac{1}{2}}$ where $V \in H^{p''}$. Put $V' = V - \sqrt{-1}\mathbf{1}$. Then we obtain

$$\begin{aligned} \|(-\Omega^{-1} - \sqrt{-1}T^{-1})z\|^2_{p',T^{-1}} &= \|T^{-\frac{1}{2}} V' T^{-\frac{1}{2}} z\|^2_{p',T^{-1}} \\ &= \sum_{i \geq 1} |\sum_{j \geq 1} V'_{i,j} \tau_j^{-\frac{1}{2}} z_j|^2 i^{p'} \\ &\leq \sum_{i \geq 1} (\sum_{j \geq 1} |V'_{i,j}|^2) i^{p'} \|z\|^2_{p',T}. \end{aligned} \tag{4.7}$$

Since $V' \in \mathcal{B}^{p''}$

$$\sum_j |V'_{ij}|^2 \leq \frac{C_{2p''}}{i^{2p''}} \|V'\|^2_{p''} < \infty. \tag{4.8}$$

Insert (4.8) into (4.7) and recall that $2p'' - p' > 1$. This implies $-\Omega^{-1}z \in \ell_{p',T^{-1}}$ concluding the proof of the corollary.□

Theorem 4.4 *Let $\Omega \in H^p_T$ be purely imaginary. For real $x \in \ell_{p,T}$, $\vartheta_T(x|\Omega)$ is not identically equal to zero, nonnegative and of positive type, i.e. for arbitrary real $x_i \in \ell_{p,T}$ and $\lambda_i \in \mathbb{C}$ $(1 \leq i \leq k)$*

$$\sum_{1\leq i,j\leq k} \bar{\lambda}_i \vartheta_T(x_i - x_j|\Omega)\lambda_j \geq 0. \tag{4.9}$$

In particular for real $x \in \ell_{p,T}$

$$0 \leq \vartheta_T(x|\Omega) \leq \vartheta_T(0|\Omega) \tag{4.10}$$

Moreover, if $p > 2$, then $\vartheta_T(x|\Omega)$ is strictly positive for all real $x \in \ell_{p,T}$.

Proof: Obviously $\vartheta_T(x|\Omega)$ is real for real x. It is also of positive type for Ω purely imaginary $\in H_{\mathrm{fn},T}$ and real $x_i \in \ell^2_{\mathrm{fn}}$, again by taking recourse to the finite dimensional case where it is obvious. Therefore it is also of positive type in the form claimed using a limiting argument and Theorem 3.1. To prove that $\vartheta_T(x|\Omega)$ is nonnegative (and even positive if $p > 2$), consider the ϑ function in two variables $z \in \ell_{p,T^{-1}}$ and $z' \in \ell_{p,T}$ defined by (compare e.g. [4] p.22 in the finite dimensional case)

$$\vartheta_T(z,z'|\Omega) = \exp\left(\frac{\pi}{4}\sqrt{-1}\,{}^t z\cdot\Omega z\right)\vartheta_T(\frac{1}{2}(z' + \Omega z)|\Omega). \tag{4.11}$$

By the methods used in the proof of Corollary 4.3 one sees that $\Omega z \in \ell_{p,T}$. In particular if $\Omega \in H_{\mathrm{fn},T}$, $z, z' \in \ell^2_{\mathrm{fn}}$, then for all large n we have

$$\begin{aligned}\vartheta_T(z,z'|\Omega) =&\vartheta_n(0|\sqrt{-1}r_n(T))^{-1}\\ &\cdot\sum_{g\in\mathbb{Z}^n}\exp\pi\sqrt{-1}\Big\{{}^t(g+\frac{1}{2}r_n(z))\cdot r_n(\Omega)(g+\frac{1}{2}r_n(z)) + {}^t g\cdot r_n(z')\Big\}.\end{aligned} \tag{4.12}$$

On the other hand, if $p > 2$ or if $\Omega \in H_{\mathrm{fn},T}$, and $z, z' \in \ell^2_{\mathrm{fn}}$, then relation (4.4) may be recast into the form (see again [4] p. 22)

$$\begin{aligned}&\vartheta_{T^{-1}}(-z', z| - \Omega^{-1})\\ &\quad= \exp\left(\frac{\pi}{2}\sqrt{-1}\,{}^t z'\cdot z\right)\cdot\det\left(\frac{T^{-\frac{1}{2}}\Omega T^{-\frac{1}{2}}}{\sqrt{-1}}\right)^{1/2}\cdot\vartheta_T(z,z'|\Omega).\end{aligned} \tag{4.13}$$

Now relation (4.12) shows that $\vartheta_{T^{-1}}(-x', 0|-\Omega^{-1})$ is positive for $\Omega \in H_{\mathrm{fn},T}$ purely imaginary and $x' \in \ell^2_{\mathrm{fn}}$ real. By relation (4.13), $\vartheta_T(x'|\Omega) = \vartheta_T(0, 2x'|\Omega)$ is then also positive. For general $x' \in \ell_{p,T}$ and $\Omega \in H_T^p$ (Ω purely imaginary) $\vartheta_T(x'|\Omega)$ is nonnegative by a limiting argument. To prove positivity in the case $p > 2$ it suffices to prove that $\vartheta_{T^{-1}}(-x', 0|\Omega)$ is strictly positive. To conclude the proof of the theorem it therefore suffices to prove that $\vartheta_T(x, 0|\Omega)$ is strictly positive for real $x \in \ell_{p',T}$ and $\Omega \in H_T^{p'}$ purely imaginary where $p' > 1$ is arbitrary.

Again for notational simplicity we consider the case $T = \mathbf{1}$ only, the general case requiring only minor modifications. Let

$$\begin{aligned} B_1(r_1) &= \{x \in \ell_{p',\mathbf{1}} \,|\, x \text{ real and } \|x\|_{p'}^2 < r_1\} \\ B_2(r_2) &= \{\Omega \in H_{\mathbf{1}}^{p'} \,|\, \Omega \text{ purely imaginary and } \|\Omega - \sqrt{-1}\mathbf{1}\|_{p'} < r_2\}. \end{aligned}$$

We will show that there is $C(p', r_1, r_2) > 0$ such that

$$\vartheta_{\mathbf{1}}(x, 0|\Omega) \geq C(p', r_1, r_2) \tag{4.14}$$

for all $x \in B_1(r_1), \Omega \in B_2(r_2)$. By continuity it suffices to consider the case $x \in B_1(r_1) \cap \ell^2_{\mathrm{fn}}, \Omega \in B_2(r_2) \cap H_{\mathrm{fn}}$. Now by (4.12) for such x and Ω, and n large we have

$$\begin{aligned} \vartheta_{\mathbf{1}}(x, 0|\Omega) = &\Big(\sum_{g\in\mathbb{Z}} \exp(-\pi g^2)\Big)^{-n} \\ &\cdot \sum_{g\in\mathbb{Z}^n} \exp\left(-\pi\Big\{\sum_{i=1}^n (g_i + \frac{1}{2}x_i)^2 + \sum_{1\leq i,j\leq n} (g_i + \frac{1}{2}x_i)U_{ij}(g_j + \frac{1}{2}x_j)\Big\}\right) \end{aligned} \tag{4.15}$$

where we have set $\Omega = \sqrt{-1}(\mathbf{1} + U)$ such that U is symmetric and real. To estimate (4.15) we first note that

$$\sum_{1\leq i\leq n} (g_i + \frac{1}{2}x_i)^2 \leq \sum_{1\leq i\leq n} g_i^2 + \frac{1}{2}\sum_{1\leq i\leq n} g_i^2 i^{-p'} + \frac{3}{4}\sum_{1\leq i\leq n} x_i^2 i^{p'}. \tag{4.16}$$

Set $r_2' = c_{p'} \cdot r_2$. Since

$$|U_{ij}| \leq \frac{\|\Omega - \sqrt{-1}\mathbf{1}\|_{p'}}{i^{p'}j^{p'}} \leq \frac{r_2}{i^{p'}j^{p'}}, \tag{4.17}$$

we have

$$\begin{aligned} \sum_{1\leq i,j\leq n} (g_i + \frac{1}{2}x_i)U_{ij}(g_j + \frac{1}{2}x_j) &\leq r_2 \left(\sum_{1\leq i\leq n} |g_i + \frac{1}{2}x_i| \frac{1}{i^{p'}}\right)^2 \\ &\leq r_2' \sum_{1\leq i\leq n} |g_i + \frac{1}{2}x_i|^2 \frac{1}{i^{p'}} \\ &\leq 2r_2' \sum g_i^2 \frac{1}{i^{p'}} + \frac{r_2'}{2}\sum x_i^2 \frac{1}{i^{p'}}. \end{aligned} \tag{4.18}$$

Let $\alpha = 2r_2' + \frac{1}{2}$ and $\beta = \left(\frac{r_2'}{2} + \frac{3}{4}\right)$. Inserting (4.16) and (4.18) into (4.15), we obtain

$$\begin{aligned} \vartheta_{\mathbf{1}}(x, 0|\Omega) \geq &\exp\Big(-\pi\beta\|x\|_{p'}^2\Big) \\ &\cdot \Big(\sum_{g\in\mathbb{Z}} \exp(-\pi g^2)\Big)^{-n} \prod_{i=1}^n \sum_{g\in\mathbb{Z}} \exp\Big(-\pi(1 + \frac{\alpha}{i^{p'}})g^2\Big). \end{aligned} \tag{4.19}$$

By the Poisson summation formula

$$\begin{aligned}\sum_{g\in\mathbb{Z}} \exp\left(-\pi(1+\frac{\alpha}{i^{p'}})g^2\right) &= (1+\frac{\alpha}{i^{p'}})^{-\frac{1}{2}} \sum_{g\in\mathbb{Z}} \exp\left(-\pi(1+\frac{\alpha}{i^{p'}})^{-1}g^2\right) \\ &\geq \left(1+\frac{\alpha}{i^{p'}}\right)^{-\frac{1}{2}} \sum_{g\in\mathbb{Z}} \exp\left(-\pi g^2\right).\end{aligned} \tag{4.20}$$

Inserting into (4.19) gives the following a priori estimate, uniformly in n:

$$\vartheta_{\mathbf{1}}(x,0|\Omega) \geq \exp\left(-\pi\beta r_1\right) \prod_{i=1}^{\infty} \left(1+\frac{\alpha}{i^{p'}}\right)^{-\frac{1}{2}} > 0 \tag{4.21}$$

which concludes the proof of Theorem 4.4 □

Remark 4.5: Corollary 4.2 and Theorem 4.4 will become relevant when we will construct new solutions of the KdV equations associated to hyperelliptic Riemann surfaces of infinite genus [10]. Since they are real these surfaces will have purely imaginary period matrices in H_T^p , $p > 2$. Corollary 4.2 will be used to write these solutions as limits of solutions corresponding to the finite genus case, which are obtained via the Its-Matveev formula. Since this involves taking the logarithm of the theta function, Theorem 4.4 becomes essential.

5 The functional equation of ϑ_T

We consider now the dependence of $\vartheta_T(z,\Omega)$ on Ω. Recall that the classical ϑ–function in n variables satisfies a functional equation with respect to the action of a subgroup of $Sp(n,\mathbb{Z})$ on both the variables z and Ω. Namely, let $\Gamma_{1,2}^n$ be the subset of $Sp(n,\mathbb{Z})$ consisting of all

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(n,\mathbb{Z})$$

such that the diagonal of ${}^tAC$ and ${}^tBD$ is even. Then $\Gamma_{1,2}^n$ turns out to be a subgroup of $Sp(n,\mathbb{Z})$ and the functional equation for $\vartheta_n(z,\Omega)$ can be stated as follows:

$$\begin{aligned}&\vartheta_n({}^t(C\Omega+D)^{-1}z \mid (A\Omega+B)(C\Omega+D)^{-1}) \\ &\quad = \zeta_\gamma \cdot \det(C\Omega+D)^{1/2} \cdot \exp\left(\sqrt{-1}\pi\ {}^tz \cdot (C\Omega+D)^{-1}C \cdot z\right) \vartheta_n(z \mid \Omega),\end{aligned} \tag{5.1}$$

where

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_{1,2}^n,$$

and ζ_γ is a certain 8^{th} root of unity which depends on γ (see e.g. [11], pp. 189). In this section we shall establish a corresponding functional equation for $\vartheta_T(z \mid \Omega)$. First we need to define the analogue of $\Gamma_{1,2}^n$. Note that for all $n \in \mathbb{N}$, there exist canonical inclusions $Sp(n,\mathbb{R}) \subset Sp(n+1,\mathbb{R})$. Let

$$Sp(\infty,\mathbb{R}) = \bigcup_{n\geq 1} Sp(n,\mathbb{R}).$$

Then $Sp(\infty,\mathbb{R})$ has a group structure induced from $Sp(n,\mathbb{R})$ and for each $n\in\mathbb{N}$ there exists a canonical injective homomorphism $i_n\colon Sp(n,\mathbb{R})\to Sp(\infty,\mathbb{R})$ and $Sp(\infty,\mathbb{R}) = \cup_{n\geq 1} i_n(Sp(n,\mathbb{R}))$. Thus, each element $\gamma\in Sp(\infty,\mathbb{R})$ can be written as

$$\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{5.2}$$

where $A,B,C,D\in M(\infty,\mathbb{R})$ are of the form

$$A = \begin{pmatrix} A_n & 0 \\ 0 & \mathbf{1} \end{pmatrix},\quad B = \begin{pmatrix} B_n & 0 \\ 0 & 0 \end{pmatrix},\quad C = \begin{pmatrix} C_n & 0 \\ 0 & 0 \end{pmatrix},\quad D = \begin{pmatrix} D_n & 0 \\ 0 & \mathbf{1} \end{pmatrix} \tag{5.3}$$

with

$$\begin{pmatrix} A_n & B_n \\ C_n & D_n \end{pmatrix} \in Sp(n,\mathbb{R}) \tag{5.4}$$

for some $n = n(\gamma) < \infty$.

Lemma 5.1 *Each* $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(\infty,\mathbb{R})$ *defines a continuous map* $H_T^p \to H_T^p$ *by*

$$\gamma(\Omega) = (A\Omega + B)\cdot(C\Omega+D)^{-1}, \quad \Omega\in H_T^p.$$

Moreover one has $\gamma_1(\gamma_2(\Omega)) = \gamma_1\gamma_2(\Omega)$ *for all* $\gamma_1,\gamma_2\in Sp(\infty,\mathbb{R})$ *and* $\Omega\in H_T^p$.

Proof. Let $n\in\mathbb{N}$ be such that $\gamma\in i_n(Sp(n,\mathbb{R}))$. Let $\Omega\in H_T^p$ and write Ω as a block matrix

$$\Omega = \begin{pmatrix} \Omega_{11} & \Omega_{12} \\ \Omega_{21} & \Omega_{22} \end{pmatrix}$$

where Ω_{11} is a $n\times n$-matrix etc. Then Ω_{11} and Ω_{22} are symmetric and have positive definite imaginary parts. In particular, Ω_{11} belongs to $\mathcal{H}^n$. By (5.4), $C_n\Omega_{11}+D_n$ is invertible. Put

$$G = (C_n\Omega_{11}+D_n)^{-1}.$$

Then we get

$$(C\Omega+D)^{-1} = \begin{pmatrix} G & -GC_n\Omega_{12} \\ 0 & \mathbf{1} \end{pmatrix} \tag{5.5}$$

and

$$\begin{aligned} \gamma(\Omega) &= (A\Omega+B)\cdot(C\Omega+D)^{-1} \\ &= \begin{pmatrix} (A_n\Omega_{11}+B_n)G & A_n\Omega_{12} - (A_n\Omega_{11}+B_n)GC_n\Omega_{12} \\ & \\ \Omega_{21}G & \Omega_{22} - \Omega_{21}GC_n\Omega_{12} \end{pmatrix}. \end{aligned} \tag{5.6}$$

Now recall that $T^{-1/2}\Omega T^{-1/2} - \sqrt{-1}\,\mathbf{1}\in\mathcal{B}^p$. Together with (5.6) this implies that

$$T^{-1/2}\gamma(\Omega)T^{-1/2} - \sqrt{-1}\,\mathbf{1}\in\mathcal{B}^p.$$

Moreover, using (5.2) – (5.4), it is easy to verify that $\gamma(\Omega)$ is symmetric with positive imaginary part. Hence $\gamma(\Omega) \in H_T^p$. The continuity of the map $\gamma\colon H_T^p \to H_T^p$ is obvious by the above remarks. The last part of the lemma is also clear. $\square$

Let $\Gamma_{1,2}^\infty$ be the subgroup of $Sp(\infty,\mathbb{Z})$ consisting of all $\gamma \in Sp(\infty,\mathbb{Z})$ of the form (5.2) safisfying

$$\text{diagonal}\,({}^tAC)\,\text{even}$$
$$\text{diagonal}\,({}^tBD)\,\text{even}.$$

The canonical inclusion $i_n\colon Sp(n,\mathbb{R}) \to Sp(\infty,\mathbb{R})$ induces inclusions $i_n\colon \Gamma_{1,2}^n \to \Gamma_{1,2}^\infty$ for all $n \in \mathbb{N}$ and by construction we have

$$\Gamma_{1,2}^\infty = \bigcup_{n\geq 1} i_n\left(\Gamma_{1,2}^n\right).$$

We now claim that for $(z,\Omega) \in \ell^2_{\mathrm{fn}} \times H_{\mathrm{fn},T}$ and $\gamma \in \Gamma_{1,2}^\infty$, the following functional equation holds

$$\begin{aligned}&\vartheta_T({}^t(C\Omega+D)^{-1}z|(A\Omega+B)(C\Omega+D)^{-1})\\ &\quad= \zeta_\gamma \det(C\Omega+D)^{1/2}\cdot\exp\{-\sqrt{-1}\pi\ {}^tz\cdot(C\Omega+D)^{-1}Cz)\}\cdot\vartheta_T(z|\Omega),\end{aligned}\tag{5.7}$$

with $\zeta_\gamma^8 = 1$. To give a meaning to and to establish (5.7) we take recourse to the definition (3.1) and apply it to the left hand side of (5.7). Now by (5.6) and (5.1), for all large n

$$\begin{aligned}&\vartheta_T({}^t(C\Omega+D)^{-1}z|(A\Omega+B)(C\Omega+D)^{-1})\\ &\quad= \frac{\vartheta_n(r_n({}^t(C\Omega+D)^{-1}z)|r_n((A\Omega+B)(C\Omega+D)^{-1})}{\vartheta_n(0|r_n(\sqrt{-1}T))}\\ &\quad= \frac{\vartheta_n({}^t(C_nr_n(\Omega)+D_n)^{-1}r_n(z)|(A_nr_n(\Omega)+B_n)(C_nr_n(\Omega)+D_n)^{-1})}{\vartheta_n(0|r_n(\sqrt{-1}T))}\\ &\quad= \zeta_{r_n(\gamma)}\det(C_nr_n(\Omega)+D_n)^{1/2}\ \exp\{-\sqrt{-1}\pi\ {}^tr_n(z)\cdot(C_nr_n(\Omega)+D_n)^{-1}C_nz\}\\ &\qquad\cdot\frac{\vartheta_n(r_n(z)|r_n(\Omega)}{\vartheta_n(0|r_n(\sqrt{-1}T))}.\end{aligned}\tag{5.8}$$

In the first relation we have also used the fact that by (5.5) $(C\Omega+D)^{-1}z \in \ell^2_{\mathrm{fn}}$. By construction the last quotient on the r.h.s. of (5.8) becomes stationary for all large n and equals $\vartheta_T(z|\Omega)$. Similarly the expression in the exponent becomes stationary for all large n and equals the corresponding expression in (5.7). Furthermore, by (5.3) $\det(C_nr_n(\Omega)+D_n)^{1/2}$ becomes stationary for all large n. We denote this expression by $\det(C\Omega+D)^{1/2}$. Thus all expressions except possibly the right eight'th root of unity $\zeta_{r_n(\gamma)}$ in (5.8) become stationary for all large n. Since (5.8) is an identity for all large n, this shows that $\zeta_{r_n(\gamma)}$ is also stationary and we write it as ζ_γ. Thus (5.7) indeed holds for $(z,\Omega) \in \ell^2_{\mathrm{fn}} \times H_{\mathrm{fn},T}$. We need a final technical lemma.

Lemma 5.2 (i) *For every $\gamma \in \Gamma_{1,2}^\infty$ the map $\Omega \mapsto \det(C\Omega+D)^{\frac{1}{2}}$ from $H_{fn,T}$ into $\mathbb{C}$ extends to a continuous map from H_T^p into $\mathbb{C}$ for each $p > 1$.*
(ii) *The map $(z,\Omega) \mapsto {}^tz\cdot(C\Omega+D)^{-1}Cz$ from $\ell^2_{fn} \times H_{fn,T}$ into $\mathbb{C}$ extends to a continuous map from $\ell_{p,T} \times H_T^p$ into $\mathbb{C}$ for all $p > 1$.*
(iii) *The map $(z,\Omega) \mapsto {}^t(C\Omega+D)^{-1}z$ of $\ell^2_{fn} \times H_{fn,T}$ into ℓ^2_{fn} extends to a continuous map from $\ell_{p,T} \times H_T^p$ into $\ell_{p',T}$ for any $p > p' > 1$.*

Proof: ad (i): By (5.5) the matrix $(C\Omega+D)$ is of the form $\mathbf{1}+$ finite rank matrix. Then the determinant is well defined and equal to $\det(r_n(C\Omega+D))$ for $n \geq n(\gamma)$. Thus the statement follows from the corresponding finite dimensional result for the action of the modular group on the Siegel's upper half plane (see e.g. [11]).

ad (ii): Set $\Omega' = (A\Omega+B)(C\Omega+D)^{-1}$ such that Ω' depends continuously on Ω by Lemma 5.1. As in the finite dimensional case (see e.g. [11] p. 177) it is easy to see that

$$ {}^t(C\Omega + D)^{-1} = -\Omega' C + A \tag{5.9} $$

such that

$$ {}^t z \cdot (C\Omega + D)^{-1} C z = {}^t({}^t(C\Omega + D)^{-1} z) \cdot Cz = {}^t((-\Omega' C + A)z) \cdot Cz $$

By (5.3)

$$ (Cz)_i = 0 \text{ and } (Az)_i = z_i \text{ for all large } i \text{ and all } z \tag{5.10} $$

and the claim (ii) follows.

ad (iii): By (5.9) we have

$$ {}^t(C\Omega + D)^{-1} z = (-\Omega' C + A) z $$

and the claim (iii) follows from lemma 5.1 and (5.10).

Theorem 3.1 and the Lemmas 5.1 and 5.2 now immediately yield the

Theorem 5.3 *For all $\gamma \in \Gamma_{1,2}^\infty$ and all $(z,\Omega) \in \ell_{p,T} \times H_T^p$, $p > 1$, the ϑ-function $\vartheta_T(z|\Omega)$ satisfies the functional equation (5.7).*

Let $\Gamma_4^\infty \subset Sp(\infty,\mathbb{Z})$ be the principal congruence subgroup of level 4, i.e., the subgroup of all $\gamma \in Sp(\infty,\mathbb{Z})$ satisfying $\gamma \equiv \mathbf{1} \bmod 4$. Then Γ_4^∞ is a subgroup of $\Gamma_{1,2}^\infty$ and it follows from [11], p.197, that the 8^{th} root of unity ζ_γ in the functional equation (5.7) satisfies $\zeta_\gamma^2 = 1$ for all $\gamma \in \Gamma_4$. Hence, by (5.7), we obtain

$$ \vartheta_T\left(0 \mid (A\Omega + B)\cdot(C\Omega + D)^{-1}\right)^2 = \det(C\Omega + D)\cdot \vartheta_T(0 \mid \Omega)^2, \quad \gamma \in \Gamma_4^\infty. $$

In analogy with the finite dimensional case we therefore may call $\vartheta_T(0 \mid \Omega)^2$ a modular form of weight 1 with respect to Γ_4^∞.

More generally, we may consider the principal congruence subgroups $\Gamma_n^\infty \subset Sp(\infty,\mathbb{Z})$ of level $n \in \mathbb{N}$ consisting of all $\gamma \in Sp(\infty,\mathbb{Z})$ satisfying $\gamma \equiv \mathbf{1} \bmod n$. As in the finite dimensional case (see [11], p.198), there are groups of intermediate levels $(n, 2n)$

$$ \Gamma_{2n}^\infty \subset \Gamma_{n,2n}^\infty \subset \Gamma_n^\infty $$

where

$$ \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} $$

belongs to $\Gamma_{n,2n}^\infty$ if $\gamma \in \Gamma_n^\infty$ and $2n$ devides the diagonals of B and C. Let n be even and suppose that $a_1, a_2 \in \ell_{p,T^{-1}}(\mathbb{Q})$ and $b_1, b_2 \in \ell_{p,T}(\mathbb{Q})$ $a_1, a_2, b_1, b_2 \in \frac{1}{n}\mathbb{Z}^\infty$. Then the theta functions (3.32) with rational characteristics a_i, b_i also satisfy a functional equation which is similar to (5.7). This functional equation is again derived from the corresponding functional equation in the case of finitely many variables by arguments as above. In this way we obtain as in [11], p. 199, that

$$\vartheta_T\begin{bmatrix}a_1\\b_1\end{bmatrix}(0 \mid \Omega) \cdot \vartheta_T\begin{bmatrix}a_2\\b_2\end{bmatrix}(0 \mid \Omega)$$

is a modular form of weight 1 w.r.t. $\Gamma^{\infty}_{n^2,2n^2}$.

References

[1] Dubrovin, B. A., Krichever, I. M., Novikov, S. P.: Integrable systems I. In: Arnold, V. I., Novikov, S. P. (eds.) Dynamical Systems IV. Encyclopaedia of Mathematical Sciences vol.4, pp. 173-280. Springer, (1990).

[2] Dunford, N., Schwartz, J. T.: Linear operators I. Interscience, New York, 1957.

[3] Feldmann, J., Knörrer, H., Trubowitz, E.: Riemann surfaces of infinite genus I: L^2-cohomology, Exhaustions with finite charge and theta series, FIM (ETH) preprint (1994).

[4] Freitag, E.: Siegelsche Modulfunktionen, Springer, Berlin, Heidelberg, New York, 1983.

[5] Glimm, J., Jaffe, A.: Quantum physics. Springer, Berlin, Heidelberg, New York, 1981.

[6] Gohberg, I.C., Krein, M.G.: Introduction to the theory of linear nonselfadjoint operators. Amer. Math. Soc., Providence, Rhode–Island, 1969.

[7] Igusa, J.: Theta functions, Springer, Berlin, Heidelberg, New York, 1972.

[8] McKean, H., Trubowitz, E.: Hill's operator and hyperelliptic function theory in the presence of infinitely many branchpoints. Commun. Pure Appl. Math. **29**, 143-226 (1976).

[9] McKean, H., Trubowitz, E.: Hill's surfaces and their theta functions. Bull. Amer. Math. Soc. **84** (6), 1052-1085 (1978).

[10] Müller, W., Schmidt, M., Schrader, R.: Hyperelliptic Riemann surfaces of infinite genus and solutions of KdV equations. in preparation.

[11] Mumford, D.: Tata lectures on theta I. Birkhäuser, Boston, Basel, Stuttgart, 1983.

[12] Schmidt, M.: Integrable systems and Riemann surfaces of infinite genus, to appear in Memoirs of the American Math. Soc.

[13] Simon, B.: Trace ideals and their applications, London Math. Soc. Lecture Note Series No 35, Cambridge Univ. Press, Cambridge, 1979.

[14] Tahara, K.: On the hyperelliptic Riemann surfaces of infinite genus with absolutely convergent Riemann's theta functions. Nagoya Math. J. **33**, 57-73 (1968).

[15] Tahara, K.: On formal theta functions of countably many variables. Nagoya Math. J. **33**, 173-194 (1968).