Quantum Electrodynamics of Confined Non-Relativistic Particles

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Abstract

We consider a system of finitely many non-relativistic, quantum mechanical electrons bound to static nuclei. The electrons are minimally coupled to the quantized electromagnetic field; but we impose an ultraviolet cutoff on the electromagnetic vector potential appearing in covariant derivatives, and the interactions between the radiation field and electrons localized very far from the nuclei are turned off. For a class of Hamiltonians we prove exponential localization of bound states, establish the existence of a ground state, and derive sufficient conditions for its uniqueness. Furthermore, we show that excited bound states of the unperturbed system become unstable and turn into resonances when the electrons are coupled to the radiation field, by using a novel renormalization transformation which acts directly on the space of Hamiltonians.

I Introduction: Description of the Mathematical Problem and Main Results

In this paper we establish mathematical results concerning physical phenomena that stood at the origin of quantum theory: that of emission and absorption of electromagnetic radiation by systems of non-relativistic, quantum mechanical matter, such as atoms and molecules.

Our analysis supplies *nonperturbative* results supporting the conventional picture of atoms and molecules interacting with the quantized radition field provided by low-order quantum mechanical perturbation theory, adding mathematical precision to it. In particular, our results concerning the structure of resonances in the fully interacting theory corresponding to excited energy levels of an atom or molecule decoupled from the radiation field are not entirely part of the conventional wisdom in this field and complement the perturbative analysis, first carried out by Bethe [7, 8], of the Lamb shift.

We conduct our analysis on a model, introduced in Subsection I.2, which is somewhat simpler than the standard model (see Subsection I.1), but which nevertheless retains all the physical features of the latter. In fact, we believe that the infrared problem in our model is more severe than in the standard one. Indeed, as stipulated by the Pauli-Fierz transformation (see Subsection I.1) and will be explained elsewhere, the particular form of the interaction in the standard model (minimal coupling, leading to gauge invariance) can be used to improve the infrared behaviour of the interaction considerably and to make it, at least formally, much more benign than the one considered in this paper. The full standard model will be considered elsewhere.

I.1 The Standard Model of Non-Relativistic Quantum Electrodynamics

The starting point of our analysis is the following *standard model of non-relativistic, quantum mechanical matter and radiation*. The system consists of a finite number of nuclei, in the following treated as static, and of electrons

treated as non-relativistic, quantum mechanical point particles coupled to the quantized radiation field. The Hilbert space of the system is given by

$$\boldsymbol{\mathcal{H}} \;=\; \boldsymbol{\mathcal{H}}_{el} \otimes \boldsymbol{\mathcal{H}}_{f} \,, \tag{I.1}$$

where

$$\boldsymbol{\mathcal{H}}_{el} \;:=\; (L^2(\mathbb{R}^3, d^3x) \otimes \mathbb{C}^2)^{\otimes_a N} \,, \tag{I.2}$$

and

$$\boldsymbol{\mathcal{H}}_{f} \equiv \boldsymbol{\mathcal{F}} \;:=\; \bigoplus_{n=0}^{\infty} (L^2(\mathbb{R}^3, d^3k) \otimes \mathbb{C}^2)^{\otimes_s n} \,. \tag{I.3}$$

In these formulae, the subscript "el" refers to *electronic* and the subscript "f" refers to *photon field*. Moreover, N denotes the total number of electrons (which is conserved), the factor $\mathbb{C}^2$ on the r.h.s. of (I.2) accounts for the spin of electrons, and $\otimes_a$ denotes an anti-symmetric tensor product, in accordance with the Pauli principle. The space $\boldsymbol{\mathcal{H}}_f = \boldsymbol{\mathcal{F}}$ is the Fock space of photons. The factor $\mathbb{C}^2$ on the r.s. of (I.3) accounts for the two possible polarizations of photons, and $\otimes_s$ denotes a symmetric tensor product appropriate for Bose-Einstein statistics.

The transverse modes of the quantized electromagnetic field are described in terms of the vector potential, $\mathbf{A}$, in the Coulomb gauge: At time $t = 0$, $\mathbf{A}$ is given by

$$\mathbf{A}(\mathbf{x}) \;:=\; \sum_{\lambda=1,2} \frac{1}{\pi} \int \frac{d^3k}{\sqrt{2\omega(\mathbf{k})}} [\, \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) \mathrm{e}^{-i\mathbf{k}\cdot\mathbf{x}} a^\dagger{}_\lambda(\mathbf{k}) + \boldsymbol{\varepsilon}_\lambda(\mathbf{k})^* \mathrm{e}^{i\mathbf{k}\cdot\mathbf{x}} a_\lambda(\mathbf{k})] \,, \tag{I.4}$$

where $\omega(\mathbf{k}) = |\mathbf{k}|$ is the energy of a photon of momentum $\mathbf{k}$; $\boldsymbol{\varepsilon}_\lambda(\mathbf{k})$, $\lambda = 1, 2$, are polarization vectors satisfying

$$\boldsymbol{\varepsilon}_\lambda(\mathbf{k})^* \cdot \boldsymbol{\varepsilon}_\mu(\mathbf{k}) = \delta_{\lambda\mu}, \quad \mathbf{k} \cdot \boldsymbol{\varepsilon}_\lambda(\mathbf{k}) = 0, \quad \lambda, \mu = 1, 2;$$

and $a^\dagger{}_\lambda(\mathbf{k})$, $a_\lambda(\mathbf{k})$ are the usual creation and annihilation operators on $\boldsymbol{\mathcal{F}}$ obeying the canonical commutation relations

$$[a^\#_\lambda(\mathbf{k}_1), a^\#_\mu(\mathbf{k}_2)] = 0, \quad [a_\lambda(\mathbf{k}_1), a^\dagger{}_\mu(\mathbf{k}_2)] = \delta_{\lambda\mu}\delta(\mathbf{k}_1 - \mathbf{k}_2), \tag{I.5}$$

where $a^\# = a$ or $a^\dagger$. The vector potential $\mathbf{A}$ and the creation and annihilation operators are unbounded, operator-valued distributions on $\boldsymbol{\mathcal{F}}$. The space $\boldsymbol{\mathcal{F}}$

contains a unique vector Ω, the vacuum vector, with the property that $a_\lambda(\mathbf{k})\Omega = 0$, for all λ and $\mathbf{k}$. The free time-evolution of the radiation field is generated by the Hamiltonian

$$\mathbb{H}_f \;:=\; \sum_{\lambda=1,2} \int d^3k\; \omega(\mathbf{k})\; {a^\dagger}_\lambda(\mathbf{k}) a_\lambda(\mathbf{k})\;. \tag{I.6}$$

The free time evolution of the quantized vector potential is given by

$$\mathbf{A}(\mathbf{x},t) \;=\; \mathrm{e}^{it\mathbb{H}_f}\,\mathbf{A}(\mathbf{x})\mathrm{e}^{-it\mathbb{H}_f}\;,$$

while the evolution of the *free* electric and magnetic fields is given by

$$\mathbf{E}(\mathbf{x},t) \;=\; \frac{\partial}{\partial t}\mathbf{A}(\mathbf{x},t), \qquad \mathbf{B}(\mathbf{x},t) \;=\; \nabla \wedge \mathbf{A}(\mathbf{x},t)\;.$$

Here we use units such that $\hbar = c = 1$, where $\hbar$ is Planck's constant and c is the velocity of light.

The Hamiltonian generating the time evolution of the total system composed of the photon field and N electrons moving in the additional electrostatic potential generated by M fixed nuclei of charges $Z_\ell\, e$ at positions $\mathbf{R}_\ell$, $\ell = 1, \ldots, M$ is given by

$$\mathbb{H} \;=\; \mathbb{H}_{el+f} \;+\; \mathbf{1}_{el} \otimes \mathbb{H}_f\;, \tag{I.7}$$

with

$$\mathbb{H}_{el+f} \;=\; \sum_{j=1}^{N} \frac{1}{2m}[\,\boldsymbol{\sigma}_j \cdot (-i\boldsymbol{\nabla}_j - e_j \mathbf{A}_\kappa(\mathbf{x}_j))]^2 + \alpha\, \mathbb{V}(x,R) \otimes \mathbf{1}_f \;\;, \tag{I.8}$$

where m is the mass of an electron, $\boldsymbol{\sigma}_j = (\sigma_j^x, \sigma_j^y, \sigma_j^z)$, σ_j^x, σ_j^y and σ_j^z are the three Pauli matrices acting on spin space of the j^{th} electron, $\alpha = \mathrm{e}^2/4\pi \cong (137)^{-1}$ is the feinstructure constant, $\mathbf{A}_\kappa(\mathbf{x}) = \int d^3y\, \check{\kappa}(\mathbf{x}-\mathbf{y})\mathbf{A}(\mathbf{y})$ is the regularized vector potential, where $\check{\kappa}$ is the Fourier transform of a smooth function, κ, with support contained in the ball $\{\mathbf{k} : |\mathbf{k}| \leq \text{const} \cdot \alpha m\}$, and $\mathbb{V}(x,R) = \sum_{j=1}^{N}\sum_{\ell=1}^{M} \frac{-Z_\ell}{|\mathbf{x}_j - \mathbf{R}_\ell|} + \sum_{1\leq i<j\leq N} \frac{1}{|\mathbf{x}_i - \mathbf{x}_j|}$. Here $x = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$, $R = (\mathbf{R}_1, \ldots, \mathbf{R}_M)$, $-\sum_{\ell=1}^{M} Z_\ell \alpha |\mathbf{x} - \mathbf{R}_\ell|^{-1}$ is the electrostatic potential at $\mathbf{x}$ corresponding to the charge distribution of the nuclei, and $\alpha/|\mathbf{x}_i - \mathbf{x}_j|$ is the repulsive Coulomb potential between the i^{th} and the j^{th} electron. We use the subscript "$el+f$" to indicate that $\mathbb{H}_{el+f}$ couples the electrons in the atom or molecule to the photon field. The Hamiltonian $\mathbb{H}_{el+f}$ describes the *standard model* of Quantum Electrodynamics for

non-relativistic particles. The smoothing operation applied to the original vector potential operator $\mathbf{A}(\mathbf{x})$ is called the *ultraviolet cutoff*. It is justified by the fact that the phenomena we study are non-relativistic in nature as far as particles are concerned. It is needed in order for the Hamiltonian to be well-defined.

To exhibit the perturbative character of the present spectral problem we change our units. To this end, we dilate the electron coordinates and photon momenta independently, $(\mathbf{x}_j, \mathbf{k}) \mapsto (\eta \mathbf{x}_j, \mu \mathbf{k})$, employing a suitable unitary operator, U_1, on $\boldsymbol{\mathcal{H}}$. Upon making the choice $\frac{1}{2m\eta^2} = \frac{\alpha}{\eta} = \mu$, we obtain

$$\begin{aligned} \mathbb{H}_1 \;&:=\; \mu^{-1}\, U_1\, \mathbb{H}\, U_1^* \qquad\qquad (\text{I}.9)\\ &=\; \sum_{j=1}^{N} \Big[\boldsymbol{\sigma}_j \cdot \Big(-i\boldsymbol{\nabla}_{\mathbf{x}_j} - 2\pi^{1/2}\alpha^{3/2}\mathbf{A}_{\bar\kappa}(\alpha \mathbf{x}_j)\Big)\Big]^2 \\ &\qquad\qquad + \mathbb{V}(x, \eta^{-1}R) \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes \mathbb{H}_f \;, \end{aligned}$$

where $\bar\kappa(\mathbf{k}) := \kappa(\mu \mathbf{k})$. This expression suggests to study the operator $\mu^{-1}U_1 H U_1^*$ as a perturbation of the operator $\left\{ \sum_{j=1}^{N} -\Delta_j + \mathbb{V}(x, \eta^{-1}R) \right\} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes \mathbb{H}_f$, because $\alpha \approx 1/137$ is small.

The problem of radiation involves the control of photons at low energies. The feature of the Hamiltonian which determines whether such a control is possible is the small $|\mathbf{k}|$ behaviour of the coupling function $g(\mathbf{k}) := \bar\kappa(\mathbf{k})(2\omega(\mathbf{k}))^{-1/2}$, showing how strongly the low energy photons are coupled to the particles. As far as the scaling dimension of the perturbation (see also Section I.3 below) is concerned, $g = \text{const}\, \omega(\mathbf{k})^{-1/2}$, as $|\mathbf{k}| \to 0$, is exactly a borderline case. Some of the results of this paper require that $\int \frac{g^2}{\omega^2} < \infty$ which the standard model for which $\bar\kappa(\mathbf{k}) \to 1$, as $|\mathbf{k}| \to 0$, misses by a wisker. However, on a formal level this problem is removed if we perform a Pauli-Fierz transformation [29]. Assuming that the charge distribution of the nuclei is concentrated around the origin in $\mathbb{R}^3$, we unitarily transform the Hamiltonian $\mathbb{H}_1$ in (I.9) by $U_2 := \exp\Big[-i\tau \mathbf{A}_{\bar\kappa}(\mathbf{0}) \cdot \Big(\sum_{j=1}^{N} \mathbf{x}_j\Big)\Big]$. This unitary conjugation leaves all terms in $\mathbb{H}$ unchanged except for

$$U_2(-i\ \boldsymbol{\nabla}_{\mathbf{x}_j})U_2^* \;=\; -i\ \boldsymbol{\nabla}_{\mathbf{x}_j} + \tau \mathbf{A}_{\bar\kappa}(\mathbf{0})\,,$$

and

$$U_2\, a^\dagger_\lambda(\mathbf{k})\, U_2^* \;=\; a^\dagger_\lambda(\mathbf{k}) \;+\; \frac{i\tau\kappa(\mathbf{k})}{\pi\sqrt{2\omega(\mathbf{k})}} \left(\sum_{j=1}^N \boldsymbol{\varepsilon}_\lambda(\mathbf{k})^* \cdot \mathbf{x}_j\right) \;.$$

Let $\mathbb{H}_2 \;:=\; U_2\, \mathbb{H}_1\, U_2^*$. Choosing $\tau := 2\sqrt{\pi}\alpha^{3/2}$ and using that

$$\Big[\boldsymbol{\sigma}\cdot\Big(-i\,\boldsymbol{\nabla}_{\mathbf{x}} - \tau\mathbf{A}_{\bar\kappa}(\alpha\mathbf{x})\Big)\Big]^2 \;=\; \Big(-i\,\boldsymbol{\nabla}_{\mathbf{x}} - \tau\mathbf{A}_{\bar\kappa}(\alpha\mathbf{x})\Big)^2 \;+\; \frac{\tau\alpha}{2}\Big(\boldsymbol{\nabla}_{\mathbf{x}}\wedge\mathbf{A}_{\bar\kappa}\Big)(\alpha\mathbf{x})\,,$$

we obtain

$$\begin{aligned}
\mathbb{H}_2 \;:=\; U_2\, \mathbb{H}_1\, U_2^* \;=\; & \qquad\qquad (I.10)\\
&\left(\mathbb{H}_{el} \;+\; 4\pi\alpha^3 c_\kappa\Big(\sum_{j=1}^N \mathbf{x}_j\Big)^2\right)\otimes\mathbf{1} + \mathbf{1}\otimes\mathbb{H}_f\\
&+ 2\sqrt{\pi}\alpha^{3/2}\,\mathbf{E}_{\bar\kappa}(0)\cdot\Big(\sum_{j=1}^N \mathbf{x}_j\Big) \;+\; \pi^{1/2}\alpha^{5/2}\sum_{j=1}^N\Big\{\boldsymbol{\sigma}_j\cdot\mathbf{B}_{\bar\kappa}(\alpha\mathbf{x}_j)\Big\}\\
&+ \sum_{j=1}^N\Big\{i4\pi^{1/2}\alpha^{3/2}\;\boldsymbol{\nabla}_{\mathbf{x}_j}\cdot[\mathbf{A}_{\bar\kappa}(\alpha\mathbf{x}_j) - \mathbf{A}_{\bar\kappa}(\mathbf{0})] \;+\; 4\pi\alpha^3[\mathbf{A}_{\bar\kappa}(\alpha\mathbf{x}_j) - \mathbf{A}_{\bar\kappa}(\mathbf{0})]^2\Big\}\;,
\end{aligned}$$

where $\mathbf{E}_{\bar\kappa}(\mathbf{x}) := i[\mathbb{H}_f\,,\,\mathbf{A}_{\bar\kappa}(\mathbf{x})$ and $\mathbf{B}_{\bar\kappa}(\mathbf{x}) := \boldsymbol{\nabla}\wedge\mathbf{A}_{\bar\kappa}(\mathbf{x})$ and

$$\mathbb{H}_{el} \;:=\; \sum_{j=1}^N -\Delta_j \;+\; \mathbb{V}(x,\eta^{-1}R)\,. \qquad (I.11)$$

Here, $c_{\bar\kappa} = \pi^{-2}\int\kappa(\mathbf{k})^2 d^3k$ is a cutoff-dependent constant. This expression shows that the singular behaviour of the coupling function $\bar\kappa(\mathbf{k})(2\omega(\mathbf{k}))^{-1/2}$ can be traded for a perturbation growing as $|\sum_{j=1}^N \mathbf{x}_j| \to \infty$. The latter does not sound so bad, at least on a conceptual level, since the processes of radiation we consider in this paper involve bound, and therefore exponentially localized, electrons (see Theorem II.1 below). Note that $\mathbb{H}_2$ can be written as

$$\begin{aligned}
\mathbb{H}_2 \;=\; & \left(\mathbb{H}_{el} \;+\; g^2 c_{\bar\kappa}\Big(\sum_{j=1}^N \mathbf{x}_j\Big)^2 \;+\; g^2\sum_{j=1}^N f(\alpha\mathbf{x}_j)\right)\otimes\mathbf{1}_f \qquad (I.12)\\
&\qquad\qquad + \mathbf{1}_{el}\otimes\mathbb{H}_f \;+\; \mathbb{W}_g\;,\\
\mathbb{W}_g \;=\; & \; g\,\mathbb{W}_1 \;+\; g^2\,\mathbb{W}_2\;, \qquad (I.13)
\end{aligned}$$

where $g := 2\sqrt{\pi}\alpha^{3/2}$, $f(\mathbf{x}) := \frac{4}{\pi^2}\int\frac{\bar\kappa^2(\mathbf{k})sin^2(\mathbf{k}\cdot\mathbf{x})d^3k}{\omega(\mathbf{k})}$, and

$$\begin{aligned}
\mathbb{W}_1 \;=\; & \int\{\,\mathbb{G}_{1,0}(\mathbf{k})\otimes a^\dagger{}_\lambda(\mathbf{k}) + \mathbb{G}_{0,1}(\mathbf{k})\otimes a_\lambda(\mathbf{k})\,\}\,d\mathbf{k}\;, \qquad (I.14)\\
\mathbb{W}_2 \;=\; & \int\Big\{\,\mathbb{G}_{2,0}(\mathbf{k},\mathbf{k}')\otimes a^\dagger{}_\lambda(\mathbf{k})a^\dagger{}_\lambda(\mathbf{k}') + \mathbb{G}_{0,2}(\mathbf{k},\mathbf{k}')\otimes a_\lambda(\mathbf{k})a_\lambda(\mathbf{k}')\\
&\qquad + \mathbb{G}_{1,1}(\mathbf{k},\mathbf{k}')\otimes a^\dagger{}_\lambda(\mathbf{k})a_\lambda(\mathbf{k}')\,\Big\}\;dkdk'\;. \qquad (I.15)
\end{aligned}$$

Here $\mathbb{G}_{m,n}$ are functions of $(\mathbf{k};\lambda)$ or $(\mathbf{k},\mathbf{k}';\lambda,\lambda')$, respectively, with values in the operators on $\boldsymbol{\mathcal{H}}_{el}$. Comparing (I.12)–(I.15) to (I.10), one easily verifies that

$$\mathbb{G}_{1,0}(\mathbf{k},\lambda) := \mathbb{G}_{0,1}(\mathbf{k},\lambda)^* := \left(\frac{\bar\kappa(\mathbf{k})}{\pi\sqrt{2\omega(\mathbf{k})}}\right)\sum_{j=1}^{N}\Big\{i\omega(\mathbf{k})\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\mathbf{x}_j + i\left(\mathrm{e}^{-i\alpha\mathbf{k}\cdot\mathbf{x}_j}-1\right)\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\nabla}_{\mathbf{x}_j} + \frac{i\alpha}{2}\mathrm{e}^{-i\alpha\mathbf{k}\cdot\mathbf{x}_j}\left[\boldsymbol{\sigma}_j\cdot(\mathbf{k}\wedge\boldsymbol{\varepsilon}_\lambda(\mathbf{k}))\right]\Big\}\ , \tag{I.16}$$

$$\mathbb{G}_{2,0}(\mathbf{k},\lambda;\mathbf{k}',\lambda') := \mathbb{G}_{0,2}(\mathbf{k},\lambda;\mathbf{k}',\lambda')^* := \left(\frac{\bar\kappa(\mathbf{k})\bar\kappa(\mathbf{k})}{2\pi^2\sqrt{\omega(\mathbf{k})\omega(\mathbf{k}')}}\right)(\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\varepsilon}_{\lambda'}(\mathbf{k}'))\sum_{j=1}^{N}\left\{\left(\mathrm{e}^{-i\alpha\mathbf{k}\cdot\mathbf{x}_j}-1\right)\left(\mathrm{e}^{-i\alpha\mathbf{k}'\cdot\mathbf{x}_j}-1\right)\right\}\ , \tag{I.17}$$

$$\mathbb{G}_{1,1}(\mathbf{k},\lambda;\mathbf{k}',\lambda') := \left(\frac{\bar\kappa(\mathbf{k})\bar\kappa(\mathbf{k})}{\pi^2\sqrt{\omega(\mathbf{k})\omega(\mathbf{k}')}}\right)(\boldsymbol{\varepsilon}_\lambda(\mathbf{k})\cdot\boldsymbol{\varepsilon}_{\lambda'}(\mathbf{k}')^*)\sum_{j=1}^{N}\left\{\left(\mathrm{e}^{-i\alpha\mathbf{k}\cdot\mathbf{x}_j}-1\right)\left(\mathrm{e}^{i\alpha\mathbf{k}'\cdot\mathbf{x}_j}-1\right)\right\}\ . \tag{I.18}$$

The Hamiltonian $\mathbb{H}_2$ in (I.12) will serve as a starting point and a motivation in constructing the model treated in this paper. The crucial difference between our model and $\mathbb{H}_2$ is the presence of a spatial cutoff (i.e., a cut off in the $\sum\mathbf{x}_j$ - variable) in the interaction term, $\mathbb{W}_g$. This saves us a hassle of controlling the large $\sum\mathbf{x}_j$ - behaviour of the interactions. While the full model will be treated in a subsequent publication, we focus in this paper on the resulting simplified model. Because we only study the interactions between electrons *bound to static nuclei* and the quantized radiation field, the large-$\mathbf{x}_j$ cutoff is not expected to change the physics.

I.2 Hamiltonians of Quantum Electrodynamics of Non-Relativistic Confined Particles

In this section, we introduce the model to be treated in this paper. It is closely related to the standard model described in the last section, especially $\mathbb{H}_2$ in Eqns. (I.12)–(I.18). Compared to the latter model, its crucial difference is that its coupling functions have a better behaviour for large $|\mathbf{x}_j|$. Careful comparison

of both models is done in Section I.6. Moreover, we introduce a few inessential simplifications which serve to simplify notations and to make the key ideas underlying our analysis more transparent.

First, we neglect electron spin and replace the photons by scalar particles. We thus redefine the Hilbert space of the system to be given by

$$\mathcal{H} \;=\; \mathcal{H}_{el} \otimes \mathcal{H}_f \,, \tag{I.19}$$

where

$$\mathcal{H}_{el} \;:=\; L^2(X, dx) \,, \tag{I.20}$$

with $X := \mathbb{R}^{3N}$, for some $N \in \mathbb{N}$, being the particle configuration space, and

$$\mathcal{H}_f \equiv \mathcal{F} \;:=\; \bigoplus_{n=0}^{\infty} L^2(\mathbb{R}^3, d^3k)^{\otimes_s n} \tag{I.21}$$

(the Fock space for scalar photons). We study Hamiltonians of the form

$$H_g \;:=\; H_0 + W_g \;:=\; H_{el} \otimes \mathbf{1} \,+\, \mathbf{1} \otimes H_f \,+\, W_g \,, \tag{I.22}$$

where H_{el} is a Schrödinger operator on $\mathcal{H}_{el}$,

$$H_{el} \;:=\; -\Delta_x \,+\, V(x) \,, \tag{I.23}$$

and the photon Hamiltonian, H_f, is now given by

$$H_f \;:=\; \int dk\, \omega(k)\, a^\dagger(k) a(k) \;:=\; \int dk\, |k|\, a^\dagger(k) a(k) \,. \tag{I.24}$$

Here, we write k, instead of $\mathbf{k}$, for vectors $k \in \mathbb{R}^3$. We adopt this notation henceforth. The interaction, W_g, is of the form

$$W_g \;:=\; g\, W_1 \,+\, g^2\, W_2 \,, \qquad \text{where} \tag{I.25}$$

$$W_1 \;:=\; \int \{\, G_{1,0}(k) \otimes a^\dagger(k) + G_{0,1}(k) \otimes a(k) \,\}\, dk \,, \tag{I.26}$$

$$\begin{aligned} W_2 \;:=\; \int \Big\{\, & G_{2,0}(k,k') \otimes a^\dagger(k) a^\dagger(k') + G_{0,2}(k,k') \otimes a(k) a(k') \\ & + G_{1,1}(k,k') \otimes a^\dagger(k) a(k') \,\Big\}\; dk dk' \,, \end{aligned} \tag{I.27}$$

where $G_{m,n}$ are functions of k or (k,k'), respectively, with values in the operators on $\mathcal{H}_{el}$.

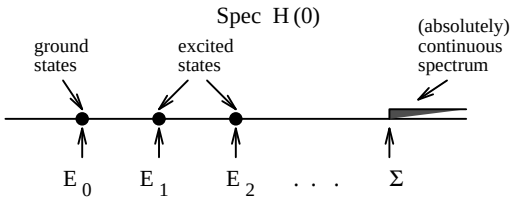

Figure I.1: The Spectrum of H_{el}

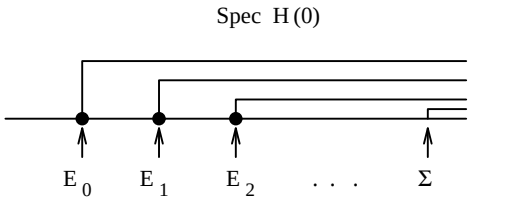

Figure I.2: The Spectrum of $H_0 = H_{el} \otimes \mathbf{1} + \mathbf{1} \otimes H_f$

Next, before adding further specifications about W_g, we discuss spectral properties of H_0 and the problem of perturbation. We formulate the main questions about mathematical models of radiation, and we briefly describe our answers to those questions.

We assume that the particle Hamiltonian H_{el} in (I.23), in the absence of the radiation field, is selfadjoint and has a standard spectrum, i.e., a continuum corresponding to the half-axis $[\Sigma, \infty)$, for some $\Sigma \leq 0$, and discrete eigenvalues $E_0, E_1, \ldots$ on the left of (below) the continuum, i.e., $E_0 < E_1 < \cdots < \Sigma$ (HVZ Theorem, see e.g. [31]). The eigenvalues, E_j, can, of course, be degenerate (see Figure I.1).

The spectrum of H_f consists of a simple eigenvalue at 0 (corresponding to the vacuum vector, $\Omega \in \mathcal{F}$) and absolutely continuous spectrum covering the half-axis $[0, \infty)$. Consequently, by separation of variables, the unperturbed Hamiltonian

$$H_0 \;=\; H_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f \,, \tag{I.28}$$

has the same point spectrum as H_{el}, i.e., $\{E_j\}$, and the continuum covers the half-axis $[E_0, \infty)$ and consists of a union of branches, $[E_j, \infty)$, starting at the eigenvalues E_j, and the branch $[\Sigma, \infty)$ (see Figure I.2).

The crucial fact is that the eigenvalues E_j lie at tips of branches of continuous spectrum of H_0, i.e., the numbers E_j play a double role as eigenvalues and

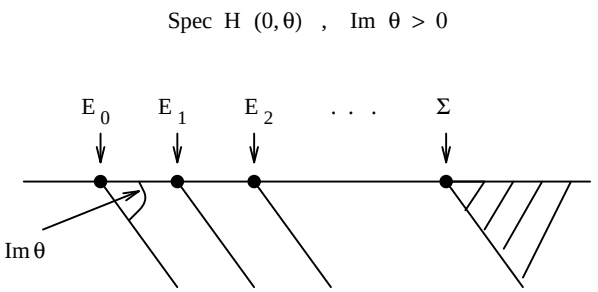

Figure I.3: The Spectrum of $H_0(\theta)$ with $\mathrm{Im}(\theta) > 0$

as thresholds of continuous spectrum. This is clarified by choosing a different projection of the Riemann surface, on which the resolvent $(H_0 - z)^{-1}$ of H_0 is defined, onto the energy plane. We shall later construct this projection explicitly by passing to a deformed Hamiltonian $H_0(\theta)$ (see Section I.5). Then

$$\sigma_{\mathrm{cont}}\,[H_0(\theta)] \;=\; \bigcup_j \{E_j + \mathrm{e}^{-\theta}\mathbb{R}_+\} \;\cup\; \{\Sigma + \mathbb{R}_+ + \mathrm{e}^{-\theta}\mathbb{R}_+\}\,,$$

and $\sigma_{\mathrm{pp}}(H_0(\theta)) = \{E_j\}$ (see Figure I.3).
The eigenfunctions of H_0 corresponding to the eigenvalues E_j have the form $\psi_j \otimes \Omega$, where ψ_j are the eigenfunctions of H_{el} corresponding to the eigenvalues E_j, and Ω is the vacuum in $\mathcal{F}, H_f\Omega = 0$. The fate of these eigenfunctions and the corresponding eigenvalues is a key concern in the theory of radiation.

Mathematically, we are faced with a problem of perturbation of eigenvalues located at thresholds of continuous spectrum. This is the most difficult situation in perturbation theory. (To our knowledge, it has not been treated in the literature, yet.) The fact that the eigenvalues $\{E_j\}$ of the atomic Hamiltonian H_{el} are also branch points of continuous spectrum of H_0 is due to the property of photons that their *rest mass vanishes*, i.e., $\omega(0) = 0$. (From this it follows that the continuous spectrum of H_f covers the entire half-axis $[0, \infty)$, and thus $[E_j, \infty)$ is a branch of continuous spectrum of H_0.) The ensuing difficulties in developing a convergent perturbation theory for the eigenvalues E_j are an aspect of the celebrated *infrared problem* of quantum electrodynamics. The fact that the number of photons is neither conserved nor bounded, for any energy interval, leads to an infinite number of degrees of freedom in the perturbation problem. In phy-

sics language, small fluctuations of energy allowed by the uncertainty principle ($\Delta E\, \Delta t \geq \hbar$) may produce an infinite number of photons whose energy is very close to 0. Such photons are called *soft photons*. A standard physical imagery is of an atom surrounded by a cloud of soft photons. The interaction between charged particles and the quantized radiation field leads to a renormalization of physical parameters. In particular, it produces the energy shifts first estimated by Bethe [7, 8] and the Lamb shift.

The main *physical* expectations underlying the theory of radiation are that there exists a stable ground state and that the excited states of the particle system, corresponding to the eigenvalues E_j, with $j > 0$, become unstable. They decay spontaneously (i.e., acquire a finite life-time) by emission of photons. Mathematically, this phenomenon is described in the language of *resonances* or, more generally, in terms of the Riemann surface of matrix elements of the resolvent, $\langle u \,|\, (H_g - z)^{-1} v \rangle$, for a universal dense set of vectors containing u, v.

Next we formulate some key mathematical problems in the theory of radiation.

(a) Is the ground state (corresponding to the eigenvalue E_0) of H_0 stable when the interactions of particles with the quantized radiation field are included?

(b) Are the excited states corresponding to the eigenvalues E_i, $i > 0$, of H_0 unstable?

(c) Do the excited states turn into resonances? If they do, what are the positions (i.e., the energies and life-times) of these resonances in the complex energy plane?

(d) What is the structure of the Riemann surface of $\langle u \,|\, (H_g - z)^{-1} v \rangle$ (for a universal dense set $\mathcal{D} \ni u, v$)?

The main results of this paper are as follows.

(i) H_g has a ground state originating from the ground state of H_0. (The coupling constant g does not need to be particularly small, for this result to hold.) The ground state is exponentially localized in the particle coordinates.

(ii) H_g has no other eigenvalues outside small neighbourhoods of the thresholds of H_{el}, besides the ground state energy. In other words, the excited states of H_g become unstable. This is proved for sufficiently small coupling constants. Moreover, the spectrum of H_g outside $O(g^2)$-neighbourhoods of the thresholds of H_{el} is *purely absolutely continuous.*

(iii) There are complex "eigenvalues" $\{E_{j,\ell}(g)\}_{\ell=1}^{N_j}$ bifurcating from each eigenvalue E_j of H_0, where N_j is the multiplicity of E_j. The energy $E_{0,1}(g)$ with the *smallest* real part bifurcating from the ground state energy E_0 of H_g is *real* and is the ground state energy of H_g (see result (i)). Under certain genericity assumptions on the potential (see (I.44)), all energies $E_{j,\ell}(g)$, $j \geq 1$, have a negative imaginary part, for $g \neq 0$. They are the complex resonance energies of H_g. (These notions will be explained more precisely, below.) We develop a constructive algorithm to calculate the functions $E_{j,\ell}(g)$.

(iv) The radiative corrections, $E_{j,\ell}(g) - E_j(0)$, are given by

$$E_{j,\ell}(g) - E_j(0) = \varepsilon_{j,\ell}\, g^2 + o(|g|^2), \tag{I.29}$$

where $\mathrm{Re}[\varepsilon_{j,\ell}]$ is given by *Bethe's formula* and $\mathrm{Im}[\varepsilon_{j,\ell}]$ is given by *Fermi's Golden Rule* (as will be explained and proven below and in [5]). For the hydrogen atom, the energy differences $\mathrm{Re}[E_{j,\ell}(g) - E_{i,k}(g)]$ reproduce the observed Lamb shifts quite accurately, (i.e., up to effects due to *the ultraviolet cutoff and relativistic corrections*).

Let $C_0(\mathbb{R}^3)$ denote the space of continuous functions on $\mathbb{R}^3$ of compact support, and let $\mathcal{F}(C_0(\mathbb{R}^3))$ be the photon Fock space over $C_0(\mathbb{R}^3)$, i.e.,

$$\mathcal{F}(C_0(\mathbb{R}^3)) = \bigoplus_{n=0}^{\infty} C_0(\mathbb{R}^3)^{\otimes_s n}\,. \tag{I.30}$$

For all $u, v \in C_0(X) \otimes \mathcal{F}[C_0(\mathbb{R}^3)]$, $\langle u|\,(H_g - z)^{-1} v\rangle$ has an analytic continuation in the variable z into the part of the lower complex half-plane depicted in Figure I.4.

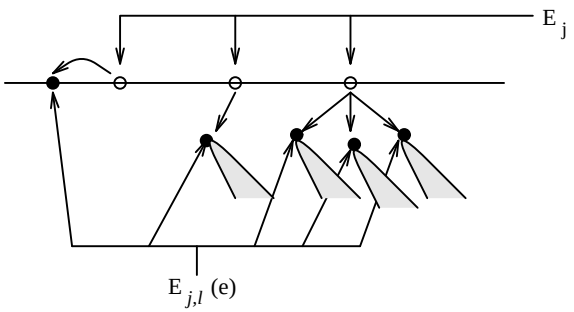

Figure I.4: A projection of the Riemann surface for $\langle u|\,(H_g - z)^{-1}v\rangle$

The complex numbers $E_{j,\ell}(g)$ are "branch points" and poles of this continuation, simultaneously. The angle of inclination of the cuts emanating from $E_{j,\ell}(g)$ is related to the choice of projection of the Riemann surface, on which $\langle u, (H_g - z)^{-1}v\rangle$ is defined, onto the energy plane. The numbers $E_{j,\ell}(g)$ are *independent* of the choice of projection. For $j \geq 1$, they correspond to resonances of H_g. Their imaginary part is the time decay rate of the unperturbed excited state, $\psi_{j,\ell}$, evolved with $\exp[-itH_g]$, as described in Subsection IV.4.1 (see Eqns. (IV.115)–(IV.116)).

We are able to give a fairly precise description of the cuts shown in Figure I.4 (see Theorem I.3). In essence, given a projection of the Riemann surface of $\langle u\,|\,(H_g - z)^{-1}v\rangle$ onto the complex energy plane, we construct a function $T_{j,\ell}(g, \mathrm{e}^{-i\varphi}\lambda)$, $0 \leq \lambda < \infty$, $\varphi > 0$, with $T_{j,\ell}(g, \mathrm{e}^{-i\varphi}\lambda) = e^{-i\varphi}\lambda\Big(1 + \mathcal{O}(|g|^2)\Big)$, such that, for z near $E_{j,\ell}(g)$, the resolvent of H_g is essentially determined, up to small error terms, by the resolvent of the operator

$$T_{j,\ell}(g, \mathrm{e}^{-i\varphi}H_f) + E_{j,\ell}(g)\,.$$

In particular, $\langle u\,|(H_g - z)^{-1}v\rangle$ is analytic in z outside a cuspidal domain around

$$E_{j,\ell}(g) + \sigma(T_{j,\ell}(g, \mathrm{e}^{-i\varphi}H_f))\,,$$

and H_f has been defined in (I.6). Our results are a manifestation of the *"infrared asymptotic freedom"* of our perturbation problem which means that the behaviour of $(H_g - z)^{-1}$, for z close to a resonance energy $E_{j,\ell}(g)$, is essentially determined by the Hamiltonian H_f of the *free* electromagnetic field.

We emphasize that the analytic properties of matrix elements of the resolvent $(H_g - z)^{-1}$ in z represent a natural and often indispensable generalization of the notion of spectrum of a selfadjoint operator H.

A word on earlier results on related problems is called for. Previous results on problems related to (a)-(d) were obtained in [11, 2, 1, 28, 16, 14, 18, 17, 22, 23, 24, 25, 3]. In these papers, various special cases of our results for simplified models were established. Usually, either an infrared cutoff (e.g. a photon mass) was introduced, or the models were considerably simpler than ours, or only partial results were proven.

As to the analytical methods involved in our analysis, the following tools will be important: Our proof of (i) involves constructive-field-theory techniques developed in [15, 11]. Our proof of (ii) involves positive-commutator techniques (see also [18, 14] and [4]) and the technique of complex dilatation, in conjunction with the isospectral Feshbach map which is a far-reaching extension of the standard Feshbach projection method. A powerful extension of the positive-commutator method appears in [4].

Our deepest results are (iii) and (iv). In order to establish them, we develop an operator-theoretic *renormalization group method* for non-selfadjoint Hamiltonians, based on an iterative use of the isospectral Feshbach map. Our renormalization map acts directly on quantum Hamiltonians, rather than on actions or on their correlation functions, as is the case with the standard approach.

As a biproduct, our renormalization group construction yields alternative proofs of results (i) and (ii), for small values of $|g|$.

I.3 Main Results

In this chapter, we present a precise formulation of our assumptions and of our main results. Recall from (I.22)–(I.24) that $H_g = H_0 + W_g$, with $H_0 = H_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes H_f$. The operator $H_f = \int d^3k\, \omega(k)\, a^\dagger(k) a(k)$, where $\omega(k) = |k|$, is a selfadjoint operator on its natural dense domain, $\mathcal{D}(H_f) \subseteq \mathcal{F}$. It is positive and

$$\sigma(H_f) \;=\; \sigma_{\rm pp}(H_f) \cup \sigma_{\rm ac}(H_f)\,,$$

where $\sigma_{\rm pp}(H_f)$ consists of the simple eigenvalue 0, corresponding to the eigenvector $\Omega \in \mathcal{F}$ (the vacuum); $\sigma_{\rm ac}(H_f)$ is the absolutely continuous spectrum of H_f. It covers the half-axis $[0,\infty)$ and has infinite multiplicity. The singular continuous spectrum of H_f is empty.

Next, $H_{el} = -\Delta_x + V(x)$ is a Schrödinger operator. Throughout the paper we assume *that H_{el} is selfadjoint and has at least one eigenvalue, E_0, below its continuum, Σ*. More specifically, we assume that H_{el} is an N-body Hamiltonian, i.e., that the potential $V(x)$ obeys:

$V(x)$ is a real function of the form $\sum_{\rm finite} V_i \circ \pi_i$, where π_i are linear maps from X ($= \mathbb{R}^{3N}$, the particle configuration space) to $\mathbb{R}^{n_i}$ and $V_i \in L^{p_i}(\mathbb{R}^{n_i}) + L^\infty(\mathbb{R}^{n_i})$, with $p_i = 2$ if $n_i \leq 3$, $p_i > 2$ if $n_i = 4$, *and $p_i = \frac{n_i}{2}$ if $n_i \geq 5$*.

This is not the most general condition that suits our purposes, but it is simple and general enough, as it includes $\mathbb{V}$ in (I.8). In particular, under this condition H_{el} is selfadjoint on the domain $\mathcal{D}(H_{el}) = \mathcal{D}(-\Delta_x)$. From our Hypothesis on $V(x)$ above and the HVZ-Theorem (see e.g. [31, 10]) follows that the particle Hamiltonian H_{el} in (I.23) has a standard spectrum: a continuum corresponding to the half-axis $[\Sigma,\infty)$, for some $\Sigma \leq 0$, and discrete eigenvalues E_j of finite multiplicity, $1 \leq N_j < \infty$, $j = 0,1,2,\ldots$ on the left of (below) the continuum, i.e., $E_0 < E_1 < \cdots < \Sigma$.

Our model is specified by conditions on the coupling functions $G_{m,n}$ in the interaction W_g defined in (I.25)–(I.27). For different results of this paper we need different conditions. We list these conditions below. For notational convenience the hypothesis presented below tend to be somewhat stronger than those used in the actual proofs. More general hypotheses can be found in the corresponding chapters; (see the end of this chapter for a guide). Here is the first set of conditions on $G_{m,n}$ that we use in Chapters II and III:

Hypothesis 1 *The coupling functions $G_{2,0} \equiv G_{0,2} \equiv G_{1,1} \equiv 0$ vanish identically, and $G_{0,1}(k) = G_{1,0}(k)^*$ is a multiplication operator in the Schrödinger representation $\mathcal{H}_{el} = L^2(X)$, i.e., for every $k \in \mathbb{R}^3$, $G_{0,1}(k)$ acts as multplication by a function which is denoted $G_x(k)$. This coupling function $G_x(k)$ obeys*

the following bounds, for some $\varepsilon \geq 0$ *:*

$$\Lambda_1 \;:=\; \sup_{x\in\mathbb{R}^{3N}} \left\{ \int \left[1+\frac{1}{\omega(k)}\right] \, |G_x(k)|^2 \, dk \right\}^{1/2} \;<\; \infty \,, \tag{I.31}$$

$$\Lambda_2(\varepsilon) \;:=\; \sup_{x\in\mathbb{R}^{3N}} \left\{ \mathrm{e}^{-\varepsilon|x|} \int \frac{|G_x(k)|^2}{\omega(k)^2} \, dk \right\}^{1/2} \;<\; \infty \,. \tag{I.32}$$

We refer to W_1 *as* W_x *.*

The second set of hypotheses is somewhat more general than Hypothesis 1. It is used in Chapters IV and V (confer to the end of Section I.1).

Hypothesis 2 *For some* $\theta_0 \geq 0$ *and all* $k, k' \in \mathbb{R}^3$ *, the maps*

$$\theta \;\mapsto\; (-\Delta_x+1)^{-1/4} \, \mathrm{e}^{3\theta/2} G_{m,n}(\mathrm{e}^{-\theta}k) \, (-\Delta_x+1)^{-1/4} \,, \tag{I.33}$$

for $m+n=1$ *, and*

$$\theta \;\mapsto\; \mathrm{e}^{3\theta} G_{m,n}(\mathrm{e}^{-\theta}k, \mathrm{e}^{-\theta}k') \,, \tag{I.34}$$

for $m+n=2$ *, are analytic as maps:* $\{|\theta| \leq \theta_0\} \to \mathcal{B}[\mathcal{H}_{el}]$ *, the bounded operators on* $\mathcal{H}_{el}$ *. Define* $J(k)$ *to be the smallest non-negative function such that, for* $m+n=1$ *,*

$$\sup_{|\theta|\leq\theta_0} \left\| (-\Delta_x+1)^{-1/4} \mathrm{e}^{3\theta/2} G_{m,n}(\mathrm{e}^{-\theta}k) (-\Delta_x+1)^{-1/4} \right\| \;\leq\; J(k) \,, \tag{I.35}$$

and, for $m+n=2$ *,*

$$\sup_{|\theta|\leq\theta_0} \left\| \mathrm{e}^{3\theta/2} G_{m,n}(\mathrm{e}^{-\theta}k, \mathrm{e}^{-\theta}k') \right\| \;\leq\; J(k)\, J(k') \,. \tag{I.36}$$

Then J *obeys*

$$\Lambda_4 \;:=\; \left\{ \int \left[1+\omega(k)^{-1-\beta}\right] \, |J(k)|^2 \, dk \right\}^{1/2} \;<\; \infty \,, \tag{I.37}$$

for some $\beta > 0$ *. Here* $m+n=1$ *represent the cases* $(m,n)=(0,1)$ *and* $(m,n)=(1,0)$ *, and* $m+n=2$ *represent the cases* $(m,n)=(0,2)$ *,* $(m,n)=(1,1)$ *, and* $(m,n)=(2,0)$ *.*

In Chapter V, where our renormalization group analysis is presented, we need the following assumption, in addition to Hypothesis 2:

Hypothesis 3 *Hypothesis 2 holds. The function* J, *defined by (I.35) and (I.36), obeys*

$$\Lambda_5 \;:=\; \sup_{k\in\mathbb{R}^3}\Big\{|k|^{\frac{1}{2}(1-\mu)}\,J(k)\Big\} \;<\; \infty\,, \tag{I.38}$$

for some $\mu > 0$.

The condition that μ be strictly positive ensures that we are dealing with a perturbation problem that is *asymptotically free* in the *infrared region* of the photon degrees of freedom. To see this, we assume, for simplicity, that the electron degrees of freedom have already been eliminated (by taking a "partial trace" over $\mathcal{H}_{el}$). The resulting effective Hamiltonian, $\widetilde{H}_{\rm eff}$, then has the form

$$\widetilde{H}_{\rm eff} \;=\; H_f \;+\; \sum_{m+n\geq 1}\int a^\dagger(k^{(m)})\;\tilde{w}_{m,n}\Big[H_f;k^{(m)};\tilde{k}^{(n)}\Big]\;a(\tilde{k}^{(n)})\,dk^{(m)}d\tilde{k}^{(n)}\,,$$

where

$$k^{(m)} \;:=\; (k_1,\ldots,k_m)\,,\quad dk^{(m)} \;:=\; \prod_{j=1}^m d^3k_j\,,\quad \text{and}\quad a^\dagger(k^{(m)}) \;:=\; \prod_{j=1}^m a^\dagger(k_j)\,.$$

Hypotheses 2 and 3 turn out to imply that

$$\sup_{r\geq 0}|\,\tilde{w}_{m,n}\Big[H_f;k^{(m)};\tilde{k}^{(n)}\Big]\,| \;\leq\; \xi^{m+n}\prod_{j=1}^m|k_j|^{\frac{1}{2}(\mu-1)}\prod_{j=1}^n|\tilde{k}_j|^{\frac{1}{2}(\mu-1)}\,, \tag{I.39}$$

for some constant ξ and all $m,n\in\mathbb{N}_0$ such that $m+n\geq 1$.

The key problem arising in our operator-theoretic renormalization group analysis is to estimate the size of matrix elements of $\widetilde{H}_{\rm eff}$ between vectors in the subspace $\chi_{H_f<\rho}\mathcal{F}$ of photon Fock space, where $\chi_{H_f<\rho}$ is the spectral projection of H_f onto energies smaller than ρ, for an *arbitrary photon energy scale* ρ, with $0<\rho\leq\rho_0$. Here, the constant ρ_0 is of order $O(1\;Rydberg)$, i.e., of order $O(1)$ in our choice of units. By dilating the photon momenta

$$k \;:=\; \frac{\rho}{\rho_0}p\,,$$

we find that $\widetilde{H}_{\rm eff}$ is unitarily equivalent to the Hamiltonian

$$H_{\rm eff} \;=\; \frac{\rho}{\rho_0}H_f \;+\; \sum_{m+n\geq 1}\left(\frac{\rho}{\rho_0}\right)^{\frac{3}{2}(m+n)}\int a^\dagger(p^{(m)})\tilde{w}_{m,n}\left[\frac{\rho}{\rho_0}H_f;\frac{\rho}{\rho_0}p^{(m)};\frac{\rho}{\rho_0}\tilde{p}^{(n)}\right]a(\tilde{p}^{(n)})dp^{(m)}d\tilde{p}^{(n)}\,,$$

and we are interested in estimating matrix elements of $H_{\rm eff}$ between vectors in the (fixed) subspace $\chi_{H_f<\rho}\,\mathcal{F}$ of photon Fock space. We define a *rescaled* Hamiltonian at an energy scale $\rho < \rho_0$ by setting

$$H_{\rm eff}^{(\rho)} \;:=\; \frac{\rho_0}{\rho} H_{\rm eff}\;.$$

Using the bounds (I.39), we find that

$$H_{\rm eff}^{(\rho)} \;=\; H_f + \tag{I.40}$$
$$\sum_{m+n\geq 1} \left(\frac{\rho}{\rho_0}\right)^{\left(1+\frac{\mu}{2}\right)(m+n)-1} \int a^\dagger(p^{(m)})\, w_{m,n}^{(\rho)}\left[H_f; p^{(m)}; \tilde{p}^{(n)}\right]\, a(\tilde{p}^{(n)})\, dp^{(m)} d\tilde{p}^{(n)}\;,$$

where the kernels $w_{m,n}^{(\rho)}$ again satisfy the bounds (I.39). Because $m+n \geq 1$, and since $\mu > 0$ (by Hypothesis 3), each term in the perturbation of H_f on the right side of (I.40) is multiplied by a factor bounded above by

$$\left(\frac{\rho}{\rho_0}\right)^{\frac{\mu}{2}(m+n)} \tag{I.41}$$

which tends to 0 exponentially fast in $m+n$, as $\rho \to 0$.

This feature will imply that the effective Hamiltonians, on a photon energy scale $\rho < \rho_0$, of the physical systems we study are perturbations of the operators

$$\varepsilon^{(\rho)}\, H_f \;+\; \Delta E^{(\rho)}\, \mathbf{1}\;, \tag{I.42}$$

the norm of the perturbation of (I.42) being bounded by $O(\rho^{\mu/2})$, and with

$$\varepsilon^{(\rho)} \;\to\; \varepsilon^{(0)}\;, \qquad \frac{\rho}{\rho_0}\Delta E^{(\rho)} \;\to\; \Delta E^{(0)}\;,$$

as $\rho \to 0$; (see Chapter V).

When $\mu = 0$, the perturbations become "marginal", but the perturbation problem is expected to remain asymptotically free in the infrared. Then the renormalization group analysis becomes somewhat more subtle; (but only the terms $\sim w_{0,1}^{(\rho)}, w_{1,0}^{(\rho)}$ require care – we defer these problems to a separate analysis).

Our results concerning the Hamiltonian H_g, defined in (I.22)–(I.27) are summarized in the following four theorems.

Theorem I.1 (Binding) *Suppose that Hypothesis 1 holds for some* $0 \leq \varepsilon < (\Sigma - E_0 - g^2 \int \frac{|G_x|^2}{\omega})^{1/2}$.

(a) *There exists a constant,* $M_\varepsilon < \infty$, *which depends only on* V *and* Λ_1 , *such that if*

$$\delta_\varepsilon \;:=\; g^2\, M_\varepsilon\, \Lambda_2(\varepsilon) \cdot \left[1 \;+\; 4g^2(\Sigma - E_0)^{-2} \sup_x \left\{ \int |G_x(k)|^2\, dk \right\}\right] \;<\; 1\,, \tag{I.43}$$

then the Hamiltonian H_g *has a ground state* ϕ .

(b) *This ground state has a non-vanishing overlap with* $P_{el} \otimes P_\Omega$, *where* $P_{el} := \chi_{H_{el} \leq \frac{1}{2}\Sigma + \frac{1}{2}E_0}$ *is the (finite dimensional) projection onto the bound states of* H_{el} *with energy below* $\frac{1}{2}\Sigma + \frac{1}{2}E_0$, *and* $P_\Omega = |\Omega\rangle\langle\Omega|$ *is the projection onto the photon vacuum. More specifically,* $\langle P_{el} \otimes P_\Omega \rangle_\phi \geq 1 - \delta_\varepsilon > 0$.

(c) *Moreover, this ground state satisfies the exponential bound*

$$\left\| \mathrm{e}^{\varepsilon|x|} \otimes \mathbf{1}_f \; \phi \right\| \;\leq\; M_\varepsilon\,. \tag{I.44}$$

(c) *As* $g \to 0$, *the ground state energy and the ground state subspace of* H_g *converge to the unperturbed ground state energy (= the ground state energy of the atom) and the ground state subspace, respectively.*

(d) *If* $\mathcal{H}_{el} = L^2(\mathbb{R}^m)$ *and* $G_x(k) = \overline{G_x(-k)}$, *for all* $k \in \mathbb{R}^3$, *then the ground state is unique.*

(e) *Assume Hypotheses 2 and 3 with* $\theta_0 = 0$ *hold. Then, for* $g \cdot (\min\{E_1, \Sigma\} - E_0)^{-1/2}$ *sufficiently small, the degeneracy of the ground state of* H_g *does not exceed the degeneracy of the ground state of* H_{el} .

Next we consider analytic continuations in z of matrix elements, $\langle u|(H_g - z)^{-1}v\rangle$, of the resolvent $(H_g - z)^{-1}$, for vectors u and v belonging to some universal dense set, $\mathcal{D} \subset \mathcal{H}$, defined as follows. Let $C_0(\mathbb{R}^3)$ be the one-photon space of continuous and compactly supported functions. Let $\mathcal{F}(C_0)$ be the bosonic Fock space over $C_0(\mathbb{R}^3)$, i.e., $\mathcal{F}(C_0(\mathbb{R}^3)) = \bigoplus_{n=0}^{\infty} C_0(\mathbb{R}^3)^{\otimes_s n}$, where $\otimes_s$ stands for the symmetric tensor product. This space is a core for H_f , and $\mathcal{D} \equiv C_0^\infty(X) \otimes \mathcal{F}(C_0)$ is a core for H_g . Let $\mathbb{C}_\pm$ denote the upper and lower halfplane, respectively.

Theorem I.2 (Instability of Excited States and Fermi's Golden Rule) *Suppose Hypothesis 2 holds, and assume that* $E_j > E_0$ *is an isolated eigenvalue of* H_{el} *of degeneracy* $N_j < \infty$ *with eigenvectors* $\psi_{j,1}, \ldots, \psi_{j,N_j}$. *Assume that the* $N_j \times N_j$ *selfadjoint matrix*

$$(A_j)_{\mu\nu} := \int_{-\infty}^{E_j - 0} \int dk \, \langle G_{10}(k)\psi_{j,\mu} \mid d\chi_{H_{el} \leq E} \, G_{10}(k)\psi_{j,\nu} \rangle \, \delta[\omega(k) - E_j + E] \quad \text{(I.45)}$$

is positive definite, i.e., $A_j \geq a_j \mathbf{1} > 0$. *Then we have that*

(a) *the operator* H_g *has absolutely continuous spectrum in each interval* $\left[\frac{2}{3}E_{j-1} + \frac{1}{3}E_j \, , \, \frac{1}{3}E_j + \frac{2}{3}E_{j+1}\right]$;

(b) *for vectors* $u, v \in \mathcal{D}$, *the matrix elements* $\langle u|(H_g - z)^{-1}v\rangle$ *of the resolvent* $(H_g - z)^{-1}$ *have an analytic continuation from* $\mathbb{C}_+$ *into*

$$\mathcal{A}_j := \left[\frac{2}{3}E_{j-1} + \frac{1}{3}E_j \, , \, \frac{1}{3}E_j + \frac{2}{3}E_{j+1}\right] + i\left(\mathbb{R}^+ - \gamma_j\right) , \quad \text{(I.46)}$$

where $\gamma_j := g^2 a_j$.

In particular, excited states of H_{el} *are unstable under the perturbation* W_g.

Theorem I.3 (Riemann surface of H_g**)** *Suppose that Hypotheses 2 and 3 hold, and pick* $\tau > 0$. *Then, for* g *sufficiently small and for* $u, v \in \mathcal{D}$, $\langle u| \, (H_g - z)^{-1}v\rangle$ *has an analytic continuation across the interval* $I := \left(E_0(g), \Sigma - \tau\right) \subset \sigma_{\rm cont}(H_g)$, *where* $E_0(g) = \inf \sigma(H_g)$, *into the domain*

$$\left\{a + ib \mid a \in I \, , \, b \geq -\eta\right\} \setminus \bigcup_{j \geq 0} \bigcup_{\ell=1}^{N_j} \left(E_{j,\ell} + \mathrm{e}^{-i\vartheta}\left\{a + ib \;\middle|\; a > 0 \, , \, |b| \leq a^{1+\frac{\mu}{2}}\right\}\right) , \quad \text{(I.47)}$$

(see Figure I.5) contained in the second Riemann sheet, for some $\eta > 0$. *Here* $E_{j,\ell}(g)$ *are complex numbers in* $\overline{\mathbb{C}_-}$ *with the following properties:*

(a) $E_{j,\ell}(g) \in \mathbb{C}_-$, *for* $j \geq 1$ *and all* $\ell = 1, \ldots, N_j$,

(b) $E_{j,\ell}(g) \to E_j$, *as* $g \to 0$, *for* $j \geq 1$ *and all* $\ell = 1, \ldots, N_j$,

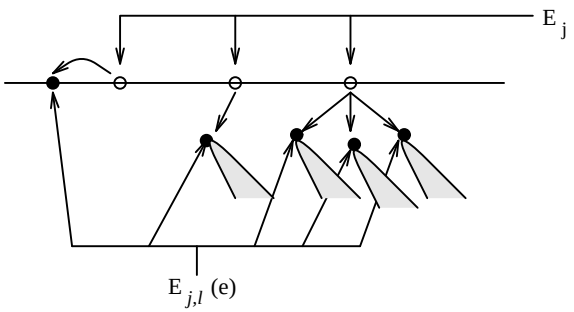

Figure I.5: A projection of the Riemann surface for $\langle u|\,(H_g - z)^{-1}v\rangle$

(c) *$E_{j,\ell}(g)$ are resonances of H_g, i.e., they are eigenvalues of a non-selfadjoint, deformed Hamiltonian $H_g(\theta)$ (arising from H_g by complex dilatation; θ is the dilatation parameter, with $\mathrm{Im}\,\theta > 0$), for given $j \geq 1$ and $\ell = 1, \ldots, N_j$, where N_j is the multiplicity of E_j. The corresponding eigenvectors have a non-vanishing overlap with $\psi_{j,\ell} \otimes \Omega$,*

(d) *$E_{j,\ell}(g)$, $j \geq 1$, $\ell = 1, \ldots, N_j$, are singularities of $\langle u|\,(H_g - z)^{-1}v\rangle$, for suitable vectors $u, v \in \mathcal{D}$ (i.e., they can be interpreted as positions of resonances of H_g),*

(e) *The numbers $E_{j,\ell}(g)$, for $j \geq 1$, $\ell = 1, \ldots, N_j$, are the unique fixpoints of explicit maps $\mathbb{C} \to \mathbb{C}$ which are given by the right side of Eqn. (V.180). $E_{j,\ell}(g)$ can be determined by iterating this map with initial value E_j and initial projection onto $\psi_{j,\ell} \otimes \Omega$.*

Before proceeding to some simple technical matters, we give an outline of the *organization* of this paper. Our paper contains five chapters. The present chapter is an introductory one. The remainder of this chapter is devoted to the derivation of some elementary bounds, proving e.g. the selfadjointness of H_g, and to the definition and discussion of the complex dilatations of H_g needed in the proof of Theorems I.2 and I.3. We will conclude this chapter with a comparison of Hypotheses 1–3 to features of the standard model of non-relativistic QED, as described by the Hamiltonian $\mathbb{H}_2$ in (I.10)–(I.18). Chapters II–V are devoted to an analysis of the operator H_g. Chapter II, Chapter III and Chapters IV–V are fairly independent of each other.

Chapter II deals with the problem of binding. Its main result is Theorem I.1(a) which is a transcription of Theorem II.8 and Corollary II.9.

Chapter III deals with the nature of the spectrum of H_g. Here, we establish positivity of certain commutators. The main result in this chapter is Corollary III.5. Improved results of this type appear in [6].

In Chapter IV, the reader will have the first glimpse at what we term the Feshbach projection method in Section IV.1. The Feshbach map is the main connection of Chapter IV with Chapter V. As an application of the Feshbach projection method, in conjunction with complex dilatation, we derive Fermi's golden rule in Section IV.2. Here, our main result is Theorem IV.3 which immediately implies Theorem I.2.

In Chapter V, the renormalization group approach to spectral problems on Fock space is developed. This theme is pursued in more detail in [5]. Except for some lengthy technical estimates, it is fairly self-contained. Technically, it is the most involved part of the paper. In Chapter V we apply techniques developed in Chapter IV to the analysis of the operator H_g and prove Theorems V.11, V.12, and Lemma V.10, which imply Theorems I.1(b) and I.3.

I.4 Bogoliubov Transformations and Selfadjointness

On $\mathcal{F}$, a special class of Bogoliubov transformations is given by canonical transformations of the form

$$a^\dagger(k) \;\to\; a^\dagger(k) + f(k) \quad \text{and} \quad a(k) \;\to\; a(k) + \bar{f}(k)$$

If $f \in L^2$ it can be unitarily implemented as follows. Define the momentum operator

$$\pi(f) \;:=\; i\Big(a^\dagger(f) \,-\, a(f)\Big) \;:=\; i\int\Big\{f(k)\,a^\dagger(k) \,-\, \bar{f}(k)a(k)\Big\}d^3k\;.$$

Then

$$e^{i\pi(f)}\,a^\dagger(h)\,e^{-i\pi(h)} \;=\; a^\dagger(h) \,-\, \langle f|h\rangle \tag{I.48}$$

and similarly for $a(h)$. The following lemma shows that these Bogoliubov transformations are related to the Pauli-Fierz transformation, that we used to transform $\mathbb{H}_1$ into $\mathbb{H}_2$.

Lemma I.4 *Assume Hypothesis 1. Then* H_g *is unitary equivalent to an operator of the same form, but with* $G_x - G_0$ *replacing* G_x *.*

Proof: We use the canonical transformation

$$a^\dagger(k) \;\to\; a^\dagger(k) \,-\, \omega(k)^{-1}\bar{G}_0(k)$$

to map H_g to

$$H_{el} \otimes \mathbf{1}_f + \int a^\dagger \omega a + \int G_{1,x} a^\dagger + \int G_{1,x}^* a - \int |G_{1,x}|^2/\omega \;,$$

where $G_{1,x} = G_x - G_0$. ∎

We remark that, as is the case for $\mathbb{H}_2$ (see (I.12)–(I.15)), the new coupling function, $G_x - G_0$, of the transformed Hamiltonian has an improved infrared behaviour at $k = 0$.

Lemma I.5 *(Lower bound) Under Hypothesis 1,*

$$H_g \;\geq\; \left(H_{el} - \int \frac{|G_x|^2}{\omega} \right) \otimes \mathbf{1}_f \;. \tag{I.49}$$

Proof: Let $f_x = G_x/\omega$. Then, omitting the integration variable k ,

$$\begin{aligned} \mathbf{1}_{el} \otimes H_f \;+\; gW_x \;&=\; \int \left[\mathbf{1}_{el} \otimes a^\dagger a \;+\; f_x \otimes a^\dagger \;+\; f_x^* \otimes a \right] \omega \, dk \\ &=\; \int \left[(\mathbf{1}_{el} \otimes a + f_x \otimes \mathbf{1}_f)^* (\mathbf{1}_{el} \otimes a + f_x \otimes \mathbf{1}_f) \right] \omega \, dk \;-\; \int \frac{|G_x|^2}{\omega} \otimes \mathbf{1}_f \;. \end{aligned}$$

Hence

$$\mathbf{1}_{el} \otimes H_f \;+\; gW_x \;\geq\; -\int \frac{|G_x|^2}{\omega} \otimes \mathbf{1} \;, \tag{I.50}$$

which implies (I.49). ∎

Lemma I.6 *(Relative bounds)*

$$\|a(f)\psi\|_{\mathcal{F}} \leq \Big(\int \frac{|f|^2}{\omega} \Big)^{1/2} \|H_f^{1/2}\psi\|_{\mathcal{F}} \tag{I.51}$$

and

$$\|a^\dagger(f)\psi\|_{\mathcal{F}}^2 \leq \int \frac{|f|^2}{\omega} \|H_f^{1/2}\psi\|_{\mathcal{F}}^2 + \int |f|^2 \|\psi\|_{\mathcal{F}}^2 \tag{I.52}$$

Proof: By the Schwarz inequality we have (dropping the subscript $\mathcal{F}$)

$$\|a(f)\psi\| \;\leq\; \int |f|\;\|a\psi\| \;\leq\; \Big(\int \frac{|f|^2}{\omega}\Big)^{1/2}\Big(\int \omega\|a\psi\|^2\Big)^{1/2}\,.$$

Since

$$\int \omega\|a\psi\|^2 \;=\; \langle\psi, H_f\psi\rangle\,,$$

this implies (I.51). Next, we use that

$$a(f)a^\dagger(f) \;=\; \int\int \bar f(p)f(k)a(p)a^\dagger(k) \;=\; \int\int \bar f(p)f(k)a^\dagger(k)a(p) \;+\; \int |f(p)|^2\,,$$

and that

$$\int\int \bar f(p)f(k)\langle a^\dagger(k)a(p)\rangle_\psi \;\leq\; \Big(\int f(p)\|a(p)\psi\|^2\Big)^2 \;\leq\; \int \frac{|f|^2}{\omega}\langle H_f\rangle_\psi\,,$$

to obtain (I.52). ∎

Since the interaction W_x can be written as

$$W_x \;=\; a^\dagger(G_x) + a(G_x)\,, \tag{I.53}$$

we have the following corollary.

Corollary I.7 *(Relative bound of W_x w.r. to $(H_f)^{1/2}$) Suppose that Hypothesis 1 holds; in particular, assume $\Lambda_1 \;=\; \sup_x \int |G_x|^2 + \sup_x \int \frac{|G_x|^2}{\omega} \;<\; \infty$. Then*

$$\|W_x\psi\|^2 \;\leq\; 2\left\|\left(\int \frac{|G_x|^2}{\omega}\right)\otimes H_f^{1/2}\;\psi\right\|^2 \;+\; \left\|\int |G_x|^2\otimes \mathbf{1}_f\;\psi\right\|^2\,. \tag{I.54}$$

Moreover,

$$|\langle W_x\rangle_\psi| \;\leq\; 2\sup_x\left\{\int \frac{|G_x|^2}{\omega}\right\}^{1/2}\|\cdot\|H_f^{1/2}\psi\|\cdot\|\psi\|\,. \tag{I.55}$$

Thus H_g is selfadjoint on the domain $D(H_{el})\otimes D(H_f)$.

It is interesting to extend this result to Hamiltonians obeying Hypothesis 2.

Corollary I.8 *(Relative bound of W_l) Assume Hypothesis 2. Then*

$$\|W_1\psi\|^2 \;\leq\; 2\Lambda_4\left\|(-\Delta+1)^{1/2}\otimes(H_f+1)^{1/2}\psi\right\|^2\,, \tag{I.56}$$

$$\|W_2\psi\|^2 \;\leq\; 2\Lambda_4^2\,\|\mathbf{1}_{el}\otimes(H_f+1)\psi\|^2\,. \tag{I.57}$$

Thus, for $g < 4\Lambda_4^{-1}$, H_g is selfadjoint on the domain $D(H_{el})\otimes D(H_f)$.

I.5 Resonances and Spectral Deformation

The natural extension of the notion of eigenvalue is the notion of (quantum) resonance. Though originally not rigorously defined, it has played a key role in physics since the birth of quantum mechanics. The first rigorous definition of a resonance was given in [34]. Resonances were the subject of intensive study ever since this work appeared; (see the review in [33]). There are three ways to generalize the notion of an eigenvalue: starting from an eigenequation, or from the space-time picture, or from the analytic structure of the resolvent. We use the third way to define a resonance, the first one as an efficient tool in investigating it, and the second one to interpret it physically.

We recall a definition from Section I.3. Let $C_0(\mathbb{R}^3)$ be the one-photon space of continuous and compactly supported functions. (In the vector model, functions on $\mathbb{R}^3$ should be replaced by transverse vector fields on $\mathbb{R}^3$, i.e., by vector fields, f, obeying $k \cdot f(k) = 0$.) Let $\mathcal{F}(C_0)$ be the bosonic Fock space over $C_0(\mathbb{R}^3)$, i.e., $\mathcal{F}(C_0) = \bigoplus_{n=0}^{\infty} C_0(\mathbb{R}^3)^{\otimes_s n}$, where $\otimes_s$ stands for the symmetric tensor product. This space is a core for H_f, while $\mathcal{D} \equiv C_0^\infty(X) \otimes \mathcal{F}(C_0)$ is a core for H_g. Consider the analytic continuation of $\langle\psi|\,(H_g - z)^{-1}\varphi\rangle$, for $\psi, \varphi \in \mathcal{D}$, from $\mathbb{C}_+$ (the upper half-plane) across the real axis into the lower half-plane, provided such a continuation exists. We define the *resonance eigenvalues* (or simply, *resonances*) as positions of singularities on the second sheet, i.e., in the lower half-plane $\mathbb{C}_-$, of the analytic continuation described above.

In Schrödinger quantum mechanics, these singularities are isolated poles. But, for the Hamiltonians studied in this paper, they are located at branch points.

The connection with eigenvalues is simple. The eigenvalues of H_g, being positions of the poles of $\langle\psi|\,(H_g - z)^{-1}\varphi\rangle$ on the real axis, can turn into resonances under perturbations. The real part of the resonance is called the resonance energy, the imaginary part is the resonance width. The resonance width is interpreted as the decay rate, its reciprocal is called the life-time of the resonance.

Next we proceed to establish the connection between our definition of resonances and an eigenvalue problem.

Theorem I.9 (i) *Under Hypothesis 2,* $\langle f|\,(z-H_g)^{-1}g\rangle$ *has an analytic continuation from* $\mathbb{C}_+$ *across the continuous spectrum of* H_g *into a complex neighbourhood of* $\mathbb{R}$ *, for all* f *and* g *in* $\mathcal{D}$ *.*

(ii) *There is a Type-A family* $H_g(\theta)$ *of operators analytic in a neighbourhood of* $\theta=0$ *, s.t.* $H_g(\theta)^* = H_g(\overline{\theta})$ *,* $\sigma\left(H_g(\theta)\right) \subset \overline{\mathbb{C}_-}$ *, for* $\operatorname{Im}\theta > 0$ *, and*

$$H_g(\theta) \;=\; U(\operatorname{Re}\theta)\, H_g(i\operatorname{Im}\theta)\, U(\operatorname{Re}\theta)^{-1}$$

for a one-parameter group, $U(\lambda)$ *, of unitary dilatations. The poles and branch points of the analytic continuation, described in (i), occur at the eigenvalues of* $H_g(\theta)$ *and fixed points of* $\sigma_{\rm cont}[H_g(\theta)]$ *under small variation of* $\operatorname{Im}\theta > 0$ *.*

Proof: We first construct the family $H_g(\theta)$ described in the theorem. Let U_θ be a dilatation transformation on the one-photon space, i.e., $U_\theta:\ f(k) \to \mathrm{e}^{-3\theta/2} f(e^{-\theta}k)$. (More generally, one can shift along a flow in $\mathbb{R}^3$, see below.) Define the spectral deformation on the Fock space by

$$U_f(\theta)\, a^\dagger(f)\, U_f(\theta)^{-1} \;=\; a^\dagger(U_\theta f)\,. \tag{I.58}$$

To apply this to H_f , we first write H_f in terms of $a^\#(\varphi_k)$, by inserting a one-photon basis $\{\varphi_k\}$:

$$H_f \;=\; \sum_{i,j} \langle \varphi_i, \omega\varphi_j\rangle\; a^\dagger(\varphi_i) a(\varphi_j)\,.$$

Applying (I.58), yields

$$H_f(\theta) \;=\; U_f(\theta)\, H_f\, U_f(\theta)^{-1} \;=\; \int dk\, \omega_\theta(k) a^\dagger(k) a(k)\,,$$

where $\omega_\theta = U_\theta\, \omega(k)\, U_\theta^{-1} = \omega(e^{-\theta}k)$.

We do not dilate the particle coordinates and hence we represent the dilatation $U_f(\theta)$ on $\mathcal{H} = \mathcal{H}_{el} \otimes \mathcal{F}$ by

$$U(\theta) \;=\; \mathbf{1}_{el} \otimes U_f(\theta)\,.$$

Applying $U(\theta)$ to W_1 and W_2 by using (I.58), one easily checks that

$$U(\theta)\, W_l\, U(\theta)^{-1} \;=\; W_l^{(\theta)}\,,$$

where $W_l^{(\theta)}$ is obtained from W_l by the replacement

$$G_{m,n}(\underline{k}) \;\to\; G_{m,n}^{(\theta)}(\underline{k}) \;:=\; \mathrm{e}^{\frac{3\theta}{2}(m+n)} G_{m,n}(\mathrm{e}^{-\theta}\underline{k}) \tag{I.59}$$

in (I.26) and (I.27), where $\underline{k} \in \mathbb{R}^{\,3(m+n)}$. Thus

$$H_g(\theta) \;:=\; U(\theta)\, H_g\, U(\theta)^{-1} \;=\; H_{el} \otimes \mathbf{1}_f \;+\; \mathrm{e}^{-\theta} \mathbf{1}_{el} \otimes H_f \;+\; W_g^{(\theta)}\,,$$

with $W_g^{(\theta)} := gW_1^{(\theta)} + g^2 W_2^{(\theta)}$. Now part (ii) of Theorem I.9 follows from Hypotheses 2.

We note that the dilatation $U(\theta)$ above dilates the photon momenta but leaves the particle coordinates unchanged. This has the strange effect that terms like $\exp(ik \cdot x)$ are transformed into $\exp(i\mathrm{e}^{-\theta} k \cdot x)$ which grow exponentially in some directions in X, unless a spatial cutoff in the particle coordinates that decays super-exponentially fast is imposed. We come back to this point in Section I.6.

It remains to show that Property (i) stated in Theorem I.9 also holds. To prove this, we use the Combes argument. Namely, by the unitarity of $U(\theta)$ for real θ,

$$\langle \psi \;\; (H_g - z)^{-1} \varphi \rangle \;=\; \Big\langle \psi(\bar{\theta}) \Big| \Big(H(\theta) - z \Big)^{-1} \varphi(\theta) \Big\rangle \;, \tag{I.60}$$

where $\psi(\theta) = U(\theta)\psi$, etc., for $\theta \in \mathbb{R}$ and $z \in \mathbb{C}_+$. Assume now that $\psi(\theta)$ and $\varphi(\theta)$ have analytic continuations into a complex neighbourhood of $\theta = 0$. Then the r.h.s. of (I.60) has an analytic continuation in θ into a complex neighbourhood of $\theta = 0$. Since Eqn (I.60) holds for real θ, it also holds in the above neighbourhood. Fix θ on the r.h.s. of (I.60), with $\mathrm{Im}\,\theta > 0$. The r.h.s. of (I.60) can be analytically extended across the real axis into the part of the resolvent set of $H_g(\theta)$ which lies in $\overline{\mathbb{C}_-}$. This yields an analytic continuation of the l.h.s. of (I.60). The thresholds of $H_g(\theta)$, i.e., the points at which branches of continuous spectrum of $H_g(\theta)$ originate, and the eigenvalues of $H_g(\theta)$ are *independent* of θ. Thus the (complex) eigenvalues of $H_g(\theta)$ are the resonances

of H. In fact, the justification of this statement is not easy, since, as we will see later, the eigenvalues in question lie at the tips of continuous spectrum of $H_g(\theta)$, and therefore adequate control of the latter is needed. See Chapters IV–V and [5].

We have assumed above that ψ and φ are such that $\psi(\theta) \equiv U(\theta)\psi$ and $\varphi(\theta) \equiv U(\theta)\varphi$ have analytic continuations into a complex neighbourhood of $\theta = 0$. The set of such vectors is dense in $\mathcal{H}_{el} \otimes \mathcal{F}$. However, it is not the set $\mathcal{D}$ or any other standard class of vectors. In order for $\psi(\theta)$, with $\psi \in \mathcal{D}$, to have an analytic continuation into a neighbourhood of $\theta = 0$, we have to modify the deformation family $U(\theta)$. We briefly sketch such a modification. Let $\psi \in \mathcal{D}$, and let v be a vector field on $\mathbb{R}^3$ supported outside the support of the one-photon part of ψ and approaching the identity vector field, $v_0(k) = k$, as $|k| \to \infty$. Define φ_θ to be either the flow generated by v (see [32]) or the shift along v, i.e., $\varphi_\theta(k) = k + \theta v(k)$, (a linear approximation to the flow above, see [19, 9]). Both definitions have their advantages. We shall appeal to the second one, since it obviously guarantees the analyticity in θ needed for our purposes. Now we define U_θ on the one-photon space by

$$U_\theta : \ \psi \ \to \ \sqrt{\mathrm{Jac}}\, \psi \circ \varphi_\theta \ , \tag{I.61}$$

where Jac is the Jacobian of the transformation $k \to \varphi_\theta(k)$. We can use a similar construction on $\mathcal{H}_{el}$ if our particle system consists of one or two particles (the hydrogen atom). In the many particle space the construction is more subtle (see e.g. [13]), and we omit it here. The resulting transformation $U(\theta)$ has the desired property that $U(\theta)\psi$ has an analytic continuation into a neighbourhood of $\theta = 0$. ∎

The key point here is that the Riemann surface of $(H_g - z)^{-1}$ is *independent* of the choice of the transformation $U(\theta)$, while the cuts (i.e., $\sigma_{\mathrm{cont}}\left(H_g(\theta)\right)$) depend only on the behaviour of $v(k)$ at infinity.

I.6 Comparison to the Standard Model

We come back to our discussion at the end of Section I.1: the coupling functions $\mathbb{G}_{m,n}$ in (I.16)–(I.18) contain terms growing linearly in $\mathbf{x}_j$. Thus, $\mathbb{H}_2$ fails to be a relatively bounded perturbation of $\mathbb{H}_{el} \otimes \mathbf{1}_f + \mathbf{1}_{el} \otimes \mathbb{H}_f$, unless we impose a spatial cutoff in the particle coordinates. Thus, together with the help of the UV-cutoff κ that regularizes the electromagnetic vector potential, $\mathbf{A} \to \mathbf{A}_{\bar{\kappa}}$ by restricting the photons to small momenta, we need to restrict the electron coordinates to small distances from the nucleus. To this end, we have introduced an analytic function $\bar{\kappa}$ obeying $\bar{\kappa}(0) = 1$ and decaying sufficiently fast at infinity. To be specific, we choose $\bar{\kappa}(\mathbf{k}) := \exp\Big(-|\mathbf{k}|^4\Big)$. So, $\bar{\kappa}(\mathrm{e}^{-\theta}\mathbf{k}/K)$ acts as an ultraviolet cutoff, cutting off the photon momenta on scale K (keeping θ fixed). Similarly, we replace

$$\mathbb{G}_{m,n} \;\to\; \mathbb{G}_{m,n;reg} \;:=\; \prod_{j=1}^{N} \bar{\kappa}(\mathbf{x}_j/r_{el})\, \mathbb{G}_{m,n} \tag{I.62}$$

in (I.16)–(I.18). Here, $\bar{\kappa}(\mathbf{x}/r_{el})$ imposes a spatial cutoff on the electron variables x at length scale r_{el}. In physics, one would choose r_{el} of the order $\max_n Z_n^{-1}$, the diameter of the atom or molecule under consideration.

As we have pointed out in the preceeding section, if we ignored the spatial cutoff functions $\bar{\kappa}(\mathbf{x}/r_{el})$ then $\mathbb{G}_{m,n;\theta}$ would grow exponentially as $|k| \to \infty$. One way of avoiding this growth is to simultaneously dilate the electron coordinates $\mathbf{x} \mapsto \mathrm{e}^{\theta}\mathbf{x}$ and the photon variable $\mathbf{k} \mapsto \mathrm{e}^{-\theta}\mathbf{k}$. In order not to unnecessarily encumber the present exposition, we refrain from doing so and, rather, damp $\|\mathbb{G}_{m,n;\theta}\|$ by a rapidly decaying cutoff, $\bar{\kappa}$.

Now, we will show that $\mathbb{G}_{m,n;theta}$ fulfills Hypotheses 2 and 3, provided we choose $\bar{\kappa}(\mathbf{k}) := \exp(-|\mathbf{k}|^4)$, for example. We remark that the choice $\tilde{\kappa}(\mathbf{k}) := \exp(-|\mathbf{k}|^2)$ would not suffice at this point because (I.65) would not hold – unless we additionally assume $gKr_{el} \ll 1$.

Theorem I.10 *Let $\bar{\kappa}(\mathbf{k}) := \exp(-|\mathbf{k}|^4)$. Then $\mathbb{G}_{m,n;\theta}$ fulfills Hypothesis 2 and 3 with*

$$J(\mathbf{k}) \;\leq\; C_0\; |\mathbf{k}|^{1/2}\; \exp[-|\mathbf{k}|^4/C_0]\,, \tag{I.63}$$

for some constant C_0 depending on $\mathbb{V}$ and ϑ_0, where $|\mathrm{Im}\theta| \leq \vartheta_0$.

Proof: We will give the proof only for $\mathbb{G}_{1,0;\theta}$ and $\mathbb{G}_{2,0;\theta}$; but the other cases are similar. For our argument we need three basic estimates

$$\left|\exp\left[-\mathrm{e}^{-4\theta}|\mathbf{k}|^4/K^4 - |\mathbf{x}|^4/r_{el}^4\right]\right| \; |\mathbf{x}| \;\leq\; \mathrm{const}\cdot r_{el}\cdot\exp\left[-\mathrm{e}^{\pi}|\mathbf{k}|^4/K^4\right], \tag{I.64}$$

$$\left|\exp\left[-\mathrm{e}^{-4\theta}|\mathbf{k}|^4/K^4 - |\mathbf{x}|^4/r_{el}^4 + \alpha\mathrm{e}^{|\theta|}|\mathbf{k}|\,|\mathbf{x}|\right]\right| \;\leq\; \mathrm{const}\cdot r_{el}\cdot\exp\left[-\mathrm{e}^{-\pi}|\mathbf{k}|^4/(2K^4) - |\mathbf{x}|^4/(2r_{el}^4)\right], \tag{I.65}$$

$$\left|\exp\left[-i\alpha\mathrm{e}^{-\theta}\mathbf{k}\cdot\mathbf{x}\right] \;-\; 1\right| \;\leq\; \alpha\cdot\mathrm{e}^{|\theta|}|\mathbf{k}|\cdot|\mathbf{x}|\;\exp\left[\alpha\mathrm{e}^{|\theta|}|\mathbf{k}|\,|\mathbf{x}|\right], \tag{I.66}$$

the proofs of which are elementary. Since $\mathbb{G}_{2,0;\theta}(\mathbf{k},\lambda;\mathbf{k}',\lambda')$ is a multiplication operator on $\boldsymbol{\mathcal{H}}_{el}$, its norm is given by the supremum of its modulus. Thus

$$\|\mathbb{G}_{2,0;\theta}(\mathbf{k},\lambda;\mathbf{k}',\lambda')\| \;\leq\; \frac{8N}{\pi}\sup_{\mathbf{x}\in\mathbb{R}^3}\left\{\, f(\mathbf{k},\mathbf{x})\, f(\mathbf{k}',\mathbf{x})\right\}, \tag{I.67}$$

where

$$\begin{aligned} f(\mathbf{k},\mathbf{x}) \;&:=\; \omega(\mathbf{k})^{-1/2}\left|\exp\left[-\mathrm{e}^{-4\theta}|\mathbf{k}|^4/K^4 - |\mathbf{x}|^4/r_{el}^4\right]\right|\cdot\left|\exp\left[-i\alpha\mathrm{e}^{-\theta}(\mathbf{k}\cdot\mathbf{x})\right] - 1\right| \\ &\leq\; \mathrm{const}\alpha\, r_{el}\mathrm{e}^{\pi/4}\,\omega(\mathbf{k})^{1/2}\exp\left[-\mathrm{e}^{\pi}|\mathbf{k}|^4/(2K^4)\right], \end{aligned} \tag{I.68}$$

successively applying (I.66), (I.65), and (I.64). The right side of (I.68) yields the desired estimate for $\mathbb{G}_{m,n;\theta}$.

Returning to the definition of $\mathbb{G}_{1,0;\theta}(\mathbf{k},\lambda)$, we observe that it is a sum of three terms corresponding to the r.s. in (I.16). Two of these are multiplication operators on $\boldsymbol{\mathcal{H}}_{el}$, and they are shown to be bounded in the same way as we have just demonstrated for $\mathbb{G}_{2,0;\theta}(\mathbf{k},\lambda;\mathbf{k}',\tau')$. So, all terms in $\mathbb{G}_{m,n;\theta}$ are bounded operators, except one which is only relatively $\left(\sum_{n=1}^{N}-\Delta_n\right)^{1/2}$–bounded. But this is sufficient for the estimate (I.35). ∎

II Binding

In this chapter we address the issue of binding, i.e., Problem (a) described in Section I.2. Theorem II.8 or, equivalently, Theorem I.1(a) represent our solution of this problem. Our approach is based on introducing an IR (infrared) cutoff, obtaining estimates on the cutoff problem that are *uniform* in the cutoff and then removing the cutoff and showing that cutoff objects converge to those of the original problem. Note that the role of the IR cutoff is to make the eigenvalues of interest isolated. The manner in which we realize the cutoff is not very important. We do it by decoupling photons of energies $\leq m$ from the electrons, so our estimates must be uniform in m. In this chapter, we require Hypothesis 1, i.e., we consider the operator $H_g = H_0 + gW_x$ with the assumption that, for some $\varepsilon \geq 0$,

$$\begin{aligned} \Lambda_1 &= \sup_x \left\{ \int \left[1 + \omega(k)^{-1}\right] |G_x(k)|^2 \, dk \right\}^{1/2} < \infty , \\ \Lambda_2(\varepsilon) &= \sup_x \left\{ \mathrm{e}^{-\varepsilon|x|} \int \left[1 + \omega(k)^{-1}\right] |G_x(k)|^2 \, dk \right\}^{1/2} < \infty . \end{aligned}$$

To ease the handling of the many sub- and superindices that we introduce below, we write $H \equiv H_g$ in this chapter.

II.1 Exponential Bounds

Theorem II.1 *Let $\Delta \subset \mathbb{R}$ be an interval with $\sup \Delta < \Sigma := \inf \sigma(H_{el}) \leq 0$ and let $\alpha, g > 0$ satisfy $\alpha^2 < \sum - \sup \Delta - g^2 \sup_x \int \frac{|G_x|^2}{\omega}$. Then there exists a constant, $M_\Delta < \infty$, depending only on V, Λ_1, and Δ, such that*

$$\| \mathrm{e}^{\alpha|x|} \otimes \mathbf{1}_f \, \chi_\Delta(H) \| \leq M_\Delta < \infty . \tag{II.1}$$

Proof: The idea of the proof is based on the fact (see e.g. [10, 21]) that, for any $\varepsilon > 0$, there is $R > 0$ such that

$$H_{el} \geq \Sigma - \varepsilon \qquad \text{on } \{|x| \geq R\} . \tag{II.2}$$

Let χ_R be the characteristic function of the set $\{|x| \geq R\}$. Denote by H_R the operator obtained from H by replacing the particle potential $V(x)$ by

$V(x)\chi_R(x)$. Then, by the inequalities $\pm gW_x \leq \varepsilon H_f + \frac{g^2\Lambda^2}{\varepsilon}$, where $\Lambda = \sup_x \left\{\int \frac{|G_x|^2}{\omega}\right\}^{1/2}$, and by an analogue of (II.2), we have that

$$H_R \geq \left[\frac{1}{2}H_f - 2g^2\Lambda^2 + \Sigma - \varepsilon\right] .$$

Hence by the condition on Δ, and for g and ε sufficiently small, $\sup\Delta < \Sigma - \varepsilon - 2g^2\lambda^2$, and therefore

$$\chi_\Delta(H_R) = 0 .$$

This implies that

$$\chi_\Delta(H) = \chi_\Delta(H) - \chi_\Delta(H_R) . \tag{II.3}$$

Without loss of generality we can replace $\chi_\Delta(\mu)$ by a smooth function which we again denote by $\chi_\Delta(\mu)$. In this case there is a smooth function f on $\mathbb{C}$ which is almost analytic [26] (i.e., $\partial_{\bar{z}} f = 0$ on $\mathbb{R}$) and compactly supported, such that

$$\chi_\Delta(A) = \int df(z)(z-A)^{-1} , \tag{II.4}$$

where $df(z) = -\frac{1}{2\pi}\partial_{\bar{z}} f(z)dxdy$, $z = x + iy$, for any selfadjoint operator A. Moreover, the support of f can be taken to be inside an arbitrarily small complex neighbourhood of the interval Δ (see [21]). Eqn. (II.3) then implies

$$\begin{aligned}\chi_\Delta(H) &= \int df(z)[(z-H)^{-1} - (z-H_R)^{-1}] \\ &= -\int df(z)(z-H_R)^{-1}\bar{\chi}_R V(z-H)^{-1} ,\end{aligned}$$

where $\bar{\chi}_R = \mathbf{1} - \chi_R$. Now, let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a smooth, convex function with $\phi \equiv 0$ on $(0,2]$ and $\phi(r) = r - 3$ for $r \geq 4$. Then $|\phi'| \leq 1$. Defining $F(x) := \alpha R\,\phi(|x|/R)$, we observe that $F \equiv 0$ on $\{|x| \leq 2R\}$ and $F(x) = \alpha|x| - 3\alpha$ for all $|x| \geq 4R$. Moreover, $|\nabla_x F| \leq \alpha$. Define $H_{R,F} = \mathrm{e}^F H_R\,\mathrm{e}^{-F}$. Then

$$\mathrm{e}^F(z-H_R)^{-1}\mathrm{e}^{-F} = (z-H_{R,F})^{-1} .$$

Since F is chosen such that $\mathrm{e}^F\bar{\chi}_R = \bar{\chi}_R$, the last two relations yield

$$\mathrm{e}^F\chi_\Delta(H) = -\int df(z)(z-H_{R,F})^{-1}\bar{\chi}_R V(z-H)^{-1} . \tag{II.5}$$

In order to estimate the first factor on the r.h.s., we write

$$H_{R,F} \;=\; H_R - |\nabla_x F|^2 - iA_F \;,$$

where $A_F = \frac{1}{2}(p\cdot\nabla_x F + \nabla_x F\cdot p)$, with $p = -i\,\nabla_x$. Having chosen $\mathrm{supp} f \ni z$ sufficiently close to Δ and ε sufficiently small, we note that $H_R - |\nabla_x F|^2 - \mathrm{Re}(z) \geq \Sigma - \varepsilon - 2g^2\Lambda^2 - \alpha^2 - \mathrm{Re}(z) \geq \mathrm{const} > 0$. Moreover, A_F is a perturbation of $Q := H_R - |\nabla_x F|^2 - \mathrm{Re}(z)$ of relative bound zero. Thus

$$\begin{aligned}\Big\|(H_{R,F}-z)^{-1}\Big\| \;&\leq\; \Big\|Q^{-1/2}\Big\|^2 \cdot \Big\|\Big(\mathbf{1} \;-\; iQ^{-1/2}(A_F+\mathrm{Im}z)Q^{-1/2}\Big)^{-1}\Big\| \\ &\leq\; \Big\|(H_R - |\nabla_x F|^2 - \mathrm{Re}z)^{-1}\Big\| \;\leq\; \mathrm{const}\;.\end{aligned}$$

This, together with (II.5), yields

$$\|\mathrm{e}^F\otimes\mathbf{1}_f\;\chi_\Delta(H)\| \;\leq\; \mathrm{const}\,\Big\|V(H+i)^{-1}\Big\|\cdot\int\Big\|\frac{H+i}{H-z}\Big\|\;|df(z)|\;.$$

Since $|\partial_{\bar z} f(z)| \leq \mathrm{const}\cdot|y|$, with $z = x+iy$, and f is compactly supported, the r.h.s. is bounded. ∎

II.2 Existence of the Ground State for H_m, $m > 0$

We begin by introducing an IR cutoff in the interaction, replacing $W \equiv W_x = \int\Big[G_x(k)\otimes a^\dagger(k) + \bar G_x(k)\otimes a(k)\Big]dk$ by

$$W_m \;:=\; \int_{\{\omega(k)\geq m\}}\Big[G_x(k)\otimes a^\dagger(k) + \bar G_x(k)\otimes a(k)\Big]dk\;. \tag{II.6}$$

We define

$$H_m \;:=\; H_{el}\otimes\mathbf{1}_f + \mathbf{1}_{el}\otimes H_f + gW_m \;=\; H_0 + gW_m \tag{II.7}$$

Our task is to prove that H_m has a ground state and then to obtain good control of the latter.

Theorem II.2 *Let $\Sigma := \inf\sigma_{cont}(H_{el})$, $E_0 := \inf\sigma(H_{el})$ and assume that*

$$\Sigma - E_0 - g^2\sup_x\Big\{\int|G_x(k)|^2\omega^{-1}(k)dk\Big\} \;\geq\; 4m > 0\;. \tag{II.8}$$

Then H_m has a ground state ϕ_m at the bottom $E_m := \inf\sigma(H_m)$ of its spectrum. Moreover, ϕ_m is unique, provided that $G_x(k) = \overline{G_x(-k)}$, for all $k\in\mathbb{R}^3$.

Proof: We begin the proof with the remark that if the one-particle Hilbert space h is a direct sum $h_1 \oplus h_2$, then the bosonic Hilbert space $\mathcal{F}$ over h is isomorphic to $\mathcal{F}_1 \otimes \mathcal{F}_2$, where $\mathcal{F}_i$ is the Fock space over h_i. Indeed, if $\{f_j\}$ is an orthonormal basis in h_1 and $\{g_j\}$ is an orthonormal basis in h_2, this isomorphism is given by

$$a^\dagger(f_1)\cdots a^\dagger(f_k)a^\dagger(g_1)\cdots a^\dagger(g_l)\Omega \;\mapsto\; a^\dagger(f_1)\cdots a^\dagger(f_k)\Omega_1 \otimes a^\dagger(g_1)\cdots a^\dagger(g_l)\Omega_2 \,.$$

In the present situation we have $h = L^2(\mathbb{R}^3) = L^2(\mathcal{A}_l) \oplus L^2(\mathcal{A}_s)$, where $\mathcal{A}_l := \{k \,|\, \omega(k) < m\}$ and $\mathcal{A}_l := \mathbb{R}^3 \backslash \mathcal{A}_s$. The isomorphism above maps H_f to $H_{f,l} \otimes \mathbf{1}_s + \mathbf{1}_l \otimes H_{f,s}$ and gW_m to $gW_m \otimes \mathbf{1}_s$. Here, $H_{f,s/l} := \int_{\mathcal{A}_{s/l}} \omega \, a^\dagger a$. In this representation, H_m appears as

$$H_m \;=\; H_{m,l} \otimes \mathbf{1}_s + (\mathbf{1}_{el} \otimes \mathbf{1}_l) \otimes H_{f,s} \,, \tag{II.9}$$

as an operator on $(\mathcal{H}_{el} \otimes \mathcal{F}_l) \otimes \mathcal{F}_s$, where

$$H_{m,l} \;:=\; H_{el} \otimes \mathbf{1}_l + \mathbf{1}_{el} \otimes H_{f,l} + gW_m \,.$$

From representation (II.9) it is obvious that H_m has a ground state $\phi_m \in (\mathcal{H}_{el} \otimes \mathcal{F}_l) \otimes \mathcal{F}_s$ if and only if $H_{m,l}$ has a ground state $\phi_{m,l} \in \mathcal{H}_{el} \otimes \mathcal{F}_l$. Indeed, in this case $\phi_m = \phi_{m,l} \otimes \Omega_s$. Thus the existence of the ground state ϕ_m follows from Lemma II.3 below. We postpone the uniqueness part of the proof after the proof of Lemma II.3.

Lemma II.3 *Under the conditions of Theorem II.2 $H_{m,l}$ has a ground state $\phi_{m,l}$ at the bottom $E_m := \inf \sigma(H_m) = \inf \sigma(H_{m,l})$ of its spectrum.*

Proof: The claim follows by proving that

$$\mathrm{dimRan}\left\{\chi_{H_{m,l} < E_m + m}\right\} \;<\; \infty \,. \tag{II.10}$$

To prove this inequality, we employ a discretization, i.e., a family of operators $H^\varepsilon_{m,l}$, for $\varepsilon > 0$, with $H^\varepsilon_{m,l} \to H_{m,l}$ in the norm resolvent sense, as $\varepsilon \to 0$. This ensures $\|\chi_{H_{m,l} \le E_m + m} - \chi_{H^\varepsilon_{m,l} \le E_m + m}\| < 1$ and, hence,

$$\mathrm{dimRan}\left\{\chi_{H_{m,l} \le E_m + m}\right\} \;=\; \mathrm{dimRan}\left\{\chi_{H^\varepsilon_{m,l} \le E_m + m}\right\}$$

for sufficiently small $\varepsilon > 0$. Therefore, (II.10) holds, provided we prove that $\dim\mathrm{Ran}\left\{\chi_{H^\varepsilon_{m,l}\le E_m+m}\right\} < \infty$, for sufficiently small $\varepsilon > 0$, which we demonstrate below.

We now construct the discretized operators $H^\varepsilon_{m,l}$. Consider $\varepsilon\omega(k)$ as a scale function on $\mathbb{R}^3$ and partition $\mathbb{R}^3$ w.r. to this function. Namely, we decompose $\mathbb{R}^3$ into a disjoint union of "cubes" $C_{\alpha,\varepsilon}$ s.t.

$$c\varepsilon\,\mathrm{dist}(C_{\alpha,\varepsilon},0) \;\le\; \mathrm{diam}(C_{\alpha,\varepsilon}) \;\le\; C\varepsilon\,\mathrm{dist}(C_{\alpha,\varepsilon},0)$$

for some universal c and C (1 and 4, say). One can find such a decomposition by breaking $\mathbb{R}^3$ first into the shells $\{\varepsilon 2^{-n-1} \le |k| < \varepsilon 2^{-n}\}$, $n \in \mathbb{Z}$, and then breaking each shell into segments by intersecting it with cones based on a partition of the unit sphere, S^2, into sets of equal size. (Alternatively, we observe that $\varepsilon|k|$ is a distance of k to the set $\{0\}$ and appeal to a general result in [35] (Chapter VI, Theorem 1), giving the desired decomposition into cubes, indeed. Note, in parentheses, that such a decomposition is locally finite, i.e. every point of $\mathbb{R}^3$ belongs to closures of a uniformly bounded number of cubes.) Define $\omega_\varepsilon(k) = \mathrm{Av}_{C_{\alpha,\varepsilon}}(\omega) \equiv (\mathrm{vol}\,C_{\alpha,\varepsilon})^{-1}\int_{C_{\alpha,\varepsilon}}\omega$, where $C_{\alpha,\varepsilon} \ni k$, and

$$H^\varepsilon_{f,l} \;=\; \int_{\mathcal{A}_l}\omega_\varepsilon a^\dagger a$$

Note that by the choice of the $C_{\alpha,\varepsilon}$'s

$$\sup\frac{|\omega-\omega_\varepsilon|}{\omega} \;\le\; \text{const}\;\cdot\varepsilon \tag{II.11}$$

and therefore

$$\pm(H_{f,l} - H^\varepsilon_{f,l}) \;\le\; \text{const}\;\cdot\varepsilon H_{f,l} \tag{II.12}$$

Similarly, we define gW^ε_m by replacing $G_x(k)$ by $G^\varepsilon_x := \mathrm{Av}_{C_{\alpha,\varepsilon}}(G_x)$ for $C_{\alpha,\varepsilon} \ni k$. We set

$$H^\varepsilon_{m,l} = H_{el}\otimes\mathbf{1}_l + \mathbf{1}_{el}\otimes H^\varepsilon_{f,l} + gW^\varepsilon_m\,. \tag{II.13}$$

Now we show that $H^\varepsilon_{m,l} \to H_{m,l}$, in norm resolvent sense, as $\varepsilon\to 0$. It is sufficient to prove, only for one $z\in\mathbb{C}$, that $\|(H^\varepsilon_{m,l}-z)^{-1} - (H_{m,l}-z)^{-1}\| \to 0$, and we pick z such that

$$\|(H^\varepsilon_{m,l}-z)^{-1}\|\;\cdot\;\|\,[\mathbf{1}_{el}\otimes(H_{f,\ell}+1)](H_{m,l}-z)^{-1}\| \;=\; \text{const} \;<\; \infty.$$

Then the second resolvent equation, Corollary I.7 and Eqn. (II.12) imply that

$$\begin{aligned} &\|(H^\varepsilon_{m,l} - z)^{-1} - (H_{m,\ell} - z)^{-1}\| \\ &\quad \leq \quad \text{const} \left\{ \varepsilon \, + \, \sup_x \int \Big(1 + \omega^{-2}(k)\Big) |G_{x,\varepsilon}(k) - G_x(k)|^2 dk \right\} \end{aligned}$$

and the right side converges to 0 by Lebesgue dominated convergence.

We complete the proof by showing that $\dim\mathrm{Ran}\left\{\chi_{H^\varepsilon_{m,l} \leq E_m + m}\right\} < \infty$ for small but non-zero $\varepsilon > 0$, i.e., that $H^\varepsilon_{m,l} - E_m - m$ has only finitely many eigenvalues. To this end, we decompose the one-photon state space into discrete and fluctuating parts as

$$L^2(\mathcal{A}_l, dk) \; = \; h^d \oplus h^f \; , \tag{II.14}$$

where h^d (the "discrete subspace") is the l^2-space of functions which are constant on the cubes $C_{\alpha,\varepsilon}$, while h^f (the "fluctuating subspace") is the orthogonal complement of h^d. As in the proof of Theorem II.2, the decomposition (II.14) leads to the representation

$$\mathcal{H}_{el} \otimes \mathcal{F}_l \; = \; \Big(\mathcal{H}_{el} \otimes \mathcal{F}_d\Big) \otimes \mathcal{F}_f \; ,$$

on which $H^\varepsilon_{m,l}$ decomposes as

$$H^\varepsilon_{m,l} \; = \; H^\varepsilon_{m,d} \otimes \mathbf{1}_f + \mathbf{1}_d \otimes H^\varepsilon_{f,f} \tag{II.15}$$

where

$$H^\varepsilon_{m,d} \; = \; H_{el} \otimes \mathbf{1}_d + \mathbf{1}_{el} \otimes H^\varepsilon_{f,d} + g W^\varepsilon_m \; . \tag{II.16}$$

Note that

$$E^\varepsilon_m \; := \; \inf \sigma(H^\varepsilon_{m,l}) \; = \; \inf \sigma(H^\varepsilon_{m,d}) \leq E_0 \; .$$

Let us denote the projection onto the vacuum in $\mathcal{F}_{d/f}$ by $P_{d/f}$, remarking that $H^\varepsilon_{f,f} P_f^\perp \geq m P_f^\perp$. Hence,

$$H^\varepsilon_{m,l} \; \geq \; \Big(H^\varepsilon_{m,d} + m\Big) \otimes P_f^\perp + H^\varepsilon_{m,d} \otimes P_f \; \geq \; H^\varepsilon_{m,d} \otimes P_f \; . \tag{II.17}$$

Next, we set $P_{el} := \chi_{H_{el} \leq \Sigma - m}$, so that $H_{el} \geq E_0 P_{el} + (\Sigma - m^*) P_{el}^\perp$. Note that P_{el} is finite dimensional. We use (I.55) which implies that for any $\lambda > 1$, we have

$$g W^\varepsilon_m \; > \; -\lambda^{-1} \mathbf{1}_{el} \otimes H^\varepsilon_{f,d} - \lambda F^\varepsilon \; , \tag{II.18}$$

where $F^\varepsilon := \sup_x \int_{\mathcal{A}_l} |G_x^\varepsilon(k)|^2 \omega_\varepsilon^{-1} dk$. Passing to sufficiently small $\varepsilon > 0$, we may assume that $F^\varepsilon \leq m^* + \sup_x \int |G_x(k)|^2 \omega^{-1}(k) dk \leq \Sigma - E_0 - 3m$, where m^* is the l.h.s. of (II.8) divided by 4. Thus

$$\begin{aligned} H_{m,d}^\varepsilon &\geq H_{el} \otimes \mathbf{1}_d - \lambda F^\varepsilon + (1-\lambda^{-1}) \mathbf{1}_{el} \otimes H_{f,d}^\varepsilon \\ &\geq P_{el}^\perp \otimes \left(\Sigma - m - \lambda F^\varepsilon \right) + P_{el} \otimes \left((1-\lambda^{-1}) H_{f,d}^\varepsilon + E_0 - \lambda F^\varepsilon \right) . \end{aligned} \tag{II.19}$$

Choosing $\lambda := 1 + m(\Sigma - E_0)^{-1}$, which ensures $\Sigma - 2m - E_m^\varepsilon - \lambda F^\varepsilon \geq 0$, and remembering that $E_0 \geq E_m^\varepsilon$, we obtain

$$H_{m,l}^\varepsilon - E_m^\varepsilon - m \;\geq\; P_{el} \otimes \left((1-\lambda^{-1}) H_{f,d}^\varepsilon - \Sigma + E_0 \right) \otimes P_f \; . \tag{II.20}$$

We conclude the proof by using the min-max principle, which implies, together with (II.20), that

$$\mathrm{dimRan} \left\{ \chi_{H_{m,l}^\varepsilon \leq E_m^\varepsilon + m} \right\} \;\leq\; \mathrm{dimRan}\{P_{el}\} \cdot \mathrm{dimRan} \left\{ \chi_{H_{f,d}^\varepsilon \leq m^{-1}(\Sigma - E_0 + m)^2} \right\} \; . \tag{II.21}$$

The right side is clearly finite, since $H_{f,d}^\varepsilon$ has discrete spectrum of finite multiplicity in any interval $[0, a]$. ∎

Proof of Theorem II.2, continued: To prove uniqueness of the ground state in the case where $\mathcal{H}_{\mathrm{at}} = L^2(\mathbb{R}^n)$ and $G_x(k) = \overline{G_x(k)}$ is fixed, we use a Perron-Frobenius argument (see [15]). We consider the Schrödinger representation, $L^2\left(\mathcal{S}'_{\mathrm{real}}(\mathbb{R}^3), d\mu_C\right)$, of the Fock space $\mathcal{F}$, where $d\mu_C$ is the Gaussian measure with mean 0 and covariance operator $C = (2\omega)^{-1}$. In the Schrödinger representation, $I = a(G_x) + a^\dagger(G_x)$ is a *real* multiplication operator. Furthermore, as follows from [15], the operators $\exp(-tH_f)$ are positivity preserving and ergodic, for all $t > 0$. Ergodicity means that, for arbitrary non-negative functions ψ and ϕ in $L^2(\mathcal{S}'_{\mathrm{real}}(\mathbb{R}^3), d\mu_C)$ of positive L^2-norm,

$$\langle \psi | \, \mathrm{e}^{-tH_f} \phi \rangle \;>\; 0 \; ,$$

for t large enough. The proof of ergodicity presented in [15] does not quite cover our case, but a variant of their proof still holds: First, note that the vacuum Ω in $L^2\left(\mathcal{S}'_{\mathrm{real}}(\mathbb{R}^3), d\mu_C\right)$ is the functional identically equal to 1, $\Omega \equiv 1$. Let P_Ω

be the orthogonal projection onto Ω, and $P_\Omega^\perp = \mathbf{1} - P_\Omega$. Then

$$\begin{aligned}\langle\psi|\mathrm{e}^{-tH_f}\phi\rangle &= \langle\psi|P_\Omega\phi\rangle - \langle\psi,\mathrm{e}^{-tH_f}P_\Omega^\perp\phi\rangle \\ &\geq \int\psi\int\phi - \|\mathrm{e}^{-\frac{t}{2}H_f}P_\Omega^\perp\psi\|\|\mathrm{e}^{-\frac{t}{2}H_f}P_\Omega^\perp\phi\|\,,\end{aligned}$$

where $\int\psi \equiv \int\limits_{\mathcal{S}'_{\mathrm{real}}(\mathbb{R}^3)} \psi(\varphi)d\mu_C(\varphi)$. Since ψ and ϕ are non-negative functions, there exists some $\delta > 0$ such that

$$\int\psi\int\phi \;\geq\; \delta^2\|\psi\|\|\phi\|$$

(of course, δ depends on ψ and on ϕ). Since the spectrum of H_f is absolutely continuous on $\mathrm{Ran}P_\Omega^\perp$, there is some $\varepsilon > 0$ (depending on ψ and on ϕ) such that

$$\|\chi_{H_f\leq\varepsilon}\,P_\Omega^\perp\,\psi\| \;\leq\; \frac{\delta}{4}\|\psi\|\,,$$

and similarly for ϕ.

Next, we choose t_0 so large that

$$\|\mathrm{e}^{-\frac{t}{2}H_f}\chi_{H_f\geq\varepsilon}\psi\| \;\leq\; \mathrm{e}^{-\frac{t\varepsilon}{2}}\|\psi\| \;\leq\; \frac{\delta}{4}\|\psi\|\,,$$

for arbitrary $t \geq t_0$, and similary for ϕ. Combining all these inequalities, we conclude that

$$\begin{aligned}\langle\psi|\,\mathrm{e}^{-tH_f}\phi\rangle &\geq \int\psi\int\phi - \|\mathrm{e}^{-\frac{t}{2}H_f}P_\Omega^\perp\psi\|\|\mathrm{e}^{-\frac{t}{2}H_f}P_\Omega^\perp\phi\| \\ &\geq \frac{3}{4}\delta^2\|\psi\|\|\phi\| \;>\; 0\,,\end{aligned}$$

for $t \geq t_0$. This proves ergodicity even for the *massless*, scalar field.

The theory of Schrödinger operators tells us that, under our assumptions on $V(x)$, $e^{-tH_{el}}$, for $t > 0$, is a positivity preserving and ergodic operator on $L^2(\mathbb{R}^n)$; (see e.g. Theorems XIII.44 and XIII.46 in [31]). Hence e^{-tH_0}, with $H_0 = H_{el}\otimes\mathbf{1}_f + \mathbf{1}_{el}\otimes H_f$, is positivity preserving and ergodic on $L^2(\mathbb{R}^n)\otimes L^2\Big(\mathcal{S}'_{\mathrm{real}}(\mathbb{R}^3), d\mu_C\Big)$, for $t > 0$. Define the bounded real multiplication operator $I_{m,k} = I\chi_{|I|\leq k}$. Since gW_m is H-bounded, we conclude that $H_0 + I_{m,k}$ converges to $H_0 + I \equiv H_m$ in the strong resolvent sense, as $k\to\infty$. Hence, by Theorems XIII.43 and XIII.45 in [31], e^{-tH_m} is positivity preserving and ergodic. The last

property implies that the ground state is strictly positive and therefore unique (see Theorem XIII.43 in [31]). This finishes the proof of Theorem II.2. ∎

Recall $\Lambda_1^2 = \sup_x \int [1+\omega(k)^{-1}]\, |G_x|^2\, dk < \infty$. Thus, due to Corollary I.7, H_m is resolvent norm continuous in g. We have shown that, for each g (including $g = 0$), the interval of length $\frac{m}{2}$ at the bottom of the spectrum of H_m consists of a finite number of eigenvalues of finite multiplicity. General perturbation theory implies that those eigenvalues are continuous in g.

Finally, we point out that W_m and thus H_m are subject to Hypothesis 1. In particular, V, Σ, Λ_1 and Λ_2 are the same as for H. Consequently, the exponential decay estimate (II.1) holds for H_m, as it does for H, with $M < \infty$ uniform in $m > 0$. We formulate this as a corollary of Theorem II.1.

Corollary II.4 *Let $\varepsilon \leq 0$ satisfy $\varepsilon^2 < \Sigma - E_0 - g^2 \sup_x \int \frac{|G_x|^2}{\omega}$, and let ϕ_m be a (normalized) ground state of H_m. Then there exists a constant, $M_\varepsilon < \infty$, depending only on V and Λ_1, uniformly in $m > 0$, such that*

$$\|\mathrm{e}^{\varepsilon|x|} \otimes \mathbf{1}_f\, \phi_m\|^2 \;\leq\; M_\varepsilon \;<\; \infty\,. \tag{II.22}$$

Proof: Clearly, $E_m \leq \langle H_m \rangle_{\psi_{0,1}\otimes\Omega} = E_0$ for $H_{el}\psi_{0,1} = E_0\psi_{0,1}$, by the variational principle. Thus, the claim follows from Theorem II.1 upon choosing $\Delta := [E_m - 1\,,\, E_0]$. ∎

II.3 Soft photons

The main result of this section is the following theorem, which yields control of the number of soft photons and is inspired by [12]:

Theorem II.5 *Let ϕ_m be a normalized ground state of H_m, and let N be the photon number operator. Then*

$$\begin{aligned} \langle \phi_m \mid \mathbf{1}_{el} \otimes N\, \phi_m \rangle \;&\leq\; g^2 \int \frac{\|G_x(k) \otimes \mathbf{1}_f\, \phi_m\|^2\, dk}{\omega(k)^2} \\ &\leq\; \|\mathrm{e}^{\varepsilon|x|} \otimes \mathbf{1}_f\, \phi_m\|^2 \cdot \sup_x \left\{ \mathrm{e}^{-\varepsilon|x|} \int \frac{|G_x(k)|^2\, dk}{\omega^2} \right\} \end{aligned} \tag{II.23}$$

where $0 \leq \varepsilon \leq (\Sigma - E_0 - g^2 \int \frac{|G_x|^2}{\omega})^{1/2}$.

Proof: By the definition of N,

$$\langle \phi_m \,|\, \mathbf{1}_{el} \otimes N\, \phi_m \rangle \;=\; \int \|\mathbf{1}_{el} \otimes a(k)\, \phi_m\|^2 \; dk \;. \tag{II.24}$$

Thus the theorem follows from Lemma II.6 below. ∎

Lemma II.6 *Let ϕ_m be a ground state of H_m. Then*

$$\|\mathbf{1}_{el} \otimes a(k)\, \phi_m\| \;\leq\; \omega(k)^{-1} \|G_x(k) \otimes \mathbf{1}_f \; \phi_m\| \;. \tag{II.25}$$

Proof: We proceed in the spirit of the virial theorem. Commuting an annihilation operator $a(k)$ through H_m, we see that

$$\mathbf{1}_{el} \otimes a(k)\, H_m - \Big(H_m + \omega(k)\Big)\, \mathbf{1}_{el} \otimes a(k) \;=\; G_x(k) \otimes \mathbf{1}_f \;. \tag{II.26}$$

Using that $H_m \phi_m = E_m \phi_m$ and equation (II.26), we obtain

$$\Big(H_m - E_m + \omega(k)\Big)\, \mathbf{1}_{el} \otimes a(k)\; \phi_m \;=\; -G_x(k) \otimes \mathbf{1}_f \; \phi_m \;.$$

Since $H_m - E_m \geq 0$ this, in turn, implies (II.25). ∎

II.4 Overlap with the Vacuum

Let P_ϕ stand for the rank-one orthogonal projection on ϕ, i.e., $P_\phi = |\phi\rangle\langle\phi|$. Our results in Section II.3 imply the following lemma.

Lemma II.7 *(Lower Bound on the Overlap) Let $\lambda \in (E_0, \Sigma)$ and $P_{el} = \chi_{H_{el} \leq \lambda}$. Let ϕ_m be a normalized ground state of H_m, $m > 0$. Suppose that $0 \leq \varepsilon \leq (\Sigma - E_0 - g^2 \int \frac{|G_x|^2}{\omega})^{1/2}$ and*

$$\delta_\varepsilon \;:=\; g^2\, M_\varepsilon\, \Lambda_2(\varepsilon) \cdot \left[1 \;+\; g^2 (\lambda - E_0)^{-2} \sup_x \left\{ \int |G_x(k)|^2 \, dk \right\}\right] \;<\; 1 \;, \tag{II.27}$$

where M_ε is the constant in Corollary II.4 and $\Lambda_2(\varepsilon) = \sup_x \left\{ \mathrm{e}^{-\varepsilon|x|} \int \frac{|G_x(k)|^2 \, dk}{\omega(k)^2} \right\}$. where M_ε is the constant in Corollary II.4. Then, for all $0 < m$, the ground state ϕ_m of H_m obeys

$$\langle \phi_m | \, (P_{el} \otimes P_\Omega) \phi_m \rangle \;\geq\; 1 - \delta_\varepsilon \;>\; 0 \;. \tag{II.28}$$

Proof: Note that (II.28) is equivalent to $\langle \phi_m, (P_{el}^{\perp} \otimes P_\Omega + \mathbf{1}_{el} \otimes P_\Omega^{\perp})\phi_m \rangle \leq \delta_\varepsilon$, where we use the notation $P^{\perp} = \mathbf{1} - P$ for a projection P. Since $P_\Omega^{\perp} \leq N$, it is sufficient to show that

$$\langle \phi_m | \, (P_{el}^{\perp} \otimes P_\Omega + \mathbf{1}_{el} \otimes N) \, \phi_m \rangle \;\leq\; \delta_\varepsilon \,, \tag{II.29}$$

in order to establish (II.28). Now we observe that

$$(P_{el}^{\perp} \otimes P_\Omega) H_m \;=\; P_{el}^{\perp} H_{el} \otimes P_\Omega \;+\; (P_{el}^{\perp} \otimes P_\Omega) a(\bar{G}_x) \,,$$

which implies

$$0 \;=\; (P_{el}^{\perp} \otimes P_\Omega)(H_m - E_m)\phi_m \;=\; [P_{el}^{\perp}(H_{el} - E_m) \otimes P_\Omega]\phi_m + (P_{el}^{\perp} \otimes P_\Omega) a(\bar{G}_x)\phi_m .$$

But, $H_{el} P_{el}^{\perp} \geq \lambda P_{el}^{\perp}$ and $E_0 \geq E_m$. Using these estimates and the Schwarz inequality, as in Lemma I.6, we obtain that

$$\begin{aligned} \langle P_{el}^{\perp} \otimes P_\Omega \rangle_{\phi_m} \;&\leq\; (\lambda - E_0)^{-1} \langle (H_{el} - E_m) P_{el}^{\perp} \otimes P_\Omega \rangle_{\phi_m} \\ &=\; -(\lambda - E_0)^{-1} \langle (P_{el}^{\perp} \otimes P_\Omega) a(\bar{G}_x) \rangle_{\phi_m} \\ &\leq\; (\lambda - E_0)^{-1} \, \langle P_{el}^{\perp} \otimes P_\Omega \rangle_{\phi_m}^{\frac{1}{2}} \, \langle a^+(G_x) a(\bar{G}_x) \rangle_{\phi_m}^{\frac{1}{2}} \,. \end{aligned}$$

Define $\alpha^2 := g^2 \sup_x \int |G_x|^2$ and $\beta^2 := g^2 \int \| G_x(k) \otimes \mathbf{1}_f \, \phi_m \|^2 \omega(k)^{-2} \, dk$. Next, we use that

$$\langle a^+(G_x) a(\bar{G}_x) \rangle_{\phi_m} \;\leq\; \alpha^2 \langle \mathbf{1}_{el} \otimes N \rangle_{\phi_m}$$

and that, due to Theorem II.5,

$$\langle \mathbf{1}_{el} \otimes N \rangle_{\phi_m} \;\leq\; \beta^2 \,.$$

The last three inequalities imply

$$\langle P_{el}^{\perp} \otimes P_\Omega \rangle_{\phi_m} \;\leq\; (\lambda - E_0)^{-2} \alpha^2 \beta^2 \,.$$

The last two inequalities together with Corollary II.4 yield (II.29), and (II.29) implies (II.28). ∎

We remark that, assuming the ground state ψ_0 of H_{el} to be unique, Eqn. (II.28) implies that

$$\| \phi_m - \phi_0 \|^2 \;\leq\; 2\Big(1 - |\langle \phi_m, \phi_0 \rangle|\Big) \;\leq\; \delta_\varepsilon \,,$$

where $\phi_0 := \psi_0 \otimes \Omega$, for a suitable choice of λ.

II.5 Existence of the Ground State

In this chapter we prove the existence of ground states of H. The result below sums up the analysis of the last three chapters.

Theorem II.8 *Suppose that Hypothesis 1 holds with* $0 \leq \varepsilon \leq (\Sigma - E_0 - g^2 \int \frac{|G_x|^2}{\omega})^{1/2}$ *and suppose that*

$$\delta_\varepsilon \;:=\; g^2\, M_\varepsilon\, \Lambda_2(\varepsilon) \cdot \left[1 \;+\; 4g^2(\Sigma - E_0)^{-2} \sup_x \left\{ \int |G_x(k)|^2\, dk \right\}\right] \;<\; 1\,, \quad \text{(II.30)}$$

where M_ε *is the constant in Corollary II.4 and* $\Lambda_2(\varepsilon) = \sup_x \left\{ \mathrm{e}^{-\varepsilon|x|} \int \frac{|G_x(k)|^2\, dk}{\omega(k)^2} \right\}$. *Then, the Hamiltonian* H *has a ground state* ϕ. *As* $g \to 0$, *the ground state energy and the ground state subspace of* H *converge to the unperturbed ground state energy (= the ground state energy of atom) and the ground state subspace. If* $\mathcal{H}_{el} = L^2(\mathbb{R}^m)$ *and* $G_x(k) = \overline{G_x(-k)}$, *then this ground state is unique. Finally,* $\langle P_{el} \otimes P_\Omega \rangle_\phi \geq 1 - \delta_\varepsilon > 0$, *where* $P_{el} := \chi_{2H_{el} \leq \Sigma + E_0}$ *the (finite dimensional) projection onto the bound states of* H_{el} *with energy below* $\frac{1}{2}\Sigma + \frac{1}{2}E_0$.

Proof: Let H_m and ϕ_m be the same as in the previous three sections. Since the unit ball in $\mathcal{H}_{el} \otimes \mathcal{F}$ is weakly compact, $\{\phi_m\}$ is weakly compact in $\mathcal{H}_{el} \otimes \mathcal{F}$. Therefore it contains a convergent subsequence which we again denote $\{\phi_m\}$. Let $\phi = w - \lim \phi_m$. By (II.28),

$$\langle \phi | (P_{el} \otimes P_\Omega) \phi \rangle \;\geq\; 1 - \delta_\varepsilon \;>\; 0\,.$$

Thus $\phi \neq 0$.

Let ψ be an arbitrary vector in $\mathcal{D}(H)$. Then

$$\begin{aligned} \langle H\psi | \phi \rangle \;&=\; \lim_{m \to 0} \langle H\psi | \phi_m \rangle \;=\; \lim_{m \to 0} \langle \psi | H \phi_m \rangle \\ &=\; \lim_{m \to 0} E_m \langle \psi | \phi_m \rangle \;+\; \lim_{m \to 0} \langle \psi | (H - H_m) \phi_m \rangle\,, \end{aligned} \quad \text{(II.31)}$$

Using $H - H_m = g(W - W_m)$ in conjunction with (I.54) in Corollary I.7, we find that

$$\begin{aligned} |\langle \psi | (H - H_m) \phi_m \rangle| \;\leq\; g \cdot \sup_x \left\{ \int_{\omega(k) \leq m} [1 + \omega(k)^{-1}]\, |G_x(k)|^2\, dk \right\}^{1/2} \\ \cdot \| \mathbf{1}_{el} \otimes (H_f + 1)^{1/2} \psi \|\, \| \phi_m \|\,, \end{aligned} \quad \text{(II.32)}$$

which converges to 0, as $m \to 0$, by Lebesgue dominated convergence. Next, we note that

$$E_m\langle\psi|\phi_m\rangle \;=\; \langle\psi|H_m\phi_m\rangle \;=\; \langle H\psi|\phi_m\rangle \;+\; \langle\psi|(H_m - H)\phi_m\rangle\,. \tag{II.33}$$

Since, for $\psi \in \mathcal{D}(H)$, $\lim_{m\to 0}\langle H\psi|\phi_m\rangle \;=\; \langle H\psi|\phi\rangle$, and since $|\langle\psi, (H_m - H)\phi_m\rangle| \to 0$, as $m \to 0$, we conclude that the sequence $\{E_m\}$ has a limit, E, and thus

$$\lim_{m\to 0} E_m\,\langle\psi|\phi_m\rangle \;=\; E\,\langle\psi|\phi\rangle\,. \tag{II.34}$$

Now, it follows from $(II.31)$ that

$$\langle H\psi\,|\,\phi\rangle \;=\; E\,\langle\psi\,|\,\phi\rangle\,, \tag{II.35}$$

for arbitrary $\psi \in \mathcal{D}(H)$. Since $\mathcal{D}(H)$ is dense, it follows that $\phi \in \mathcal{D}(H^*) = \mathcal{D}(H)$ and that $H\phi = E\phi$. This proves the existence of the ground state for H. Passing in (II.28) to the limit as $m \to 0$, we conclude that $\langle P_{el}\otimes P_\Omega\rangle_\phi \geq 1 - \delta_\varepsilon$.

Finally, as in the proof of Theorem II.2, $H \equiv H_g$ is resolvent norm continuous in g, by Corollary I.7. Hence general perturbation theory implies that the ground state energy E_g of H_g $(= \inf\sigma(H_g))$ is continuous in g and, in particular, converges to E_0, the ground state energy of $H_{g=0}$. Let P_g denote the projection onto the ground states of H_g. The norm convergence of P_g to $P_{g=0} \;=\; \chi_{H_{el}=E_0}\otimes P_\Omega$ (the projection onto groundstates of H_{el} times P_Ω) follows from our estimates in Chapter II.4 which proves the second statement of Theorem II.8.

Again if $\mathcal{H}_{el} = L^2(\mathbb{R}^m)$ then as in the proof of Theorem II.2 (see the end of the proof), the Perron-Frobenious argument implies that the ground state of H is unique, provided $G_x(k) = \overline{G_x(-k)}$. ∎

Theorem II.8 $(\phi := w - \lim \phi_m)$ together with Theorem II.1 implies

Corollary II.9 *The ground state, ϕ, of H verifies the exponential bound* $\|\mathrm{e}^{\alpha|x|}\otimes \mathbf{1}_f\;\phi\| \;\leq\; \mathrm{const}\,\delta^{-1/2}\,\|\phi\|$, *provided* $\delta \equiv \Sigma - E_0 - g^2\int \frac{|G_x|^2}{\omega})^{1/2} - \alpha^2 > 0$.

Note that the exponential bound for the ground state can also be proved directly. Also, we remark that the assumption $\delta_\varepsilon < 1$ in Theorem II.8 can be weakened, at the expense of making a more involved argument necessary.

III Positive Commutators

In this chapter we begin the study of the continuous spectrum of H_g. To this end, we estimate from below the commutator of H_g with an anti-selfadjoint operator $A = -A^*$. This method, known as "Positive Commutators" or "Mourre Estimate", originates from [27, 30]. We refer the reader to [10] for a textbook exposition and remark that the method has been applied to the Spin-Boson model with a positive photon mass, which is related to the model treated here, in [16, 18, 14]. In a forthcoming paper [6], we will present a refined version of the material in this chapter.

Our first choice below for A is $A_1 := \mathbf{1}_{el} \otimes A_f$, where A_f is the second quantization of the dilatation generator on the one-photon space, $L^2(\mathbb{R}^3)$. The estimate of $[A_1, H_g]$ from below allows us to conclude that the spectrum of H_g is absolutely continuous, with no point spectrum, outside a neighbourhood of $\sigma(H_{el})$.

The second choice for A is $A_2 := A_1 + A_{el} \otimes \mathbf{1}_f$, where A_{el} is the dilatation generator on the particle Hilbert space, $L^2(\mathbb{R}^{3N})$. Then we estimate $[A_2, H_g]$ from below in terms of $[A_{el}, H_{el}]$, for small values of g. Our estimate is such that if $[A_{el}, H_{el}]$ is positive, then so is $[A_2, H_g]$. Since, typically, $[A_{el}, H_{el}]$ is positive away from T_{el}, the set of eigenvalues and thresholds of H_{el}, this estimate allows us to conclude that the spectrum of H_g is absolutely continuous, with no point spectrum, outside a small neighbourhood of T_{el}.

In contrast to what we present in Chapter IV, the real axis away from T_{el} covers parts of the half-axis (Σ, ∞). In Chapter IV, we establish absolute continuity of the spectrum of H_g in the interval $[E_0\ ,\ \Sigma - Cg]$, for some constant, C, assuming that the interaction is analytic with respect to dilatation of the photon momenta.

We come to the assumptions and notation used in this chapter. To begin with, we assume Hypothesis 1. Recall that this means that $H_g = H_0 + gW_x$ with

coupling functions $G_x(k)$ in the interaction $W_x = a^\dagger(G_x) + a(G_x)$ that obey

$$\Lambda_1 \;:=\; \sup_{x\in\mathbb{R}^{3N}} \left\{ \int \left[1+\frac{1}{\omega(k)}\right] \, |G_x(k)|^2 \, dk \right\}^{1/2} \;<\; \infty \,. \tag{III.1}$$

Next, we make the definitions of A_1 and A_2 precise. We set

$$A_1 \;\equiv\; \mathbf{1}_{el} \otimes A_f \;:=\; \int a^\dagger(k) \frac{1}{2}(k\cdot\nabla_k + \nabla_k \cdot k) a(k) dk \,, \tag{III.2}$$

which is the second quantization, $d\Gamma(d)$, of the dilatation generator in the one-photon space, $d := \frac{1}{2}(k\cdot\nabla_k + \nabla_k\cdot k)$. Furthermore,

$$A_2 \;=\; A_1 \;+\; A_{el}\otimes \mathbf{1}_f \,, \tag{III.3}$$

where A_{el} is the dilatation generator on the atomic Hilbert space $\mathcal{H}_{el}$. Finally, we recall that $E_0, E_1, \ldots, E_M$ denote the eigenvalues of H_{el} (possibly $M=\infty$), and we set $E_{M+1} := \Sigma$ provided $M<\infty$ and, furthermore, $E_{-1} := -\infty$.

Theorem III.1 *Fix $j \geq -1$ and pick E_s and E_l such that $E_j < E_s < E_l < E_{j+1}$. Assume Hypothesis 1 and, additionally, that*

$$\Lambda_3^{(1)} \;:=\; \sup_{x\in\mathbb{R}^{3N}} \left\{ \int \frac{|\, k\cdot\nabla_k G_x(k)\,|^2}{\omega(k)^2} \, dk \right\}^{1/2} \;<\; \infty \tag{III.4}$$

holds. Denote $\Delta := [E_s, E_l]$ and assume that

$$\delta \;:=\; \mathrm{dist}\{\,\Delta\,,\,\sigma(H_{el})\} \;=\; \min\{E_s - E_j\,,\,E_{j+1}-E_l\} \;\geq\; g^{2/5} \,. \tag{III.5}$$

Then, for $g>0$ sufficiently small,

$$\chi_\Delta(H_g)\,[A_1, H_g]\,\chi_\Delta(H_g) \;\geq\; \delta\,\left[1 - O(g^{4/5})\right]\,\chi_\Delta(H_g)^2 \,. \tag{III.6}$$

Proof: During the proof, we will not display trivial factors in the tensor product $\mathcal{H}_{el}\otimes\mathcal{F}$ and simply write H_f instead of $\mathbf{1}_{el}\otimes H_f$ and H_{el} instead of $H_{el}\otimes\mathbf{1}_f$, etc.. We begin the proof with the remark that the second quantization respects commutators, i.e., for given operators Q, Q' on the one-photon space $L^2(\mathbb{R}^3)$ and given $f\in L^2(\mathbb{R}^3)$, the following relations for their second quantization holds,

$$[\,d\Gamma(Q)\,,\,d\Gamma(Q')\,] \;=\; d\Gamma\,(\,[Q,Q']\,)\,, \tag{III.7}$$

$$\left[\,d\Gamma(Q)\,,\,a^\dagger(f)\,\right] \;=\; a^\dagger(Qf)\,, \tag{III.8}$$

as quadratic forms on a suitably chosen domain. Thus, we find by computation,

$$[A_1\,,\,H_g] \;=\; H_f \;+\; g\widetilde{W}_x\,, \tag{III.9}$$

where $\widetilde{W}_x = a^\dagger(\widetilde{G}_x) + a(\widetilde{G}_x)$, with $\widetilde{G}_x = k\cdot\nabla_k G_x + \frac{3}{2}G_x$. By (I.55), we have that

$$\pm gW_x \;\leq\; bH_f \;+\; \frac{g^2\,\Lambda_1^2}{b}\,, \tag{III.10}$$

for any $b > 0$. The last two equations show that we have to estimate H_f from below on an appropriate set of vectors. We proceed to do this: Writing $H_g = H_{el} + H_f + gW_x$ and using (III.10), we obtain

$$H_g \;\leq\; H_{el} \;+\; (1+b)H_f \;+\; \frac{g^2\,\Lambda_1^2}{b}\,,$$

which implies that

$$\langle H_f\rangle_\phi \;\geq\; (1+b)^{-1}\left\langle H_g - H_{el} - \frac{g^2\,\Lambda_1^2}{b}\right\rangle_\phi\,.$$

From now on we assume that $\phi \in \mathrm{Ran}\,\chi_\Delta(H_g)$. For such vectors we estimate the r.h.s. of the last inequality from below. The key idea of the forthcoming estimate is to use estimates originating in a quantum version of the classical energy conservation law. We have that

$$\langle H_f\rangle_\phi \;\geq\; (1+b)^{-1}\left\langle E_s - \frac{g^2\,\Lambda_1^2}{b} - H_{el}\right\rangle_\phi\,,$$

where $E_s = \inf\Delta$. Next, we note that

$$\mathbf{1} \;=\; \chi_{H_{el}\leq E_j} \;+\; \chi_{H_{el}\geq E_{j+1}}\,,$$

and, using this decomposition, we obtain the bound

$$\langle H_f\rangle_\phi \;\geq\; (1+b)^{-1}\left(E_s - E_j - \frac{g^2\,\Lambda_1^2}{b}\right)\|\phi\|^2 \;-\; (1+b)^{-1}\left\langle f(H_{el})^2\right\rangle_\phi\,, \tag{III.11}$$

where $f(\mu) = (\mu - E_j)^{1/2}\,\chi_{\mu\geq E_{j+1}}$.

We claim that

$$\left|\left\langle f(H_{el})^2\right\rangle_\phi\right| \;\leq\; \frac{Cg^2\Lambda_1^2}{\delta}\,. \tag{III.12}$$

To demonstrate (III.12), we point out that it suffices to prove (III.6) for a smooth function, $\widetilde{\chi}_\Delta$ instead of χ_Δ, which obeys $\widetilde{\chi}_\Delta \equiv 1$ on Δ and such that

$$\mathrm{supp}\{\widetilde{\chi}_\Delta\} \subset \left(\frac{E_j + E_s}{2} , \frac{E_l + E_{j+1}}{2} \right) . \tag{III.13}$$

We may require without loss of generality that $|\partial^n \widetilde{\chi}_\Delta| \leq C_n \delta^{-n}$. For notational convenience, we simply assume χ_Δ to have these properties, henceforth. Then, a standard operator calculus estimate using almost analytic functions as in (II.4) (see e.g. [21]) yields

$$\begin{aligned} \left\| (H_g + i) \left[f(H_{el}), \chi_\Delta(H_g) \right] \right\| &\leq \frac{C\, g}{\delta} \left\| \left[f(H_{el}), W_x \right] (H_g + i)^{-1} \right\| \qquad \text{(III.14)} \\ &\leq \frac{C\, g}{\delta^2} \left(\| f(H_{el})\, W_x\, (H_g + i)^{-1} \| + \| W_x\, f(H_{el})\, (H_g + i)^{-1} \| \right) . \end{aligned}$$

Moreover, since both $f(H_{el})$ and H_g are clearly relatively H_0-bounded, the two norms on the right side of (III.14) are bounded by a constant and we obtain

$$\left\| (H_g + i) \left[f(H_{el}), \chi_\Delta(H_g) \right] \right\| \leq \frac{C\, g\, \Lambda_1}{\delta} . \tag{III.15}$$

Next, we denote $\widehat{E} := \frac{1}{2}(E_l + E_{j+1})$ and observe that (III.13) implies that

$$\chi_\Delta(H_g)\, \chi_{H_g \geq \widehat{E}} = 0 . \tag{III.16}$$

These two relations, (III.15) and (III.16), show that

$$\langle\, f(H_{el})\, \chi_{H_g \geq \widehat{E}}\, f(H_{el})\, \rangle_\phi = O(g\, \delta^{-1}\, \Lambda_1) , \tag{III.17}$$

$$\langle\, f(H_{el})\, \chi_{H_g \geq \widehat{E}}\, H_g\, f(H_{el})\, \rangle_\phi = O(g\, \delta^{-1}\, \Lambda_1) , \tag{III.18}$$

which, in turn, implies that

$$\langle\, f(H_{el})\, \chi_{H_g \leq \widehat{E}}\, f(H_{el})\, \rangle_\phi = \left\langle\, f(H_{el})^2 \,\right\rangle_\phi + O(g\, \delta^{-1}\, \Lambda_1) , \tag{III.19}$$

$$\langle\, f(H_{el})\, \chi_{H_g \leq \widehat{E}}\, H_g\, f(H_{el})\, \rangle_\phi = \langle\, f(H_{el})\, H_g\, f(H_{el})\, \rangle_\phi + O(g\, \delta^{-1}\, \Lambda_1) \tag{III.20}$$

Using these two equations, we estimate

$$\begin{aligned} \widehat{E} \left\langle\, f(H_{el})^2 \,\right\rangle_\phi &= \widehat{E} \left\langle f(H_{el})\, \chi_{H_g \leq \widehat{E}}\, f(H_{el}) \right\rangle_\phi + O(g\, \delta^{-1}\, \Lambda_1) \\ &\geq \langle f(H_{el})\, \chi_{H_g \leq \widehat{E}}\, H_g\, f(H_{el}) \rangle_\phi + O(g\, \delta^{-1}\, \Lambda_1) \\ &= \langle f(H_{el})\, H_g\, f(H_{el}) \rangle_\phi + O(g\, \delta^{-1}\, \Lambda_1) . \end{aligned}$$

Due to (III.10) with $b = 1$, we have that $H_f + gW_x \geq -g^2\,\Lambda_1^2$. Using this fact, (III.19), (III.20), and that $f(\mu) \neq 0$ only for $\mu \geq E_{j+1}$, we derive the bound

$$\begin{aligned}
\widehat{E}\,\left\langle\, f(H_{el})^2\,\right\rangle_\phi &= \widehat{E}\,\left\langle f(H_{el})\,\chi_{H_g\leq\widehat{E}}\,f(H_{el})\right\rangle_\phi \,+\, O(g\,\delta^{-1}\,\Lambda_1) \\
&\geq \left\langle f(H_{el})\,H_g\,\chi_{H_g\leq\widehat{E}}\,f(H_{el})\,\right\rangle_\phi \,+\, O(g\,\delta^{-1}\,\Lambda_1) \\
&= \langle f(H_{el})\,H_g\,f(H_{el})\,\rangle_\phi \,+\, O(g\,\delta^{-1}\,\Lambda_1) \\
&\geq \left\langle f(H_{el})\,\left(H_{el} - g^2\Lambda_1\right)\,f(H_{el})\right\rangle_\phi \,+\, O(g\,\delta^{-1}\,\Lambda_1) \\
&\geq \left(E_{j+1} - g^2\Lambda_1\right)\left\langle\, f(H_{el})^2\,\right\rangle_\phi \,+\, O(g\,\delta^{-1}\,\Lambda_1)\,. \qquad \text{(III.21)}
\end{aligned}$$

Choosing g sufficiently small such that $4g^2\Lambda_1^2 \leq Cg \leq \delta$, we obtain $E_{j+1} - \widehat{E} - g^2\Lambda_1 > \delta/4$, which implies (III.12). Together with (III.11), (III.12) yields that

$$\langle H_f\rangle_\phi \;\geq\; \left(\frac{1}{1+b}\right)\left(\delta - \frac{g^2\Lambda_1^2}{b} - \frac{Cg^2\Lambda_1^2}{\delta^2}\right)\|\phi\|^2\,. \qquad \text{(III.22)}$$

Recalling the definition of $\widetilde{W}_x$ (see line after Eqn. (III.9)) and using (I.55), we obtain

$$|\langle g\widetilde{W}_x\rangle_\phi| \;\leq\; \tilde{b}\langle H_f\rangle_\phi \,+\, \frac{g^2}{\tilde{b}}\left(\Lambda_3^{(1)}\right)^2\|\phi\|^2\,, \qquad \text{(III.23)}$$

for any $\tilde{b} > 0$. Adding up (III.22) and (III.23) according to (III.9), we arrive at

$$\begin{aligned}
\frac{\langle H_f + g\widetilde{W}_x\rangle_\phi}{\|\phi\|^2} &\geq \left(\frac{1-\tilde{b}}{1+b}\right)\left(\delta - \frac{g^2\Lambda_1^2}{b} - \frac{Cg^2\Lambda_1^2}{\delta^2}\right) \,-\, \frac{g^2}{\tilde{b}}\left(\Lambda_3^{(1)}\right)^2 \\
&= \delta\left[1 \,+\, O\left(\frac{g^2}{b\delta} + \frac{g^2}{\tilde{b}\delta} + \frac{g^2}{\delta^3} + b\right)\right]\,. \qquad \text{(III.24)}
\end{aligned}$$

The claim (III.6) follows now from choosing $b := \tilde{b} := g^{4/5}$, since we assumed in (III.5) that $\delta \geq g^{2/5}$. ∎

Note that estimate (III.6) is more elementary than what is known as Mourre estimate; (the energy intervals are not shrunk in order to handle the contribution of the interaction).

In order to derive the desired results from Theorem III.1, we use Mourre theory. We verify the applicability of this theory: Eqns. (III.9) and (III.23) imply that $[A_1, H_g]$ is H_g-bounded. Next, we iterate (III.9) and obtain the second commutator,

$$\Big[\,A_1\,,[A_1,H_g]\Big] \;=\; H_f + a^\dagger\Big((k\cdot\nabla_k + 3/2)^2 G_x\Big) + a\Big((k\cdot\nabla_k + 3/2)^2 G_x\Big)\,. \qquad \text{(III.25)}$$

An analogue of (III.23) now implies the H_g-boundedness of this second commutator. Note that only for this estimate, we need $\sup_x |(k\cdot\nabla_k)^2 G_x(k)|/\omega(k)^{1/2} \in L^2(\mathbb{R}^3)$. Also, the relative boundedness of $[A_1, H_g]$ requires only $\sup_x |k\cdot\nabla_k G_x(k)|/\omega(k)^{1/2} \in L^2(\mathbb{R}^3)$, and for Theorem III.1, it would actually suffice to assume $\sup_x \left\{ \mathrm{e}^{-\varepsilon|x|} |k\cdot\nabla_k G_x(k)|/\omega(k)^{1/2} \right\} \in L^2(\mathbb{R}^3)$, for some $\varepsilon > 0$ sufficiently small.

The relative H_g-boundedness of $[A_1, H_g]$ and $\big[A_1\,,\,[A_1, H_g]\big]$ and Theorem III.1 yield (see e.g. [21]).

Lemma III.2 *Assume that Hypothesis 1 holds and, additionally, that*

$$\Lambda_3^{(2)} \;:=\; \sup_{x\in\mathbb{R}^{3N}} \left\{ \int \frac{|\,(k\cdot\nabla_k)^2 G_x(k)\,|^2}{\omega(k)^2}\, dk \right\}^{1/2} \;<\; \infty\,. \tag{III.26}$$

Then, for sufficiently small g, the spectrum of H_g in $\Omega := \{\lambda \in \mathbb{R} \,|\, \mathrm{dist}\{\lambda, \sigma(H_{el})\} > g^{2/5}\}$ is absolutely continuous. Moreover, for any $\Delta \subset \Omega$, the limiting absorption principle holds: $(H-z)^{-1}\,\chi_\Delta(H_g)$, as a map from $\langle A_1\rangle^{-\mu}\mathcal{H}$ to $\langle A_1\rangle^{\mu}\mathcal{H}$, $\mu > \frac{1}{2}$, is bounded and norm continuous, as z approaches the real axis.

Next, we consider $\sigma_{\mathrm{cont}}(H_{el})$. Recall that A_{el} is the dilatation generator in the atomic Hilbert space $\mathcal{H}_{el}$. We require Hypothesis 1 and assume that the Mourre estimate holds for H_{el} and A_{el}. Specifically, we assume that, given $\mu \in \sigma_{\mathrm{cont}}(H_{el}) \setminus T_{el}$, and given $\varepsilon > 0$, there is some $\delta > 0$ such that

$$\overline{\chi}_\Delta(H_{el})\,[A_{el}, H_{el}]\,\overline{\chi}_\Delta(H_{el}) \;\geq\; (\theta_{el} - \varepsilon)\,\overline{\chi}_\Delta(H_{el})^2\,, \tag{III.27}$$

provided $\Delta = \mu + [-\delta\,,\,\delta]$. Here,

$$\theta_{el} \;:=\; \mathrm{dist}\{\Delta\,,\,T_\Delta\}\,, \quad T_\Delta \;:=\; T_{el} \cap (-\infty, \sup\Delta)\,, \tag{III.28}$$

where T_{el} are the thresholds and eigenvalues of H_{el}, and

$$\overline{\chi}_\Delta(H_{el}) \;:=\; \chi_{\Delta\cap[\Sigma,\infty)} \;=\; \chi_\Delta(H_{el}) P_{el}^{\mathrm{cont}}\,.$$

Note that, by passing from δ to $\delta/2$, we may assume that

$$\overline{\chi}_{\Delta-\nu}(H_{el})\,[A_{el}, H_{el}]\,\overline{\chi}_{\Delta-\nu}(H_{el}) \;\geq\; (\theta_{el} - \varepsilon)\,\overline{\chi}_{\Delta-\nu}(H_{el})^2\,, \tag{III.29}$$

for any $-\delta \leq 2\nu \leq \delta$.

Theorem III.3 *Assume that Hypothesis 1 holds and, additionally, that*

$$\widehat{\Lambda}_3^{(1)} \;:=\; \sup_{x\in\mathbb{R}^{3N}} \left\{ \int \frac{|\,(k\cdot\nabla_k - x\cdot\nabla_x)G_x(k)\,|^2}{\omega(k)^2}\,dk \right\}^{1/2} \;<\; \infty\,. \quad \text{(III.30)}$$

Let μ, ε, Δ, *and* θ_{el} *be as specified in (III.29). Set* $\Delta' := \mu + \frac{1}{2}[-\delta,\delta]$. *Then*

$$\chi_{\Delta'}(H_g)\,[A_2, H_g]\,\chi_{\Delta'}(H_g) \;\geq\; \theta'\,\chi_{\Delta'}(H_g)^2\,, \quad \text{(III.31)}$$

where

$$\theta' \;=\; \min\left\{\theta_{el}\,,\,\frac{\delta}{4}\right\} \;-\; \varepsilon \;-\; Cg^2 \;-\; \frac{C\,g}{\delta}\,. \quad \text{(III.32)}$$

Proof: Using (III.7) and (III.8), we find that

$$[A_2, H_g] \;=\; [A_{el}, H_{el}] \;+\; H_f \;+\; g\widehat{W}_x\,, \quad \text{(III.33)}$$

where

$$\widehat{W}_x \;=\; \widetilde{W}_x + [A_{el}, W_x] \;=\; a^\dagger(F_x) + a(F_x) \quad \text{(III.34)}$$

with

$$F_x \;=\; k\cdot\nabla_k G_x \;-\; x\cdot\nabla_x G_x \;+\; \frac{3}{2}G_x\,. \quad \text{(III.35)}$$

Using (III.10), we obtain

$$[A_2, H_g] \;\geq\; [A_{el}, H_{el}] \;+\; \frac{1}{2}H_f \;-\; 2g^2\left(\Lambda_1 + \widehat{\Lambda}_3^{(1)}\right)^2.$$

By multiplying this inequality from the left and the right by $\chi_\Delta(H_0)$, where $H_0 = H_{el} + H_f$, and using the fiber integral representation of the r.h.s., we obtain

$$\chi_\Delta(H_0)\,[A_2, H_g]\,\chi_\Delta(H_0) \;\geq\; \int_{\nu\geq 0}^{\oplus} \chi_{\Delta-\nu}(H_{el})\left([A_{el}, H_{el}] + \frac{\nu}{2}\right)\chi_{\Delta-\nu}(H_{el})\,d\nu$$
$$-2g^2\left(\Lambda_1 + \widehat{\Lambda}_3^{(1)}\right)^2\chi_\Delta(H_0)^2 \quad \text{(III.36)}$$

Now, we break up the integral on the r.h.s. of (III.36) into one over $[0,\delta]$ and one over $[\delta,\infty]$. Due to (III.29), the first integral is bounded from below by

$$\int_{0\leq\nu\leq\delta}^{\oplus} \chi_{\Delta-\nu}(H_{el})\left([A_{el}, H_{el}] + \frac{\nu}{2}\right)\chi_{\Delta-\nu}(H_{el})d\nu \;\geq\; (\theta_{el}-\varepsilon)\int_{0\leq\nu\leq\delta}^{\oplus}\chi^2_{\Delta-\nu}(H_{el})d\nu, \quad \text{(III.37)}$$

while the second one is bounded by

$$\int_{\nu\geq\delta}^{\oplus} \chi_{\Delta-\nu}(H_{el})\,\Big([A_{el},H_{el}]+\frac{\nu}{2}\Big)\,\chi_{\Delta-\nu}(H_{el})\,d\nu \;\geq\; \Big(\frac{\delta}{4}-\varepsilon\Big)\int_{0\leq\nu\leq\delta}^{\oplus}\chi^2_{\Delta-\nu}(H_{el})\,d\nu\,. \tag{III.38}$$

Thus, we obtain

$$\chi_\Delta(H_0)\,[A_2,H_g]\,\chi_\Delta(H_0) \;\geq\; \Big(\min\Big\{\theta_{el}\,,\,\frac{\delta}{4}\Big\}\;-\;\varepsilon\;-\;2g^2\,\Big(\Lambda_1+\widehat{\Lambda}_3^{(1)}\Big)^2\Big)\;\chi_\Delta(H_0)^2\,. \tag{III.39}$$

Let $\Delta' := \mu+\frac{1}{2}[-\delta,\delta]$ such that $\mathrm{dist}\{\Delta'\,,\,\mathbb{R}\setminus\Delta\}=\delta/2$. Using operator calculus (see e.g. [21]), we find that

$$\Big\|\chi_{\Delta'}(H_g)\Big(1-\chi_\Delta(H_0)\Big)\Big\| \;\leq\; \frac{C\,g\,\Lambda_1}{\delta}\,.$$

This, together with (III.39), yields

$$\begin{aligned}
&\chi_{\Delta'}(H_g)\,[A_2,H_g]\,\chi_{\Delta'}(H_g)\\
&\quad\geq\;\chi_{\Delta'}(H_g)\,\chi_\Delta(H_0)\,[A_2,H_g]\,\chi_\Delta(H_0)\,\chi_{\Delta'}(H_g)\;-\;\frac{C\,g\,\Lambda_1}{\delta}\,\chi^2_{\Delta'}(H_g)\\
&\quad\geq\;\Big(\min\Big\{\theta_{el}\,,\,\frac{\delta}{4}\Big\}\;-\;\varepsilon\;-\;2g^2\,\Big(\Lambda_1+\widehat{\Lambda}_3^{(1)}\Big)^2\Big)\;\chi_{\Delta'}(H_g)\,\chi_\Delta(H_0)^2\,\chi_{\Delta'}(H_g)\\
&\qquad\qquad-\frac{C\,g\,\Lambda_1}{\delta}\,\chi^2_{\Delta'}(H_g)\\
&\quad\geq\;\Big(\min\Big\{\theta_{el}\,,\,\frac{\delta}{4}\Big\}\;-\;\varepsilon\;-\;2g^2\,\Big(\Lambda_1+\widehat{\Lambda}_3^{(1)}\Big)^2\;-\;\frac{C\,g\,\Lambda_1}{\delta}\Big)\;\chi^2_{\Delta'}(H_g)\,.
\end{aligned} \tag{III.40}$$

This inequality implies (III.31) upon replacing δ by $\delta/2$. ∎

As in the discussion of Lemma III.2 above, A_2 maps a core of H_g into itself, and $[A_2,H_g]$ and $\big[A_2,[A_2,H_g]\big]$ are H_g-bounded, provided $\widehat{\Lambda}_3^{(2)}$ (defined below) is finite. Thus, Theorem III.3 implies (see [21])

Theorem III.4 *Assume that Hypothesis 1 holds and, additionally, that*

$$\widehat{\Lambda}_3^{(2)} \;:=\; \sup_{x\in\mathbb{R}^{3N}}\Big\{\int\frac{|\,(k\cdot\nabla_k-x\cdot\nabla_x)^2G_x(k)\,|^2}{\omega(k)^2}\,dk\Big\}^{1/2} \;<\;\infty\,. \tag{III.41}$$

Pick $\mu\in[\sigma,\infty)$ away from the thresholds of H_{el} and $\varepsilon>0$. Let δ, Δ, and θ_{el} be as specified in (III.29). Set $\Delta':=\mu+\frac{1}{2}[-\delta,\delta]$. Then, for $g\cdot\max\{1,\delta^{-2}\}>0$ sufficiently small, the spectrum of H_g in Δ' is absolutely continuous. Moreover,

the limiting absorption principle holds: $(H_g - z)^{-1}\,\chi_{\Delta'}(H_g)$, *as a map from* $\langle A_2\rangle^{-\mu}\mathcal{H}$ *to* $\langle A_2\rangle^{\mu}\mathcal{H}$, $\mu > \frac{1}{2}$, *is bounded and norm continuous as* z *approaches the real axis.*

Additionally assuming that $\delta \geq c\,\theta_{el}$ for some small constant, $c > 0$, independent of μ and ε, we obtain the following corollary

Corollary III.5 *Assume that Hypothesis 1 and (III.41) hold. Pick* $\varepsilon > 0$ *and suppose there exists a small contant,* $c > 0$, *such that (III.27)–(III.28) holds for any interval,* Δ, *with* $|\Delta| \geq c\,\theta_{el}$. *Then there exists a constant,* C, *such that the spectrum of* H_g *in* $\mathcal{A}_g := \mathbb{R} \setminus (T_{el} + [-Cg, Cg])$ *is absolutely continuous.*

For more precise results along the same lines see [6].

Theorem III.4 above completes our study of the spectrum of H_g outside a small neighbourhood of the eigenvalues and thresholds of H_{el}. We proceed now to investigating the nature of the spectrum of H_g in $(E_0\,,\,\Sigma - O(g))$ This part is harder and makes use of some elements of the renormalization group construction. In the next chapter, we begin with developing some basic technical tools.

IV Feshbach's Projection Method and Fermi's Golden Rule

In this chapter we define (in an abstract context) the Feshbach map and establish a key "isospectral property". This map will provide a tool for studying the spectrum of the Hamiltonian H near the atomic energies E_j.

IV.1 The Feshbach Map

Let $\mathcal{H}$ be a separable Hilbert space, and let P be a bounded, but not necessarily orthogonal projection on $\mathcal{H}$ (i.e., $P^2 = P$, but possibly $P^* \neq P$). We set $\overline{P} = \mathbf{1} - P$. For any densely defined, closed operator H on $\mathcal{H}$, whose domain contains the range of P, we define

$$H_P \;:=\; PHP \qquad \text{and} \qquad H_{\overline{P}} \;:=\; \overline{P}H\overline{P}\,. \tag{IV.1}$$

For an operator A, let $\rho(A)$ denote its resolvent set, i.e., the set of complex numbers z such that $A-z\mathbf{1}$ has a bounded inverse. We view $H_{\overline{P}}$ as an operator on $\overline{P}\mathcal{H}$ and assume that $0 \in \rho(H_{\overline{P}})$, i.e.,

$$(H_{\overline{P}})^{-1} \text{ exists on } \overline{P}\mathcal{H} \text{ and is bounded,} \tag{IV.2}$$

and, furthermore,

$$\|\overline{P}(H_{\overline{P}})^{-1}\overline{P}HP\| \,<\, \infty\,, \qquad \text{and} \qquad \|PH\overline{P}(H_{\overline{P}})^{-1}\overline{P}\| \,<\, \infty\,. \tag{IV.3}$$

We define the *Feshbach map*, $f_P(H)$, by

$$f_P(H) \;=\; \Big(PHP \,-\, PH\overline{P}(H_{\overline{P}})^{-1}\overline{P}HP\Big)\Big|_{\mathrm{Ran}P}\,, \tag{IV.4}$$

provided $0 \in \rho(H_{\overline{P}})$.

Next we define

$$S_P \;=\; P - \overline{P}(H_{\overline{P}})^{-1}\overline{P}HP\,. \tag{IV.5}$$

Then

$$\mathrm{Ker}\{S_P\} \;=\; \mathrm{Ker}\{P\} \tag{IV.6}$$

Indeed, we have the identity

$$P\varphi \;=\; [\mathbf{1} + \overline{P}(H_{\overline{P}})^{-1}\overline{P}HP]S_P\varphi\,. \tag{IV.7}$$

Hence, $S_P\varphi = 0$ if and only if $P\varphi = 0$. Shifting $H \mapsto H - z$ by a spectral parameter, $z \in \mathbb{C}$, we have

Theorem IV.1 *Under assumption (IV.3), and for $z \in \rho(H_{\overline{P}})$, we have that*

$$z \in \sigma_{\#}(H) \;\Leftrightarrow\; 0 \in \sigma_{\#}(H-z) \;\Leftrightarrow\; 0 \in \sigma_{\#}[f_P(H-z)]\,, \tag{IV.8}$$

where $\sigma_{\#} = \sigma, \sigma_{\mathrm{pp}}$. Moreover, eigenfunctions of $H - z$ and $f_P(H-z)$ are related by

$$\mathrm{Ker}\{(H-z)S_P(z)\} \;=\; \mathrm{Ker}\{f_P(H-z)\}\,, \tag{IV.9}$$

where $S_P(z) := P - \overline{P}(H_{\overline{P}} - z)^{-1}\overline{P}HP$, and

$$P\,\mathrm{Ker}(H-z) \;=\; \mathrm{Ker}\{f_P(H-z)\}\,. \tag{IV.10}$$

These relations imply, in particular, that

$$\mathrm{dimKer}\{H-z\} \;=\; \mathrm{dimKer}\{f_P(H-z)\}\,. \tag{IV.11}$$

Proof: Statements (IV.8) and (IV.9) follow from the following identities:

$$(H-z)\,S_P(z) \;=\; f_P(H-z)\,P\,, \tag{IV.12}$$

$$P(H-z)^{-1}P \;=\; [\,f_P(H-z)\,]^{-1}\,, \tag{IV.13}$$

on $\mathrm{Ran}\{P\}$, and

$$\begin{aligned}(H-z)^{-1} \;=\;& [\,f_P(H-z)\,]^{-1}P \;-\; [\,f_P(H-z)\,]^{-1}PH\overline{P}(H_{\overline{P}}-z)^{-1}\overline{P}\\ &-\overline{P}(H_{\overline{P}}-z)^{-1}\overline{P}HP(f_P(H)-z)^{-1} \;+\; \overline{P}(H_{\overline{P}}-z)^{-1}\overline{P}\\ &+\overline{P}(H_{\overline{P}}-z)^{-1}\overline{P}HP(f_P(H)-z)^{-1}PH\overline{P}(H_{\overline{P}}-z)^{-1}\overline{P}\end{aligned} \tag{IV.14}$$

These identities are proved by elementary algebraic manipulations based on the second resolvent identify and on the representation

$$H \;=\; H_P + H_{\overline{P}} + PH\overline{P} + \overline{P}HP\,.$$

Identity (IV.12) implies (IV.9) and identities (IV.13) and (IV.14) imply (IV.8). In the latter case, the argument proceeds as follows: Let $z \in \rho(H) \cap \rho(H_{\overline{P}})$. Then one shows that the l.h.s. of (IV.13) defines the inverse of $f_P(H-z)$ and therefore $0 \in \rho[f_P(H-z)]$. Next, suppose that $0 \in \rho[f_P(H-z)]$ and $z \in \rho(H_{\overline{P}})$. Then the r.h.s. of (IV.14) defines the inverse of $H-z$, so $z \in \rho(H)$. It remains to prove (IV.10). Let $z \in \sigma_{\mathrm{pp}}(H)$ and $0 \neq \psi \in \mathrm{Ker}\{H-z\}$. Projecting the equation $(H-z)\psi = 0$ on $\mathrm{Ran}\,P$ and on $\mathrm{Ran}\,\overline{P}$, we obtain

$$\begin{aligned}(H_P-z)P\psi - PH\overline{P}\psi \;&=\; 0\,,\\ (H_{\overline{P}}-z)\overline{P}\psi \;&=\; -\overline{P}HP\psi\,.\end{aligned}$$

Solving the second equation for $\overline{P}\psi$, we obtain that

$$\overline{P}\psi \;=\; -\overline{P}(H_{\overline{P}}-z)^{-1}\overline{P}HP\psi\,.$$

Substituting this identity into the first equation, we find that $f_P(H-z)\,P\psi \;=\; 0$ and hence

$$P\,\mathrm{Ker}\{H-z\} \;\subseteq\; \mathrm{Ker}\{f_P(H-z)\}\,.$$

Conversely, if $0 \neq \varphi = P\varphi \in \mathrm{Ker}\{f_P(H-z)\}$ then $S_P(z)\varphi \in \mathrm{Ker}\{H-z\}$, by (IV.12). Thus $P\varphi = P\,S_P(z)\,\varphi \in P\,\mathrm{Ker}\{H-z\}$. ∎

Theorem IV.1 establishes the basic properties of the Feshbach map crucial for the methods we develop in subsequent sections. We summarize Theorem IV.1 by saying that the map f_P is isospectral.

Let us mention a further important property of the Feshbach map which follows directly from (IV.13):

$$f_{P_1} \circ f_{P_2} \;=\; f_{P_1 P_2}\,, \tag{IV.15}$$

provided $[P_1, P_2] = 0$. In Chapter V, we use Identity (IV.15) with $P_1 P_2 = P_2 P_1 = P_2$, in which case it may be interpreted as a semigroup property.

IV.2 Instability of Excited States and Fermi's Golden Rule

Our first application of the Feshbach map to the operator H_g from (I.22) is to eliminate the particle degrees of freedom. Simultaneously, we shall eliminate the degree of freedom of the photon field corresponding to field energies $\geq \rho_0$, for some $\rho_0 > 0$ to be specified below. This reduces our spectral problem on $\mathcal{H}_{el} \otimes \mathcal{F}$ to one on the Hilbert space $\mathbb{C}^{N_j} \otimes \{\chi_{H_f<\rho_0}\mathcal{F}\}$, where $\chi_{H_f<\rho_0}\mathcal{F} = \mathrm{Ran}\{\,\chi_{H_f<\rho_0}\,\}$ and $\mathbb{C}^{N_j} = \mathrm{span}\{\psi_{j,\ell} \mid 1 \leq \ell \leq N_j\}$ is the subspace in H_{el} spanned by the eigenfunctions $\psi_{j,\ell}$ of H_{el} corresponding to the energy E_j, i.e., $H_{el}\psi_{j,\ell} = E_j\psi_{j,\ell}$.

We recall from (I.22) that H_g is given by

$$H_g \;=\; H_{el} \otimes \mathbf{1}_f \;+\; \mathbf{1}_{el} \otimes H_f \;+\; W_g\,, \tag{IV.16}$$

where $W_g = gW_1 + g^2W_2$, and

$$W_1 \;=\; \int \Big\{G_{10}(k) \otimes a^\dagger(k) \;+\; G_{01}(k) \otimes a(k)\Big\}\; dk\,, \tag{IV.17}$$

$$W_2 \;=\; \int \Big\{G_{20}(k,k') \otimes a^\dagger(k)a^\dagger(k') \;+\; G_{02}(k,k') \otimes a(k)a(k') + G_{11}(k,k') \otimes a^\dagger(k)a(k')\Big\}\; dk\,. \tag{IV.18}$$

Here, $G_{m,n}$ are operator-valued functions whose specification we recall below. Our purpose is to prove the following theorem.

Theorem IV.2 (a) *For $g > 0$ sufficiently small, the operator H_g has absolutely continuous spectrum in each interval $\left[\frac{2}{3}E_{j-1} + \frac{1}{3}E_j\ ,\ \frac{1}{3}E_j + \frac{2}{3}E_{j+1}\right]$, where $j \geq 1$ is such that the $N_j \times N_j$ selfadjoint matrix*

$$(A_j)_{\mu\nu} := \int_{-\infty}^{E_j - 0} \int dk \, \langle G_{10}(k)\psi_{j,\mu} \,|\, d\chi_{H_{el} \leq E}\, G_{10}(k)\psi_{j,\nu} \rangle \; \delta[\omega(k) - E_j + E] \tag{IV.19}$$

is positive definite, i.e., $A_j \geq a_j \mathbf{1} > 0$;

(b) *for vectors u, v belonging to a certain dense set, the matrix elements $\langle u|(H_g - z)^{-1}v\rangle$ of the resolvent $(H_g - z)^{-1}$ have an analytic continuation from $\mathbb{C}_+$ into*

$$\mathcal{A}_j := \left[\frac{2}{3}E_{j-1} + \frac{1}{3}E_j\ ,\ \frac{1}{3}E_j + \frac{2}{3}E_{j+1}\right] + i\left(\mathbb{R}^+ - \gamma_j\right)\ , \tag{IV.20}$$

where $\gamma_j := g^2 a_j$.

Clearly, Theorem IV.2(a) is a consequence of Theorem IV.2(b) which, in turn, is a consequence of Theorem IV.3 below. For its proof, we use the method of complex dilatations described in Section I.5.

IV.2.1 Complex Dilatations

We recall from (I.59) that a complex dilatation $U_f(\theta)$ transforms H_g into $H_g(\theta) \equiv U_f(\theta) H_g U_f(\theta)^{-1}$, given by

$$H_g(\theta) \;=\; H_{el} \otimes \mathbf{1} + \mathrm{e}^{-\theta}(\mathbf{1} \otimes H_f) + W_g(\theta)\,, \tag{IV.21}$$

where $W_g(\theta) = gW_1(\theta) + g^2 W_2(\theta)$ results from (IV.17), (IV.18) by the replacement $G^{(\theta)}_{m,n}(\underline{k}) := \mathrm{e}^{\frac{3\theta}{2}(m+n)} G_{m,n}(\mathrm{e}^{-\theta}\underline{k})$ with $1 \leq m+n \leq 2$ and $\underline{k} \in \mathbb{R}^{3(m+n)}$.

Throughout this chapter we make Hypothesis 2, i.e., for $m+n = 1$ and $k \in \mathbb{R}^3$, we assume $G^{(\theta)}_{mn}(k)$ to be analytic functions with values in the quadratic forms defined on $D\left[(-\Delta_x)^{1/4}\right]$, and for $m+n = 2$ and $\underline{k} \in \mathbb{R}^3 \times \mathbb{R}^3$, we assume $G^{(\theta)}_{mn}(\underline{k})$ to be analytic functions with values in the bounded operators on H_{el}. Moreover, Hypothesis 2 states that the smallest non-negative function J such that for all $k, k' \in \mathbb{R}^3$

$$\sup_{|\theta| \leq \theta_0} \left\| (-\Delta_x + 1)^{-1/4} G^{(\theta)}_{m,n}(k) (-\Delta_x + 1)^{-1/4} \right\| \;\leq\; J(k)\,, \tag{IV.22}$$

for $m+n=1$, and, for $m+n=2$,

$$\sup_{|\theta|\leq\theta_0} \left\| G^{(\theta)}_{m,n}(k,k') \right\| \;\leq\; J(k)\,J(k')\,, \tag{IV.23}$$

obeys

$$\Lambda_4(\beta) \;:=\; \left\{ \int \left[1+\omega(k)^{-1-\beta}\right] \,|J(k)|^2\,dk \right\}^{1/2} \;<\; \infty\,, \tag{IV.24}$$

for some $\beta>0$.

Theorem IV.3 *We require Hypothesis 2 and assume that $E_j > E_0$ is an isolated eigenvalue of H_{el} of degeneracy $N_j<\infty$ with eigenvectors $\psi_{j,1},\ldots,\psi_{j,N_j}$. Assume that A_j in (IV.19) is positive definite, i.e., $A_j \geq a_j\mathbf{1}>0$, and define $\gamma_j := g^2 a_j$. Then, for $g>0$ sufficiently small, the set*

$$\mathcal{A}_j \;:=\; \left[\frac{2}{3}E_{j-1}+\frac{1}{3}E_j\,,\,\frac{1}{3}E_j+\frac{2}{3}E_{j+1}\right] \;-\; i\gamma_j \;+\; i\mathbb{R}^+ \;+\; O\left(g^{2+\frac{\beta}{2+\beta}}\right)\,, \tag{IV.25}$$

is contained in the resolvent set $\rho[H_g(\theta)]$ of $H_g(\theta)$.

Before we proceed to the proof of Theorem IV.3 (which we give in Subsection IV.4.1), we introduce some more notation and derive some preliminary lemmata which pave the way to Theorem IV.3.

IV.3 Elimination of Particle Degrees of Freedom

In this subsection, we verify the hypotheses of Theorem IV.1 which allows us to apply the isospectral Feshbach map (see Corollary IV.5), after introducing some more notation and proving a preliminary technical estimate.

To begin with, we fix $\theta =: i\vartheta$ to be some small purely imaginary number, in spite of the fact that the connection between Theorem IV.2(b) above and Theorem IV.3 is given by the analytic continuation of matrix elements of the resolvent as a function of θ. Thus, we shall not display the θ-dependence of $G^{(\theta)}_{m,n}$ anymore and simply write $G_{m,n}$, instead. Moreover, our estimates below do not reflect the dependence on $\vartheta>0$, and they are not uniform as $\vartheta\to 0$.

Now, we formulate our assumption of isolation of the eigenvalue more precisely: Let

$$\delta \;:=\; \min\Big\{ |E_j-E_{j-1}|\,,\,|E_{j+1}-E_j| \Big\} \;>\; 0\,, \tag{IV.26}$$

where possibly $E_{j+1} = \Sigma$. Since V is a perturbation of $-\Delta$ with relative bound zero, both operators

$$(\,|H_{el} - E_j| + \mathbf{1}\,)\,(-\Delta + \mathbf{1})^{-1} \qquad \text{and} \qquad (-\Delta + \mathbf{1})^{-1}\,(\,|H_{el} - E_j| + \mathbf{1}\,)\;,$$

are bounded on $\mathcal{H}_{el}$. Thus, by (IV.22), there exists a constant C_1 such that, for all $k, k' \in \mathbb{R}^3$ and $m + n = 1$,

$$\Big\|\,(|H_{el} - E_j| + \mathbf{1})^{-1/4}\,G_{m,n}(k)\,(|H_{el} - E_j| + \mathbf{1})^{-1/4}\,\Big\| \;\leq\; C_1\,J(k)\;, \tag{IV.27}$$

and, for $m + n = 2$,

$$\|\,G_{mn}(k,k')\,\| \;\leq\; J(k)\,J(k')\;. \tag{IV.28}$$

Next, we choose some $0 < \tau < 1$ and fix

$$\rho_0 \;:=\; g^{2-2\tau}\;, \tag{IV.29}$$

such that, for $g > 0$ sufficiently small,

$$\rho_0 \;\leq\; \frac{\delta}{2}\,\sqrt{1 - \cos\vartheta}\;. \tag{IV.30}$$

We define

$$P_0 \;:=\; P_{el,j} \otimes \chi_{H_f < \rho_0} \;\equiv\; \sum_{\ell=1}^{N_j} |\psi_{j,\ell}\rangle\langle\psi_{j,\ell}| \otimes \chi_{H_f < \rho_0}\;. \tag{IV.31}$$

Then

$$\overline{P}_0 \;:=\; \mathbf{1} - P_0 \;=\; \overline{P}_0^{(1)} \,+\, \overline{P}_0^{(2)}\;, \tag{IV.32}$$

where

$$\overline{P}_0^{(1)} \;=\; P_{el,j} \otimes \chi_{H_f \geq \rho_0}\;, \tag{IV.33}$$

$$\overline{P}_0^{(2)} \;=\; \overline{P}_{el,j} \otimes \mathbf{1}_f\;. \tag{IV.34}$$

In order to control the spectrum of $H_g(\theta)$ we appeal to Theorem IV.1:

$$z \in \sigma[H_g(\theta)] \cap D(E_j,\,\rho_0/2) \;\Leftrightarrow\; 0 \in \sigma[f_{P_0}(H_g(\theta) - z)] \cap D(E_j - z,\,\rho_0/2)\,, \tag{IV.35}$$

where f_{P_0} is the Feshbach map defined in (IV.4). The equivalence (IV.35) is valid provided z belongs to the resolvent set of $\overline{P}_0 H_g(\theta) \overline{P}_0$ (see (IV.2)) and Inequalities (IV.3) hold. To verify this hypotheses, we require the following lemma.

Lemma IV.4 *Suppose that* $z \in D(E_j,\, \rho_0/2)$, *i.e., that* $|E_j - z| \leq \rho_0/2$, *and assume that* $C_1, \Lambda_4(\beta), \delta < \infty$, *and* $0 < \vartheta < \pi/2$. *Then , for some constant* $C_0 \equiv C_0(C_1, \Lambda_4(\beta), \delta, \vartheta) < \infty$ *and* $l = 1, 2$,

$$\left\| \, |H_0 - z|^{-1/2} \overline{P}_0 \, W_l \, \overline{P}_0 |H_0 - z|^{-1/2} \, \right\| \;\leq\; C_0 \cdot \rho_0^{-1/2} \,, \tag{IV.36}$$

$$\left\| \, |H_0 - z|^{-1/2} \overline{P}_0 \, W_l \, P_0 \, \right\| \;\leq\; C_0 \,, \tag{IV.37}$$

$$\left\| \, P_0 \, W_l \, \overline{P}_0 |H_0 - z|^{-1/2} \, \right\| \;\leq\; C_0 \,. \tag{IV.38}$$

Proof: First, we introduce

$$\widehat{H}_0 \;:=\; |H_{el} - E_j| \otimes \mathbf{1}_f \;+\; \mathbf{1}_{el} \otimes H_f \;+\; \rho_0 \tag{IV.39}$$

and observe that

$$\left\| \overline{P}_0 \frac{\widehat{H}_0}{H_0 - z} \right\| \;=\; \max \left\{ \left\| \overline{P}_0^{(1)} \frac{\widehat{H}_0}{H_0 - z} \right\| \,,\, \left\| \overline{P}_0^{(2)} \frac{\widehat{H}_0}{H_0 - z} \right\| \right\} \,,$$

and

$$\left\| \overline{P}_0^{(1)} \, \frac{\widehat{H}_0}{H_0 - z} \right\| \;=\; \sup_{r > \rho_0} \left| \frac{r + \rho_0}{\mathrm{e}^{-i\vartheta} r + E_j - z} \right| \;\leq\; \sup_{r > \rho_0} \left\{ \frac{r + \rho_0}{r - |E_j - z|} \right\} \;\leq\; 4 \,, \tag{IV.40}$$

$$\left\| \overline{P}_0^{(2)} \, \frac{\widehat{H}_0}{H_0 - z} \right\| \;\leq\; \sup_{r > 0 \,, |t| \geq \delta} \left\{ \frac{|t| + r + \rho_0}{|t + \mathrm{e}^{-i\vartheta} r| \;-\; |E_j - z|} \right\} \,. \tag{IV.41}$$

Now, $|t + \mathrm{e}^{-i\vartheta} r|^2 = t^2 + r^2 - 2tr \cos\vartheta \geq (1 - \cos\vartheta)(t^2 + r^2)$, and

$$|E_j - z| \;\leq\; \frac{\rho_0}{2} \;\leq\; \frac{\delta}{4} \sqrt{1 - \cos\vartheta} \;\leq\; \frac{|t|}{4} \sqrt{1 - \cos\vartheta} \,. \tag{IV.42}$$

Thus,

$$\left\| \overline{P}_0^{(2)} \, \frac{\widehat{H}_0}{H_0 - z} \right\| \;\leq\; \frac{\mathrm{const}}{\sqrt{1 - \cos\vartheta}} \,,$$

and hence, for some constant C_0',

$$\left\| \overline{P}_0 \, \frac{\widehat{H}_0}{H_0 - z} \right\| \;\leq\; C_0' \,. \tag{IV.43}$$

By inserting (IV.43) into the l.s. of (IV.36)–(IV.38), we obtain the bounds

$$\left\| \, |H_0 - z|^{-1/2} \overline{P}_0 \, W_l \, \overline{P}_0 |H_0 - z|^{-1/2} \, \right\| \;\leq\; C_0'' \left\| \widehat{H}_0^{-1/2} \, W_l \, \widehat{H}_0^{-1/2} \right\| \,, \tag{IV.44}$$

$$\left\| \, |H_0 - z|^{-1/2} \overline{P}_0 \, W_l \, P_0 \, \right\| \;\leq\; C_0'' \, \rho_0^{1/2} \, \left\| \widehat{H}_0^{-1/2} \, W_l \, \widehat{H}_0^{-1/2} \right\| \tag{IV.45}$$

$$\left\| \, P_0 \, W_l \, \overline{P}_0 |H_0 - z|^{-1/2} \, \right\| \;\leq\; C_0'' \, \rho_0^{1/2} \, \left\| \widehat{H}_0^{-1/2} \, W_l \, \widehat{H}_0^{-1/2} \right\| \tag{IV.46}$$

for some constant $C_0'' < \infty$. Thus, it remains to be shown that

$$\left\| \widehat{H}_0^{-1/2} W_l \widehat{H}_0^{-1/2} \right\| \leq C_0''' \cdot \rho_0^{-1/2} , \tag{IV.47}$$

for some constant C_0''' because in this case we obtain (IV.36)–(IV.38) from $C_0 := C_0'' \cdot C_0'''$. We demonstrate (IV.47) only for

$$W_{0,q} := \int G_{0,q}(k_1, \ldots, k_q) \otimes a(k_1) \cdots a(k_q)\, dk_1 \ldots dk_q , \tag{IV.48}$$

where $q = 1$ or $q = 2$. To this end, we pick a normalized vector $\psi \in \mathcal{H}$ and estimate, using $(k_1, \ldots, k_q) =: \underline{k}$, $,\omega(\underline{k}) = \omega(k_1) + \ldots + \omega(k_q)$, and the Pull-Through Formulae (IV.66)–(IV.67),

$$\begin{aligned} & \left\| \widehat{H}_0^{-1/2} W_{0,q} \widehat{H}_0^{-1/2} \psi \right\| \qquad\qquad (\text{IV.49}) \\ &= \left\| \widehat{H}_0^{-1/2} \int \Big(G_{0,q}(\underline{k}) \otimes \mathbf{1}_f \Big) \Big(\widehat{H}_0 + \omega(\underline{k}) \Big)^{-1/2} (\mathbf{1}_{el} \otimes a(k_1) \cdots a(k_q)) \psi \, dk_1 \cdots dk_q \right\| \\ &\leq \left\{ \int \left\| \widehat{H}_0^{1/2} \Big(G_{0,q}(\underline{k}) \otimes \mathbf{1}_f \Big) \Big(\widehat{H}_0 + \omega(\underline{k}) \Big)^{1/2} \mathbf{1}_{el} \otimes \Big(H_f + \omega(\underline{k}) \Big)^{q/2} \right\|^2 \right. \\ &\qquad\qquad \left. \frac{dk_1}{\omega(k_1)} \cdots \frac{dk_q}{\omega(k_q)} \right\}^{1/2} \cdot B_q^{1/2}(\psi) , \end{aligned}$$

where

$$\begin{aligned} B_q(\psi) &:= \int \| \mathbf{1}_{el} \otimes (H_f + \omega(\underline{k}))^{-\frac{q}{2}} a(k_1) \cdots a(k_q)\, \psi \|^2 \, \omega(k_1) dk_1 \cdots \omega(k_q) dk_q \\ &\leq \|\psi\|^2 = 1 . \end{aligned} \tag{IV.50}$$

Thus

$$\left\| \widehat{H}_0^{-1/2} W_{0,q} \widehat{H}_0^{-1/2} \right\| \leq \left\{ \int M_q(\underline{k})^2 \, \frac{dk_1}{\omega(k_1)} \cdots \frac{dk_q}{\omega(kq)} \right\}^{1/2} , \tag{IV.51}$$

where

$$\begin{aligned} M_q(\underline{k}) &:= \left\| \widehat{H}_0^{-1/2} \Big(G_{0,q}(\underline{k}) \otimes \mathbf{1}_f \Big) \Big(\widehat{H}_0 + \omega(\underline{k}) \Big)^{-1/2} \mathbf{1}_{el} \otimes \Big(H_f + \omega(\underline{k}) \Big)^{q/2} \right\| \\ &= \sup_{r>0} \left\| \Big(|H_{el} - E_j| + r + \rho_0 \Big)^{-1/2} G_{0,q}(\underline{k}) \right. \qquad (\text{IV.52}) \\ &\qquad \left. \Big(|H_{el} - E_j| + r + \omega(\underline{k}) + \rho_0 \Big)^{-1/2} \Big(r + \omega(\underline{k}) \Big)^{q/2} \right\|_{\mathcal{H}_{el}} . \end{aligned}$$

using the spectral theorem for H_f. In view of (IV.51), it suffices to prove that

$$M_q(\underline{k}) \;\leq\; \frac{\text{const}}{\rho_0^{1/2}} \prod_{j=1}^{q} (1+\omega(k_j))^{1/2}\, J(k_j)\,. \tag{IV.53}$$

Now, we distinguish the cases $q=1$ and $q=2$. For $q=1$, we insert (IV.27) into (IV.52) and get

$$\begin{aligned} M_1(\underline{k}) & \\ &\leq \sup_{r,t,t'>0} \left\{ \frac{C_1\, J(k)\, (t+1)^{1/4}\, (t'+1)^{1/4}\, (r+\omega(k))^{1/2}}{[t+r+\rho_0]^{1/2}\, [t'+r+\rho_0+\omega(k)]^{1/2}} \right\} \\ &\leq \sup_{r>0} \left\{ \frac{C_1\, J(k)\, (r+\rho_0+1)^{1/4}\, (r+\rho_0+\omega(k)+1)^{1/4}\, (r+\omega(k))^{1/2}}{[r+\rho_0]^{1/2}\, [r+\rho_0+\omega(k)]^{1/2}} \right\} \\ &\leq \sup_{r>0} \left\{ \frac{C_1\, J(k)\, (r+\rho_0+1)^{1/4}\, (r+\rho_0+\omega(k)+1)^{1/4}}{[r+\rho_0]^{1/2}} \right\} \\ &\leq (\rho_0+1)^{1/4}\, (\rho_0+\omega(k)+1)^{1/4}\, \rho_0^{-1/2}\, C_1\, J(k) \\ &\leq (\rho_0+1)^{1/2}\, \rho_0^{-1/2}\, [1+\omega(k)]^{1/4}\, C_1\, J(k)\,, \end{aligned} \tag{IV.54}$$

by minimizing over t, t' and $r>0$.

For $q=2$, we have

$$\begin{aligned} M_2(\underline{k}) &\leq \sup_{r>0} \left\{ \frac{J(k_1)\, J(k_2)\, [r+\omega(\underline{k})]}{[r+\rho_0]^{1/2}\, [r+\rho_0+\omega(\underline{k})]^{1/2}} \right\} \\ &\leq [\rho_0+\omega(\underline{k})]^{1/2}\, \rho_0^{-1/2}\, J(k_1)\, J(k_2) \\ &\leq (\rho_0+1)^{1/2}\, \rho_0^{-1/2} \prod_{j=1}^{2} [1+\omega(k_j)]^{1/4}\, J(k_j)\,, \end{aligned} \tag{IV.55}$$

and, hence, arrive at (IV.53) which, inserted into (IV.51), gives (IV.47) and thus proves the claim (IV.36)–(IV.38). ∎

Corollary IV.5 *Let* $z \in D(E_j,\, \rho_0/2) \equiv \{E_j+\zeta|\ |\zeta| \leq \rho_0/2\}$. *Then, for* $g>0$ *sufficiently small,* $z \in \sigma[H_g]$ *if and only if* $0 \in \sigma[f_{P_0}(H_g-z)]$, *where*

$$\begin{aligned} f_{P_0}(H_g-z) \;&=\; P_{el,j} \otimes \left\{ \chi_{H_f \leq \rho_0} (E_j - z + \mathrm{e}^{-i\vartheta} H_f) \right\} \;+\; g P_0 W_1 P_0 \\ &\quad + g^2 P_0 W_2 P_0 \;-\; \sum_{l,l'=1}^{2} g^{l+l'} P_0 W_l \overline{P}_0 \left(\overline{P}_0 H_g \overline{P}_0 - z \right)^{-1} \overline{P}_0 W_{l'} P_0\,. \end{aligned} \tag{IV.56}$$

Proof: We assume $z \in D(E_j,\, \rho_0/2)$. We verify the hypotheses (IV.2) and (IV.3) of Theorem IV.1. In order to prove that $\overline{P}_0 H_g \overline{P}_0 - z$ is invertible on $\mathrm{Ran}\{\overline{P}_0\}$, for $|z - Ej| \leq \rho_0/2$, we establish absolute convergence of the Neumann series expansion.

$$(\overline{P}_0 H_g \overline{P}_0 \,-\, z)^{-1} \overline{P}_0 \;=\; |H_0 - z|^{-1/2} \overline{P}_0 \qquad \text{(IV.57)}$$
$$\sum_{n=0}^{\infty} U^* \left(-|H_0 - z|^{-1/2} \overline{P}_0 W_g \overline{P}_0 |H_0 - z|^{-1/2} U^* \right)^n \overline{P}_0 |H_0 - z|^{-1/2} \,,$$

where $U^* = U^{-1} = |H_0 - z|(H_0 - z)^{-1} \overline{P}_0$ is the unitary on $\mathrm{Ran}\{\overline{P}_0\}$ that results from the Polar decomposition of $H_0 - z$. The series (IV.57) converges in norm, since by Lemma IV.4,

$$\left\| \, |H_0 - z|^{1/2} \, \overline{P}_0 \, W_g \, \overline{P}_0 \, |H_0 - z|^{-1/2} \, \right\| \;\leq\; \frac{C_0 \, g}{\rho_0^{1/2}} \;=\; C_0 \, g^{\tau} < \frac{1}{2} \,, \qquad \text{(IV.58)}$$

for $g > 0$ sufficiently small, using the definition (IV.29) of $\rho_0 = g^{2-2\tau}$.

Moreover, Lemma IV.4 also establishes the boundedness of $\|R_{\overline{P}_0}(z) W_g P_0\|$ and $\|P_0 W_g R_{\overline{P}_0}(z)\|$. Thus the hypotheses of Theorem IV.1 are satisfied. ∎

IV.4 The Spectrum of the Feshbach Hamiltonian, $f_{P_0}(H_g(\theta) - z)$

As a consequence of Corollary IV.5 in the preceeding subsection, $\sigma[H_g] \cap D(E_j,\, \rho_0/2)$ is given by those $z \in D(E_j,\, \rho_0/2)$ for which $0 \in \sigma[f_{P_0}(H_g - z)]$, where

$$f_{P_0}(H_g - z) \;=\; P_{el,j} \otimes \left\{ \chi_{H_f < \rho_0}(E_j - z + \mathrm{e}^{-i\vartheta} H_f) \right\} \;+\; g P_0 W_1 P_0 \qquad \text{(IV.59)}$$
$$+ \, g^2 P_0 W_2 P_0 \;-\; \sum_{l,l'=1}^{2} g^{l+l'} P_0 W_l \overline{P}_0 \left(\overline{P}_0 H_g \overline{P}_0 - z \right)^{-1} \overline{P}_0 W_{l'} P_0 \,.$$

Our goal is to show that if $z \in D(E_j, \rho_0/2)$ and $0 \in \sigma[f_{P_0}(H_g - z)]$ then

$$z \;\in\; E_j \;-\; \Delta E_j(g) \;+\; \mathrm{e}^{-i\vartheta} R_+ \;+\; D\left(0, g^{2+\frac{\beta}{2+\beta}}\right) \,, \qquad \text{(IV.60)}$$

where $\mathrm{Im}\{\Delta E_j(g)\} = -\Gamma_j$. This will imply Theorem IV.3. The proof of (IV.60) is accomplished in a sequence of lemmata.

Lemma IV.6

$$\left\| \sum_{l,l'=1}^{2} g^{l+l'} P_0 W_l \overline{P}_0 \left[(\overline{P}_0 H_g \overline{P}_0 - z)^{-1} \; - \; (\overline{P}_0 H_0 \overline{P}_0 - z)^{-1} \right] \overline{P}_0 W_l P_0 \right\| \\ = \; O\left(g^{2+\tau}\right) \; . \qquad \text{(IV.61)}$$

Proof: By the 2^{nd} resolvent equation, the left side of (IV.61) is bounded by

$$\sum_{l_1,l_2,l_3=1}^{2} g^{l_1+l_2+l_3} \left\| P_0 W_{l_1} \overline{P}_0 |H_0 - z|^{-1/2} \right\| \; \left\| \, |H_0 - z|^{-1/2} \overline{P}_0 W_{l_2} \overline{P}_0 |H_0 - z|^{-1/2} \right\| \\ \left\| \, |H_0 - z|^{-1/2} \overline{P}_0 W_{l_3} P_0 \right\| \; \left\| \, |H_0 - z|^{1/2} \overline{P}_0 (\overline{P}_0 H_g \overline{P}_0 - z)^{-1} \overline{P}_0 |H_0 - z|^{1/2} \right\| \\ \leq \; 16 \, C_0^3 \, g^3 \, \rho_0^{-1/2} \; , \qquad \text{(IV.62)}$$

using Lemma IV.4 and (IV.57), (IV.58). ∎

Lemma IV.7 *Let*

$$\mathrm{Rem}_1(g, \rho_0) \; := \; \sum_{l,l'=1}^{2} g^{l+l'} P_0 W_l \overline{P}_0 \left(\overline{P}_0 H_g \overline{P}_0 - z \right)^{-1} \overline{P}_0 W_{l'} P_0 \\ - g^2 P_0 W_1 \left(\frac{\overline{P}_0}{H_0 - z} \right) W_1 P_0 \, . \qquad \text{(IV.63)}$$

Then

$$\| \mathrm{Rem}_1(g, \rho_0) \| \; = \; O\left(g^{2+\tau}\right) \; . \qquad \text{(IV.64)}$$

Proof: By Lemma IV.6, it suffices to show that

$$\left\| \sum_{l+l' \geq 3} g^{l+l'} P_0 W_l \left(\frac{\overline{P}_0}{H_0 - z} \right) W_{l'} P_0 \right\| \; \leq \; O\left(g^3\right) \; , \qquad \text{(IV.65)}$$

which follows directly from (IV.37), (IV.38) in Lemma IV.4. ∎

Next, we use the following Pull-Through formula.

Lemma IV.8 (Pull-Through Formulae) *Let $F : \mathbb{R}_+ \to \mathbb{C}$ be a borel function with $F[r] = O(r+1)$. Then $F[H_f]$, defined by the spectral theorem for H_f, is defined on the domain of H_f, and it obeys the following intertwining relations:*

$$F[H_f] \, a^\dagger(k) \; = \; a^\dagger(k) \, F[H_f + \omega(k)] \, , \qquad \text{(IV.66)}$$

$$a(k) \, F[H_f] \; = \; F[H_f + \omega(k)] \, a(k) \, . \qquad \text{(IV.67)}$$

Eqns. (IV.66) and (IV.67) follow immediately from $F[H_f]\prod_{i=1}^N a^\dagger(k_i)\Omega = F\left[\sum_{i=1}^N \omega(k_i)\right]\prod_{i=1}^N a^\dagger(k_i)\Omega$. We apply the Pull-Through formulae (IV.66)–(IV.67) and rewrite

$$\begin{aligned} P_0 W_1 \left(\frac{\overline{P}_0}{H_0 - z}\right) W_1 P_0 \;&=\; Q \qquad\qquad (\text{IV.68}) \\ &+ \int P_0 \left[G_{1,0}(k)\otimes a^\dagger(k)a^\dagger(k')\right]\left[\frac{\overline{P}(\omega(k'))}{H_0 + \mathrm{e}^{-i\vartheta}\omega(k') - z}\right][G_{1,0}(k')\otimes \mathbf{1}_f]\, P_0\, dkdk' \\ &+ \int P_0 \left[G_{0,1}(k)\otimes \mathbf{1}_f\right]\left[\frac{\overline{P}_0(\omega(k))}{H_0 + \mathrm{e}^{-i\vartheta}\omega(k) - z}\right][G_{0,1}(k')\otimes a(k)a(k')]\, P_0\, dkdk' \\ &+ \int P_0 \left[G_{1,0}(k)\otimes a^\dagger(k)\right]\left[\frac{\overline{P}_0}{H_0 - z}\right][G_{0,1}(k')\otimes a(k')]\, P_0\, dkdk' \\ &+ \int P_0 \left[G_{0,1}(k)\otimes a^\dagger(k')\right]\left[\frac{\overline{P}_0(\omega(k)+\omega(k'))}{H_0 + \mathrm{e}^{-i\vartheta}(\omega(k)+\omega(k')) - z}\right] \\ &\qquad\qquad [G_{10}(k')\otimes a(k)]\, P_0\, dkdk' \,, \end{aligned}$$

where

$$Q \;:=\; \int P_0\,[G_{1,0}(k)\otimes \mathbf{1}_f]\left[\frac{\overline{P}_0(\omega(k))}{H_0 + \mathrm{e}^{-i\vartheta}\omega(k) - z}\right]\left[G_{01}^{(0)}(k)\otimes \mathbf{1}_f\right] P_0\, dkdk', \quad (\text{IV.69})$$

and $\overline{P}_0(\omega) := \mathbf{1} - P_0(\omega)$ with $P_0(\omega) := P_{el,j}\otimes \chi_{H_f+\omega<\rho_0}$.

Lemma IV.9 *Let*

$$\mathrm{Rem}_2(g,\rho_0) \;:=\; g^2 P_0 W_1 \left[\frac{\overline{P}_0}{H_0 - z}\right] W_1 P_0 \;-\; g^2\, Q\,. \qquad (\text{IV.70})$$

Then $\mathrm{Rem}_2(g,\rho_0)$ *is a bounded operator from* $P_0(\mathcal{H}_{el}\otimes\mathcal{F})$ *to itself, with*

$$\|\mathrm{Rem}_2(g,\rho_0)\| \;=\; O\left(g^{2+2\beta(1-\tau)}\right)\,. \qquad (\text{IV.71})$$

Proof: The operator $\mathrm{Rem}_2(g,\rho_0)$ is given by the last four terms on the right side of (IV.68). Obviously all these terms map the subspace $P_0(\mathcal{H}_{el}\otimes\mathcal{F})$ to itself. We estimate the norms of all four terms separately. The estimates on the norms of the operators proportional to $a^\dagger(k)a^\dagger(k')$ and to $a(k)a(k')$, respectively, are analogous; we only examine the one proportional to $a(k)a(k')$. Condition (IV.27) states that

$$\left\| \left(|H_{el} - E_j| + \mathbf{1}\right)^{-1/4} G_{m,n}(k) \left(|H_{el} - E_j| + \mathbf{1}\right)^{-1/4} \right\| \;\le\; C_1\, J(k)\,,$$

for $m+n=1$. Thus, for an arbitrary vector ψ,

$$\begin{aligned}\left\| \int P_0 \left[G_{0,1}(k)\otimes \mathbf{1}_f\right] \left[\frac{\overline{P}_0(\omega(k))}{H_0+\mathrm{e}^{-i\vartheta}\omega(k)-z}\right] \left[G_{0,1}(k')\otimes a(k)a(k')\right]\, P_0\psi\, dkdk' \right\| \\ \leq \; C_1^2 \int J(k)J(k') \left\| \frac{\overline{P}_0(\omega(k))(|H_{el}-E_j|+\mathbf{1})^{1/2}}{H_{el}\otimes\mathbf{1}_f+\mathrm{e}^{-i\vartheta}\mathbf{1}_{el}\otimes(H_f+\omega(k))\,-z}\right\| \\ \cdot \left\| \mathbf{1}_{el}\otimes a(k)a(k')\chi_{H_f<\rho_0}\psi\right\| \, dkdk' \,. \qquad \text{(IV.72)}\end{aligned}$$

We recall from Eqs. (IV.33) and (IV.34) that

$$\overline{P}_0(\omega) \;=\; \overline{P}_0^{(1)}(\omega) \;+\; \overline{P}_0^{(2)} \;,$$

where

$$\overline{P}_0^{(1)}(\omega) \;=\; P_{el,j}\otimes\chi_{H_f+\omega\geq\rho_0} \;, \qquad \text{(IV.73)}$$

$$\overline{P}_0^{(2)} \;=\; \overline{P}_{el,j}\otimes\mathbf{1}_f \;. \qquad \text{(IV.74)}$$

Since $|E_j-z|\leq\rho_0/2$, we have that

$$\begin{aligned}\left\| \frac{\overline{P}_0^{(1)}(\omega(k))\,(|H_{el}-E_j|+\mathbf{1})^{1/2}}{H_{el}\otimes\mathbf{1}_f+\mathrm{e}^{-i\vartheta}\mathbf{1}_{el}\otimes(H_f+\omega(k)))\,-z}\right\| \;&=\; \left\| \frac{\chi_{H_f+\omega(k)\geq\rho_0}}{\mathrm{e}^{-i\vartheta}(H_f+\omega(k))+E_j-z}\right\|_{\mathcal{F}} \\ \leq \;\sup_{r\geq\max\{0,\rho_0-\omega(k)\}}\left\{\frac{1}{r+\omega(k)-|E_j-z|}\right\} \;&\leq\; \frac{4}{\omega(k)+\rho_0}\,. \qquad \text{(IV.75)}\end{aligned}$$

Furthermore, using that $|E_j-z|\leq\rho_0/2\leq\delta\sqrt{1-\cos\vartheta}/4$, by (IV.30), we find that

$$\begin{aligned}&\left\| \frac{\overline{P}_0^{(2)}\,(|H_{el}-E_j|+\mathbf{1})^{1/2}}{H_{el}\otimes\mathbf{1}_f+\mathrm{e}^{-i\vartheta}\mathbf{1}_{el}\otimes(H_f+\omega(k))\,-z}\right\| \\ &\qquad\leq\;\sup_{t\geq\delta,\,r>0}\left|\frac{4\,(t+1)^{1/2}}{4|t+\mathrm{e}^{-i\vartheta}(r+\omega(k))|\;-\;\delta\sqrt{1-\cos\vartheta}}\right| \qquad \text{(IV.76)} \\ &\qquad\leq\;\sup_{t\geq\delta}\left\{\frac{8\sqrt{2}\,[2t+2+\omega(k)]^{1/2}}{3\sqrt{1-\cos\vartheta}\,[t+\omega(k)]}\right\} \;\leq\; \frac{4\,[1+\delta]^{1/2}}{\sqrt{1-\cos\vartheta}\,[\delta+\omega(k)]}\,.\end{aligned}$$

Combining (IV.75) and (IV.76), we obtain the bound

$$\left\| \frac{\overline{P}_0(\omega(k))\,(|H_{el}-E_j|+\mathbf{1})^{1/2}}{H_{el}\otimes\mathbf{1}_f+\mathrm{e}^{-i\vartheta}\mathbf{1}_{el}\otimes(H_f+\omega(k))\,-z}\right\| \;\leq\; \frac{C_2}{\omega(k)+\rho_0}\,, \qquad \text{(IV.77)}$$

where $C_2 := 4(1+\delta)^{1/2}(1-\cos\vartheta)^{-1/2}$. We may now complete our bound on the right side of (IV.72): Using (IV.77) we find that

$$\begin{aligned} \text{R.S. of (IV.72)} &\leq C_1^2\, C_2 \int \frac{J(k)\, J(k')}{\omega(k)+\rho_0} \left\| \mathbf{1}_{el} \otimes a(k)a(k')\chi_{H_f<\rho_0}\, \psi \right\| dkdk' \\ &\leq C_1^2\, C_2\, \rho_0^{\beta-1} \left(\int_{|k|\leq\rho_0} \frac{J(k)^2}{\omega(k)^{1+\beta}} dk \right) \left\| \mathbf{1}_{el} \otimes H_f\, \chi_{H_f<\rho_0}\, \psi \right\| \\ &\leq C_1^2\, C_2\, \Lambda_4(\beta)^2\, \rho_0^{\beta}\, \|\psi\| \,. \end{aligned} \tag{IV.78}$$

The estimates on the remaining two terms in $\mathrm{Rem}_2(g,\rho_0)$ are carried out in the same fashion; the resulting bounds are similar to (IV.78). This completes the proof of Lemma IV.9. ∎

Lemma IV.10

$$\left\| g^2 P_0 W_2 P_0 \right\| \;\leq\; 3\, g^{2+2(1+\beta)(1-\tau)}\, \Lambda_4(\beta)^2 \,. \tag{IV.79}$$

Proof: By Eq. (IV.18), $P_0 W_2 P_0$ is the sum of three terms. We estimate them separately, but all three estimates have the same structure. As an example we estimate the term proportional to $G_{1,1}(k,k') \otimes a^\dagger(k)a(k')$: For arbitary vectors ϕ and ψ in $H_{el} \otimes F$,

$$\begin{aligned} &\left| \int \left\langle P_0\psi \,\middle|\, G_{11}(k,k') \otimes a^\dagger(k)a(k') P_0\psi \right\rangle dkdk' \right| \\ &\qquad \leq \int J(k)J(k') \|a(k)P_0\phi\| \cdot \|a(k')P_0\psi\| dkdk' \\ &\qquad \leq \rho_0^{\beta} \Lambda_4(\beta)^2 \cdot \left\| \mathbf{1}_{el} \otimes H_f^{1/2} P_0\phi \right\| \cdot \left\| \mathbf{1}_{el} \otimes H_f^{1/2} P_0\psi \right\| \,, \\ &\qquad \leq \rho_0^{1+\beta}\, \Lambda_4(\beta)^2 \cdot \|\phi\| \cdot \|\psi\| \,. \blacksquare \end{aligned} \tag{IV.80}$$

The results in Lemmata IV.6 through IV.10 may be summarized as follows: If $f_{P_0}(H_g - z)$ denotes the effective Hamiltonian (Feshbach map of $H_g - z$) at a photon energy scale ρ_0, as defined in Eq. (IV.56), then

$$\begin{aligned} &f_{P_0}(H_g - z) \;= \\ &\qquad P_0 \mathbf{1}_{el} \otimes (E_j - z + \mathrm{e}^{-i\vartheta} H_f) P_0 \;+\; g P_0 W_1 P_0 \;+\; g^2 Q \;+\; \sum_{\mu=1}^{3} \mathrm{Rem}_\mu(g,\rho_0) \,, \end{aligned} \tag{IV.81}$$

with

$$\|\mathrm{Rem}_1(g,\rho_0)\| \;=\; O\left(g^{2+\tau}\right) \,, \tag{IV.82}$$

by Lemma IV.7, and

$$\|\mathrm{Rem}_2(g,\rho_0)\| \;=\; O\left(g^{2+2\beta(1-\tau)}\right)\;, \tag{IV.83}$$

by Lemma IV.9, where $\mathrm{Rem}_1(g,\rho_0)$ is given by (IV.63), and $\mathrm{Rem}_2(g,\rho_0)$ is given by (IV.70). Furthermore,

$$\|\mathrm{Rem}_3(g,\rho_0)\| \;=\; O\left(g^{2+2(1+\beta)(1-\tau)}\right)\;,$$

with

$$\mathrm{Rem}_3(g,\rho_0) \;:=\; g^2\rho_0 W_2 \rho_0\;, \tag{IV.84}$$

by Lemma IV.10, all O-symbols in (IV.83)–(IV.84) represent explicitly computable constants which possibly depend on ϑ, δ, and β.

Our next task is to analyze the term g^2Q on the right side of (IV.81). We observe that, by (IV.69),

$$\begin{aligned} Q \;=\; & \int P_{el,j}\,(G_{0,1}(k)\otimes \mathbf{1}_f)\Big[\overline{P}_{el,j}\,(H_{el}\otimes\mathbf{1}_f) + \mathrm{e}^{-i\vartheta}\mathbf{1}_{el}\otimes(H_f+\omega(k)) - z\Big]^{-1} \\ & \qquad \cdot\,(G_{1,0}(k)\otimes\mathbf{1}_f)\,P_{el,j}\otimes\chi_{H_f<\rho_0}dk \\ & +\int (P_{el,j}G_{0,1}(k)P_{el,j}G_{1,0}(k)P_{el,j}\otimes\mathbf{1}_f) \\ & \qquad \cdot\Big[\mathrm{e}^{-i\vartheta}\mathbf{1}_{el}\otimes(H_f+\omega(k)) + E_j - z\Big]^{-1}\,\chi_{H_f+w(k)\geq\rho_0}\,\chi_{H_f<\rho_0}dk\;. \end{aligned} \tag{IV.85}$$

Lemma IV.11 *For* $|z-E_j|\leq\rho_0/2$, *define* Rem_4 *by*

$$\begin{aligned} & \int P_{el,j}\,(G_{0,1}(k)\otimes\mathbf{1}_f)\,\Big[\overline{P}_{el,j}\,(H_{el}\otimes\mathbf{1}_f) + \mathrm{e}^{-i\vartheta}\,(H_f+\omega(k)) - z\Big]^{-1} \\ & \qquad\qquad (G_{1,0}(k)\otimes\mathbf{1}_f)\,P_{el,j}\otimes\chi_{H_f<\rho_0}dk \\ & \quad = \Big\{\int P_{el,j}G_{0,1}(k)\,\Big[\overline{P}_{el,j}H_{el} - E_j + \mathrm{e}^{-i\vartheta}\omega(k)\Big]^{-1} \\ & \qquad\qquad G_{10}P_{el,j}dk\Big\}\,\chi_{H_f<\rho_0} + g^{-2}\mathrm{Rem}_4\,(g,\rho_0)\;. \end{aligned} \tag{IV.86}$$

Then

$$\|\mathrm{Rem}_4(g,\rho_0)\| \;=\; O\left(g^{4-2\tau}\right)\;. \tag{IV.87}$$

Proof: Using the 2^{nd} resolvent equation, we find

$$\begin{aligned} & g^{-2}\,\mathrm{Rem}_4\,(g,\rho_0) \;= \\ & \quad \int P_{el,j}\,(G_{0,1}(k)\otimes\mathbf{1}_f)\,\Big[\overline{P}_{el,j}\,(H_{el}\otimes\mathbf{1}_f) + \mathrm{e}^{-i\vartheta}\,(H_f+\omega(k)) - z\Big]^{-1} \\ & \qquad \left[\frac{\mathrm{e}^{-i\vartheta}\mathbf{1}_{el}\otimes H_f\chi_{H_f<\rho_0} \;+\; E_j - z}{\overline{P}_{el,j}H_{el}\otimes\mathbf{1}_f - E_j + \mathrm{e}^{-i\vartheta}\omega(k)}\right](G_{1,0}(k)\otimes\mathbf{1}_f)\,P_{el,j}\otimes\chi_{H_f<\rho_0}dk\;. \end{aligned} \tag{IV.88}$$

Since

$$\left\| \frac{\left(|\overline{P}_{el,j} H_{el} - E_0| + \mathbf{1}_{el} \right)^{1/4}}{\overline{P}_{el,j} H_{el} \otimes \mathbf{1}_f + \mathrm{e}^{-i\vartheta} \mathbf{1}_{el} \otimes (H_f + \omega(k)) - z} \right\| \;\leq\; C_3 \, \delta^{-1} \,,$$

uniformly in $|z - E_j| \leq \rho_0/2$, we may bound the norm of $\mathrm{Rem}_4(g, \rho_0)$ by

$$C_3^2 \, \delta^{-2} \, \left\| \mathrm{e}^{-i\vartheta} \mathbf{1}_{el} \otimes H_f \chi_{H_f < \rho_0} \;+\; E_j - z \right\| \int J(k)^2 \, dk \;\leq\; \left(\frac{3 C_3^2 \, \Lambda_4(\beta)^2}{\delta^2} \right) \rho_0 \,. \blacksquare$$

Next, we note that in the first term on the right side of (IV.86) we may rotate the k-integration to the subspace $e^{i\vartheta} \mathbb{R}^3$, so as to find that

$$\begin{aligned} Z_j^{\mathrm{od}} \;:=\; & \int P_{el,j} G_{0,1}(k) \left[\overline{P}_{el,j} H_{el} - E_j + \mathrm{e}^{-i\vartheta} \omega(k) \right]^{-1} G_{1,0}(k) P_{el,j} \, dk \\ & \int P_{el,j} G_{0,1}(k) \left[\overline{P}_{el,j} H_{el} - E_j + \omega(k) - i0 \right]^{-1} G_{1,0}(k) P_{el,j} \, dk \,, \end{aligned} \tag{IV.89}$$

and hence

$$\mathrm{Im}\{ Z_j^{\mathrm{od}} \} \;=\; -A_j \,, \tag{IV.90}$$

where the $N_j \times N_j$ matrix A_j has been defined in Eq. (IV.19). Here we use the notation

$$\mathrm{Re}\{Z\} \;=\; \frac{1}{2}(Z + Z^*) \quad \text{and} \quad \mathrm{Im}\{Z\} \;=\; \frac{1}{2i}(Z - Z^*) \,, \tag{IV.91}$$

where Z is a matrix.

Next, we deal with the second term in Q on the right side of Eq. (IV.85). Similar to Z_j^{od}, we introduce

$$Z_j^{\mathrm{d}} \;:=\; \int P_{el,j} \, G_{0,1}(k) \, P_{el,j} \, G_{1,0}(k) \, P_{el,j} \, \frac{dk}{\mathrm{e}^{-i\vartheta} \omega(k)} \tag{IV.92}$$

Our bound on the second term in Q on the right side of Eq. (IV.85) is contained in the following lemma.

Lemma IV.12 *Let*

$$\mathcal{G}(k) \;:=\; P_{el,j} \, G_{0,1}(k) \, P_{el,j} \, G_{1,0}(k) \, P_{el,j} \,, \tag{IV.93}$$

and

$$\begin{aligned} & \mathrm{Rem}_5(g, \rho_0) \;:= \\ & g^2 \left\{ \int \mathcal{G}(k) \otimes \left[\mathrm{e}^{-i\vartheta} \, (H_f + \omega(k)) + E_j - z \right]^{-1} \chi_{H_f + \omega(k) \geq \rho_0} dk \;-\; Z_j^{\mathrm{d}} \right\} \chi_{H_f < \rho_0} \,. \end{aligned} \tag{IV.94}$$

Then, for $|z - E_j| \leq \rho_0/2$,

$$\|\mathrm{Rem}_5(g, \rho_0)\| \;\leq\; 7C_1^2 \Lambda_4(\beta)^2 g^{2+2\beta(1-\tau)} \,. \tag{IV.95}$$

Proof: First, we observe that

$$\begin{aligned} \int \mathcal{G}(k) \otimes \Big\{ \left[\mathrm{e}^{-i\vartheta} \left(H_f + \omega(k)\right) + E_j - z \right]^{-1} \chi_{H_f+\omega(k)\geq\rho_0} \\ - \frac{1}{\mathrm{e}^{-i\vartheta}\omega(k)} \Big\} \, dk \, \chi_{H_f<\rho_0} \\ = \; \Big\{ \int \mathcal{G}(k) \otimes \frac{-(\mathrm{e}^{-i\vartheta} H_f + E_j - z)}{(\mathrm{e}^{-i\vartheta}\left(H_f + \omega(k)\right) + E_j - z)\,\mathrm{e}^{-i\vartheta}\omega(k)} \, \chi_{H_f+\omega(k)\geq\rho_0} dk \\ + \int \mathcal{G}(k) \otimes \left[\chi_{H_f+\omega(k)\geq\rho_0} - 1\right] \frac{dk}{\mathrm{e}^{-i\vartheta}\omega(k)} \Big\} \chi_{H_f<\rho_0} \end{aligned} \tag{IV.96}$$

Using (IV.93), (IV.27) and the definition of $P_{el,j}$, we find that the norm of the right side is bounded by

$$C_1^2 \int J(k)^2 \left\{ \frac{6\rho_0}{[\rho_0 + \omega(k)]\omega(k)} \;+\; \frac{\chi_{\omega(k)<\rho_0}}{\omega(k)} \right\} dk \;\leq\; 7C_1^2 \delta_j^2 \rho_0^\beta \Lambda_4(\beta)^2 \,. \blacksquare$$

IV.4.1 Proof of Theorem IV.3

To begin the proof of Theorem IV.3, we recall from Corollary IV.5 in Section IV.3 that $\sigma[H_g] \cap D(E_j,\ \rho_0/2)$ is given by those $z \in D(E_j,\ \rho_0/2)$ for which $0 \in \sigma[f_{P_0}(H_g - z)]$, where

$$\begin{aligned} f_{P_0}(H_g - z) \;&=\; P_{el,j} \otimes \left\{ \chi_{H_f<\rho_0} (E_j - z + \mathrm{e}^{-i\vartheta} H_f) \right\} \;+\; g P_0 W_1 P_0 \\ &\quad + g^2 P_0 W_2 P_0 \;-\; \sum_{l,l'=1}^{2} g^{l+l'} P_0 W_l \overline{P}_0 \left(\overline{P}_0 H_g \overline{P}_0 - z \right)^{-1} \overline{P}_0 W_{l'} P_0 \,. \end{aligned} \tag{IV.97}$$

We prove Theorem IV.3 in two steps. First, we show that

$$\{z | 0 \in \sigma[f_{P_0}(H_g - z)] \cap D(E_j, \rho_0/2) \;\subseteq\; E_j - \Delta E_j(g) + \mathrm{e}^{-i\vartheta} R_+ + D\left(0, g^{2+\frac{\beta}{2+\beta}}\right) , \tag{IV.98}$$

where $\mathrm{Im}\{\Delta E_j(g)\} = -\Gamma_j$. This implies that

$$\mathcal{A}_j^{(1)} \;:=\; \left\{ z \in D(E_j,\ \rho_0/2) \mid \mathrm{Im}\{z\} > -a_j g^2 + C_5\, g^{2+\frac{\beta}{2+\beta}} \right\} \;\subseteq\; \rho[H_g] \,, \tag{IV.99}$$

where $\rho[H_g]$ is the resolvent set of $H_g = H_g(\theta)$. Secondly, we show that

$$\mathcal{A}_j^{(2)} \;:=\; \mathcal{A}_j \setminus \mathcal{A}_j^{(1)} \;\subseteq\; \rho[H_g]\,, \tag{IV.100}$$

where $\mathcal{A}_j$ was defined in (IV.25) to be the set

$$\mathcal{A}_j \;:=\; \left[\frac{2}{3}E_{j-1} \,+\, \frac{1}{3}E_j \,,\; \frac{1}{3}E_j \,+\, \frac{2}{3}E_{j+1}\right] \;-\; i\gamma_j \;+\; i\mathbb{R}^+ \;+\; O\left(g^{2+\frac{\beta}{2+\beta}}\right)\,.$$

Theorem IV.3 then follows by putting together (IV.99) and (IV.100).

In order to establish (IV.98), we first remark that $Z_j^{\rm d}$ in (IV.92) is hermitian, i.e., $Z_j^{\rm d} = {\rm Re}\{Z_j^{\rm d}\}$. We find this by deforming the k-integration on the right side of (IV.92) into the complex domain. We set

$$\Delta E_j \;:=\; g^2\, Z_j^{\rm d}\,, \tag{IV.101}$$

$$\Gamma_j \;:=\; g^2\, {\rm Im}\{Z_j^{\rm od}\} \;=\; g^2 A_j\,, \tag{IV.102}$$

where $Z_j^{\rm d}$ and $Z_j^{\rm od}$ are defined in (IV.92) and (IV.89), respectively. Returning to expression (IV.81) for the effective Hamiltonian $f_{P_0}[H_g - z]$ at photon energy scale ρ_0 and using (IV.83)–(IV.84), Lemma IV.11 and Lemma IV.12, we conclude that

$$\begin{aligned} f_{P_0}(H_g - z) \;&=\; P_0 \left(E_j - z + [\Delta E_j - i\Gamma_j] \otimes \mathbf{1}_f + {\rm e}^{-i\vartheta}\mathbf{1}_{el} \otimes H_f\right) P_0 \\ &\quad + g P_0 W_1 P_0 \;+\; \sum_{\mu=1}^{5} {\rm Rem}_\mu\,(g, \rho_0)\,, \end{aligned} \tag{IV.103}$$

where ΔE_j and Γ_j are $O\,(g^2)$, and

$$\left\| \sum_{\mu=1}^{5} {\rm Rem}_\mu\,(g, \rho_0) \right\| \;=\; O\left(g^{4-2\tau} + g^{2+\tau} + g^{2+2(1+\beta)(1-\tau)}\right)\,. \tag{IV.104}$$

Furthermore, it is easy to show that

$$\|g P_0 W_1 P_0\| \;\leq\; C_4\, g\, \rho_0^{\frac{1}{2}(1+\beta)} \;=\; O\left(g^{1+(1-\tau)(1+\beta)}\right)\,. \tag{IV.105}$$

Given $\beta > 0$ such that $\Lambda_4(\beta) < \infty$, we choose

$$\tau \;:=\; \frac{\beta}{2+\beta} \qquad \text{and hence} \qquad \rho_0 \;=\; g^{2-\frac{2\beta}{2+\beta}} \tag{IV.106}$$

in (IV.29). The coupling constant $g \geq 0$ has to be so small, that both Condition (IV.30) and (IV.58) are satisfied. Adding up the error terms in (IV.104) and (IV.105), we obtain

$$\left\| g P_0 W_1 P_0 \; + \; \sum_{\mu=1}^{5} \mathrm{Rem}_\mu(g\,,\rho_0) \right\| \; = \; O\left(g^{2+\frac{\beta}{2+\beta}} \right) \,. \tag{IV.107}$$

Assuming that (IV.19) in Theorem IV.2(a) is valid, i.e.,

$$\Gamma_j \; = \; g^2\, A_j \; \geq \; g^2\, a_j\, \mathbf{1} \; > \; 0\,, \tag{IV.108}$$

we conclude from (IV.103), (IV.107) and (IV.108) that, for g small enough and a sufficiently large constant, $C_5 \equiv C_5(\vartheta,\delta,\beta)$, the operator $f_{P_0}(H_g - z)$ has a bounded inverse whenever

$$|z - E_j| \; \leq \; \frac{\rho_0}{2} \quad \text{and} \quad \mathrm{Im}\{z\} \; > \; -a_j g^2 \; + \; C_5\, g^{2+\frac{\beta}{2+\beta}}\,, \tag{IV.109}$$

which is equivalent to (IV.99).

To prove (IV.100), we first note that all $z \in \mathcal{A}_j^{(2)}$ obey $-\pi + \vartheta/2 \leq \arg\{z - E_j\} \leq -\vartheta/2$, provided that

$$\Gamma_j \; = \; a_j g^2 \; \leq \; \frac{\rho_0}{6}\, \sin(\vartheta/2)\,. \tag{IV.110}$$

Thus, for any $z \in \mathcal{A}_j^{(2)}$,

$$\begin{aligned} \left\| \frac{\widehat{H}_0(P_{el,j} \otimes \mathbf{1})}{H_0 - z} \right\| \; &= \; \sup_{r>0} \left\{ \frac{r + \rho_0}{|E_j - z + \mathrm{e}^{-i\vartheta} r|} \right\} \\ &\leq \; \sup_{r>0} \left\{ \frac{r + \rho_0}{|(\rho_0/2) + \mathrm{e}^{-i\vartheta/2} r|} \right\} \; \leq \; \frac{\sqrt{8}}{\sqrt{1 - \cos(\vartheta/2)}}\,. \end{aligned} \tag{IV.111}$$

On the other hand, for any $z \in \mathcal{A}_j^{(2)}$,

$$\left\| \frac{\widehat{H}_0(\overline{P}_{el,j} \otimes \mathbf{1})}{H_0 - z} \right\| \; = \; \sup_{r>0,\; |t| \geq \delta} \left\{ \frac{|t| + r + \rho_0}{|t + E_j - z + \mathrm{e}^{-i\vartheta} r|} \right\} \; \leq \; \frac{16}{\sqrt{1 - \cos\vartheta}}\,, \tag{IV.112}$$

as is easily verified by separately examining the cases $r > |t|/6$ and $r \leq |t|/6$, using (IV.110) and the fact that $\sin\vartheta \geq 1 - \cos\vartheta$. From (IV.111) and (IV.112) we conclude that

$$\sup_{z \in \mathcal{A}_j^{(2)}} \left\| \frac{\widehat{H}_0}{H_0 - z} \right\| \; \leq \; \frac{16}{\sqrt{1 - \cos\vartheta}}\,. \tag{IV.113}$$

So, finally we obtain the invertibility of $H_g - z$ and thus (IV.100) from an expansion in a Neumann series as in (IV.57). This series is norm convergent since, by (IV.113),

$$\begin{aligned} \Big\| \, |H_0 - z|^{-1/2} W_g |H_0 - z|^{-1/2} \, \Big\| \;&\leq\; \left\| \frac{\widehat{H}_0}{H_0 - z} \right\| \cdot \left\| \widehat{H}_0^{-1/2} W_g \widehat{H}_0^{-1/2} \right\| \\ &\leq\; \frac{16\, C_0\, g^{\beta/(2+\beta)}}{\sqrt{1 - \cos\vartheta}} \;<\; 1\,, \end{aligned} \tag{IV.114}$$

assuming again a sufficiently small choice of $g > 0$. This completes the proof of Theorem IV.3. ∎

In physics, the number $1/\gamma_j$ in Theorem IV.3 is called the *life time* of $\psi_{j,\ell} \otimes \Omega$. This name refers to *Fermi's Golden Rule* which says that

$$\Big| \Big\langle \psi_{j,\ell} \otimes \Omega \;\Big|\; \mathrm{e}^{-itH_g}\, \psi_{j,\ell} \otimes \Omega \Big\rangle \Big| \;\sim\; \mathrm{e}^{-\gamma_j\, t}\,, \qquad \text{as } \; g \to 0\,, \tag{IV.115}$$

for t in an arbitrary, but fixed compact interval. In [20], one finds a mathematically rigorous framework for the derivation of such an estimate on the *dynamics* e^{-itH_g} as in (IV.115) from *spectral information* on $H_g(\theta)$, given in Theorem IV.3. The main idea is to use the integral representation

$$\mathrm{e}^{-itH_g} \;=\; \lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{\mathbb{R}} \mathrm{e}^{-it\lambda}\, \mathrm{Im} \left\{ (\lambda + i\varepsilon - H_g)^{-1} \right\} d\lambda\,, \tag{IV.116}$$

to convert matrix elements of e^{-itH_g} into those of $(H_g - \lambda - i\varepsilon)^{-1}$. By Theorem IV.2 we can continue the latter into the lower half plane and may deform the integration contour in an interval about E_j, shifting the path of integration close to $\mathrm{Im}\{\lambda\} = -\gamma_j$. Then we observe that the magnitude of $\mathrm{e}^{-it\lambda}$ is bounded by $\mathrm{e}^{-\gamma_j t}$ up to small corrections, indeed. We do not give details on the precise derivation of (IV.115).

V Effective Hamiltonian and Renormalization Group

The purpose of this chapter is to analyze the flow of effective Hamiltonians under renormalization transformations, lowering the photon energy scale. Our analysis exhibits the infrared asymptotic freedom of QED as described by the Hamiltonian H_g, described in (I.22). As an application of our methods, we shall prove the existence of a ground state, with an estimate on its multiplicity, and the existence of resonances as eigenvalues of the complex dilated Hamiltonian, $H_g(\theta)$ in (IV.21)).

This chapter is organized as follows. In Section V.1 we describe the general strategy of our renormalization group analysis. In Section V.2 we eliminate the electron degrees of freedom with the help of the isospectral Feshbach map. In Section V.3 we outline the proof of the fact that the renormalization map, applied to the effective model derived in Section V.2, is a contraction. From this we obtain information on the spectrum of $H_g(\theta)$. As we demonstrate in Section V.4, the fixed point of the renormalization map gives rise to an eigenvalue of $H_g(\theta)$, the resonance sought for.

V.1 The General Strategy of the Renormalization Group Construction

In this section, we describe the key ideas underlying our renormalization group construction of resonances for the Hamiltonian $H_g(\theta)$ considered in Eq. (IV.21), (with Hypotheses 2 and 3 of Section I.3 required throughout this chapter). Technical estimates will be supplied in subsequent sections.

We start from the effective Hamiltonian

$$\widetilde{H}_{\text{eff}}[z] + E_j - z := f_{P_0}\Big(H_g(\theta) - z\Big) , \tag{V.1}$$

where the r.h.s. is defined in (IV.56). Recall that

$$P_0 = P_{el,j} \otimes \chi_{H_f<\rho_0} , \tag{V.2}$$

where $P_{el,j}$ is the orthogonal projection onto the eigenspace of the atomic Hamiltonian H_{el} corresponding to the eigenvalue E_j, and $\chi_{H_f<\rho_0}$ is the spectral projection of H_f onto the subspace of vectors in Fock space with field energy $< \rho_0$; furthermore z belongs to the disk $D(E_j,\, \rho_0/2)$, i.e., $|z - E_j| \leq \rho_0/2$. For simplicity we assume that the eigenvalue E_j is *simple*, so that $P_{el,j}$ is a one-dimensional projection. Then we can view $\widetilde{H}_{\rm eff}[z]$ as an operator on the Hilbert space $\chi_{H_f<\rho_0}\mathcal{F}$.

We introduce some convenient notation. As in Section I.3, we define

$$k^{(m)} \,:=\, (k_1, \ldots, k_m) \,\in\, \mathbb{R}^{\,3m}\,, \qquad dk^{(m)} \,:=\, \textstyle\prod_{i=1}^m d^3k_i\,, \tag{V.3}$$

$$a^\dagger(k^{(m)}) \,:=\, \textstyle\prod_{i=1}^m a^\dagger(k_i)\,, \qquad \omega(k^{(m)}) \,:=\, \sum_{i=1}^m \omega(k_i)\,. \tag{V.4}$$

In Section V.2, we shall show that, for all $z \in D(E_j,\, \rho_0/2)$,

$$\widetilde{H}_{\rm eff}[z] \;=\; \chi_{H_f<\rho_0}\Big(\tilde{T}_{\rm eff}[H_f, z] \,+\, \widetilde{W}_{\rm eff}[z] \,+\, \tilde{E}_{\rm eff}[z]\cdot \mathbf{1}\Big)\chi_{H_f<\rho_0}\,, \tag{V.5}$$

for $\theta = i\vartheta$, with $\vartheta > 0$, and $\tilde{T}_{\rm eff}[r, z]$ is a function of $r \in [0, \rho_0]$ with

$$\Big|\tilde{T}_{\rm eff}[r, z] - r\Big| \;<\; O\left(g^2\rho_0^{-1}\right)\,, \tag{V.6}$$

$$\left|\frac{\partial}{\partial r}\tilde{T}_{\rm eff}[r, z] - 1\right| \;<\; O\left(g^2\rho_0^{-1}\right)\,, \tag{V.7}$$

$\tilde{E}_{\rm eff}[z]$ is a complex number with

$$\tilde{E}_{\rm eff}[z] \;=\; O\left(g^2\rho_0^{-1}\right), \tag{V.8}$$

and $\widetilde{W}_{\rm eff}[z]$ is a bounded operator on $\chi_{H_f<\rho_0}\mathcal{F}$ of the following form:

$$\widetilde{W}_{\rm eff}[z] \;=\; \sum_{M+N\geq 1} \widetilde{W}_{M,N}[z]\,, \tag{V.9}$$

where M and N are nonnegative integers, and

$$\widetilde{W}_{M,N}[z] \;=\; \int dk^{(m)} d\tilde{k}^{(N)}\, a^\dagger(k^{(M)})\, \tilde{w}_{M,N}\left[H_f; z; k^{(M)}, \tilde{k}^{(N)}\right]\, a(\tilde{k}^{(N)}). \tag{V.10}$$

Here $\tilde{w}_{M,N}\left[H_f; z; k^{(M)}, \tilde{k}^{(N)}\right]$ is an operator on $\chi_{H_f<\rho_0}\mathcal{F}$ whose norm (for $z \in D(E_j,\, \rho_0/2)$, $k^{(M)} \in \mathbb{R}^{\,3M}$, and $\tilde{k}^{(N)} \in \mathbb{R}^{\,3N}$) is bounded by

$$\left\|\tilde{w}_{M,N}\left[H_f; z; k^{(M)}, \tilde{k}^{(N)}\right]\right\| \;\leq\; \xi_0^{M+N} \prod_{i=1}^M |k_i|^{\frac{\mu-1}{2}} \prod_{j=1}^N |\tilde{k}_j|^{\frac{\mu-1}{2}}\,, \tag{V.11}$$

where ξ_0 is a constant depending on the coupling constant g, with $\xi_0 = O(g)$, and μ describes the strength of the *infrared singularity* in the interaction term $W_g := gW_1 + g^2W_2$ of $H_g(\theta)$; see (I.35), (I.36) and Hypothesis 3 in Section I.3.

Next, we rescale photon momenta by means of the unitary dilatation $U_f(-\ln \rho_0)$,

$$k \;\to\; \rho_0 k, \tag{V.12}$$

passing from $\widetilde{H}_{\rm eff}[z]$ to a unitarily equivalent Hamiltonian on the Hilbert space $\chi_{H_f<1}\mathcal{F}$. We set

$$H_{\rm eff}[z] \;:=\; \rho_0^{-1}\, U_f(-\ln \rho_0)\, \widetilde{H}_{\rm eff}[z]\, U_f(-\ln \rho_0)^*\,, \tag{V.13}$$

for all $z \in D(E_j,\, \rho_0/2)$. The effect of rescaling photon momenta, $k \to \rho_0 k$, and multiplying the resulting Hamiltonian by $1/\rho_0$ has been described in Section I.3 (see (I.39)–(I.41)). The main consequence of this operation is that the coefficients, $w_{M,N}$, of the rescaled perturbation still obey a bound of the form of inequality (V.11), *but with the constant* ξ_0 *replaced by* $\rho_0^{\frac{\mu}{2}}\xi_0$, (which is $< \xi_0$, for $\mu > 0$ and $\rho_0 < 1$; see Eq. (I.41) in Section I.3).

Following the photon momenta, we rescale the spectral parameter, as well, The operator $H_{(0)}[\hat{z}]$, defined by

$$H_{(0)}[\hat{z}] \;:=\; {\rm e}^{i\vartheta}\, H_{\rm eff}\left[\frac{{\rm e}^{-i\vartheta}\rho_0}{2}\hat{z} \,+\, E_j\right], \tag{V.14}$$

for $\hat{z} \in D_1 := \{|z| \leq 1\}$ (and hence $\frac{{\rm e}^{-i\vartheta}\rho_0}{2}\hat{z} + E_j \in D(E_j, \rho_0/2)$), belongs to a certain *Banach space*, V_μ, of Hamiltonians $H[z]$ on $\mathcal{H}_{\rm red} := \chi_{H_f<1}\mathcal{F}$ which we now define:

Elements of V_μ are analytic families of bounded operators, $H : D_1 \to \mathcal{B}[\mathcal{H}_{\rm red}]$, parametrized by vectors in the linear space

$$V'_\mu \;:=\; \mathbb{C} \,\oplus\, \mathcal{T} \,\oplus\, \mathcal{W}_\mu\,, \tag{V.15}$$

where $\mathcal{T}$ is the space of *Lipshitz-continuous* functions on the interval $[0,1]$ *vanishing* at the origin, and

$$\mathcal{W}_\mu \;:=\; \bigoplus_{M+N\geq 1} \mathcal{W}_\mu(M,N)\,, \tag{V.16}$$

where M and N are nonnegative integers, with $M+N \geq 1$, and

$$\mathcal{W}_\mu(M,N) \;:=\; \Big\{ F_{M,N} : [0,1] \times B_1^M \times B_1^N \to \mathbb{C} \Big\} \;, \tag{V.17}$$

with $B_r = \Big\{ k \in \mathbb{R}^3 \big|\; |k| \leq r \Big\}$. We denote elements of $\mathcal{W}_\mu$ by $\{w\} \equiv \{ w_{M,N} \in \mathcal{W}_{M,N} \,|\, M+N \geq 1 \}$. To give a meaning to $\partial_r w_{M,N}$, we assume $r \mapsto w_{M,N}\Big[r; k^{(M)}, \tilde{k}^{(N)}\Big]$ to be *Lipshitz continuous* on the interval $[0,1]$ a.e. $B_1^M \times B_1^N$.

To define a norm $\| \cdot \|^{(\infty,1)}$ on $\mathcal{W}_\mu$, we pick two additional constants $0 < \rho, \xi < 1$ and set

$$\|w_{M,N}\|^{(\infty,1)} \;:=\; \max\Big\{ \xi^{-(M+N)} \|w_{M,N}\|^{(\infty)} \;,\quad (4\pi\xi\rho^{1/2})^{-(M+N)} \|\partial_r w_{M,N}\|^{(1)} \Big\} \;, \tag{V.18}$$

where

$$\|w_{M,N}\|^{(\infty)} \;:=\; \sup_{[0,1]\times B_1^M \times B_1^N} \left\{ \Big| w_{M,N}\Big[r; k^{(M)}, \tilde{k}^{(N)}\Big] \Big| \prod_{i=1}^M |k_i|^{\frac{1-\mu}{2}} \prod_{j=1}^N |\tilde{k}_j|^{\frac{1-\mu}{2}} \right\} \tag{V.19}$$

and

$$\|\partial_r w_{M,N}\|^{(1)} \;:=\; \sup_{[0,1]} \left\{ \int \Big| \partial_r w_{M,N}\Big[r; k^{(M)}, \tilde{k}^{(N)}\Big] \Big| \prod_{i=1}^M \frac{d^3 k_i}{|k_i|^{\frac{3+\mu}{2}}} \prod_{j=1}^N \frac{d^3 \tilde{k}_j}{|k_j|^{\frac{3+\mu}{2}}} \right\} \;. \tag{V.20}$$

The linear space V'_μ is a Banach space with norm given by

$$\Big\| (E, T, \{w\}) \Big\|' \;:=\; \max\Big\{ \,|E|\,,\; \|T'\|_\infty \,,\; \max_{M+N\geq 1} \|w_{M,N}\|^{(\infty,1)} \Big\} \,, \tag{V.21}$$

for $(E, T, \{w\}) \in V'_\mu$.

To every element $(E, T, \{w\}) \in V'_\mu$ we assign the operator

$$H \;:=\; \chi_{H_f<1} \Big(T[H_f] \,+\, W \,+\, E \cdot \mathbf{1} \Big) \chi_{H_f<1} \tag{V.22}$$

on $\mathcal{H}_{\rm red} = \chi_{H_f<1}\mathcal{F}$, where $W = \sum_{M+N\geq 1} W_{M,N}$, with

$$W_{M,N} \;=\; \int dk^{(M)} d\tilde{k}^{(N)}\, a^\dagger(k^{(M)})\, w_{M,N}\Big[H_f; k^{(M)}, \tilde{k}^{(N)}\Big]\, a(\tilde{k}^{(N)}) \,. \tag{V.23}$$

The Banach space V_μ consists of analytic families of bounded operators, $H : D_1 \to \mathcal{B}[\mathcal{H}_{\rm red}]$, parametrized by elements $(E[z], T[z], \{w[z]\}) \in V'_\mu$ with the

property that

$$\Big\| H[\cdot] \Big\| \;:=\; \sup_{z\in D_1} \Big\| \big(E[z]\,,\, T[H_f;z]\,,\, \{w[z]\}\big) \Big\|' \;<\; \infty\,. \tag{V.24}$$

A ball, $\mathcal{B}_\varepsilon$, of operators in V_μ is defined by setting

$$\begin{aligned} \mathcal{B}_\varepsilon \;&=\; \Big\{ H[\,\cdot\,] \in V_\mu \;|\; \sup_{D_1} \|H[z] - H_f\|' \leq \varepsilon \Big\} \\ &=\; \Big\{ H[\,\cdot\,] \in V_\mu \;|\; \sup_{D_1} \|(E[z], T[r,z]-r, \{w[z]\})\|' \leq \varepsilon \Big\}\,. \end{aligned} \tag{V.25}$$

In the following, we define a *renormalization map*, $\mathcal{R}$, on the ball $\mathcal{B}_\varepsilon \subset V_\mu$, for some sufficiently small $\varepsilon > 0$ (and for a convenient choice of the parameter ξ and ρ introduced in (V.18)). The renormalization map $\mathcal{R}$ is the composition of three operations, Dec, Resc, and Comp, i.e.,

$$\mathcal{R} \;:=\; \mathrm{Comp} \circ \mathrm{Resc} \circ \mathrm{Dec} \tag{V.26}$$

which we now define.

V.1.1 Decimation of Degrees of Freedom

For

$$z \in \mathcal{U}^{(\mathrm{in})} := \Big\{ z \in D_1 \;\Big|\; |z - E[z]| \leq \rho/2 \Big\} \tag{V.27}$$

and $H \in \mathcal{B}_\varepsilon$, we define

$$\mathrm{Dec}(H[z]) - z \;:=\; f_{\chi_\rho}\Big(H[z] - z\Big)\chi_\rho \tag{V.28}$$

where $\chi_\rho \equiv \chi_\rho[H_f] := \chi_{H_f<\rho}$ and f_{χ_ρ} is the corresponding Feshbach map; (see Eq. (IV.4) and Theorem IV.1). Recall that

$$f_{\chi_\rho}\Big(H[z] - z\Big) \;=\; \chi_\rho \left(H[z] \;-\; H[z]\overline{\chi}_\rho (H_{\overline{\chi}_\rho}[z] - z)^{-1} \overline{\chi}_\rho H[z] \;-\; z \right) \chi_\rho\,, \tag{V.29}$$

where

$$H_{\overline{\chi}_\rho}[z] \;=\; \overline{\chi}_\rho\, H[z]\, \overline{\chi}_\rho\,, \tag{V.30}$$

and $\overline{\chi}_\rho = \mathbf{1} - \chi_\rho = \chi_{H_f \geq \rho}$. For $z \in \mathcal{U}^{(\mathrm{in})}$ the operator $(H_{\overline{\chi}_\rho}[z] - z)^{-1}$ is a bounded operator on $\overline{\chi}_\rho\, \chi_{H_f<1}\mathcal{F} = \chi_{\rho \leq H_f < 1}\mathcal{F}$, and the Feshbach map f_{χ_ρ} is well defined.

We study the form of $f_{\chi_\rho}\Big(H[z]-z\Big)$ by expanding $\overline{\chi}_\rho(H_{\overline{\chi}_\rho}[z]-z)^{-1}\overline{\chi}_\rho$ in a *Neumann series* in the "perturbation" $W[z]$ of $T[H_f;z]$ defined in Eqs. (V.22) and (V.23). Because all states in $\chi_{\rho\leq H_f<1}\mathcal{F}$ have a field energy between ρ and 1, and since z has been assumed to be contained in $\mathcal{U}^{(\mathrm{in})}$, we have on $\chi_{H_f\geq\rho}\mathcal{F}$ that

$$|\,T[H_f;z]+E[z]-z\,|\;\geq\;|\,T[H_f;z]\,|\;-\;(\rho/2)\;\geq\;\frac{1-2\varepsilon}{4}\Big(H_f+\rho\Big)\,. \qquad \text{(V.31)}$$

Hence, this Neumann series is *norm-convergent*. Using the *"Pull-Through" formulae*

$$f[H_f]\,a^\dagger(k) \;=\; a^\dagger(k)\,f[H_f+\omega(k)]\,, \qquad \text{(V.32)}$$

$$a(k)\,f[H_f] \;=\; f[H_f+\omega(k)]\,a(k)\,, \qquad \text{(V.33)}$$

where $f[r]=O(r)$ is a borel function on $[0,\infty]$, and additionally using Wick's theorem, we can rewrite all terms in the Neumann series expansion of $\overline{\chi}_\rho(H_{\overline{\chi}_\rho}[z]-z)^{-1}\overline{\chi}_\rho$ in a Wick-ordered *normal form*; (see Subsect. V.2, Eqs. (V.49)–(V.59) and Sect. V.3); the term $\chi_\rho H[z]\chi_\rho$ in the definition of $f_{\chi_\rho}\Big(H[z]-z\Big)$ already has the desired normal form. This concludes our sketch of the operation Dec, and we proceed to defining the operations Resc and Comp.

V.1.2 Rescaling of the Effective Hamiltonian and its Spectral Parameter

Secondly, we rescale the photon momenta,

$$k\;\to\;\rho k\,, \qquad \text{(V.34)}$$

which defines a unitary map, denoted by $U_f(-\ln\rho)$, from $\chi_{H_f<\rho}\mathcal{F}$ onto $\chi_{H_f<1}\mathcal{F}$. We then define

$$\mathrm{Resc}\Big(\mathrm{Dec}\Big(H[z]\Big)\Big)\;:=\;\rho^{-1}\,U_f(-\ln\rho)\,\mathrm{Dec}\Big(H[z]\Big)\,U_f(-\ln\rho)^*\,. \qquad \text{(V.35)}$$

Finally, we rescale the spectral parameter z by means of

$$\widehat{Z}:\mathcal{U}^{(\mathrm{in})}\;\to\;D_1\,,\quad z\;\mapsto\;\frac{2}{\rho}\Big(z-E[z]\Big)\,, \qquad \text{(V.36)}$$

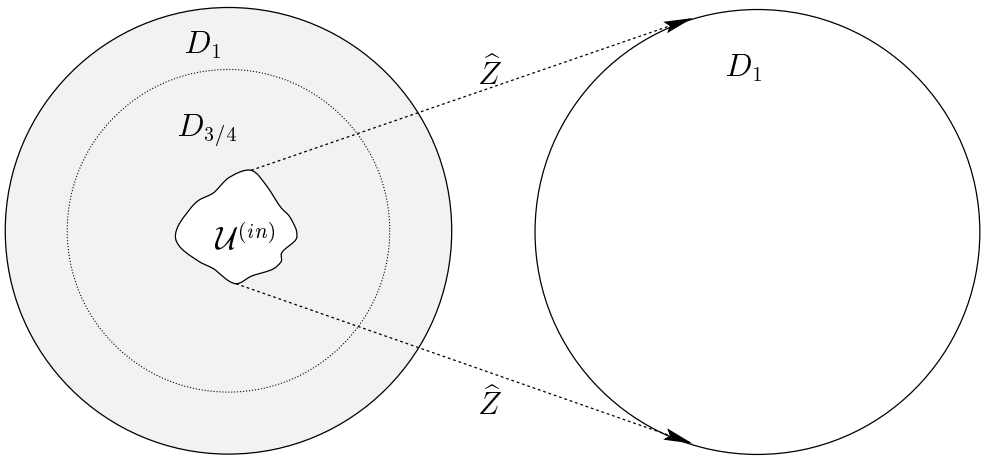

Figure V.1: Rescaling of the Spectral Parameter

(see Figure V.1).

We note that $H \in \mathcal{B}_\varepsilon$ and $z \in \mathcal{U}^{(\mathrm{in})}$ implies that $|z| \leq \rho/2 + |E[z]| \leq 1/2 + \varepsilon$ and thus

$$\mathcal{U}^{(\mathrm{in})} \subseteq D_{3/4}\,, \tag{V.37}$$

for $\varepsilon \leq 1/4$. Moreover, $\widehat{Z}$ is a *bijection*, as follows from (V.37) and *Cauchy's estimate* which yields

$$\sup_{\mathcal{U}^{(\mathrm{in})}} \Big| \partial_z E[z] \Big| \;\leq\; 4 \sup_{D_1} \Big| E[z] \Big| \;\leq\; 4\varepsilon \;<\; 1/2\,, \tag{V.38}$$

for $\varepsilon < 1/8$. Actually, the same estimate applied to the other coefficients of H implies that

$$\sup_{\mathcal{U}^{(\mathrm{in})}} \Big\| \left(\partial_z E[z],\, \partial_z T[\cdot\,; z],\, \{\partial_z w[z]\}\right) \Big\|' \;\leq\; 4\varepsilon\,. \tag{V.39}$$

Using the inverse map $\widehat{Z}^{-1} : D_1 \to \mathcal{U}^{(\mathrm{in})}$, we finally define the renormalization map, $\mathcal{R}$, to be

$$\mathcal{R}\Big(H\Big)[\hat{z}] \;:=\; \mathrm{Comp} \circ \mathrm{Resc} \circ \mathrm{Dec}\Big(H\Big)[\hat{z}]\,, \;:=\; \mathrm{Resc} \circ \mathrm{Dec}\left(H\left[\widehat{Z}^{-1}[\hat{z}]\right]\right)\,. \tag{V.40}$$

Note that $\mathcal{R}$ is isospectral in the sense that

$$0 \in \sigma\Big\{\, \mathcal{R}\Big(H\Big)\Big[\widehat{Z}[z]\Big] \;-\; \widehat{Z}[z] \,\Big\} \;\Leftrightarrow\; 0 \in \sigma\Big\{\, H[z] \;-\; z \,\Big\}\,, \tag{V.41}$$

provided we can justify the application of the isospectral Feshbach map f_{χ_ρ} to define Dec and thus $\mathcal{R}$.

The punchline of our analysis is to show that, for sufficiently small $\varepsilon > 0$, the non-linear map $\mathcal{R} : H \mapsto \mathcal{R}(H)$ defined in (V.40) is well-defined on $\mathcal{B}_\varepsilon$, and the image of $\mathcal{B}_\varepsilon$ is contained in the $\mathcal{B}_\varepsilon$, again; (see Section V.3). In fact, we will show that our assumption that $\mu > 0$ implies that $\mathcal{R} : \mathcal{B}_\varepsilon \to \mathcal{B}_\varepsilon$ is a contraction. It is quite easy to describe the geometrical properties of $\mathcal{R}$.

(1) The one-dimensional subspace $\mathbb{C}$ of V'_μ on the right side of Eq. (V.15) determines a one-dimensional, *unstable* manifold $\mathcal{U} \subset V_\mu$ for $\mathcal{R}$,

$$\mathcal{R}(H[\cdot] = E[\cdot] \cdot \mathbf{1})[\hat{z}] \;=\; \frac{1}{\rho} E\left[\hat{Z}^{-1}(\hat{z})\right] . \tag{V.42}$$

The expansion rate of $\mathcal{R}$ along $\mathcal{U}$ is $1/\rho > 1$.

(2) The one-dimensional subspace

$$\{T\,[H_f; \hat{z}] \;=\; \lambda \cdot H_f \mid \lambda \in \mathbb{C}\}\,, \tag{V.43}$$

for $\hat{z} \in D_1$, is a manifold, $\mathcal{M}$, of fixed points of $\mathcal{R}$:

$$R(\lambda H_f)[\hat{z}] \;=\; \lambda H_f\,. \tag{V.44}$$

(3) The piece of the stable manifold $\mathcal{S}$ of $\mathcal{R}$ contained in the ball $\mathcal{B}_\alpha$ has codimension 2 (see Figure V.2). The tangent space to $\mathcal{S}$ at a point contained in $\mathcal{S} \cap \mathcal{M} \cap \mathcal{B}_\alpha$ is given by analytic families of operators with values in the space $\mathcal{W}_\mu$ introducted in (V.15) and (V.16). The definition (V.35) of Resc and the simple calculations in Section I.3 (Eqs. (I.40) and (I.41)) show that the contraction rate of $\mathcal{R}$ along $\mathcal{S}$ near $\mathcal{M} \cap \mathcal{B}_\alpha$ is given by $\rho^{\mu/2} < 1$; (recall that $\mu > 0$ and $\rho < 1$).

The significance of the map $\mathcal{R}$ in the study of the spectrum of the operator $H_g(\theta)$ can be understood by recalling Theorem IV.1, Eq. (IV.8), from Section IV.1: On the disk $D(E_j,\, \rho_0/2)$, z belongs to the spectrum of the Hamiltonian $H_g(\theta)$ if and only if $0 \in \sigma(\widetilde{H}_{\text{eff}}[z])$. The spectrum of $\widetilde{H}_{\text{eff}}[z]$, for $z \in D(E_j, \rho_0/2)$, in turn, is determined by the spectrum of $H_{(0)}[z]$, inside the disk D_1.

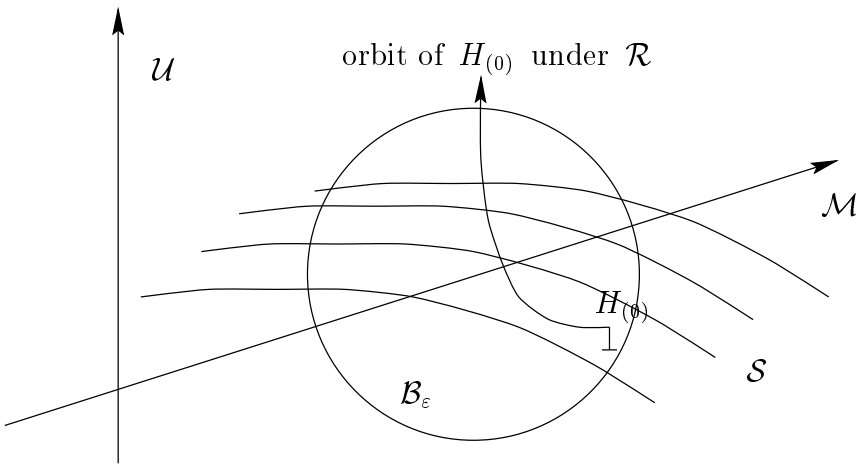

Figure V.2: Orbit under $\mathcal{R}$ starting at H_0

We have noted that $H_{(0)}$ belongs to the Banach space V_μ defined in (V.15)–(V.24). In fact, it belongs to the ball $\mathcal{B}_\varepsilon$ defined in (V.25). By definition of the renormalization map $\mathcal{R}$, the spectrum of $H_{(0)}[z]$ inside the deformed disk

$$\mathcal{U}_{(0)}^{(\mathrm{in})} \;:=\; \Big\{ z \in D_1 \,\Big|\, |z - E_{(0)}[z]| \leq \rho/2 \Big\}\,, \tag{V.45}$$

where

$$E_{(0)}[z] \;:=\; \frac{\mathrm{e}^{i\vartheta}}{\rho_0}\, \widetilde{E}_{\mathrm{eff}} \left[\frac{\mathrm{e}^{i\vartheta}\rho}{2} z \,+\, E_j \right] \tag{V.46}$$

coincides with the spectrum of $\mathcal{R}\Big(H_{(0)}\Big)[\hat{z}]$ inside the disk D_1.

V.2 The Effective Hamiltonian at Energy Scale ρ_0

To motivate the abstract renormalization group approach, we return to the effective Hamiltonian $f_{P_0}(H_g(\theta) - z)$ investigated in the preceeding chapter in (IV.56). For simplicity, we assume $P_{el,j} = |\varphi_{el,j}\rangle\langle\varphi_{el,j}|$ to be one-dimensional, so that $f_{P_0}\Big(H_g - z\Big)$ (again, we do not display the dependence on θ) can be viewed as an operator on $\chi_{H_f<\rho_0}\mathcal{F}$. Recall that

$$\begin{aligned} f_{P_0}\Big(H_g - z\Big) \;&=\; \chi_{H_f<\rho_0}\Big(E_j - z + \mathrm{e}^{-i\vartheta}H_f\Big) \\ &\quad + \chi_{H_f<\rho_0}\Big\langle W_g \;-\; W_g\overline{P}_0\Big(\overline{P}_0 H_g \overline{P}_0 - z\Big)^{-1}\overline{P}_0 W_g\Big\rangle_{el,j}\chi_{H_f<\rho_0}\,, \end{aligned} \tag{V.47}$$

where $W_g = gW_1 + g^2W_2$, omitting the trivial factor $P_{el,j}$. Here, we project an operator $A \in \mathcal{B}(H_{el} \otimes F)$ onto $\langle A \rangle_{el,j} \in \mathcal{B}(F)$ by means of

$$P_{el,j} \otimes \langle A \rangle_{el,j} \;:=\; (P_{el,j} \otimes \mathbf{1})\; A\; (P_{el,j} \otimes \mathbf{1})\;. \tag{V.48}$$

Restricting to $1 \leq m+n \leq 2$, we make use of the notation (V.3)–(V.4) in the following definition.

$$W_{m,n} \;:=\; \int G_{m,n}\left(k^{(m)}, \tilde{k}^{(n)}\right) \otimes a^\dagger(k^{(m)}) a(\tilde{k}^{(n)})\, dk^{(m)}\, d\tilde{k}^{(n)}\;. \tag{V.49}$$

V.2.1 Decimating and Rescaling of the effective Hamiltonian, $f_{P_0}(H_g - z)$

Next, we may expand $f_{P_0}\Big(H_g - z\Big)$ in a Neumann series

$$\begin{aligned} f_{P_0}\Big(H_g - z\Big) \;=\; & \chi_{H_f<\rho_0}\Big(E_j - z + \mathrm{e}^{-i\vartheta} H_f\Big) \;-\; \sum_{L=1}^{\infty} (-1)^L \\ & \sum_{\substack{m_l+n_l=1 \\ l=1,\ldots,L}}^{2} g^{\left(\sum_{l=1}^{L} m_l+n_l\right)}\, \chi_{H_f<\rho_0} \left\langle W_{m_1,n_1} \overline{R}_0[H_f] \cdots \overline{R}_0[H_f] W_{m_L,n_L} \right\rangle_{el,j} \chi_{H_f<\rho_0}\;, \end{aligned} \tag{V.50}$$

where,

$$\begin{aligned} \overline{R}_0[H_f] \;&:=\; \Big(H_{el} \otimes \mathbf{1} + \mathrm{e}^{-i\vartheta} \mathbf{1} \otimes H_f - z\Big)^{-1} \overline{P}_0[H_f]\,, \\ P_0[H_f] \;&:=\; P_{el,j} \otimes \chi_{H_f<\rho_0}\;. \end{aligned} \tag{V.51}$$

We then convert $f_{P_0}\Big(H_g - z\Big)$ into a Wick-ordered "normal" form, moving all creation operators $a^\dagger(k)$ to the left of functions of H_f, and moving all annihilation operators $a(\tilde{k})$ to the right of functions of H_f. This amounts to determining coefficients

$$\tilde{w}_{M,N} : \mathbb{R}^+ \times D(E_j, \rho_0/2) \times B_{\rho_0}^{3M} \times B_{\rho_0}^{3N} \;\to\; \mathbb{C}\;, \tag{V.52}$$

for all $M+N \geq 0$, such that, for $z \in D(E_j,\, \rho_0/2) = \{|E_j - z| \leq \rho_0/2\}$,

$$f_{P_0}\Big(H_g - z\Big) \;=\; \widetilde{H}_{\text{eff}}[z] + E_j - z\;, \tag{V.53}$$

where

$$\widetilde{H}_{\text{eff}}[z] \;=:\; \chi_{\rho_0} \left(\widetilde{T}_{\text{eff}}[H_f; z] \;+\; \widetilde{W}_{\text{eff}}[z] \;+\; \widetilde{E}_{\text{eff}}[z] \right) \chi_{\rho_0}\;, \tag{V.54}$$

$$\widetilde{E}_{\rm eff}[z] \;=\; \Delta\tilde{w}_{0,0}[0;z]\,, \tag{V.55}$$

$$\widetilde{T}_{\rm eff}[H_f;z] \;=\; T[H_f;z] \,+\, \Delta\tilde{w}_{0,0}[H_f;z] \,-\, \Delta\tilde{w}_{0,0}[0;z]\,, \tag{V.56}$$

$$\Delta\tilde{w}_{0,0}[s;z] \;:=\; {\rm e}^{i\vartheta}\,\Big\langle\, f_{\chi_\rho}\Big(H_g + {\rm e}^{-i\vartheta}s - z\Big) \,-\, E_j + z\,\Big\rangle_\Omega\,, \tag{V.57}$$

and

$$\widetilde{W}_{\rm eff}[z] \;=\; \sum_{M+N\geq 1} \widetilde{W}_{M,N}[z]\,, \tag{V.58}$$

with

$$\widetilde{W}_{M,N}[z] \;=\; \int dk^{(M)} d\tilde{k}^{(N)}\; a^\dagger(k^{(M)})\,\tilde{w}_{M,N}\left[H_f;z,k^{(M)},\tilde{k}^{(N)}\right] a(\tilde{k}^{(N)})\,. \tag{V.59}$$

Here $\tilde{w}_{M,N}[H_f;z;k^{(M)},\tilde{k}^{(N)}]$ defines an operator on $\chi_{H_f<\rho_0}\mathcal{F}$, by the functional calculus for H_f, for each point $\Big(z;k^{(M)};\tilde{k}^{(N)}\Big)$.

Our next goal is to show that if Hypotheses 2 and 3 hold then, for sufficiently small $g\rho_0^{-1/2} > 0$ and $M+N \geq 0$,

$$\begin{aligned}&\Big|\tilde{w}_{M,N}\left[r\,,z\,,k^{(M)}\,,\tilde{k}^{(N)}\right]\Big| \\ &\qquad\leq\; \left(\frac{C\,g}{\rho_0^{1/2}}\right)^{M+N+2\delta_{(M+N),0}}\;\rho_0^{1-\frac{1}{2}(M+N)}\cdot\prod_{j=1}^{M}|k_j|^{\frac{\mu-1}{2}}\prod_{j=1}^{N}|\tilde{k}_j|^{\frac{\mu-1}{2}}\end{aligned} \tag{V.60}$$

(see Lemma V.3) and to verify that, after passing from $\tilde{w}_{M,N}$ to rescaled coupling functions, $w_{M,N}$, this implies that, for $M+N\geq 1$ and $0<\rho\leq 1$,

$$\begin{aligned}&\Big\|(H_f+\rho)^{-1/2}\,\chi_{H_f<1}\,\widetilde{W}_{M,N}\,\chi_{H_f<1}\,(H_f+\rho)^{-1/2}\Big\| \\ &\qquad\leq\; \left(\frac{Cg}{\rho_0^{1/2}}\right)^{\frac{1}{2}(M+N+1)}\left(\frac{\rho_0^{1+\mu}}{\rho}\right)^{\frac{1}{2}(M+N)}\end{aligned} \tag{V.61}$$

(see Corollary V.4). To this end, we first determine the exact form of $\tilde{w}_{M,N}$ by a Wick-ordering procedure.

Lemma V.1 *For $g>0$ sufficiently small, the coefficients, $\tilde{w}_{M,N}$, defined through (V.54) – (V.59) are given by*

$$\begin{aligned}\tilde{w}_{M,N}\left[r;\,z;\,k^{(M)};\,\tilde{k}^{(N)}\right] \;=\; &\sum_{L=1}^{\infty}(-1)^{L-1}\sum_{\substack{m_l+p_l+n_l+q_l=1,2;\\ l=1,\ldots,L}}\delta_{\sum_{l=1}^L m_l\,,M}\;\delta_{\sum_{l=1}^L n_l\,,N} \\ &\prod_{l=1}^{L}\left\{\binom{m_l+p_l}{p_l}\binom{n_l+q_l}{q_l}\right\}\Big\{\widetilde{D}_L\Big[r;\{W^{m_l,n_l}_{p_l,q_l};k_l^{(m_l)};\tilde{k}_l^{(n_l)}\}_{l=1}^L\Big]\Big\}^{symm}_{M,N}\,,\end{aligned} \tag{V.62}$$

where

$$\widetilde{D}_L\left[r;\,\{W^{m_l,n_l}_{p_l,q_l};k^{(m_l)}_l;\tilde{k}^{(n_l)}_l\}^L_{l=1}\right]\;:= \tag{V.63}$$
$$\mathrm{e}^{i\vartheta}\prod_{l=1}^{L}(-\lambda)^{m_l+p_l+n_l+q_l}\cdot\Big\langle\varphi_{el}\otimes\Omega\,\Big|\,W^{m_1,n_1}_{p_1,q_1}\left[k^{(m_1)}_1;\tilde{k}^{(n_1)}_1\right]\,\overline{R}_0\left[H_f+\mu_1\right]\cdots$$
$$\cdots\,\overline{R}_0\left[H_f+\mu_{L-1}\right]\,W^{m_L,n_L}_{p_L,q_L}\left[k^{(m_L)}_L;\tilde{k}^{(n_L)}_L\right]\,\varphi_{el}\otimes\Omega\Big\rangle\;,$$

$$W^{m_l,n_l}_{p_l,q_l}\left[k^{(m_l)}_l;\tilde{k}^{(n_l)}_l\right]\;:= \tag{V.64}$$
$$\int dx^{(p_l)}_l d\tilde{x}^{(q_l)}_l G_{m_l+p_l,n_l+q_l}\left[k^{(m_l)}_l,x^{(p_l)}_l\,;\,\tilde{k}^{(n_l)}_l,\tilde{x}^{(q_l)}_l\right]\otimes a^\dagger(x^{(p_l)}_l)a(\tilde{x}^{(q_l)}_l),$$

$$\mu_l\;:=\;r\;+\sum_{j=1}^{l}\omega(\tilde{k}^{(n_j)}_j)+\sum_{j=l+1}^{L}\omega(k^{(m_j)}_j)\,, \tag{V.65}$$

$\overline{R}_0$ *is defined in (V.51), and*

$$\left\{A[k^{(M)};\tilde{k}^{(N)}]\right\}^{symm}_{M,N}\;:=\;\frac{1}{M!\,N!}\sum_{\pi\in\mathcal{S}_M}\sum_{\sigma\in\mathcal{S}_N}A[k_{\pi(1)},\ldots,k_{\pi(M)}\,,\;\tilde{k}_{\sigma(1)},\ldots,\tilde{k}_{\sigma(N)}]\,. \tag{V.66}$$

Proof: First, by (IV.57) and (IV.58), the Neumann series (V.50) is absolutely convergent in norm, provided g is sufficiently small (see (IV.58)). Next we use the identity

$$W_{M_1,N_1}\,\overline{R}_0[H_f]\,W_{M_2,N_2}\,\overline{R}_0[H_f]\cdots\overline{R}_0[H_f]\,W_{M_L,N_L}\;= \tag{V.67}$$
$$\sum_{m_1=0}^{M_1}\sum_{n_1=0}^{N_1}\cdots\sum_{m_L=0}^{M_L}\sum_{n_L=0}^{N_L}\int\prod_{l=1}^{L}\left\{dk^{(M_l-m_l)}_l d\tilde{k}^{(N_l-n_l)}_l\binom{M_l}{m_l}\binom{N_l}{n_l}\right\}$$
$$a^\dagger(k^{(m)})\left\{\,D_L\left[H_f\,;\,\{W^{m_l,n_l}_{M_l-m_l,N_l-n_l};k^{(m_l)}_l;\tilde{k}^{(n_l)}_l\}^L_{l=1}\right]\right\}^{symm}_{m,n}\;a(\tilde{k}^{(n)})\,,$$

which follows from the Pull-Through formula

$$f[H_f]\,a^\dagger(k)\;=\;a^\dagger(k)\,f[H_f+\omega(k)]\,, \tag{V.68}$$
$$a(k)\,f[H_f]\;=\;f[H_f+\omega(k)]\,a(k)\,, \tag{V.69}$$

and Wick's Theorem

$$\prod_{j=1}^{N}a^{\sigma_j}(k_j)\;=\;\sum_{Q\subseteq\{1,\ldots,N\}}\left\langle\prod_{j\in\{1,\ldots,N\}\setminus Q}a^{\sigma_j}(k_j)\right\rangle_\Omega\;\vdots\prod_{j\in Q}a^{\sigma_j}(k_j)\vdots\,, \tag{V.70}$$

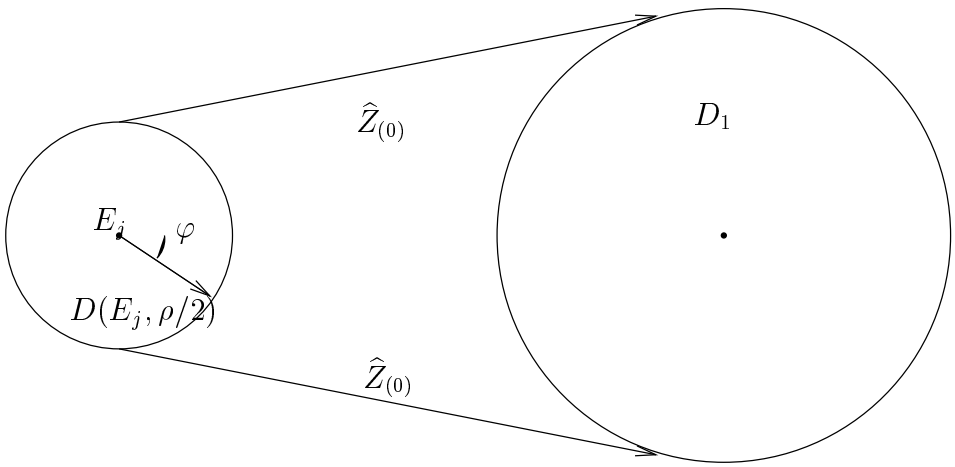

Figure V.3: Rescaling of the Spectral Parameter

where $a^+ := a^\dagger$, $a^- := a$, and

$$:\prod_{j=1}^{N} a^{\sigma_j}(k_j):\ :=\ \prod_{\substack{j=1,\\ \sigma_j=+}}^{N} a^\dagger(k_j)\ \prod_{\substack{j=1,\\ \sigma_j=-}}^{N} a(k_j) \tag{V.71}$$

(see [5] for details). From (V.67) we can directly read off the coefficients $\tilde{w}_{M,N}$.∎

Next we rescale our effective Hamiltonian $\widetilde{H}_{\text{eff}}$, by means of the unitary dilatation $U_f(\tau_0)$, where $\tau_0 := -\ln \rho_0 \in \mathbb{R}$. It is easily verified that

$$U_f(\tau_0)\, H_f\, U_f(\tau_0)^* \ =\ \rho_0\, H_f\ , \tag{V.72}$$

$$U_f(\tau_0)\, \chi_{H_f<\rho_0}\, U_f(\tau_0)^* \ =\ \chi_{H_f<1}\ , \tag{V.73}$$

$$U_f(\tau_0)\left(\int dk^{(M)} d\tilde{k}^{(N)}\, a^\dagger(k^{(M)})\tilde{w}_{(M,N)}\left[H_f; z; k^{(M)}; \tilde{k}^{(N)}\right] a(\tilde{k}^{(N)})\right) U_f(\tau_0)^* \tag{V.74}$$

$$=\ \int dk^{(M)}\, d\tilde{k}^{(N)}\ a^\dagger(k^{(M)})\left\{\rho_0^{\frac{3}{2}(M+N)}\tilde{w}_{M,N}\left[\rho_0 H_f;\ z;\ \rho_0 k^{(M)};\ \rho_0 \tilde{k}^{(N)}\right]\right\}\ a(\tilde{k}^{(N)})\ .$$

As a result, we obtain $H_{\text{eff}}[z] = \rho_0^{-1} U_f(\tau_0)\, \widetilde{H}_{\text{eff}}[z]\, U_f(\tau_0)^*$.

Furthermore, we define a rescaled spectral parameter (see Figure V.3), using the bijection

$$\widehat{Z}_{(0)} : D(E_j,\ \rho_0/2)\ \to\ D_1\ , \qquad z \mapsto \frac{2\mathrm{e}^{i\vartheta}}{\rho_0}(z - E_j)\ . \tag{V.75}$$

Now, we pass from $H_{\text{eff}}(z)$ to a new analytic family of rescaled Hamiltonians, $H_{(0)} : D_1 \to \mathcal{B}[\chi_{H_f<1}\mathcal{F}]$, by setting, for each $\hat{z} \in D_1$,

$$H_{(0)}[\hat{z}]\ :=\ \mathrm{e}^{i\vartheta}\, H_{\text{eff}}\left[\widehat{Z}_{(0)}^{-1}[\hat{z}]\right]\ =\ \frac{\mathrm{e}^{i\vartheta}}{\rho_0}\, U_f\,(\tau_0)\ \widetilde{H}_{\text{eff}}\left[\widehat{Z}_{(0)}^{-1}[\hat{z}]\right] U_f\,(\tau_0)^*\ . \tag{V.76}$$

By comparing (V.76) to (V.54) – (V.59), we read off the form of $H_{(0)}[\hat{z}]$, for each $\hat{z} \in D_1$:

$$H_{(0)}[\hat{z}] \;=\; \chi_{H_f<1}\Big(\,T_{(0)}[H_f;\hat{z}] \;+\; W_{(0)}[\hat{z}] \;+\; E_{(0)}[\hat{z}]\,\Big)\,\chi_{H_f<1}\,, \tag{V.77}$$

where

$$\begin{aligned}
T_{(0)}[H_f;\hat{z}] \;&=\; H_f \;+ \qquad\qquad\qquad\

where $\Gamma(x+1) = x\,\Gamma(x)$ *is the Gamma function,* C_μ *is some* μ*-dependent constant, and*

$$W_{M,N} \;:=\; \int dk^{(M)} d\tilde{k}^{(N)}\; a^\dagger(k^{(M)})\, w_{M,N}\left[H_f; k^{(M)}, \tilde{k}^{(N)}\right]\, a(\tilde{k}^{(N)})\,. \tag{V.85}$$

Proof: We pick $\phi = \chi_{H_f<1}\phi$, $\psi = \chi_{H_f<1}\psi \in \chi_{H_f<1}\mathcal{F}$ and consider

$$\begin{aligned} A^2(\phi,\psi) \;&:=\; \left|\left\langle \phi \;\middle|\; \chi_{H_f<1}\, W_{M,N}\, \chi_{H_f<1}\,\psi \right\rangle\right|^2 \qquad\qquad (V.86)\\ &=\; \left|\int dk^{(M)} d\tilde{k}^{(N)}\; \left\langle a(k^{(M)})\phi \;\middle|\; w_{M,N}\left[H_f; k^{(M)}; \tilde{k}^{(N)}\right]\, a(\tilde{k}^{(N)})\psi \right\rangle\right|^2 . \end{aligned}$$

Remembering $a(k^{(M)}) \equiv \prod_{j=1}^M a(k_j)$ and $\omega(k^{(M)}) \equiv \sum_{j=1}^M \omega(k_j)$, the Pull-Through formulae (IV.66)–(IV.67) imply that

$$\begin{aligned} a(k^{(M)})\chi_{H_f<1} \;&=\; \chi[H_f + \omega(k^{(M)}) < 1]\; a(k^{(M)})\; \chi_{H_f<1} \qquad\qquad (V.87)\\ &=\; \chi[\omega(k^{(M)}) < 1]\; a(k^{(M)})\; \chi_{H_f<1}\,, \end{aligned}$$

which, together with Schwarz' inequality and $\chi_{r<1} =: \chi_1[r]$, yields

$$\begin{aligned} &A^2(\phi,\psi) \;\le\; B^{(M)}(\phi)\cdot B^{(N)}(\psi) \qquad\qquad (V.88)\\ &\cdot \int \chi_1[\omega(k^{(M)})]\,\chi_1[\omega(\tilde{k}^{(N)})] \left\{ \sup_{0<r<1} \left| w_{M,N}\left[r; k^{(M)}; \tilde{k}^{(N)}\right]\right|^2 \right\} \prod_{j=1}^M \frac{dk_j}{\omega(k_j)} \prod_{j=1}^N \frac{d\tilde{k}_j}{\tilde{\omega}(k_j)}, \end{aligned}$$

where $B^{(0)}(\phi) := \|\phi\|^2$ and

$$B^{(M)}(\phi) \;:=\; \int \left\| a(k^{(M)})\chi_1\phi \right\|^2 \prod_{j=1}^M \omega(k_j) dk_j\,. \tag{V.89}$$

Another application of the Pull-Through formulae (IV.66)–(IV.67) then gives

$$\begin{aligned} B^{(M)}(\phi) \;&=\; \int \left\langle \prod_{j=1}^{M-1} a(k_j)\chi_1\phi \right| H_f \prod_{j=1}^{M-1} a(k_j)\chi_1\phi \right\rangle \prod_{j=1}^{M-1} \omega(k_j) dk_j \qquad\qquad (V.90)\\ &\le\; \int \left\langle \prod_{j=1}^{M-1} a(k_j)\chi_1\phi \right| \left(H_f + \omega(k^{(M-1)})\right) \prod_{j=1}^{M-1} a(k_j)\chi_1\phi \right\rangle \prod_{j=1}^{M-1} \omega(k_j) dk_j\\ &=\; \int \left\| a(k^{(M-1)})\chi_1\, H_f^{1/2}\phi \right\|^2 \prod_{j=1}^{M-1} \omega(k_j) dk_j \;=\; B^{(M-1)}(H_f^{1/2}\phi)\\ &\le\; B^{(M-2)}(H_f\phi) \;\le\; \ldots \;\le\; B^{(0)}(H_f^{M/2}\,\chi_{H_f<1}\,\phi) \;=\; \left\| H_f^{M/2}\,\chi_{H_f<1}\,\phi \right\|^2 , \end{aligned}$$

denoting $\chi_1\phi := \chi_{H_f<1}\phi$. Inserting the assumption on $w_{M,N}$, we estimate the integral on the right side of (V.88) by

$$\begin{aligned}
&\int \chi_1[\omega(k^{(M)})]\,\chi_1[\omega(\tilde{k}^{(N)})]\,\left\{\sup_{0<r<1}\left|w_{M,N}\left[r;k^{(M)};\tilde{k}^{(N)}\right]\right|^2\right\}\,\prod_{j=1}^{M}\frac{dk_j}{\omega(k_j)}\,\prod_{j=1}^{N}\frac{d\tilde{k}_j}{\omega(\tilde{k}_j)} \\
&\quad\leq\ \alpha^2\left(\int \chi_1[\omega(k^{(M)})]\,\prod_{j=1}^{M}\frac{dk_j}{\omega(k_j)^{2-\mu}}\right)\cdot\left(\int \chi_1[\omega(\tilde{k}^{(N)})]\,\prod_{j=1}^{N}\frac{d\tilde{k}_j}{\omega(\tilde{k}_j)^{2-\mu}}\right) \\
&\quad\leq\ \alpha^2\left(\frac{4\pi}{3}\right)^{M+N}\cdot\left(\int \chi[\omega_1+\cdots+\omega_M<1]\,\prod_{j=1}^{M}\frac{d\omega_j}{\omega_j^{-\mu}}\right) \qquad \text{(V.91)} \\
&\qquad\qquad\cdot\left(\int \chi[\omega_1+\cdots+\omega_N<1]\,\prod_{j=1}^{N}\frac{d\omega_j}{\omega_j^{-\mu}}\right) \\
&\quad=\ \frac{\alpha^2\,((4\pi/3)\,\Gamma[\mu+1])^{M+N}}{\Gamma[(\mu+1)M+1]\,\Gamma[(\mu+1)N+1]}\,. \qquad \text{(V.92)}
\end{aligned}$$

Here, $\Gamma[x]$ denotes the Gamma function for $x>0$ and the last equality is derived in [5]. From inserting (V.91) and (V.90) into (V.88), we obtain

$$\begin{aligned}
&\left\|(H_f+\rho)^{-M/2}\,\chi_{H_f<1}\,W_{M,N}\,\chi_{H_f<1}\,(H_f+\rho)^{-N/2}\right\| \\
&\qquad\leq\ \frac{\alpha^2\,((4\pi/3)\,\Gamma[\mu+1])^{\frac{1}{2}(M+N)}}{\Gamma[(\mu+1)M+1]^{1/2}\,\Gamma[(\mu+1)N+1]^{1/2}}\,.
\end{aligned}$$

From here, the assertion follows trivially from

$$\left\|\frac{[H_f+\rho]^{1/2}}{[H_f+\rho]^{M/2}}\,\chi_{H_f<1}\right\|\ =\ \sup_{0\leq r<1}\left\{(r+\rho)^{(1-M)/2}\right\}\ \leq\ \sqrt{2}\,\rho^{-\delta_{M0}/2}\,. \blacksquare \qquad \text{(V.93)}$$

To adapt Lemma V.2 to our present situation, we now show that the effective interaction in (V.58), (V.59) satisfies its hypothesis. We first consider the corresponding coupling functions $\tilde{w}_{M,N}$ of $\widetilde{W}_{M,N}$ in $\widetilde{W}_{\text{eff}}$.

Lemma V.3 *Assume that Hypothesis 3 holds in addition to Hypothesis 2, i.e., the function J obeys the bound*

$$J(k)\ \leq\ |k|^{\frac{\mu-1}{2}}\,. \qquad \text{(V.94)}$$

Then, for sufficiently small $g\rho_0^{-1/2}>0$ and $M+N\geq 0$,

$$\begin{aligned}
&\left|\tilde{w}_{M,N}\left[r\,,z\,,k^{(M)}\,,\tilde{k}^{(N)}\right]\right| \qquad \text{(V.95)} \\
&\qquad\leq\ \left(\frac{C\,g}{\rho_0^{1/2}}\right)^{M+N+2\delta_{(M+N),0}}\ \rho_0^{\,1-\frac{1}{2}(M+N)}\cdot\prod_{j=1}^{M}|k_j|^{\frac{\mu-1}{2}}\,\prod_{j=1}^{N}|\tilde{k}_j|^{\frac{\mu-1}{2}}\,.
\end{aligned}$$

Proof: For the proof of (V.95), we need the following slight generalization of (IV.36) in Lemma IV.16:

$$\begin{aligned} &\left\| \widehat{H}_0^{-1/2} W^{m,n}_{m+p,n+q}\left(k^{(m)};\tilde{k}^{(n)}\right) \widehat{H}_0^{-1/2} \right\| \\ &\qquad \leq \; C_0\, g^{m+n+p+q} \rho_0^{-\frac{1}{2}(1+\delta_{p+q,0})} \cdot \prod_{j=1}^{m} J(k_j) \cdot \prod_{j=1}^{n} J(\tilde{k}_j)\,, \end{aligned} \tag{V.96}$$

where

$$\begin{aligned} &W^{m,n}_{m+p,n+q}\left(k^{(m)};\tilde{k}^{(n)}\right) \;:= \\ &\qquad g^{m+n+p+q} \int dx^{(p)} dx^{(q)}\, G^{(\theta)}_{m+p,n+q}\left(k^{(m)}, x^{(p)}; \tilde{k}^{(n)}, \tilde{x}^{(q)}\right) \otimes a^\dagger(x^{(p)}) a(\tilde{x}^{(q)})\,, \end{aligned} \tag{V.97}$$

for $m+p+n+q = 1$ or $= 2$ (see (IV.17) and (IV.18)). The proof goes along the same lines as the one of Lemma IV.4.

Using the fact that $\psi_j \otimes \Omega = \rho_0^{1/2}\, \widehat{H}_0^{-1/2} \psi_j \otimes \Omega$, we estimate

$$\begin{aligned} &\left| \widetilde{D}_L\left[r; \{W^{m_l,n_l}_{m_l+p_l,n_l+q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^{L}\right] \right| \\ &= \; \rho_0 \left| \left\langle \varphi_j \otimes \Omega \;\middle|\; \left(\widehat{H}_0^{-1/2} W_1 \left(\widehat{H}_0^{-1/2} + \mu_1\right)^{-1/2}\right) \left(\overline{R}_0\left[H_f + \mu_1\right]\left(\widehat{H}_0 + \mu_1\right)\right) \right.\right. \\ &\quad \left.\left. \cdots \left(\overline{R}_0\left[H_f + \mu_{L-1}\right]\left(\widehat{H}_0 + \mu_{L-1}\right)\right) \cdot \left(\left(\widehat{H}_0 + \mu_{L-1}\right)^{-1/2} W_L\left(\widehat{H}_0^{-1/2}\right)\right) \varphi_j \otimes \Omega \right\rangle \right| \\ &\leq \; \rho_0 \cdot \prod_{l=1}^{L} \left\| \left(\widehat{H}_0 + \mu_{l-1}\right)^{-1/2} W_l \left(\widehat{H}_0 + \mu_l\right)^{-1/2} \right\| \cdot \prod_{l=1}^{L-1} \left\| \overline{R}_0\left[H_f + \mu_l\right]\left(\widehat{H}_0 + \mu_l\right) \right\| \\ &\leq \; \rho_0 \prod_{l=1}^{L} \left(\frac{C\, g^{m_l+p_l+n_l+q_l}}{\rho_0^{\frac{1}{2}+\frac{1}{2}\delta_{p_l+q_l,0}}} \right) \cdot \prod_{j=1}^{M} |k_j|^{\frac{\mu-1}{2}} \prod_{j=1}^{N} |\tilde{k}_j|^{\frac{\mu-1}{2}}\,, \end{aligned} \tag{V.98}$$

where we set $\mu_0 = \mu_L = 0$, and in the last inequality we make use of (V.96), (V.94) and (V.75). Now, we observe that $m_l + n_l + p_l + q_l \geq 1$ implies that $1 + \delta_{(p_l+q_l),0} \leq 2(m_l+n_l) + p_l + q_l$ and hence

$$\frac{g^{m_l+n_l+p_l+q_l}}{\rho_0^{\frac{1}{2}+\frac{1}{2}\delta_{p_l+q_l,0}}} \;\leq\; \left(\frac{g}{\rho_0^{1/2}}\right)^{m_l+n_l+p_l+q_l} \cdot \rho_0^{-\frac{1}{2}(m_l+n_l)}\,. \tag{V.99}$$

Thus, for any $a > 0$ and g sufficiently small such that $2C\,a^2\, g \leq \rho_0^{1/2}$,

$$\begin{aligned} &\left| \widetilde{D}_L\left[r; \{W^{m_l,n_l}_{m_l+p_l,n_l+q_l}; k_l^{(m_l)}; \tilde{k}_l^{(n_l)}\}_{l=1}^{L}\right] \right| \;\leq \\ &\left(\frac{2C\,a^2\, g}{\rho_0^{1/2}}\right)^{\max\{M+N,\, L\}} \rho_0^{1-\frac{1}{2}(M+N)}\, a^{-L}\, (2a)^{-m_l-n_l-p_l-q_l} \prod_{j=1}^{M} |k_j|^{\frac{\mu-1}{2}} \prod_{j=1}^{N} |\tilde{k}_j|^{\frac{\mu-1}{2}}\,. \end{aligned} \tag{V.100}$$

We insert this estimate into (V.62) and observe that $\max\{M+N,\, L\} = M+N+2\delta_{(M+N),0}$, since $L \geq 2$ for $M+N=0$. Additionally estimating the binomial coefficients in (V.62) by $2^{m_l+p_l+n_l+q_l}$, we can sum up the series

$$\begin{aligned}\Big|\tilde{w}_{M,N}\Big[r;\, z;\, k^{(M)};\, \tilde{k}^{(N)}\Big]\Big| \Big(\frac{2C\,a^2\,g}{\rho_0^{1/2}}\Big)^{-M-N-2\delta_{(M+N),0}} \rho_0^{\frac{1}{2}(M+N)-1} \prod_{j=1}^{M} |k_j|^{\frac{1-\mu}{2}} \prod_{j=1}^{N} |\tilde{k}_j|^{\frac{1-\mu}{2}} \\ \leq \sum_{L=1}^{\infty} a^{-L} \sum_{\substack{m_l+p_l+n_l+q_l=1,2;\\ l=1,\dots,L}} \prod_{l=1}^{L} \left\{ \binom{m_l+p_l}{p_l} \binom{n_l+q_l}{q_l} \right\} (2a)^{-m_l-n_l-p_l-q_l} \\ \leq \sum_{L=1}^{\infty} \left(\frac{a^4}{(a-1)^5}\right)^{-L}, \qquad \text{(V.101)}\end{aligned}$$

arriving at (V.95), upon the choice $a := 5$. ∎

By insertion of (V.95) into (V.82), we obtain the following corollary:

Corollary V.4 *We require Hypothesis 2 and 3. Then, for sufficiently small* $g\rho_0^{-1/2} > 0$ *and* $M+N \geq 0$,

$$\Big|\, w^{(0)}_{M,N}\Big[r; z; k^{(M)}; \tilde{k}^{(N)}\Big]\,\Big| \;\leq\; \Big(\frac{C\,g}{\rho_0^{1/2}}\Big)^{\frac{1}{2}(M+N+1)} \cdot \rho_0^{\frac{1+\mu}{2}(M+N)} \cdot \prod_{j=1}^{M} |k_j|^{\frac{\mu-1}{2}} \prod_{j=1}^{N} |k_j|^{\frac{\mu-1}{2}}, \qquad \text{(V.102)}$$

and, for $M+N \geq 1$ *and* $0 < \rho \leq 1$,

$$\begin{aligned}\Big\| (H_f+\rho)^{-1/2}\, \chi_{H_f<1}\, W^{(0)}_{M,N}\, \chi_{H_f<1}\, (H_f+\rho)^{-1/2} \Big\| \qquad \text{(V.103)} \\ \leq \Big(\frac{C\,g}{\rho_0^{1/2}}\Big)^{\frac{1}{2}(M+N+1)} \Big(\frac{\rho_0^{1+\mu}}{\rho}\Big)^{\frac{1}{2}(M+N)}.\end{aligned}$$

Corollary V.4 shows how an estimate on the L^∞ norm of $w^{(0)}_{M,N}$ turns into a bound on the operator norm of $W^{(0)}_{M,N}$. We remark that for the derivation of (V.103) from (V.102), we additionally used that $M+N \geq 1$ implies $\delta_{M0}+\delta_{N0} \leq 1$ and that $\Gamma[\mu+1]^m \leq \Gamma[(\mu+1)m+1]$. This last estimate appears to be rather rough, as we do not make use of the superexponential growth of the Gamma function. Indeed, in the renormalization scheme we present in the next section, this superexponential growth becomes important, but for our present consideration the weaker estimate (V.103) is sufficient. Since $W_{(0)} = \sum_{M+N\geq 1} W^{(0)}_{M,N}$, we obtain from (V.103)

Lemma V.5 *Assume Hypotheses 2 and 3, suppose that* $g\rho_0^{-1/2}$ *is sufficiently small and that* $0 < \rho_0^{1+\mu} \leq \rho_1 < \rho_0 \leq 1$. *Then*

$$\left\| (H_f + \rho_1)^{-1/2} \, \chi_{H_f<1} \, W_{(0)} \, \chi_{H_f<1} \, (H_f + \rho_1)^{-1/2} \right\| \; \leq \; \left(\frac{C \, g \, \rho_0^{\mu/2}}{\rho_1^{1/2}} \right) \tag{V.104}$$

Note the fact that μ must be strictly positive to ensure that $\rho_1 < \rho_0$. This makes the heuristic discussion given in the introduction precise.

V.2.3 Invertibility of $H_{(0)}$ for $z \in D_1$

Returning to (V.77), we investigate the invertibility of $H_{(0)}[z]$, i.e., whether $\frac{1}{2}\mathrm{e}^{i\vartheta}\rho_0 z + E_j$ belongs to the resolvent set of $H_g(\theta)$. Recall from (V.45) that

$$\mathcal{U}_{(0)}^{(\mathrm{in})} \; := \; \left\{ z \in D_1 \,\middle|\, |z - E_{(0)}[z]| \leq \rho/2 \right\}, \tag{V.105}$$

and define

$$\mathcal{U}_{(0)}^{(\mathrm{out})}(\varepsilon') \; := \; \left\{ z \in D_1 \setminus \mathcal{U}_{(0)}^{(\mathrm{in})} \,\middle|\, |\arg\left(z - E_{(0)}[z]\right)| \geq \varepsilon' \right\}, \tag{V.106}$$

for $\varepsilon' > 0$.

Lemma V.6 *Assume Hypotheses 2, 3, assume that* $0 < \rho_0^{1+\mu} \leq \rho_1 < \rho_0 \leq 1$. *Then, there exists a constant,* C_1, *such that*

$$\mathcal{U}_{(0)}^{(\mathrm{out})}(C_1 g \rho_0^{-1/2}) \; \subseteq \; \rho[H_g(\theta)], \tag{V.107}$$

the resolvent set of $H_g(\theta)$, *provided that* $g\rho_0^{-1/2}$ *is sufficiently small.*

Proof: According to (V.77) and (V.104), $H_{(0)}[z]$ is invertible, provided we can prove that $T_{(0)}[H_f; z] + E_{(0)}[z] - z$ is m-sectorial and that

$$\left\| \frac{(H_f + \rho_1) \; \chi_{H_f<1}}{T_{(0)}[H_f; z] + E_{(0)}[z] - z} \right\| \; = \; \sup_{0 \leq r < 1} \left| \frac{r + \rho_1}{T_{(0)}[r; z] + E_{(0)}[z] - z} \right| \; < \; \frac{\rho_0^{1/2}}{C_2 g}, \tag{V.108}$$

for a constant, $C_2 > C$, where C is the constant in (V.104). Indeed, in this case the series

$$\begin{aligned} \left(H_{(0)}[z]\right)^{-1} \; &= \; \left|Q_{(0)}[H_f; z]\right|^{-1/2} \, U^* \\ &\cdot \sum_{n=0}^{\infty} \left(- \left|Q_{(0)}[H_f; z]\right|^{-1/2} \, W_{(0)}[z] \, \left|Q_{(0)}[H_f; z]\right|^{-1/2} U^* \right)^n \left|Q_{(0)}[H_f; z]\right|^{-1/2}, \end{aligned} \tag{V.109}$$

where $Q_{(0)}[H_f;z] := T_{(0)}[H_f;z] + E_{(0)}[z] - z$, is readily seen to be convergent since, by (V.104) and (V.108),

$$\left\| \left|T_{(0)} + E_{(0)} - z\right|^{-1/2} W_{(0)} \left|T_{(0)} + E_{(0)} - z\right|^{-1/2} \right\| \leq \frac{C}{C_2} < 1 . \tag{V.110}$$

We derive (V.108) from the Lipshitz continuity of $T_{(0)}$ and the bound

$$\sup_{0\leq r<1} \left|\Big(\partial_r T_{(0)}[r;z]\Big) - 1\right| = \sup_{0\leq r<1} \left|\partial_r w^{(0)}_{0,0}[r;z]\right| \leq \frac{C_3\, g\, \rho_0^{\mu/2}}{\rho_1^{1/2}} , \tag{V.111}$$

uniformly in $z \in \mathcal{U}^{(\mathrm{out})}_{(0)}$. In case that $8r \geq \rho_1$ and $C_1 \geq 2C_3$, we get

$$\left|T_{(0)} + E_{(0)} - z\right| \geq r\,(C_1 - C_3) \geq \frac{C_1}{18}(r + \rho_1) , \tag{V.112}$$

while for $8r < \rho_1$, we trivially have

$$\left|T_{(0)} + E_{(0)} - z\right| \geq \frac{\rho_1}{2} - \frac{\rho_1}{8}\left[1 + \frac{C_3 g}{\rho_0^{1/2}}\right] \geq \frac{r + \rho_1}{8} . \tag{V.113}$$

From here the assertion follows immediately, ignoring the fact that we have not given a derivation of (V.111). This estimate is more subtle and is sketched below, after Eqn. (V.159). For details on (V.111), we refer the reader to [5]. ∎

The idea is now to iterate the Feshbach map, substituting $\chi_{H_f<\rho_0}$ by $\chi_{H_f<\rho\cdot\rho_0}$, with $\rho < 1$ in order to analyze the spectrum of $H_g(\theta)$ in $\mathcal{U}^{(\mathrm{in})}_{(0)} \subseteq D_{3/4}$. Subsequently, we will apply the same step to lower the photon energy scales from $\rho_{(1)} := \rho \cdot \rho_0 < \rho_0$ to $\rho_{(2)} := \rho^2 \cdot \rho_0 < \rho_{(1)} < \rho_0$, and so on. In the next chapter, we will describe this induction in detail.

V.3 Passing from Energy Scale $\rho_{(n)}$ to the lower Energy Scale $\rho_{(n+1)} < \rho_{(n)}$

In this section, we present the key tool for our analysis: an inductive renormalization group construction. The idea underlying our construction is that by lowering the photon energy scale, thereby decimating the degrees of freedom of the system, the dynamics of the remaining degrees of freedom is approximated ever better by the free photon dynamics H_f. Our construction is realized in terms of a renormalization map, $\mathcal{R}$, consisting of a decimation of degrees of freedom, followed

by rescaling by a factor of ρ^{-1}. The renormalization map associates to every operator $H[z]$ on the subspace $\mathcal{H}_{\rm red} \equiv {\rm Ran}\left\{\chi_{H_f<1}\right\} \subseteq \mathcal{F}$ an operator $\mathcal{R}(H)[\hat{z}]$ on $\mathcal{H}_{\rm red}$ with the property that $H[z]$ is invertible if and only if $\mathcal{R}(H)[\hat{z}]$ is, for z belonging to a small neighbourhood of a certain part of $\sigma(H)$ described below.

V.3.1 Construction of the Renormalization Map $\mathcal{R}$

The renormalization map $\mathcal{R}$ depends on the scale parameter $0 < \rho < 1$, which we fix once and for all. It is defined on a certain subset of a linear space, V_μ, of Hamiltonians on $\mathcal{H}_{\rm red}$, where $\mu > 0$ is the same parameter as the $\mu > 0$ in Hypothesis 3. Our definition of V_μ, given in (V.15)–(V.24), is modelled on the example of $H_{(0)}$ in (V.77), which, we recall, is given by

$$H_{(0)}[z] \;=\; \chi_{H_f<1}\left(T_{(0)}[H_f;z] + W_{(0)}[z] + E_{(0)}[z]\right)\chi_{H_f<1}\,. \tag{V.114}$$

The elements of V_μ are analytic families of operators $H[z]$, with $z \in D_1$, parametrized by vectors in the linear space

$$V'_\mu \;:=\; \mathbb{C} \;+\; \mathcal{T} \;+\; \mathcal{W}_\mu\,. \tag{V.115}$$

We identify the elements of V'_μ with operators $H \in \mathcal{B}\left(\mathcal{H}_{\rm red}\right)$ via the correspondence

$$(E[z], T[z], \{w[z]\}) \;\mapsto\; H[z] \;:=\; \chi_1\left(T[H_f;z] + W[z] + E[z]\right)\chi_1\,, \tag{V.116}$$

where $\chi_1 := \chi_{H_f<1}$ and $W = \sum_{M+N\geq 1} W_{M,N}$,

$$W_{M,N} = \int dk^{(M)} d\tilde{k}^{(N)}\, a^\dagger(k^{(M)})\, w_{M,N}\left[H_f; k^{(M)}; \tilde{k}^{(N)}\right] a(\tilde{k}^{(N)})\,. \tag{V.117}$$

The subset $\mathcal{B}_\varepsilon \subseteq V_\mu$ on which the renormalization map $\mathcal{R}$ is defined is given by analytic families $D_1 \ni z \mapsto (E[z],\, T[z],\, \{w[z]\}) \in V'_\mu$ such that, for some sufficiently small $\varepsilon \geq 0$ to be chosen below,

$$\sup_{D_1}\Big\|(E[z],\, T[r;z] - r,\, \{w[z]\})\Big\|' \;=\; \sup_{D_1}\Big\|H[z] - (H_f - z)\Big\|' \;\leq\; \varepsilon\,. \tag{V.118}$$

In analogy to (V.106), we define

$$\mathcal{U}^{(\rm out)}(\varepsilon') \;:=\; \Big\{z \in D_1 \setminus \mathcal{U}^{(\rm in)} \;\Big|\; |\arg(z - E[z])| \geq \varepsilon'\Big\}\,, \tag{V.119}$$

and we recall from (V.37) that

$$\mathcal{U}^{(\mathrm{in})} \;=\; \Big\{ z \in D_1 \;\Big|\; |z - E[z]| \leq \rho/2 \Big\} \;\subseteq\; D_{3/4}\,, \tag{V.120}$$

for $\varepsilon \leq 1/4$.

Lemma V.7 *Let* $H = \chi_1(T + W + E)\chi_1 \in V_\mu$ *with* $\sup_{D_1} \|H[z] - H_f\|' \leq \varepsilon$ *and assume that* $\varepsilon\rho^{-1/2} \leq 1/12$ *and that* $\xi \leq 1/5$.

(a) *There exists a constant,* C_1', *such that* $H[z] - z$ *is invertible for any* $z \in \mathcal{U}^{(\mathrm{out})}(C_1'\,\varepsilon\rho^{-1/2})$,

(b) *For any* $z \in \mathcal{U}^{(\mathrm{in})}$, *the operator* $H_{\overline{\chi}_\rho}[z] - z := \overline{\chi}_\rho H[z]\overline{\chi}_\rho - z$ *is invertible on* $\overline{\chi}_\rho\mathcal{H}_{\mathrm{red}}$, *where* $\overline{\chi}_\rho := 1 - \chi_\rho$ *and* $\chi_\rho := \chi_{H_f<\rho}$. *Moreover, the Feshbach map* $f_{\chi_\rho}(H[z] - z)$ *is defined and is invertible on* $\chi_{H_f<\rho}\mathcal{F}$ *if and only if* $H[z] - z$ *is invertible on* $\mathcal{H}_{\mathrm{red}}$.

Proof: The proof of (a) is similar to the proof of $E_j + \frac{1}{2}\mathrm{e}^{-i\vartheta}\rho_0\mathcal{U}_0^{(\mathrm{out})} \subseteq \rho\,[H_g(\theta)]$ in Lemma V.6. To prove (b), we observe that by (V.31), for $z \in \mathcal{U}^{(\mathrm{in})}$,

$$\left\| \frac{(H_f + \rho)\overline{\chi}_\rho}{T[H_f; z] + E[z] - z} \right\| \;\leq\; 8\,, \tag{V.121}$$

while Lemma V.2 with $\alpha := \varepsilon\xi^{M+N}$ implies that

$$\left\| (H_f + \rho)^{-1/2}\; \chi_1 W \chi_1\; (H_f + \rho)^{-1/2} \right\| \;\leq\; \frac{12(4\pi/3)^{1/2}\,\xi\,\varepsilon}{\rho^{1/2}}\,. \tag{V.122}$$

Thus, the Neumann series for $\Big(\overline{\chi}_\rho H[z]\overline{\chi}_\rho - z\Big)^{-1}\overline{\chi}_\rho$ converges, provided ε is chosen such that $\varepsilon\,\rho^{-1/2} < 5(12)^{-1}(4\pi/3)^{-1/2}$. Now, (b) follows from Theorem IV.1. ∎

Recall from (V.28) that

$$\mathrm{Dec}(H[z]) - z \;:=\; f_{\chi_\rho}\Big(H[z] - z\Big)\chi_\rho\,. \tag{V.123}$$

The next step is to rewrite $\mathrm{Dec}(H[z])$, for $z \in \mathcal{U}^{(\mathrm{in})}$, in a Wick-ordered form

$$\mathrm{Dec}\Big(H[z]\Big) \;=:\; \chi_\rho\,\Big(\,\widetilde{T}[H_f; z] \,+\, \widetilde{W}[z] \,+\, \widetilde{E}[z]\,\Big)\,\chi_\rho\,, \tag{V.124}$$

where

$$\begin{aligned} \widetilde{E}[z] &= E[z] + \Delta\tilde{w}_{0,0}[0;z]\,, && \text{(V.125)} \\ \widetilde{T}[H_f;z] &= T[H_f;z] + \Delta\tilde{w}_{0,0}[H_f;z] - \Delta\tilde{w}_{0,0}[0;z]\,, && \text{(V.126)} \\ \Delta\tilde{w}_{0,0}[s;z] &:= \Big\langle f_{\chi_\rho}\Big(H[s;z]-z\Big) + z\Big\rangle_\Omega\,, && \text{(V.127)} \end{aligned}$$

where $s \in [0,1]$, and $H[s;z]$ results from replacing H_f by $H_f + s$ in $H[z]$, i.e.,

$$H[s;z] := \Big(E[z]\,;\, T[(\cdot)+s\,;\,z]\,;\, \{w[(\cdot)+s\,;\,z\,;\,(\cdot),(\cdot)]\}\Big)\,. \qquad \text{(V.128)}$$

Furthermore,

$$\widetilde{W}[z] = \sum_{M+N\geq 1} \widetilde{W}_{M,N}[z]\,, \qquad \text{(V.129)}$$

with

$$\widetilde{W}_{M,N}[z] = \int dk^{(M)} d\tilde{k}^{(N)}\; a^\dagger(k^{(M)})\, \tilde{w}_{M,N}\Big[H_f;z,k^{(M)},\tilde{k}^{(N)}\Big]\, a(\tilde{k}^{(N)})\,. \qquad \text{(V.130)}$$

Applying the Wick-ordering procedure together with the Pull-Through formulae described in the proof of Lemma V.1, we see that the coupling functions $\tilde{w}_{M,N}$ are given by the same formula as (V.62)–(V.65) in Lemma V.1, except that now the summation $\sum^2_{m_l+n_l+p_l+q_l=1}$ is replaced by $\sum_{m_l+n_l+p_l+q_l\geq 1}$.

Then we rescale $\widetilde{H}[z]$ by $U_f(-\ln\rho)$ and apply the map

$$\widehat{Z} : \mathcal{U}^{(\mathrm{in})} \to D_1\,, \qquad z \mapsto \frac{2}{\rho}\Big(z - E[z]\Big)\,, \qquad \text{(V.131)}$$

which is bijective since $\sup_{\mathcal{U}^{(\mathrm{in})}} |\partial_z E[z] - 1| \leq 1/2$ (see Figure V.4), by Cauchy's estimate.

By rescaling according to (V.40) (see also (V.76)), we now obtain a new analytic family, $\mathcal{R}(H)[\hat{z}]$, for $\hat{z} \in D_1$, as follows.

$$\mathcal{R}(H)[\hat{z}] := \rho^{-1}\, U_f(-\ln\rho)\, \widetilde{H}\Big[\widehat{Z}^{-1}[\hat{z}]\Big]\, U_f(-\ln\rho)^*\,. \qquad \text{(V.132)}$$

As in (V.77)–(V.82), we identify $\mathcal{R}(H)$ and $\Big(\widehat{E},\widehat{T},\{\hat{w}\}\Big) \in V_\mu$ by

$$\mathcal{R}(H)[\hat{z}] =: \chi_1\Big(\widehat{T}[H_f;\hat{z}] + \widehat{W}[\hat{z}] + \widehat{E}[z]\Big)\,\chi_1\,, \qquad \text{(V.133)}$$

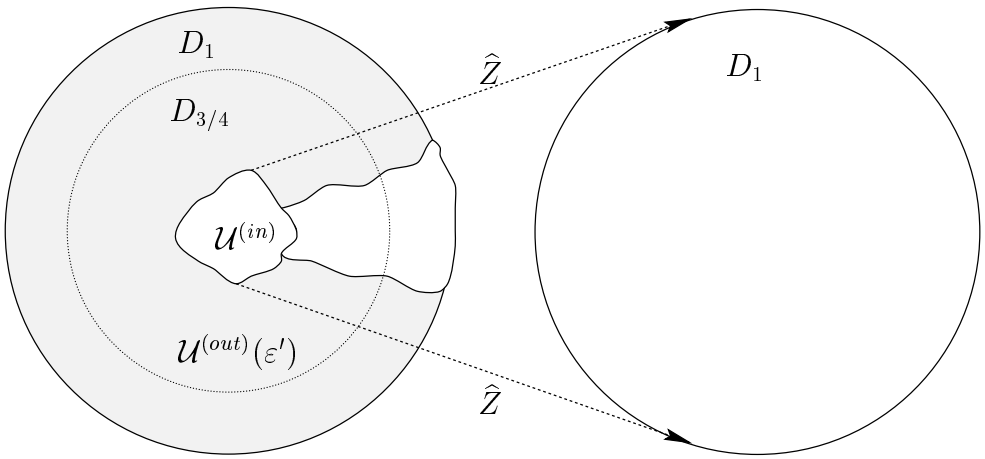

Figure V.4: Rescaling of the Spectral Parameter

and we obtain

$$\widehat{T}[H_f;\hat{z}] := \rho^{-1}\,T\left[\rho H_f;\widehat{Z}^{-1}[\hat{z}]\right] + \rho^{-1}\left(\Delta\tilde{w}_{0,0}\left[\rho H_f;\widehat{Z}^{-1}[\hat{z}]\right] - \Delta\tilde{w}_{0,0}\left[0;\widehat{Z}^{-1}[\hat{z}]\right]\right)\,, \tag{V.134}$$

$$\widehat{E}[\hat{z}] := E\left[\widehat{Z}^{-1}[\hat{z}]\right] - \rho^{-1}\,\Delta\tilde{w}_{0,0}\left[0;\widehat{Z}^{-1}[\hat{z}]\right]\,, \tag{V.135}$$

where, similar to (V.127) and (V.128),

$$\Delta\tilde{w}_{0,0}[s;z] := \Big\langle\, f_{\chi_\rho}\Big(H[s;z]-z\Big)\,\Big\rangle_\Omega\,, \quad \text{with } z\in\mathcal{U}^{(\mathrm{in})} \text{ and} \tag{V.136}$$

$$H[s;z] := \Big(E[z]\,;\,T[(\cdot)+s\,;\,z]\,;\,\{w[(\cdot)+s\,;\,z\,;\,(\cdot),(\cdot)]\}\Big)\,, \tag{V.137}$$

$$\widehat{W}[\hat{z}] := \sum_{M+N\geq 1}\widehat{W}_{M,N}[\hat{z}]\,, \tag{V.138}$$

$$\widehat{W}_{M,N}[\hat{z}] := \int dk^{(M)}d\tilde{k}^{(N)}\,a^\dagger(k^{(M)})\,\hat{w}_{M,N}\left[H_f;\hat{z};k^{(M)},\tilde{k}^{(N)}\right]\,a(\tilde{k}^{(N)})\,, \tag{V.139}$$

$$\hat{w}_{M,N}\left[r;\hat{z};k^{(M)};\tilde{k}^{(N)}\right] := \rho^{\frac{3}{2}(M+N)-1}\,\tilde{w}_{M,N}\left[\rho H_f;Z^{-1}[\hat{z}];\rho k^{(M)},\rho\tilde{k}^{(N)}\right]\,. \tag{V.140}$$

Note that the renormalization map $\mathcal{R}: V_\mu \to V_\mu$ is equivalently defined by

$$\mathcal{R}\,(E,\,T,\,\{w\}) := \left(\widehat{E},\,\widehat{T},\,\{\hat{w}\}\right)\,. \tag{V.141}$$

V.3.2 The Contraction Property of the Renormalization Map $\mathcal{R}$

We come to the heart of the matter: the contraction property of $\mathcal{R}$. Concretely, we show that

$$\sup_{z\in D_1}\left\|H[z]-T_0[H_f;z]\right\|' \;\leq\; \varepsilon\,, \tag{V.142}$$

for some T_0 with $|\partial_r T_0 - 1| \leq 1/8$, implies that

$$\sup_{z\in D_1}\left\|\mathcal{R}(H)[z]-T^T[H_f;z]\right\| \;\leq\; \hat{\varepsilon} \;\ll\; \varepsilon\,, \tag{V.143}$$

with $|\partial_r T^T - 1| \leq \sup_{r,z}|\partial_r T_0 - 1| + \varepsilon$. Obviously, this implication enables us to iterate the renormalization map, yielding a sequence of analytic families in V_μ which converges to the (trivial) fixpoint of $\mathcal{R}$,

$$(0,\, r,\, \{0\}) = \mathcal{R}\,(0,\, r,\, \{0\})\ . \tag{V.144}$$

Note that $(0,\, r,\, \{0\})$ corresponds to $H[z] \equiv H_f$, the free photon Hamiltonian.

Theorem V.8 *Fix* $\mu > 0$, $0 < \rho^{\mu/2} \leq 1/8$, $0 < 2\sqrt{\pi}\xi \leq \min\{1/12,\ \rho^{\frac{3+\mu}{4}}\}$ *and assume that* $H = \chi_1(T+W+E)\chi_1 \in V_\mu$, *with*

$$\sup_{z\in D_1}\left\|H[z]\;-\;T_0[H_f;z]\right\|' \;\leq\; \varepsilon\,, \tag{V.145}$$

for some T_0 *with* $|\partial_r T_0 - 1| \leq 1/8$. *Then, for* $\varepsilon\rho^{-1/2}$ *sufficiently small,*

$$\sup_{z\in D_1}\left\|\mathcal{R}(H)[z]\;-\;T^T[H_f;z]\right\| \;\leq\; 4\rho^{\mu/2}\varepsilon\left[1\;+\;O\left(\frac{\varepsilon}{\rho^{1/2}}\right)\right]\,, \tag{V.146}$$

where $T^T[H_f;\hat{z}] := \rho^{-1}\,T\left[\rho H_f;\ \widehat{Z}^{-1}[\hat{z}]\right]$, *obeying* $|\partial_r T^T - 1| \leq \sup_{r,z}|\partial_r T_0 - 1| + \varepsilon$.

We remark that Theorem V.8 shows that for ε small enough, the renormalization map $\mathcal{R}$ is a contraction on the ball of radius ε about the fix point $(0, r, \{0\})$ in V_μ.

Proof: We only sketch some key ideas of the proof, which can be found in [5] in full detail. First, we consider, for $\left(k^{(M)}, \tilde{k}^{(N)}\right) \in B_\rho^M \times B_\rho^N$,

$$\tilde{w}_{M,N}\left[r;z;k^{(M)},\tilde{k}^{(N)}\right] \;=\; \tilde{w}^T_{M,N}\left[r;z;k^{(M)},\tilde{k}^{(N)}\right] \;+\; \Delta\tilde{w}_{M,N}\left[r;z;k^{(M)},\tilde{k}^{(N)}\right]\,, \tag{V.147}$$

where $\tilde{w}^T_{M,N} := w_{M,N}$ and $\Delta\tilde{w}_{M,N}$ is given by

$$\Delta\tilde{w}_{M,N}\left[r;\, z;\, k^{(M)};\, \tilde{k}^{(N)}\right] \;=\; \sum_{L=2}^{\infty}(-1)^{L-1}\sum_{\substack{m_l+p_l+n_l+q_l=1,2;\\ l=1,\dots,L}}\delta_{\sum_{l=1}^{L} m_l\,,M}\,\delta_{\sum_{l=1}^{L} n_l\,,N} \tag{V.148}$$
$$\prod_{l=1}^{L}\left\{\binom{m_l+p_l}{p_l}\binom{n_l+q_l}{q_l}\right\}\left\{\widetilde{D}_L\Big[r;\{W^{m_l,n_l}_{p_l,q_l};k^{(m_l)}_l;\tilde{k}^{(n_l)}_l\}_{l=1}^{L}\Big]\right\}^{symm}_{M,N}\,,$$

in analogy to (V.63). We remark that the representation (V.148) is valid for $\Delta w_{0,0}$, as well. From (V.145), we obtain

$$\left|w_{M,N}\left[r;z;k^{(M)};\tilde{k}^{(N)}\right]\right| \;\le\; \varepsilon\,\xi^{M+N}\prod_{j=1}^{M}|k_j|^{\frac{\mu-1}{2}}\prod_{j=1}^{N}|\tilde{k}_j|^{\frac{\mu-1}{2}}\,, \tag{V.149}$$

and we estimate $|\widetilde{D}_L[\cdots]|$ in a similar way as in the proof of Lemma V.3, using that $\xi \le 1/5 \le 1/4$:

$$\begin{aligned}
&\left|\widetilde{D}_L\Big[r;\{W^{m_l,n_l}_{p_l,q_l};k^{(m_l)}_l;\tilde{k}^{(n_l)}_l\}_{l=1}^{L};\{\overline{R}_0\}_{l=1}^{L-1}\Big]\right| \qquad\qquad (V.150)\\
&\le\; \rho\prod_{l=1}^{L}\frac{C\,\varepsilon\,\xi^{m_l+n_l+p_l+q_l}}{\rho^{\frac12+\frac12\delta_{p_l+q_l,0}}}\prod_{j=1}^{M}|k_j|^{\frac{\mu-1}{2}}\prod_{j=1}^{N}|\tilde{k}_j|^{\frac{\mu-1}{2}}\\
&\le\; \rho\cdot\left(\frac{4C\,\varepsilon}{\rho}\right)^{L}\cdot\left(\frac{4\,\xi}{\rho^{1/2}}\right)^{M+N}\cdot\prod_{l=1}^{L}\left(\frac14\right)^{m_l+n_l+p_l+q_l}\cdot\prod_{j=1}^{M}|k_j|^{\frac{\mu-1}{2}}\prod_{j=1}^{N}|\tilde{k}_j|^{\frac{\mu-1}{2}}\,.
\end{aligned}$$

Moreover,

$$\sum_{\substack{m_l+\dots+q_l\ge 1\\ l=1,\dots,L}}\prod_{l=1}^{L}\left\{\binom{m_l+p_l}{p_l}\binom{n_l+q_l}{q_l}\left(\frac14\right)^{m_l+n_l+p_l+q_l}\right\} \;\le\; \left(\sum_{m=0}^{\infty}2^{-m}\right)^{4L} \;=\; (16)^L\,. \tag{V.151}$$

Thus, putting together (V.150) and (V.151) and summing over $L \ge 2$ with $16C\varepsilon < \rho^{1/2}$, we obtain

$$\left|(\tilde{w}_{M,N}-w_{M,N})\left[r;z;k^{(M)};\tilde{k}^{(N)}\right]\right| \;\le\; C\,\varepsilon^2\left(\frac{4\,\xi}{\rho^{1/2}}\right)^{M+N}\cdot\prod_{j=1}^{M}|k_j|^{\frac{\mu-1}{2}}\prod_{j=1}^{N}|\tilde{k}_j|^{\frac{\mu-1}{2}}\,.$$

We define in accordance with (V.140),

$$\hat{w}^T_{M,N}\left[r;\hat{z};k^{(M)},\tilde{k}^{(N)}\right] \;:=\; \rho^{\frac32(M+N)-1}\tilde{w}^T_{M,N}\left[\rho r;Z^{-1}\left(\frac{\rho}{2}\hat{z}\right);\rho k^{(M)},\rho\tilde{k}^{(N)}\right]\,, \tag{V.152}$$

$$\Delta\hat{w}^T_{M,N}\left[r;\hat{z};k^{(M)},\tilde{k}^{(N)}\right] \;:=\; \rho^{\frac32(M+N)-1}\Delta\tilde{w}_{M,N}\left[\rho r;Z^{-1}\left(\frac{\rho}{2}\hat{z}\right);\rho k^{(M)},\rho\tilde{k}^{(N)}\right]\,, \tag{V.153}$$

so that $\hat{w}_{M,N} = \hat{w}^T_{M,N} + \Delta\hat{w}_{M,N}$, and we obtain from (V.149)

$$\left|\hat{w}^T_{M,N}\left[r;\hat{z};k^{(M)},\tilde{k}^{(N)}\right]\right| \;\leq\; \frac{\varepsilon}{\rho}\left(\xi\,\rho^{1+\frac{\mu}{2}}\right)^{M+N}\cdot\prod_{j=1}^{M}|k_j|^{\frac{\mu-1}{2}}\prod_{j=1}^{N}|\tilde{k}_j|^{\frac{\mu-1}{2}}\,. \qquad \text{(V.154)}$$

From (V.151), we get

$$\left|\Delta\hat{w}_{M,N}\left[r;\hat{z};k^{(M)};\tilde{k}^{(N)}\right]\right| \;\leq\; C\,\varepsilon^2\,\left(4\,\xi\,\rho^{\frac{1}{2}+\frac{\mu}{2}}\right)^{M+N}\prod_{j=1}^{M}|k_j|^{\frac{\mu-1}{2}}\prod_{j=1}^{N}|\tilde{k}_j|^{\frac{\mu-1}{2}}. \quad \text{(V.155)}$$

Since $\rho^{\mu/2} < 1/8$ (actually, $\rho^{\frac{1+\mu}{2}} \leq 1/4$ would suffice here, but not for the estimate on $\partial_r\hat{w}_{M,N}$), by assumption, Estimate (V.155) yields, for $M+N \geq 1$,

$$\left|\hat{w}_{M,N}\left[r;\hat{z};k^{(M)};\tilde{k}^{(N)}\right]\right| \;\leq\; \rho^{\frac{\mu}{2}}\,\varepsilon\,[1\,+\,C\,\varepsilon]\cdot\xi^{M+N}\prod_{j=1}^{M}|k_j|^{\frac{\mu-1}{2}}\prod_{j=1}^{N}|\tilde{k}_j|^{\frac{\mu-1}{2}}\,. \quad \text{(V.156)}$$

We observe that the requirement $\mu > 0$ is forced upon us by the behaviour of $\hat{w}^T_{0,1}$ and $\hat{w}^T_{1,0}$ under renormalization – all other terms renormalize to 0 with a higher power of ρ. Separately, we note that (V.155) yields

$$|\Delta\hat{w}_{0,0}\,[r;\,\hat{z}]| \;\leq\; C\,\varepsilon^2 \;\leq\; C\,\rho^{1/2}\,\varepsilon\,. \qquad \text{(V.157)}$$

Thus, for $\hat{z} \in D_1$

$$|\widehat{E}[\hat{z}]|\,,|\widehat{T}[r;\hat{z}]| \;\leq\; C\,\rho^{1/2}\,\varepsilon\,, \qquad \text{(V.158)}$$

which, by Cauchy's estimate and (V.120), implies

$$|\partial_{\hat{z}}\widehat{E}[\hat{z}]|\,,|\partial_{\hat{z}}\widehat{T}[r;\hat{z}]| \;\leq\; 4C\,\rho^{1/2}\,\varepsilon \;<\; 1/6\,, \qquad \text{(V.159)}$$

for all $\hat{z} \in \mathcal{U}^{(\mathrm{in})} \subseteq D_{3/4}$ and a sufficiently small choice of $\varepsilon\,\rho^{1/2}$.

Now we turn to the estimate of $|\partial_r\widehat{T} - 1|$. Let us return to (V.121). At a first glance, it seems that all that is required to prove (V.121) is a bound on $|r^{-1}\widehat{T}[r;\hat{z}] - 1|$. Indeed, for $r \geq \rho$, we may estimate,

$$\begin{aligned}\left|\,\widehat{T}[r;\hat{z}] - \widehat{E}[\hat{z}] - \hat{z}\,\right| \;&\geq\; r\,\left(1 - \sup_{[0,1]}|r^{-1}\widehat{T}[r;\hat{z}] - 1|\right) \;-\; \frac{\rho}{2}\\ &\geq\; r\,\left(\frac{1}{2} - \sup_{[0,1]}|r^{-1}\widehat{T}[r;\hat{z}] - 1|\right) \;>\; 0\,,\end{aligned}$$

provided $|r^{-1}\widehat{T}[r;\hat{z}]-1|$ is sufficiently small. However, a bound on $|r^{-1}\widehat{T}[r;\hat{z}]-1|$ requires control of $|\partial_r T[r;z] - 1|$ and of $|\partial_r w_{M,N}[r,\ldots]|$ for arbitary values of

r in the interval $[0,1]$, as we show in an example below. Thus to ensure that $\mathcal{R}$ maps a space of Hamiltonians into itself, this space has to be equipped with a norm that yields control over the derivatives $|\partial_r T - 1|$ and $|\partial_r w_{M,N}|$. This is reflected in our choice of the norms $\| \cdot \|^{(\infty)}$ and $\| \cdot \|^{(1)}$, defined in (V.68) and (V.69). To illustrate the key idea of the proof we study one contribution to $\Delta\tilde{w}_{0,0}$ given by (surpressing the z-dependence in our notation)

$$\begin{aligned} S(r) &:= \left\langle W_{0,1}^{0,0}\, \overline{R}_0\, [H_f + r]\; W_{1,0}^{0,0} \right\rangle_\Omega \\ &= \int w_{0,1}[r;x] \frac{\chi[r+\omega(x) \geq \rho]}{T[r+\omega(x)] - Z} w_{1,0}[r;x]\, dx\,. \end{aligned} \tag{V.160}$$

First, we observe that the derivative of S at $r = 0$ depends on the derivative of T at an arbitrary point, $\omega(x) \in [0,1]$. Next, we compute the derivative of S:

$$\begin{aligned} \partial_r S(r) &= \int w_{0,1}[r\,,x] \left(\frac{\delta(r+\omega(x)-\rho)}{T[\rho]+E-z} \right) w_{1,0}[r;x]\, dx \\ &\quad - \int w_{0,1}[r\,,x] \left(\frac{\partial_r T[r+\omega(x)] \chi[r+\omega(x)\geq\rho]}{(T[r+\omega(x)]+E-z)^2} \right) w_{1,0}[r;x]\, dx \\ &\quad + \int \partial_r w_{0,1}[r;x] \left(\frac{\chi[r+\omega(x)\geq\rho]}{T[r+\omega(x)]+E-z} \right) w_{1,0}[r;x]\, dx \\ &\quad + \int w_{0,1}[r;x] \left(\frac{\chi[r+\omega(x)\geq\rho]}{T[r+\omega(x)]+E-z} \right) \partial_r w_{1,0}[r;x]\, dx\,. \end{aligned} \tag{V.161}$$

From (V.161), we draw the following conclusions:

(a) The first term on the right side of (V.161) is given by

$$\int_{S^2=\{|n|=1\}} w_{0,1}[r;\,(\rho-r)n]\, w_{1,0}[r;\,(\rho-r)n] \frac{(\rho-r)^2\, d^2n}{T[\rho]+E-z} \tag{V.162}$$

and is bounded by $C\varepsilon^2\xi^2\rho^\mu$, for all $0 \leq r \leq \rho$.

(b) The second term is bounded by $C\,\varepsilon^2\,\xi^2\,\rho^{-1}$.

(c) We bound the third term of (V.161) as follows:

$$\begin{aligned} &\left| \int \partial_r w_{0,1}[r;x] \left(\frac{\chi[r+\omega(x)\geq\rho]}{T[r+\omega(x)]+E-z} \right) w_{1,0}[r;x]\, dx \right| \\ &\qquad \leq \|w_{1,0}\|^{(\infty)} \cdot \rho^{-1} \cdot \int \frac{|\partial_r w_{0,1}[r;x]|\, dx}{|x|^{\frac{\mu-1}{2}}} \\ &\qquad \leq \rho^{-1} \cdot \|w_{1,0}\|^{(\infty)} \cdot \|\partial_r w_{0,1}\|^{(1)} \;\leq\; C\varepsilon^2\xi\rho^{-1}(4\pi\xi\rho^{1/2})\,. \end{aligned} \tag{V.163}$$

The fourth term is bounded similarly.

From summing the terms in (a)–(c) we obtain

$$|\partial_r S| \;\leq\; C\left(\frac{\varepsilon}{\rho^{1/2}}\right)^2 \xi^2\,, \tag{V.164}$$

for all $0 \leq r \leq \rho$. To bound a general contribution

$$\partial_r \left\langle W^{m_1,m_1}_{p_1,q_1} \overline{R}_0[H_f+\mu_1]\cdots\overline{R}_0[H_f+\mu_{L-1}]W^{m_L,n_L}_{p_L,q_L}\right\rangle_\Omega \tag{V.165}$$

to $\Delta\tilde{w}_{M,N}$, we evaluate the derivative using Leibniz' rule and estimate the resulting terms as in (a)–(c) above. Here, an additional problem arises from the large number of terms generated by writing each contribution to (V.165) of the form

$$\begin{aligned}\Big\langle \Big(W_1\overline{R}_0[H_f+\mu_1]\cdots\overline{R}_0[H_f+\mu_{l-2}]W_{l-1}\overline{R}_0[H_f+\mu_{l-1}]\Big)\;\partial_{H_f}\Big(W_l\overline{R}_0[H_f+\mu_l]\Big)\\ \Big(W_{l+1}\overline{R}_0[H_f+\mu_{l+1}]\cdots\overline{R}_0[H_f+\mu_{L-1}]W_L\Big)\Big\rangle_\Omega\Big|_{r=H_f}\,,\end{aligned} \tag{V.166}$$

with $W_j := W^{m_j,n_j}_{p_j,q_j}$, as a sum of Wick-contractions between the factor to the left of $\partial_{H_f}\left(W_l\overline{R}_0\left[H_f+\mu_l\right]\right)$ and the factor to its right. This problem is solved by using (V.85): If $n!$ terms are generated by n contracted creation and annihilation operators, then each term is bounded by $\Gamma[(\mu+1)n+1]^{-1} \leq (n!)^{-1}$. At this point, we also need to make use of the assumption $\rho^{\mu/2} \leq 1/8$ which illustrates that our method does not yield uniform control over $\mathcal{R}$ as $\mu \to 0$. Further details on this estimate can be found in [5].

We now summarize the result of our analysis. Assuming that $2\sqrt{\pi}\xi \leq \min\{1/12,\ \rho^{\frac{3+\mu}{4}}\}$, $\rho^{\mu/2} \leq 1/8$ and that $\varepsilon\rho^{-1/2} \leq c$, for some sufficiently small constant, $c>0$, we conclude that, for all $\hat{z} \in D_1$,

$$\left\|\left(\widehat{E}[\hat{z}],\, \widehat{T}[\hat{z}]-T^T[\hat{z}],\ ,\{\hat{w}\}\right)\right\| \;\leq\; 4\rho^{\mu/2}\,\varepsilon\,\left[1\;+\;O\left(\frac{\varepsilon}{\rho^{1/2}}\right)\right]\,, \tag{V.167}$$

and summing up the contributions to $|\partial_r\Delta\hat{w}_{0,0}|$ as in (V.164), we additionally obtain $|\partial_r\widehat{T}_0 - 1| \leq \sup_{r,z}|\partial_r T_0 - 1| + \varepsilon$. ∎

Let us define a sequence $\rho_{(n)} := \rho_0\cdot\rho^n$, where ρ is as in Theorem V.8 and $\rho_0 := g^{2-2\tau} \leq \min\left\{1,\ \frac{1}{2}\sqrt{1-\cos\vartheta}\,\delta\right\}$ as in Chapter IV, Equation (IV.30),

for some $0 < \tau < 1$. Our initial condition for the iteration of $\mathcal{R}$ is given by $H_{(0)} = \left(E_{(0)}\,, T_{(0)}\,, \left\{w^{(0)}\right\}\right) \in V_\mu$ from (V.77)–(V.82). Indeed, for all $M+N \geq 0$, $z \in D_1$ and sufficiently small $g\rho_0^{-1/2}$, we verified that

$$\left\|\partial_r w^{(0)}_{M,N}\right\|^{(1)}\,,\ \ \left\|w^{(0)}_{M,N}\right\|^{(\infty)} \ \leq\ \left(\frac{C\,g}{\rho_0^{1/2}}\right)^{2\delta_{(M+N),0}} \left(C\,g\,\rho_0^{\mu/2}\right)^{M+N} \tag{V.168}$$

in Corollary V.4, concerning $\left\|w^{(0)}_{M,N}\right\|^{(\infty)}_{\mu}$, and in [5], for $\left\|\partial_r w^{(0)}_{M,N}\right\|^{(1)}_{\mu}$. For $M = N = 0$, (V.168) implies that, in particular,

$$\sup_{z\in D_1}\left|\partial_r\left(T_{(0)}[r;z]-r\right)\right| \ =\ \sup_{z\in D_1}\left|\partial_r w^{(0)}_{0,0}[r;z]\right| \ \leq\ C\,g^2\,\rho_0^{-1}\,, \tag{V.169}$$

$$\sup_{z\in D_1}\,|\,Z_{(0)}[z]\,| \ =\ \sup_{z\in D_1}|w^{(0)}_{0,0}[r=0;z]| \ \leq\ C\,g^2\,\rho_0^{-1}\,. \tag{V.170}$$

Putting together (V.168)–(V.170), we obtain

$$\sup_{D_1}\left\|\,H_{(0)}[z]\ -\ H_f\,\right\|' \ \leq\ \varepsilon^{(0)}\,, \tag{V.171}$$

choosing

$$\rho_0 \ \leq\ g^{3/2}\,,\ \ \varepsilon^{(0)}\ :=\ C\,g^{1/2}\ \ \text{and}\ \ \ \ 2\sqrt{\pi}\xi\ :=\ \min\{1/12\,,\rho^{\frac{3+\mu}{4}}\}\,. \tag{V.172}$$

We point out that our analysis in Chapter IV, in particular (IV.98),gives us control over all $z \notin D(E_j,\ \rho_0/2)$, provided $\rho_0 \gg g^2$. We choose $\rho_0 := g^{3/2}$ because it is convenient.

Theorem V.9 *Let $\mu > 0$ and $0 < \rho^{\mu/2} < 1/8$. Assume that $\rho_0 \leq g^{3/2}$ and that $g > 0$ is chosen such that*

$$g^{2+3\mu}\cdot\rho^{-(5+\mu)} \tag{V.173}$$

is sufficiently small. Define, for $n = 1\,, 2\,, \ldots,$

$$H_{(n)}\ :=\ \mathcal{R}^n(H_{(0)})\ \equiv\ (E_{(n)}, T_{(n)}, \{w^{(n)}\})\,. \tag{V.174}$$

Then, for all $n \geq 1$,

$$\left\|\,H_{(n)}\ -\ T^T_{(n)}\,\right\| \ \leq\ \left(8\rho^{\mu/2}\right)^n\cdot\varepsilon^{(0)}\,, \tag{V.175}$$

where $T^T_{(n)}[\hat{z}] := \rho^{-1}T_{(n-1)}\left[Z^{-1}[\hat{z}]\right]$

Proof: Theorem V.9 follows from Theorem V.8 and (V.171), by induction. ∎

V.3.3 Cuspidal Domains of (possible) Spectrum and Fixpoints of $\mathcal{R}$

Next, we investigate the convergence of the composition of the bijections $\widehat{Z}^{-1}_{(n)}$ which, according to (V.174), map

$$\widehat{Z}^{-1}_{(n)} \,:\, D_1 \,\to\, \mathcal{U}^{(\mathrm{in})}_{(n)} := \Big\{ z \in D_1 \;\Big|\; |z - E_{(n)}[z]| \leq (\rho/2) \Big\} \,. \tag{V.176}$$

We recall from (V.75) that $\widehat{Z}_{(0}[z] := \mathrm{e}^{i\vartheta}(z - E_j)$ and define

$$\mathcal{Z}^{-1}_{(n)} \;:=\; Z^{-1}_{(0)} \circ Z^{-1}_{(1)} \circ Z^{-1}_{(2)} \circ \cdots \circ Z^{-1}_{(n)} \quad \text{and} \quad S_{(n)} \;:=\; \mathcal{Z}^{-1}_{(n)}\,(D_1) \;. \tag{V.177}$$

Lemma V.10 *For all $n \in N$,*

$$S_{(0)} \supseteq S_{(1)} \supseteq \ldots \supseteq S_{(n)} \,, \tag{V.178}$$

$$\rho_0 \left(\frac{\rho}{4}\right)^n \;\leq\; \text{inner rad}\left(S_{(n)}\right) \;\leq\; \text{outer rad}\left(S_{(n)}\right) \;\leq\; \rho_0\,(4\rho)^n \;, \tag{V.179}$$

where inner rad(A) *is the supremum of the radii of all balls contained in a set, A, and* outer rad(A) *is the infimum of the radii of all balls containing A. Moreover, there exists a unique $E_j(g) \in \mathbb{C}$ such that $\{E_j(g)\} = \bigcap_{n\in N} S_{(n)}$. This number satisfies*

$$\Big| \mathcal{Z}^{-1}_{(n)}[E_j(g)] \,-\, E_j(g) \Big| \;\leq\; \Big(8\rho^{\mu/2}\Big)^n \,\varepsilon_{(0)} \,. \tag{V.180}$$

Proof: The uniqueness of $E_j(g)$ and (V.180) is an immediate consequence of (V.179) and (V.178). In turn, (V.179) and (V.178) follow directly from (V.176), (V.177) and the fact that $|\partial_z E_{(n)}| \leq 1/2$ on $\mathcal{U}^{(\mathrm{in})}_n$, resulting from (V.175). ∎

Theorem V.11 *The spectrum of $H_g(\theta)$ in the region*

$$\left[\frac{2}{3}E_{j-1} + \frac{1}{3}E_j \;,\;\; \frac{1}{2}E_j + \frac{2}{3}E_{j+1}\right] \;+\; i[-C\,,C] \,, \tag{V.181}$$

for $C > 0$, $\mathrm{Im}(\theta) > 0$, and $g > 0$ sufficiently small, is contained in the cuspidal domain

$$E_j(g) + \mathrm{e}^{-i\vartheta} \Big\{ a + ib \;\Big|\; a > 0 \,,\; |b| \leq a^{1+\frac{\mu}{2}} \Big\} \;. \tag{V.182}$$

Proof: For $|z - E_j| \geq \frac{1}{2}g^{3/2}$, we appeal to (IV.98) in Subsection IV.4.1, noticing that $\mu > 0$ implies that $\beta \geq 1$.

It remains to consider $z \in D(E_j, \frac{1}{2}g^{3/2})$. From a Neumann series expansion as in (V.109)–(V.106), we obtain that, for any $z \in \mathcal{U}_{(n)}^{(\mathrm{out})}$, where

$$\mathcal{U}_{(n)}^{(\mathrm{out})} := \left\{ z \in D_1 \setminus \mathcal{U}_{(n)}^{(\mathrm{in})} \;\middle|\; \left| \arg\left(z - E_{(n)}[z] \right) \right| \geq C\varepsilon^{(0)}(8\rho^{\mu/2})^n \right\}, \qquad \text{(V.183)}$$

$H^{(n)}[z]$ is invertible. By the property of the Feshbach map (see Theorem IV.1), this implies that

$$\mathcal{Z}_{(n)}^{-1}\left(\mathcal{U}_{(n)}^{(\mathrm{out})} \right) \subseteq \rho\left[H_g(\theta) \right], \qquad \text{(V.184)}$$

the resolvent set of $H_g(\theta)$. The cuspidal domain in (V.182) is then easily seen to be contained in the complement of

$$\bigcup_{n=1}^{\infty} \mathcal{Z}_{(n)}^{-1}\left(\mathcal{U}_{(n)}^{(\mathrm{out})} \right). \qquad \blacksquare \qquad \text{(V.185)}$$

V.4 Existence of Resonances

Our last topic is the proof of the existence of a resonance in the vicinity of E_j. More precisely, we now prove

Theorem V.12 *Let $E_j(g)$ be the number constructed in Lemma V.10 and assume that g is sufficiently small. Then $E_j(g)$ is an eigenvalue of $H_g(\theta)$ with normalized eigenvector, $\psi_g(\theta) \in H_{el} \otimes F$, that has a non-vanishing overlap with $\varphi_{el,j} \otimes \Omega$.*

Proof: The key element of our proof consists in estimating the overlap of $\psi_g(\theta)$ with $\varphi_{el,j} \otimes \Omega$. We obtain $\psi_g(\theta)$ as a limit of a sequence, $\{\psi_m\}_{m>0}$, of eigenvectors of regularized Hamiltonians, H^m, with infrared cutoff, $m > 0$. The operators H^m are defined by replacing $G_{M,N}^{(\theta)}\left(k^{(M)}, \tilde{k}^{(N)} \right)$ in (IV.21) by

$$G_{M,N;m}^{(\theta)}\left(k^{(M)}; \tilde{k}^{(N)} \right) := G_{M,N}^{(\theta)}\left(k^{(M)}; \tilde{k}^{(N)} \right) \prod_{j=1}^{M} \chi\left[|k_j| \geq m \right] \cdot \prod_{j=1}^{N} \chi\left[|\tilde{k}_j| \geq m \right]. \qquad \text{(V.186)}$$

This is the same idea as the one used in Section II.2. Now we apply the analysis of the previous section to H^m. Obviously, $H_{(0)}^m$ obeys the same bounds as $H_{(0)}$

does: identifying $H^m_{(0)}$ with $\left(E^m_{(0)},\, T^m_{(0)},\, \{w^{(0),m}\}\right)$, we have

$$\left\|\left(E^m_{(0)},\, T^m_{(0)},\, \{w^{(0),m}\}\right)\right\| \;\leq\; \varepsilon^{(0)}\,, \tag{V.187}$$

and obtain a sequence of Hamiltonians, $\{\, H^m_{(n)}\,\}_{n\in\mathbb{N}_0}$, which we represent by $\left(E^m_{(n)},\, T^m_{(n)},\, \{w^{(n),m}\}\right) := \mathcal{R}^n\left(H^m_{(0)}\right)$. By Theorem 2, we have

$$\left\|\left(E^m_{(n)},\, T^m_{(n)} - T^{T,m}_{(n-1)},\, \{w^{(n),m}\}\right)\right\| \;\leq\; \left(8\rho^{\mu/2}\right)^n \varepsilon^{(0)}\,. \tag{V.188}$$

Moreover, the scaling relation (V.140) implies that

$$w^{(n),m}_{M,N} \;\equiv\; 0\,, \tag{V.189}$$

for all $M+N\geq 1$, provided

$$\rho_{(n)} \;=\; \rho_0\,\rho^n \;<\; m\,. \tag{V.190}$$

Thus, for all $n > \frac{ln(m)-ln(\rho_0)}{ln(\rho)}$,

$$H^m_{(n)}[z] \;=\; T^m_{(n)}\,[H_f; z] + E^m_{(n)}[z]\,, \tag{V.191}$$

and

$$\left(H^m_{(n)}[z] + E^m_{(n)}[z] - z\right)\Omega \;=\; \left(E^m_{(n)}[z] - z\right)\cdot\Omega \;=\; 0\,, \tag{V.192}$$

provided that $E^m_{(n)}[z] = z$. Now since $|\partial_z E^m_{(n)}[z]| \leq 1/2$, the equation $E^m_{(n)}[z] = z$ has a unique solution, $\lambda^m_{(n)} \in D_1$, by Rouche's Theorem, and hence

$$\mathrm{Ker}\left\{H^m_{(n)}\left(\lambda^m_{(n)}\right)\lambda^m_{(n)}\right\} \;=\; \mathbb{C}\cdot\Omega\,. \tag{V.193}$$

By Theorem IV.1, we thus obtain

$$\mathrm{Ker}\left\{H^m_{(n-1)}\left(\lambda^m_{(n-1)}\right) - \lambda^m_{(n-1)}\right\} \;=\; \mathbb{C}\cdot\tilde{\psi}^m_{(n-1)}\,. \tag{V.194}$$

where $\lambda^m_{(n-1)} := \left(\widehat{Z}^m_{(n)}\right)^{-1}\left[\lambda^m_{(n)}\right]$ and

$$\tilde{\psi}^m_{(n-1)} \;= \tag{V.195}$$
$$\left(\chi_\rho + \overline{\chi}_\rho\left[\overline{\chi}_\rho H^m_{(n-1)}\left(\lambda^m_{(n-1)}\right)\overline{\chi}_\rho - \lambda^m_{(n-1)}\right]^{-1}\overline{\chi}_\rho W^m_{(n-1)}\chi_\rho\right) U_f[-\ln\rho]^*\;\tilde{\psi}^m_{(n)}\,,$$

where $\tilde{\psi}^m_{(n)} := \Omega$. By Theorem V.9 and Lemma V.2, we have that

$$\left\| \overline{\chi}_\rho \left[\overline{\chi}_\rho H^m_{(n-1)} \overline{\chi}_\rho - \lambda^m_{(n-1)} \right]^{-1} \overline{\chi}_\rho W^m_{(n-1)} \chi_\rho \right\| \;\leq\; \left(8\rho^{\mu/2} \right)^n \varepsilon^{(0)} \rho^{-1/2} \,. \tag{V.196}$$

Thus

$$\|\tilde{\psi}^m_{(n-1)}\| \;\leq\; \left(1 + \varepsilon^{(0)} \rho^{-1/2} (8\rho^{\mu/2})^{n-1} \right) \cdot \|\tilde{\psi}^m_{(n)}\| \,. \tag{V.197}$$

Proceeding recursively, we obtain from (V.195) and (V.197) an eigenvector $\tilde{\psi}^m$ of H^m, corresponding to the eigenvalue $\lambda^m := \left(\mathcal{Z}^m_{(n)} \right)^{-1} \left[\lambda^m_{(n)} \right]$, whose norm is bounded by

$$\|\tilde{\psi}^m\| \;\leq\; \prod_{j=0}^{n} \left(1 + \varepsilon^{(0)} \rho^{-1/2} (8\rho^{\mu/2})^{n-1} \right) \;\leq\; \exp\left[\frac{C\varepsilon^{(0)}}{\rho^{1/2}} \left(\frac{1}{1 - (8\rho^{\mu/2})} \right) \right] < \infty \,. \tag{V.198}$$

Moreover, $|\langle \tilde{\psi}^m | \varphi_{el,j} \otimes \Omega \rangle| = 1$. Passing to $\psi^m := \|\tilde{\psi}^m\|^{-1} \tilde{\psi}^m$, we conclude that H^m has the eigenvalue λ^m with normalized eigenvector ψ^m. This eigenvector has a non-vanishing overlap

$$|\langle \psi^m \,|\, \varphi_{el,j} \otimes \Omega \rangle| \;\geq\; \exp\left[\frac{-C\varepsilon^{(0)}}{\rho^{1+\frac{\mu}{2}}} \left(\frac{1}{1 - \rho^{\mu/2}} \right) \right] > 0 \,, \tag{V.199}$$

uniformly in $m \to 0$. Now, we proceed in analogy to Section II.5. Indeed, we appeal to the proof of Theorem II.8 to establish the analogues of (II.31)–(II.35), and we refer the reader to [5] for details. Essentially, it remains to show that $m \mapsto H^m$ is a norm-resolvent continuous family of operators for $m \in [0, m_0]$ and $m_0 > 0$ sufficiently small. We define $\psi_g(\theta) := w - \lim_{m \to 0} \psi^m$ for a suitable subsequence $\{\psi^m\}_{m>0}$, noting that $\psi_g(\theta) \neq 0$, by (V.199). Thus, $\psi_g(\theta) \neq 0$ is the desired eigenvector of $H_g(\theta) = H_{m=0}$ since, by norm-resolvent continuity, $H_g(\theta)\psi_g(\theta) = E_j(g)\psi_g(\theta)$, for $E_j(g) := \lim_{m \to 0} \lambda^m$, indeed. ∎

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