A CELLULAR BRAID ACTION AND THE YANG - BAXTER EQUATION

MIRKO LÜDDE

Abstract. Using a theorem of Schechtman - Varchenko on integral expressions for solutions of Knizhnik - Zamolodchikov equations we prove that the solutions of the Yang - Baxter equation associated to complex simple Lie algebras belong to the class of generalised Magnus representations of the braid group. Hence they can be obtained from the homology of a certain cell complex, or equivalently as group homology of iterated free groups.

Contents

1. Introduction 1
2. The braid action on cell complexes 2
3. Monodromy of Knizhnik - Zamolodchikov equations 6
References 8

1. Introduction

The braid valued Burau representation $B_n \hookrightarrow \mathrm{GL}_n(\mathbb{Z}[B_n \ltimes F_n])$, where B_n is Artin's braid group on n strings and F_n is the free group of rank n, has been introduced as a generalisation of the classical Burau representation in [CL92]. Here we give a construction of it in terms of elementary topology. Moreover we prove that the Yang - Baxter representations of the braid group associated to the quasitriangular Hopf algebras $U_q(\mathfrak{g})$, $\mathfrak{g}$ a complex simple Lie algebra, can be obtained from the braid valued Burau matrices via a homological construction. Therefore these Yang - Baxter matrices belong to the class of generalised Magnus representations introduced in [Lüd92, Lüd96].

By a theorem of E. Artin the braid group B_n faithfully acts on the free group F_n. This action can be understood by the lifting of paths in the Faddell - Neuwirth fibration of configuration spaces. In this way, an action of the braid group on certain cell complexes can be obtained as well. This action is algebraically described by the braid valued Burau matrices. The cellular action when projected to homology yields the monodromy of the natural flat connection (Gauss - Manin connection) of homology fiber bundles associated to the Faddell - Neuwirth fibration. This might be compared with a theorem of Kohno on the reduced Burau representation describing the monodromy of the Jordan - Pochhammer local system [Koh88].

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In [SV91] there is constructed an imbedding of the flat Knizhnik - Zamolodchikov bundle associated to a Kac - Moody algebra into the flat cohomology bundle. We combine this result with the Drinfeld - Kohno theorem, cf. [Koh88, Dri90, Kas95], on the monodromy of the Knizhnik - Zamolodchikov connection. This shows that the Yang - Baxter matrices associated to complex simple Lie algebras can be obtained from the cellular braid action. Hence these Yang - Baxter braid representations can be understood in purely topological terms. To put it algebraically, they are quotients of tensor products of the braid valued Burau modules.

However, in order to obtain an explicit description of the Yang - Baxter matrices, one explicitly had to compute the homology (with local coefficients) of complexes, the investigation of which had been initiated by Vl. Arnold, E. Brieskorn, P. Orlik, L. Solomon and others, see [OT92, SV91].

Nevertheless the presented setting might serve as a step towards an understanding of the topological meaning of solutions of the Yang - Baxter equation and of link invariants constructed from them. A similar approach to the Jones polynomial was made in [Law93, CL95]. In particular, it would be interesting to know whether universal Vassiliev invariants, as constructed from a particular Knizhnik - Zamolodchikov connection, can be understood in this way, too.

The present work goes back to ideas in [CF87, GS91, RRRA91] in the context of conformal field theory and it is a continuation of the investigation started in [CL92, Lüd92, Lüd96]. Closely related work was done in [Law90].

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2. The braid action on cell complexes

We compute the action of the braid group on a 1 - dimensional cell complex. From this action we obtain the braid valued Burau matrices, as defined (differently) in [CL92, Lüd92, Lüd96]. They describe the natural connection of homology bundles associated to the Faddell - Neuwirth fibration.

Proofs of facts on the braid group that are used without further explanation can be found in [Bir74, BZ85].

For every positive integer k let $X_k := \{(x_1, \dots, x_k) \in (\mathbb{R}^2)^k; x_i \neq x_j \text{ if } i \neq j\}$ be the configuration space of k points in the euclidean plane. By a theorem of Faddell and Neuwirth, the projection $p : X_{m+n} \to X_n$ is a topological fiber bundle with fibers $Y_x := p^{-1}(x) = \{(y_1, \dots, y_m) \in X_m; y_i \neq x_j \text{ for } i \in \{1, \dots, m\}, j \in \{1, \dots, n\}\}$. For $m = 1$ the fibers are n - fold punctured euclidean planes.

Given a point $x \in X_n$, we construct a subgroupoid of the fundamental groupoid of the base space X_n. The objects are the permuted points $\{\pi(x); \pi \in S_n\}$ (S_n the permutation group on n elements) and the morphisms are the path classes between these points. Since $B_n \simeq \pi_1(X_n/S_n, x)$, there is a functor mapping this groupoid onto the braid group, by identifying all the objects of G.

By a general fact on fibrations there is another functor mapping the groupoid to the category of topological spaces and homotopy classes of maps. It sends a point $x \in X_n$ to the fiber Y_x over the point and sends a path class in X_n from x to x' to a homotopy class of deformations of Y_x into $Y_{x'}$. These deformations can be chosen so as to map the base points of the fibers Y_x into each other.

In the case $m = 1$ we have the following lemma.

Lemma 1 (Braiding isotopies). *Let $\gamma := (\gamma_1, \dots, \gamma_n) : I \to X_n$ ($I := [0,1]$) be a continuous path (a representative of a braid) between points x, x' with $x = \pi(x')$, $\pi \in S_n$ a permutation.*

1. *There is an isotopy $\psi : I \times \mathbb{R}^2 \to \mathbb{R}^2$ (a continuous family of homeomorphisms), connecting the identity $\mathrm{Id} : \mathbb{R}^2 \to \mathbb{R}^2$ with a homeomorphism $\psi(1) : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\gamma(t) = (\psi(t)(x_1), \dots, \psi(t)(x_n))$.*
2. *Any two such isotopies related to homotopic paths are homotopy equivalent.*
3. *Let $D \subset \mathbb{R}^2$ be a closed subset, homeomorphic to a disc and containing the set $\{\gamma_1(0), \dots, \gamma_n(0)\}$. Then the class of isotopies associated to γ contains a representative ψ such that $\psi \mid (\mathbb{R}^2 \backslash D)$ is the identity mapping.*

We use this topological fact in order to construct a braid action on a cell complex and on its edge - path groupoid.

Lemma 2 (Braid action on a free groupoid in the plane). *Let G be a free groupoid defined by the set of objects $\{0+\epsilon, 1\pm\epsilon, \dots, n\pm\epsilon\}$ for fixed $0 < \epsilon < 1/2$, with base point $P := 0+\epsilon$ and with generating set of morphisms $\{w_0, \dots, w_{n-1}\} \cup \{l_1^\pm, \dots, l_n^\pm\}$, where w_k has domain $k+\epsilon$ and target $k+1-\epsilon$, such that $w_k \in \mathrm{Hom}_G(k+\epsilon, k+1-\epsilon)$, whereas $l_i^\pm \in \mathrm{Hom}_G(i \mp \epsilon, i \pm \epsilon)$. Then there is an imbedding $B_n \hookrightarrow \mathrm{Aut}(G, P)$ of the braid group B_n into the group of automorphisms of G preserving the base point. Under this imbedding a braid generator τ_i acts on the morphisms according to*

$$\tau_i : \quad w_j \mapsto \begin{cases} w_j, \\ w_{i-1} l_i^+ w_i l_{1+i}^+, \\ w_i^{-1}, \\ (l_i^-)^{-1} w_i (l_{1+i}^-)^{-1} w_{1+i}, \end{cases} \qquad l_j^\pm \mapsto \begin{cases} l_j^\pm, & j \notin \{i-1, i, 1+i\}, \\ l_j^\pm, & j = i-1, \\ l_{1+i}^\mp, & j = i, \\ l_i^\mp, & j = 1+i. \end{cases}$$

Proof. We consider the 1 - dimensional cell complex imbedded into the complex plane. The 0 - cells are precisely the objects of G. The 1 - cells are the positively oriented half circles $l_k^\pm := k + S^1(\epsilon)^\pm = \{z \in \mathbb{C}; \mathrm{abs}(z-k) = \epsilon, \Im(z) \leq (\geq) 0\}$ of radius ϵ for $k \in \{1, \dots, n\}$ and the closed intervals $w_m := \{[m+\epsilon, 1+m-\epsilon]; m \in \{0, \dots, n-1\}\}$ of length $1-2\epsilon$. A braid generator τ_i ($i \in \{1, \dots, n-1\}$) defines a class of isotopies of the plane. There is a representative isotopy mapping the points i and $1+i$ into each other and being the identity outside an open disc around the point $i + (1/2)$ of radius R with $(1/2)+\epsilon < R < 1$. The isotopy joins the identity homeomorphism with a self homeomorphism of the disc which can be chosen cellular. In this way the described action on the groupoid emerges. One can check that the braid relations $\tau_i\tau_j = \tau_j\tau_i$ for $\mathrm{abs}(i-j) \geq 2$ and $\tau_i\tau_{1+i}\tau_i = \tau_{1+i}\tau_i\tau_{1+i}$ hold. The map $B_n \to \mathrm{Aut}(G, P)$ is a monomorphism, since the fundamental group $\pi_1(G, P) \leq G$ is a free group F_n of rank n imbedded in G. Its generators are the loops $f_j := \gamma_j l_j \gamma_j^{-1}$ around the points $j \in \{1, \dots, n\}$ with paths $\gamma_j := w_0 (l_1^-)^{-1} w_1 (l_2^-)^{-1} \cdots w_{j-1}$ from P to $j - \epsilon$ and circles $l_j := l_j^+ l_j^-$ around j. (There is a spanning tree consisting of all 1 - cells without the cells l_j^+. The paths γ_j are the unique paths in the spanning tree running from the base point to $j-\epsilon$. The cells l_j^+ therefore are in bijective correspondence with the generators of the fundamental group via $l_j^+ \mapsto \gamma_j l_j^+ (\gamma_j (l_j^-)^{-1})^{-1}$.) The action of the braid group restricted to this subgroup is precisely the action of Artin's theorem on the imbedding $B_n \hookrightarrow \mathrm{Aut}(F_n)$. Indeed, we get

$$\tau_i: \quad \gamma_j \mapsto \begin{cases} \gamma_j, \\ \gamma_{j-1} l_{j-1}^+, \\ \gamma_j l_j^+ w_j l_{1+j}^+, \end{cases} \qquad l_j \mapsto \begin{cases} l_j, & i \notin \{j-1, j\}, \\ l_{j-1}^- l_{j-1}^+, & i = j-1, \\ l_{1+j}^- l_{1+j}^+, & i = j. \end{cases}$$

This combines to

$$\tau_i : f_j \mapsto \begin{cases} f_j, & i \notin \{j-1, j\}, \\ f_{j-1}, & i = j-1, \\ f_j f_{1+j} f_j^{-1}, & i = j, \end{cases}$$

an action that is known to be faithful. □

Given a free connected groupoid (G, P) with base point P such that G is freely generated by a subset $S \subset \cup_{x,y} \mathrm{Hom}_G(x, y)$ we can construct a free left $\pi_1(G, P)$ module M. It is freely generated by elements $[s]$ bijectively corresponding to the generators $s \in S$. For every object x of G we choose a path $\gamma_x \in \mathrm{Hom}_G(P, x)$ joining the base point P with the point x. If $E \leq \mathrm{End}(G, P)$ is a monoid of endomorphisms of G preserving the base point, the tensor product $\mathbb{Z}[E \ltimes \pi_1(G, P)] \otimes_{\pi_1(G,P)} M$ (the semidirect product $E \ltimes \pi_1(G, P)$ is defined by the action of E on G) obtains a structure as right E module by setting

$$(1 \otimes [s])e := e \otimes (e(\gamma_x) \cdot \gamma_{e(x)}^{-1})\delta(e(s)),$$

where $s \in \mathrm{Hom}_G(x, z)$, $e(s) \in \mathrm{Hom}_G(e(x), e(z))$, $e \in E$ and δ is the extended Fox derivation for groupoids. It is uniquely determined by the equations

$$\begin{aligned} \delta(s) &= [s], \\ \delta(gg') &= \delta(g) + (\gamma_x \cdot g \cdot \gamma_y^{-1})\delta(g'), \end{aligned}$$

with $g \in \mathrm{Hom}_G(x, y)$, $g' \in \mathrm{Hom}_G(y, z)$. (Notice that this implies, $\delta(\mathrm{Id}_x) = 0$ and $\delta(g^{-1}) = -\gamma_y \cdot g^{-} \cdot \gamma_x^{-1}\delta(g)$, analogous to the case of the common Fox derivation.) The properties of δ are precisely the properties of a lifting map, which maps paths in the cell complex to the corresponding chains in the chain complex of the universal covering, see [BZ85]. Hence we have lifted the action of E on the groupoid G to an action on the chain complex of the universal covering of G.

In this way, from the cellular action of the braid group on our cell complex we obtain modules over the braid group.

Lemma 3 (Braid valued Burau modules from groupoid)**.** 1. *The $\mathbb{Z}[B_n \ltimes F_n]$ - $\mathbb{Z}[B_n]$ bimodule induced by the cellular braid action projects to a quotient carrying a braid representation in terms of the matrices (which map to the classical reduced Burau matrices in $\mathrm{GL}_n(\mathbb{Z}[t, t^{-1}])$ for $\tau_i \mapsto 1$, $f_j \mapsto t$)*

$$\tau_j \begin{pmatrix} \mathbf{1}_{j-2} & 0 & 0 & 0 & 0 \\ 0 & 1 & f_j & 0 & 0 \\ 0 & 0 & -f_j & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_{n-j-1} \end{pmatrix}$$

in $\mathrm{GL}_n(\mathbb{Z}[B_n \ltimes F_n])$ (where the semidirect product $B_n \ltimes F_n$ is defined by the previously computed action of B_n onto $\pi_1(G, P) \simeq F_n$ and $\mathbf{1}_k$ is the k by k unit matrix).

2. *It also contains a submodule with braid action described by the matrices (which map to the classical unreduced Burau matrices)*
$$\tau_i \begin{pmatrix} \mathbf{1}_{i-1} & 0 & 0 & 0 \\ 0 & 1 - f_i f_{i+1} f_i^{-1} & f_i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{n-i-1} \end{pmatrix}.$$

Proof. We choose paths $\gamma_j^+ := w_0(l_1^-)^{-1} w_1 (l_2^-)^{-1} \cdots w_{j-1}(l_j^-)^{-1}$ from P to $j+\epsilon$ and $\gamma_j^- := w_0(l_1^-)^{-1} w_1(l_2^-)^{-1} \cdots w_{j-1}$ from P to $j-\epsilon$, $j \in \{1, \ldots, n\}$. These paths under the braid action transform as
$$\tau_i : \quad \gamma_j^- \mapsto \begin{cases} \gamma_j^-, \\ f_{j-1}\gamma_{j-1}^+, \\ f_j f_{1+j} \gamma_{1+j}^+, \end{cases} \qquad \gamma_j^+ \mapsto \begin{cases} \gamma_j^+, & i \notin \{j-1, j\}, \\ \gamma_{j-1}^-, & i = j-1, \\ f_j \gamma_{1+j}^-, & i = j. \end{cases}$$
The representation that emerges on the free module is then given by

$\tau_i : 1 \otimes_{\pi_1(G,P)} [w_j] \mapsto$
$$\tau_i \otimes_{\pi_1(G,P)} \begin{cases} \tau_i(\gamma_j^+)(\gamma_j^+)^{-1}\delta(w_j), & j \notin \{i-1, i, 1+i\}, \\ \tau_i(\gamma_j^+)(\gamma_j^+)^{-1}\delta(w_j l_{1+j}^+ w_{1+j} l_{2+j}^+), & j = i-1, \\ \tau_i(\gamma_j^+)(\gamma_{1+j}^-)^{-1}\delta(w_j^{-1}), & j = i, \\ \tau_i(\gamma_j^+)(\gamma_{j-1}^-)^{-1}\delta((l_{j-1}^-)^{-1} w_{j-1} (l_j^-)^{-1} w_j), & j = 1+i \end{cases}$$
which can be calculated to be

$\tau_i : 1 \otimes_{\pi_1(G,P)} [w_j] \mapsto$
$$\tau_i \otimes_{\pi_1(G,P)} \begin{cases} [w_j], & j \notin \{i-1, i, 1+i\}, \\ [w_j] + [l_{1+j}^+] + f_{1+j}[w_{1+j}] + f_{1+j}[l_{2+j}^+], & j = i-1, \\ -f_j[w_j], & j = i, \\ -[l_{j-1}^-] + [w_{j-1}] - [l_j^-] + [w_j], & j = 1+i. \end{cases}$$
By a similar computation, we can find the transformation of the remaining generators. It is clear already from our previous lemma, that the submodule of $[l_j^\pm]$ cells is invariant under the braid action. We therefore can consider the quotient by this submodule and obtain a module of chains $[w_j]$ alone, which yields the given matrix representation. (This corresponds to the limit $\epsilon \to 0$, in which the length of the cells $l_j^\pm$ approaches zero.) The second matrix representation is obtained similarly by considering the braid action on the module generated by elements $\delta(f_j)$ corresponding to the free group $\pi_1(G, P)$. □

The braid valued Burau matrices can be iterated. There is an imbedding $B_n \ltimes F_n \hookrightarrow B_{1+n}$ such that the matrices can be considered to have values in $\mathrm{GL}_n(\mathbb{Z}B_{1+n})$. Hence the matrix elements themselves can be represented by braid valued Burau matrices, and so on. The presented construction suggests that the m - fold iteration describes the braid action on m - dimensional cell complexes homotopic to the $2m$ - dimensional fibers Y_x of the fiber bundle $p : X_{n+m} \to X_n$. Using the fact that $\pi_1(Y_x, y) \simeq F_n \ltimes \cdots \ltimes F_{n+m-1}$ (the semidirect products are defined by the action of the image of $B_n \ltimes F_n \hookrightarrow B_{1+n}$ onto F_{1+n}) and that all higher homotopy groups vanish, this indeed has been proven in [Lüd96]. More precisely, the (iterations of the) braid valued Burau matrices describe a braid action on a free resolution for $\pi_1(Y_x, y)$ and therefore on the (group-) homology $H_*(Y_x; L) \simeq H_*(\pi_1(Y_x, y); L)$ for suitable local coefficient systems L.

We summarise the content of the present section.

1 (Homology bundle with flat connection)**.** *Let L be a locally constant sheaf of vector spaces on X_{n+m}.*

1. *There is a vector bundle over X_n with fibers $H_*(Y_x, i_x^*(L))$, possessing a natural flat connection (Gauss - Manin connection in homology).*
2. *The monodromy of the connection is described (on the level of cell complexes) by the (iterations of the) braid valued Burau modules.*

Proof. Consider the sheaf associated to the presheaf $U \mapsto H_k(V, L \mid V)$, where $U \subset X_n$ is any open set and $V := p^{-1}(U) \subset X_{n+m}$. For suitable neighborhoods U of $x \in X_n$, $H_k(V, L \mid V) \simeq H_k(U \times Y_x, L) \simeq H_k(Y_x, i_x^*(L))$, since L is locally constant. This determines the stalks of the sheaf to be $H_k(Y_x, i_x^*(L))$. It also shows, it is a locally constant sheaf of vector spaces. Hence it gives rise to a flat vector bundle. The second part of the assertion follows from our previous construction and remarks. □

The iteration procedure of the braid valued Burau matrices seems to produce a huge amount of braid representations (these representations yield the generalised Magnus representations of [Lüd96]), which, however, appear to be difficult to handle. Both facts will be illustrated in the next section, where two strong theorems are needed to show that we may obtain a lot of solutions of the Yang - Baxter equation from it.

3. Monodromy of Knizhnik - Zamolodchikov equations

In [CF87, SV91] there have been given integral expressions for flat sections of the Knizhnik - Zamolodchikov vector bundles. The families of homology cycles over which the integrals are taken are flat sections of the natural flat connection of the homology bundles of the Faddell - Neuwirth fibration. Combined with the Drinfeld - Kohno theorem this shows that the Yang - Baxter matrices of complex simple Lie algebras can be understood topologically: they are induced from the cellular action of the braid group described by the braid valued Burau matrices.

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra with Cartan subalgebra $\mathfrak{h} \leq \mathfrak{g}$, with simple roots $\alpha_i \in \mathfrak{h}^*$ and with Chevalley raising (lowering) operators e_i (f_i), $i \in \{1, \ldots, r\}$, respectively. Let $\Lambda_1, \ldots, \Lambda_n \in \mathfrak{h}^*$, and let $L(\Lambda_i)$ be the irreducible heighest weight module of weight Λ_i. Let $C \in \mathfrak{g} \otimes \mathfrak{g}$ be the canonical element w. r. t. the Killing form $K : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ and let $C_{i,j}$ be the representation of C on the i - th and j - th factor in $L := L(\Lambda_1) \otimes \cdots \otimes L(\Lambda_n)$. Then the Knizhnik - Zamolodchikov equation with respect to $\mathfrak{g}, L$ and $\kappa \in \mathbb{C}\backslash\{0\}$ is the equation $0 = d(s) - \kappa \sum_{1 \leq i < j \leq n} \frac{C_{i,j}(s) d(x_i - x_j)}{(x_i - x_j)}$ for a local section $s : U \to L$ of the trivial vector bundle $L \times X_n \to X_n$, see [KZ84, Koh88].

Let $\lambda := \sum_{i=1}^r k_i \alpha_i \in \mathfrak{h}^*$ be a positive root, $\sum k_i = m$ and let $L_{\Lambda - \lambda} \subset L$ be the corresponding weight space, where $\Lambda := \sum_{i=1}^n \Lambda_i$. It has been shown in [SV91] how to construct solutions of the equation with values in the subspace $V_\lambda := \bigcap_{i=1}^r \mathrm{Ker}(e_i : L_{\Lambda - \lambda} \to L)$. For definiteness, we shortly review the construction.

Consider the Faddell - Neuwirth fiber bundle $p : Y \to X$, $Y := X_{m+n}$, $X := X_n$, $(y_1, \ldots, y_{m+n}) \mapsto (x_1, \ldots, x_n) := (y_1, \ldots, y_n)$ with fibers $i_x : Y_x \hookrightarrow Y$. We choose a basepoint $y_0 := (1, \ldots, n+m) \in Y$ and identify $\pi_1(Y, y_0) \simeq P_{n+m}$, where P_{n+m} is the pure braid group on $n + m$ strings. Its generators $\{\vartheta_{i,j} := \vartheta_{j,i}, 1 \leq i < j \leq$

$n+m\}$ are represented by closed paths in Y, where the point j encircles the point i once in a positive orientation and does not encircle any other components of y_0.

We define a homomorphism $L \in \mathrm{Hom}(\pi_1(Y, y_0), \mathbb{C}\backslash\{0\})$ by setting $L(\vartheta_{i,j}) := \exp(2\sqrt{-1}\pi a_{i,j})$ for exponents

$$a_{i,j} := \kappa \begin{cases} K(\Lambda_i, \Lambda_j), & 1 \leq i, j \leq n, \\ -K(\alpha_{\pi(i)}, \Lambda_j), & n+1 \leq i \leq n+m; 1 \leq j \leq n, \\ K(\alpha_{\pi(i)}, \alpha_{\pi(j)}), & n+1 \leq i, j \leq n+m. \end{cases}$$

Here we have set $\pi : \{1, \dots, m\} \to \{1, \dots, r\}$ such that $\pi(j) = i$ if $\sum_{p=1}^{i-1} k_p < j \leq \sum_{p=1}^{i} k_p$. The local system equivalently is defined by the function $l_\lambda := \prod_{1 \leq i < j \leq n+m} (y_i - y_j)^{a_{i,j}}$.

Let (Ω_Y^*, d) and (Ω_X^*, d) be the complexes of sheaves of holomorphic differential forms on Y and X, respectively. Similarly, let Ω_L^* be the complex of sheaves of holomorphic differential forms on Y with values in the flat vector bundle (local system) $\mathbb{C} \times_L \tilde{Y}$ (where $\tilde{Y}$ is the universal covering) determined by L. Furthermore, $\Omega_{L,p}^* := \Omega_L^* / (p^*(\Omega_X^1) \wedge \Omega_L^*)$ is the quotient of the Ω_Y^* - module of L - valued forms by the submodule generated by the pulled back 1 - forms on X, i.e. the relative de Rham complex of sheaves of p - vertical forms. Finally, $\Omega^* := p_*(\Omega_{L,p}^*)$ is the direct image sheaf on the base space X. Let $\mathfrak{H}^*$ be the cohomology sheaf of the complex Ω^* and let $\mathfrak{H}_*$ be the sheaf of sections on X of the homology bundle induced by the Faddell - Neuwirth fibration and the dual local system L^*. There is the de Rham map $\int : \mathfrak{H}_* \otimes (V_\lambda \otimes \mathfrak{H}^*) \to V_\lambda$, integrating vertical differential forms over homology classes of the fibers.

In [SV91] it was shown that there are holomorphic functions ϕ_I on Y, being rational in $\mathbb{C}^{m+n} \supset Y$, and vectors $v_I \in L_{\Lambda - \lambda}$ (for I belonging to a suitable index set) such that for $\eta_I := (2\sqrt{-1}\pi)^{-m} \phi_I l_\lambda dy_{1+n} \wedge \dots \wedge dy_{m+n}$ the function $\eta := \sum_I \eta_I \otimes v_I$ has the following property (which still holds for $\mathfrak{g}$ being a Kac - Moody algebra of a symmetrisable generalised Cartan matrix).

2 (Solutions of Knizhnik - Zamolodchikov equation)**.** *Let $\sigma \in \mathfrak{H}_m(U)$ be a family of cycles that is flat with respect to the natural connection. Then the map $U \to V_\lambda$, $x \mapsto \int_{\sigma(x)} \eta(x)$ satisfies the Knizhnik - Zamolodchikov equation. For sufficiently small* $\mathrm{abs}(\kappa) \neq 0$ *the set of these functions for varying σ is a complete set of solutions.*

If $\Lambda_1 = \dots = \Lambda_n$, the local system L defined above is invariant under the permutation of the coordinates $(y_1, \dots, y_n)$ of the base space. Therefore the holonomy of the natural connection in homology yields a representation of the braid group $B_n \simeq \pi_1(X_n/S_n, x)$ on the cycles $\sigma(x) \in H_m(Y_x; L^*)$. Knowing also the action of B_n on the function η, we obtain a braid representation from the monodromy of the integral expressions.

Corollary 1 (K. - Z. monodromy from braid valued Burau matrices)**.** *Let V be an irreducible module over the quasitriangular Hopf algebra $U_q(\mathfrak{g})$. Then the braid group representations $B_n \to \mathrm{Aut}(V^{\otimes n})$ induced from the universal R - matrix of $U_q(\mathfrak{g})$ can be obtained as group homology modules $H_m(\pi_1(Y_x; y), i_x^*(L))$ and with braid action induced from the braid valued Burau module.*

Proof. On the one hand, the monodromy of the Knizhnik - Zamolodchikov equation associated to V yields the representation of B_n coming from the universal R

- matrix of $U_q(\mathfrak{g})$. This follows from the Drinfeld - Kohno theorem, see e.g [Kas95, sect. XIX.4, pp. 458]. On the other hand, the solutions of the Knizhnik - Zamolodchikov equation for small κ are given as integrals over cycles in $H_m(Y_x; i_x^*(L))$. This homology can be computed as group homology of $\pi_1(Y_x, y_0)$ and it carries the braid representation induced from the braid valued Burau matrices [Lüd96]. Hence for small values of κ, both representations coincide. But from Chen's iterated integral expression for the monodromy of the Knizhnik - Zamolodchikov equation we know it is an entire function in the variable κ, see [Koh88]. Similarly, the braid representation deduced from the braid valued Burau matrices is given by polynomial functions in $\exp(2\sqrt{-1}\pi\kappa)$, hence as well is entire in κ. Therefore both representations coincide as functions of κ. $\square$

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Institut für Reine Mathematik, Humboldt Universität zu Berlin, Ziegelstrasse 13a, D-10099 Berlin, Germany

E-mail address: `luedde@mathematik.hu-berlin.de`