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Fusion of q-tensor operators : quasi-Hopf-algebraic point of view.

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Abstract

Tensor operators associated with a given quantum Lie algebra $U_q(\mathcal{J})$ admit a natural description in the R-matrix language. Here we employ the R-matrix approach to discuss the problem of fusion of tensor operators. The most interesting case is provided by the quantum WZNW model, where, by construction, we deal with sets of linearly independent tensor operators. In this case the fusion problem is equivalent to construction of an analogue $\mathcal{F}(p)$ of the twisting element $\mathcal{F}$ which is employed in Drinfeld's description of quasi-Hopf algebras. We discuss the construction of the twisting element $\mathcal{F}(p)$ in a general situation and give illustrating calculations for the case of the fundamental representation of $U_q(sl(2))$.

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I. INTRODUCTION

§ 1.1. Motivations and notations

The theory of tensor operators has arisen originally as a result of a group-theoretical treatment of quantum mechanics [1]. And conversely, the further development of the representation theory was inspired by the physical interpretation of its mathematical content. Relatively recent appearance of the theory of quantum groups [2] has led to development of the theory of q-deformed tensor operators [3, 4]. The latter turned out to be not purely mathematical construction; it is employed, in particular, in the description of the quantum WZNW model [5, 6, 7].

In the present paper we discuss some aspects of the *fusion* procedure for (deformed) tensor operators in its R-matrix formulation [8]. We consider the special case of the fusion scheme – a construction of a set of basic tensor operators for given irrep ρ^K if we are given those for two other irreps ρ^I and ρ^J (and ρ^K appears in decomposition of $\rho^I \otimes \rho^J$). It turns out that this problem is closely related to Drinfeld's construction of quasi-Hopf algebras [9]. Our aim is to obtain exact prescriptions applicable in practice, but to formulate the problem precisely we need to give first rather detailed introduction to the subject.

We suppose that the reader is familiar with the notion of Hopf algebra. The latter is an associative algebra $\mathcal{G}$ equipped with unit $e \in \mathcal{G}$, a homomorphism $\Delta : \mathcal{G} \mapsto \mathcal{G} \otimes \mathcal{G}$ (the *co-product*), an anti-automorphism $S : \mathcal{G} \mapsto \mathcal{G}$ (the *antipode*) and a one-dimensional representation $\epsilon : \mathcal{G} \mapsto \mathbb{C}$ (the *co-unit*) which obey a certain set of axioms [10]. A quasi-triangular Hopf algebra [2] possesses in addition an invertible element $R \in \mathcal{G} \otimes \mathcal{G}$ (the *universal R-matrix*) obeying certain relations which, in particular, imply the Yang-Baxter equation.

Throughout the paper we shall use so-called R-matrix formalism [11, 12]. Let us recall that its main ingredients are operator-valued matrices (L-operators)

$$L^I_+ = (\rho^I \otimes id)\, R_+ \ , \quad L^I_- = (\rho^I \otimes id)\, R_- \tag{1.1}$$

and numerical matrices (R-matrices)

$$R^{IJ}_+ = (\rho^I \otimes \rho^J)\, R_+ \ , \quad R^{IJ}_- = (\rho^I \otimes \rho^J)\, R_- \ , \tag{1.2}$$

with ρ^I, ρ^J being irreps of $\mathcal{G}$ and $R_+ = R$, $R_- = (R')^{-1}$. Here and further on $'$ stands for permutation in $\mathcal{G} \otimes \mathcal{G}$.

Our consideration is restricted to the case of $\mathcal{G} = U_q(\mathcal{J})$ with $|q| = 1$ and with $\mathcal{J}$ being a semi-simple Lie algebra (the case of $\mathcal{G}$ being an arbitrary semi-simple modular Hopf algebra needs some additional technique; see the discussion in [7]). For simplicity we assume also that q is not a root of unity.

We perform all explicit computations only in the case of $U_q(sl(2))$, but they can be certainly repeated for, at least, $U_q(sl(n))$. Let us underline also that, although we deal with deformed tensor operators and keep index q in some formulae, the classical (i.e., non-deformed) theory is recovered in the limit $q = 1$ and, therefore, it does not need special comments.

§ 1.2. (Deformed) tensor operators, generating matrices

Let the given quasi-triangular Hopf algebra $\mathcal{G}$ be a symmetry algebra for some physical model. This means that the operators corresponding to the physical variables in this model are classified with respect to their transformation properties under the adjoint action of $\mathcal{G}$. Recall that if $\mathcal{H}$ is a certain Hilbert space such that $\mathcal{G} \subset \mathrm{End}\,\mathcal{H}$, then ($q$-deformed) adjoint

action of an element $\xi \in \mathcal{G}$ on some element $\eta \in \mathrm{End}\,\mathcal{H}$ is defined as follows [3, 4]:

$$(\mathrm{ad}_q\xi)\ \eta\ =\ \sum_k\ \xi^1_k\ \eta\ S(\xi^2_k), \tag{1.3}$$

where ξ^a_k are the components of the co-product $\Delta\xi = \sum\limits_k \xi^1_k \otimes \xi^2_k \in \mathcal{G}\otimes\mathcal{G}$ and $S(\xi) \in \mathcal{G}$ is the image of ξ under action of the antipode.

From the physical point of view, the space $\mathcal{H}$ in (1.3) is the Hilbert space of the model in question. For $\mathcal{G}$ being a (quantum) Lie algebra it is often chosen as the corresponding model space, $\mathcal{M} = \bigoplus_I \mathcal{H}_I$ (I runs over all highest weights and each subspace $\mathcal{H}_I$ appears with multiplicity one).

Let $\rho^J : \mathcal{G} \mapsto \mathrm{End}\ V^J$ be a highest weight J irrep of $\mathcal{G}$ with the carrier vector space V^J of the dimension δ_J. The set of operators $\{T^J_m\}^{\delta_J}_{m=1}$ acting on the Hilbert space $\mathcal{H}$ is called a *tensor operator* (of highest weight J) if

$$(\mathrm{ad}_q\xi)\ T^J_m\ =\ \sum_n\ T^J_n\ \Big(\rho^J(\xi)\Big)_{nm} \quad \text{for all } \xi \in \mathcal{G}\ . \tag{1.4}$$

If $\{T^I_m\}^{\delta_I}_{m=1}$ and $\{T^J_{m'}\}^{\delta_J}_{m'=1}$ are tensor operators acting on the same Hilbert space, then, using corresponding (deformed) Clebsch-Gordan coefficients, we can construct tensor operator of weight K as follows [3, 4]:

$$T^K_{m''}\ =\ \sum_{m,m'}\ \left\{\begin{array}{ccc} I & J & K \\ m & m' & m'' \end{array}\right\}_q\ T^I_m\ T^J_{m'}\ . \tag{1.5}$$

This formula describes *fusion* of tensor operators.

In the case of $\mathcal{G} = U_q(sl(2))$ tensor operators, $\{T^J_m\}^J_{m=-J}$, are labeled by spin J and the definition (1.4) acquires the form:

$$\begin{array}{rcl} X^\pm\ T^J_m\ q^H - q^{H\mp1}\ T^J_m\ X^\pm & = & \sqrt{[J\mp m][J\pm m+1]}\,T^J_{m\pm1}, \\ q^H\ T^J_m\ q^{-H} & = & q^m\ T^J_m\ , \end{array} \tag{1.6}$$

where $[x] = (q^x - q^{-x})/(q - q^{-1})$ is a q-number and $X_\pm$, H are the generators with the commutation relations

$$[\,H\,,\,X_\pm\,] = \pm X_\pm\ , \quad [\,X_+\,,\,X_-\,] = [\,2\,H\,]\ .$$

An example of (deformed) tensor operator (of spin 1) is provided by the following set of combinations of the generators:

$$T^1_1\ =\ q^{-H}\,X_+\ , \quad T^1_0\ =\ (q^{-1}\,X_-\,X_+ - q\,X_+\,X_-)/\sqrt{[2]}\ , \quad T^1_{-1}\ =\ -q^{-H}\,X_-\ . \tag{1.7}$$

Notice, however, that this is a rather special case because, generally speaking, components of tensor operators act on the model space as shifts between different subspaces $\mathcal{H}_I$, whereas for the components T^1_m in (1.7) $\mathcal{H}_I$ are invariant subspaces.

Let us remark that along with the tensor operator of *covariant* type introduced in (1.4) one can define a *contravariant* tensor operator as the set of operators $\{\overline{T}^J_m\}^{\delta_J}_{m=1}$ obeying the following relations:

$$(\mathrm{ad}_q\xi)\ \overline{T}^J_m = \sum_n \Big(\rho^J(S(\xi))\Big)_{mn}\ \overline{T}^J_n \quad \text{for all } \xi \in \mathcal{G}\ . \tag{1.8}$$

Further we shall consider only the covariant case since the theory and computations for the contravariant case are quite analogous.

In the case of quasi-triangular Hopf algebra we can describe tensor operators using R-matrix language. Let ρ^J be a highest weight J irrep of $\mathcal{G}$ with the carrier space V^J of dimension δ_J. Let $U^J \in \mathrm{End}\, V^J \otimes \mathrm{End}\, \mathcal{H}$ be a matrix obeying the following R-matrix relations:

$$\overset{1}{L}{}^I_{\pm}\ \overset{2}{U}{}^J = \overset{2}{U}{}^J R^{IJ}_{\pm}\ \overset{1}{L}{}^I_{\pm}\ , \tag{1.9}$$

where $L^I_{\pm}$ and $R^{IJ}_{\pm}$ are defined as in (1.1)-(1.2). Equations (1.9) are equivalent [8] to the statement that each row of U^J satisfies (1.4); that is, all rows of U^J are tensor operators of weight J. We shall refer to U^J as *generating matrices* because, according to the Wigner-Eckart theorem (see, e.g., the comments in [8]), matrix elements of entries of U^J evaluated on vectors from $\mathcal{H}$ give the (q-deformed) Clebsch-Gordan coefficients. Notice that if U^J obeys (1.9) and M is a matrix with entries commuting with all the elements from $\mathcal{G}$, then

$$\widetilde{U}^J \ = \ M\, U^J \tag{1.10}$$

also obeys (1.9), i.e., $\widetilde{U}^J$ also is a generating matrix.

The matrix U^J in (1.9) may have an arbitrary number of rows. However, it is more natural to consider the case of U^J being a square matrix; therefore, from now on we shall regard it as $\delta_J \times \delta_J$ matrix.

Now let U^I and U^J be two generating matrices. The fusion formula (1.5) can be written in the R-matrix language as follows [8]:

$$U^{IJ}_K \ = \ P^{IJ}_K\, F^{IJ}\ \overset{2}{U}{}^J\ \overset{1}{U}{}^I\, P^{IJ}_K \quad \in \mathrm{End}\,(V^I \otimes V^J) \otimes \mathrm{End}\, \mathcal{H}\ , \tag{1.11}$$

where the l.h.s. is a new generating matrix of weight K written in the basis of $V^I \otimes V^J$. Here F^{IJ} is an arbitrary $(\delta_I\, \delta_J) \times (\delta_I\, \delta_J)$ matrix whose entries commute with all elements of $\mathcal{G}$; and $P^{IJ}_K \in \mathrm{End}\,(V^I \otimes V^J)$ stands for the projector (i.e., $(P^{IJ}_K)^2 = P^{IJ}_K$) onto the subspace in $V^I \otimes V^J$ corresponding to the representation ρ^K (cf. §3.2).

One can rewrite (1.11) in the conventional basis of the space V^K:

$$U^K_{mn} \ = \ e^t_m\, U^{IJ}_K\, e_n, \quad m,n \ = \ 1,..,\delta_K. \tag{1.12}$$

Here $\{e_n\}$ is an orthonormal set of the eigenvectors of the projector P^{IJ}_K; that is, $e^t_m\, e_n = \delta_{mn}$ and $P^{IJ}_K = \sum_{n=1}^{\delta_K} e_n \otimes e^t_n$.

Formula (1.11) resembles the fusion formula for R-matrices [12]:

$$\overset{1,32}{R}{}^{LK}_{\pm} = \overset{23}{P}{}^{IJ}_K\ \overset{13}{R}{}^{LJ}_{\pm}\ \overset{12}{R}{}^{LI}_{\pm}\ \overset{23}{P}{}^{IJ}_K\ , \tag{1.13}$$

where the l.h.s. stands for $R^{LK}_{\pm}$ written in the basis of $V^L \otimes V^I \otimes V^J$ and we use notations of [12]. Of course, the origin of both eqs. (1.11) and (1.13) is the Hopf structure of $\mathcal{G}$.

The fusion formulae given above are of direct practical use since they allow to construct corresponding objects (generating matrices and R-matrices) for higher representations starting with those for the fundamental irreps.

Let us now introduce the Clebsch-Gordan maps, $\mathrm{C}[IJK] : V^I \otimes V^J \mapsto V^K$ and $\mathrm{C}'[IJK] : V^K \mapsto V^I \otimes V^J$. They are given by[1]

$$\mathrm{C}[IJK] = \sum\nolimits_{n=1}^{\delta_K} \widehat{e}_n \otimes e^t_n\ , \quad \mathrm{C}'[IJK] = \sum\nolimits_{n=1}^{\delta_K} e_n \otimes \widehat{e}^{\,t}_n\ , \tag{1.14}$$

[1] $\mathrm{C}[IJK]$ and $\mathrm{C}'[IJK]$ in (1.14) can be regarded as rectangular matrices with numerical entries if vectors in V^I, V^J, V^K are realized as usual numerical vectors. They act on these vectors by matrix multiplication from the left. For instance, the result of action $\mathrm{C}'[IJK]$ on a vector $a = \sum_n a_n e_n \in V^K$ is $\mathrm{C}'[IJK]\, a = \sum_n a_n \widehat{e}_n \equiv \widehat{a}$ – the same vector but written in the basis of the tensor product $V^I \otimes V^J$.

where $\widehat{e}_n$ stands for the vector e_n rewritten in the basis of the space V^K (cf. §3.2). The main properties of the CG maps are:

$$\mathrm{C}[IJK]\,\Big((\rho^I\otimes\rho^J)\,\Delta(\,\xi\,)\Big)\;\mathrm{C}'[IJK]\;=\;\rho^K(\xi)\quad\text{for any }\xi\in\mathcal{G}, \tag{1.15}$$

$$\sum_K \mathrm{C}'[IJK]\,\mathrm{C}[LMK]=\delta_{IL}\,\delta_{JM},\qquad \mathrm{C}[IJK]\,\mathrm{C}'[IJL]=\delta_{KL}\;. \tag{1.16}$$

With the help of the CG maps we can rewrite eqs. (1.12)-(1.13) in the following form:

$$U^K \;=\; \mathrm{C}[IJK]\,F^{IJ}\;\overset{2}{U}{}^J\;\overset{1}{U}{}^I\,\mathrm{C}\,'[IJK]\;, \tag{1.17}$$

$$R_\pm^{LK} \;=\; \overset{23}{\mathrm{C}}\,[IJK]\;\overset{13}{R}{}_\pm^{LJ}\;\overset{12}{R}{}_\pm^{LI}\;\overset{23}{\mathrm{C}}\,'[IJK]\;. \tag{1.18}$$

§ 1.3. Exact generating matrices

For generating matrices, as they have been defined above, rows are not necessarily linearly independent tensor operators. However, in the case of $\mathcal{G}=U_q(sl(n))$ (and very probably even in the case of $U_q(\mathcal{J})$ for any semi-simple $\mathcal{J}$) there exists a scheme which allows to obtain an example of generating matrix with all rows being linearly independent tensor operators. Actually, this scheme has been developed in studies of the quantum WZNW model [5, 6, 7, 13]. Let us now describe it (remember that we deal with the case of $|q|=1$).

Let $\mathcal{J}$ be a semi-simple Lie algebra of rank n. Introduce n-dimensional vector $\vec{p}=2J+\rho$, where J runs over all highest weights and ρ is the sum of the positive roots of $\mathcal{J}$. Let $\mathcal{C}$ be a commutative algebra of functions on the weight space of $U_q(\mathcal{J})$, i.e., an algebra of functions depending on the components of $\vec{p}$.

Next, let us introduce two auxiliary objects: $D=q^{2\vec{H}\otimes\vec{p}}\in\mathcal{G}\otimes\mathcal{C}$ and $\Omega=q^{4\vec{H}\otimes\vec{H}}\in\mathcal{G}\otimes\mathcal{G}$, where $\vec{A}\otimes\vec{B}$ is understood as $\sum_{i=1}^n A_i\otimes B_i$; and H_i are the basic generators of the Cartan subalgebra of $\mathcal{G}$. Finally, we define the homomorphism $\sigma:\mathcal{C}\mapsto\mathcal{G}\otimes\mathcal{C}$, such that

$$\sigma(\vec{p})\;=\;e\otimes\vec{p}+2\vec{H}\otimes e. \tag{1.19}$$

Now we can look for objects $\mathcal{R}_\pm(\vec{p})\in\mathcal{G}\otimes\mathcal{G}\otimes\mathcal{C}$ [obeying the standard relation $\mathcal{R}_-(\vec{p})=(\mathcal{R}'_+(\vec{p}))^{-1}$, where $'$ means permutation of the first two tensor factors] which are solutions of the following equations:

$$\overset{12}{\mathcal{R}}_\pm(\vec{p})\;\overset{13}{\mathcal{R}}_\pm(\vec{p}_2)\;\overset{23}{\mathcal{R}}_\pm(\vec{p})\;=\overset{23}{\mathcal{R}}_\pm(\vec{p}_1)\;\overset{13}{\mathcal{R}}_\pm(\vec{p})\;\overset{12}{\mathcal{R}}_\pm(\vec{p}_3)\;, \tag{1.20}$$

$$[\,\mathcal{R}_\pm(\vec{p})\,,\,q^{H_i}\otimes q^{H_i}\otimes e\,]\;=\;0\quad\text{ for all }\;i\;, \tag{1.21}$$

$$\mathcal{R}_-(\vec{p})\,(e\otimes D)\;=\;(\Omega\otimes e)\,(e\otimes D)\,\mathcal{R}_+(\vec{p})\;, \tag{1.22}$$

$$\mathcal{R}^*_\pm(\vec{p})\;=\;\mathcal{R}^{-1}_\pm(\vec{p})\;. \tag{1.23}$$

The subscript $i=1,2,3$ of the argument of $\mathcal{R}_\pm(\vec{p})$ means that this argument is shifted according to (1.19) and the $\vec{H}$-term appears in the i^{th} tensor component.[2] It is easy to verify that for $|q|=1$ eq. (1.23) [unitarity of $\mathcal{R}_\pm(\vec{p})$] is consistent[3] with eqs. (1.20)-(1.22). Eq. (1.21) is the same symmetry condition which is known[4] for the standard R-matrices of $U_q(\mathcal{J})$.

[2] To make these shifts more transparent, let us introduce the element $Q=e^{2\vec{H}\otimes\vec{\mathrm{x}}}$, where components of $\vec{\mathrm{x}}$ are such that $[p_i,\mathrm{x}_j]=\delta_{ij}$. Then $\overset{12}{\mathcal{R}}_\pm(\vec{p}_3)=\overset{3}{Q}{}^{-1}\,\overset{12}{\mathcal{R}}_\pm(\vec{p})\,\overset{3}{Q}$, etc. Notice that the shifted matrices belong to $\mathcal{G}\otimes\mathcal{G}\otimes\mathcal{G}\otimes\mathcal{C}$.

[3] The conjugation of an object belonging to m-fold tensor product $\mathcal{G}^{\otimes m}$ is understood as follows: $(\xi_1\otimes\xi_2\ldots\otimes\xi_m)^*=\xi_1^*\otimes\xi_2^*\ldots\otimes\xi_m^*$.

[4] Recall that the quantization $U(\mathcal{J})\to U_q(\mathcal{J})$ does not deform the co-multiplication for elements of the Cartan subalgebra of $\mathcal{J}$.

In general, for $U_q(\mathcal{J})$ there exists a family of solutions of eq. (1.20). A solution $\mathcal{R}^{IJ}_{\pm}(\vec{p})$ obeying, in addition, the conditions (1.21)-(1.23) is most remarkable among them [5, 6, 15, 21, 7]. Entries of such $\mathcal{R}^{IJ}_{\pm}(\vec{p})$ coincide with the corresponding (deformed) $6j$-symbols.

Let us now consider an element $U \in \mathcal{G} \otimes \mathrm{End}\,\mathcal{H}$ which obeys the equations

$$\mathcal{R}_{\pm}(\vec{p})\ \overset{2}{U}\ \overset{1}{U} \;=\; \overset{1}{U}\ \overset{2}{U}\ R_{\pm}\ , \tag{1.24}$$

$$\overset{1}{U}\ (e \otimes D) \;=\; (\Omega \otimes e)\,(e \otimes D)\ \overset{1}{U}\ , \tag{1.25}$$

$$U^{-1}\,D\,U \;=\; q^{C \otimes e}\,L_{+}\,L_{-}^{-1}\ , \tag{1.26}$$

where $C \in \mathcal{G}$ and $\rho^J(C) = 2J(J+\rho)$. It can be shown [5, 6, 7, 13] that such U (if it exists) is a generating matrix[5] for $\mathcal{G} = U_q(\mathcal{J})$. Notice that (1.20)-(1.22) are nothing but consistency conditions for (1.24)-(1.25). The relation (1.25) is a matrix form of the equation

$$U\,\vec{p} \;=\; \sigma(\vec{p})\,U. \tag{1.27}$$

Eq. (1.26) plays the role of a normalization condition.

Observe that from the group of transformations (1.10) only the following subgroup

$$U \mapsto D^{\alpha}\,U\ ,\quad \alpha \in \mathbb{R} \tag{1.28}$$

survives for the solution U of eqs. (1.24)-(1.26). Additionally, the rescaling

$$U \mapsto (e \otimes f)\,U\,(e \otimes f)^{-1} \tag{1.29}$$

with an arbitrary element $f \in \mathcal{C}$ is allowed.

Let us explain why the generating matrix obeying (1.24)-(1.27) is of particular interest from the point of view of the theory of tensor operators. Notice that the property (1.27) ensures that rows of $U^J = (\rho^J \otimes id)U$ are linearly independent tensor operators.[6] In other words, if for a given irrep ρ^J and a given vector $|I,m\rangle$ in the model space $\mathcal{M}$ of $\mathcal{G}$ we consider the set of vectors $U^J_{ij}|I,m\rangle \;\; i,j = 1,..,\delta_J$, then all non-vanishing vectors in this set are pairwise linearly independent. In particular, if J is a fundamental representation, then entries of U^J provide a set of basic shifts on $\mathcal{M}$. Thus, solutions of (1.24)-(1.26) present very special but, in fact, the most interesting case of generating matrices. We shall call them *exact* generating matrices.

An important from the practical point of view property of exact generating matrices is that the matrix elements $\langle K,m''|U^J_{ij}|I,m'\rangle$ coincide up to some p-dependent factors allowed according to (1.28)-(1.29) with the Clebsch-Gordan coefficients $\left\{ \begin{smallmatrix} I & J & K \\ m & m' & m'' \end{smallmatrix} \right\}_q$ (with the weights I, J, K restricted by the triangle inequality).

Let us tell briefly about the physical content of the relations given above. Equation (1.20) has appeared in various forms in studies of quantum versions of the Liouville [16, 17], Toda [18] and Calogero-Moser [19] models. In these models $\mathcal{R}(\vec{p})$ is interpreted as a dynamical R-matrix. From the point of view of the theory of tensor operators relations (1.19)-(1.27) most closely connected with a quantization of the WZNW model [5, 6, 7]. Here $\mathcal{R}(\vec{p})$ plays a

[5] To be more precise, U and $\mathcal{R}_{\pm}(\vec{p})$ are not matrices but so called *universal* objects. If we fix representations of their $\mathcal{G}$-parts: $U^J = (\rho^J \otimes id)U$, $\mathcal{R}^{IJ}_{\pm}(\vec{p}) = (\rho^I \otimes \rho^J)\mathcal{R}_{\pm}(\vec{p})$, we obtain a generating matrix and $\mathcal{C}$-valued counterparts of the standard R-matrices.

[6] To clarify this statement, we can rewrite eq. (1.27) in the following form: $[U^J, p_i] = 2H_i{}^J U^J$, $i = 1,..,n$. In the conventional basis (where the generators H_i of the Cartan subalgebra are diagonal) the last relation for k-th row of U^J becomes: $[U^J_k, p_i] = 2(H_i{}^J)_{kk}U^J_k$, $i = 1,..,n$. Since the elements H_i are linearly independent, we infer that the rows of U^J generate linearly independent shifts on the space $\mathcal{C}$.

role of the braiding matrix and eqs. (1.24)-(1.26) with appropriate dependence on the spatial coordinate (or its discretized version) describe vertex operators. Let us mention that in the WZNW theory the quantum-group parameter of $\mathcal{G} = U_q(\mathcal{J})$ is given by $q = e^{i\gamma\hbar}$, where $\hbar > 0$ is the Planck constant and the deformation parameter $\gamma > 0$ is interpreted as a coupling constant. This provides the motivation to study the case of $|q| = 1$.

II. FUSION OF EXACT GENERATING MATRICES

§ 2.1. Formulation of the problem

Suppose we are given two generating matrices, U^I and U^J, for some irreps, ρ^I and ρ^I, of $\mathcal{G}$. Then by formula (1.11) we can build up a generating matrix U^K for every irrep ρ^K which appears in the decomposition of $\rho^J \otimes \rho^J$. For the sake of shortness we shall call them *descendant* matrices. However, as we have explained before, it is natural to deal not with all possible generating matrices but only with exact ones, i.e., with those which obey additional equations (1.24)-(1.27) with $\mathcal{R}_{\pm}(\vec{p})$, D and Ω introduced above. Thus, if U^I and U^J are exact generating matrices, it is natural to look for such a matrix F^{IJ} that descendant generating matrices U^K obtained by the formula (1.11) would be also exact.

Let us underline that this problem would not arise if eq. (1.24) contained in the l.h.s. the standard R-matrix instead of $\mathcal{R}(\vec{p})$. Indeed, for an operator-valued matrix $g^J \in \mathrm{End}\, V^J \otimes \mathcal{G}$ (it may be regarded as L-operator type object[7]) which obeys the usual quadratic relations[8]

$$R^{IJ}\ \overset{2}{g}{}^J\ \overset{1}{g}{}^I\ = \overset{1}{g}{}^I\ \overset{2}{g}{}^J\ R^{IJ}\ , \tag{2.1}$$

the fusion formula is well known [eq. (1.13) is its particular realization]:

$$(g^K)_{mn}\ =\ e^t_m\ \overset{2}{g}{}^J\ \overset{1}{g}{}^I\ e_n\ , \tag{2.2}$$

where e_n, $n = 1, .., \delta_K$ are the eigenvectors of the projector P_K^{IJ} introduced in §1.2. For example, in the case of $\mathcal{G} = U_q(sl(2))$, starting with $g^{\frac{1}{2}}$ and applying (2.2) iteratively, one obtains matrices g^J for any spin J:

$$g^0 = 1\,,\quad g^{\frac{1}{2}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\quad g^1 = \begin{pmatrix} a^2 & q^{-\frac{1}{2}}\sqrt{[2]}\,ab & b^2 \\ q^{-\frac{1}{2}}\sqrt{[2]}\,ac & ad + q^{-1}bc & q^{-\frac{1}{2}}\sqrt{[2]}\,bd \\ c^2 & q^{-\frac{1}{2}}\sqrt{[2]}\,cd & d^2 \end{pmatrix}, \ldots \tag{2.3}$$

For generating matrices the fusion problem is more complicated because $\mathcal{R}(\vec{p})$ in eqs. (1.20) and (1.24) is an attribute not of Hopf algebra but of quasi-Hopf algebra. In this section we shall discuss some general aspects of the fusion problem in the quasi-Hopf case. In the next section we shall consider an example – the case of $U_q(sl(2))$.

It should be also underlined that the fusion problem (as it is formulated above) does not appear if the language of *universal* objects (see, e.g., [20, 7]) is used instead of the language of operator-valued matrices. For example, instead of the set of matrices $g^J \in \mathrm{End}\, V^J \otimes \mathcal{G}$ obeying (2.1) we could introduce the element $g \in \mathcal{G} \otimes \mathcal{G}$ and fix its *functoriality* relation as follows:

$$(\Delta \otimes id)\, g\ = \overset{2}{g}\,\overset{1}{g}\ . \tag{2.4}$$

[7] For $\mathcal{G}$ replaced by its dual $\mathcal{G}'$ the matrix g^J is regarded as the quantum group-like element. The fusion formulae (2.2)-(2.3) are also valid in this case.

[8] We prefer this order of auxiliary spaces in (2.1) since it is the same as in (1.24).

Then both quadratic relations (2.1) and fusion formula (2.2) can be obtained from (2.4) with the help of the axioms of quasi-triangular Hopf algebra. In fact, in this approach we actually do not need the fusion formula because each g^J can be obtained simply by evaluation of the universal element g in the corresponding representation: $g^J = (\rho^J \otimes id)g$.

Similarly, we could introduce the universal object $U \in \mathcal{G} \otimes \mathrm{End}\,\mathcal{H}$ with the functoriality relation [7]:

$$(\Delta \otimes id)\, U \;=\; \mathcal{F}\;\overset{2}{U}\,\overset{1}{U}\;, \tag{2.5}$$

where $\mathcal{F}$ obeys a certain set of axioms. Then quadratic relations (1.24) [with $\mathcal{R}(\vec{p})$ constructed from $\mathcal{F}$ and R according to (2.12)] would be consequences of (2.5). Again, fixing representation of $\mathcal{G}$-part of the universal element U, we obtain a generating matrix $U^J = (\rho^J \otimes id)U$ and, therefore, we do not need the fusion formula.

Although the language of universal objects is more convenient in abstract theoretical constructions, in practice we usually do not have explicit formulae for involved universal objects [for instance, the universal R-matrices are known only for $U_q(sl(2))$ and $U_q(sl(3))$]. Therefore, in the present paper we intentionally have adopted the matrix language to discuss how to construct exact generating matrices for an arbitrary irrep from those for given irreps if explicit universal formulae are unknown.

§ 2.2. Quasi-Hopf features

Let us remind that an associative algebra $\mathcal{G}$ equipped with co-product, co-unit and antipode is said to be quasi-Hopf algebra [9] if its co-multiplication is "quasi-coassociative"; that is, for all $\xi \in \mathcal{G}$ we have

$$((id \otimes \Delta)\,\Delta(\xi))\,\Phi \;=\; \Phi\,((\Delta \otimes id)\,\Delta(\xi))\;, \quad (\epsilon \otimes id)\,\Delta(\xi) \;=\; (id \otimes \epsilon)\,\Delta(\xi) \;=\; \xi. \tag{2.6}$$

Here $\Phi \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G}$ is an invertible element (the *co-associator*) which must satisfy certain equations. For quasi-triangular quasi-Hopf algebra it is additionally postulated that there exists an invertible element $\mathcal{R} \in \mathcal{G} \otimes \mathcal{G}$ (the *twisted* R-matrix) such that

$$\mathcal{R}\,\Delta(\xi) \;=\; \Delta'(\xi)\,\mathcal{R} \qquad \text{for all } \xi \in \mathcal{G}\;, \tag{2.7}$$

$$(\Delta \otimes id)\,\mathcal{R} = \Phi_{312}\,\overset{13}{\mathcal{R}}\,\Phi^{-1}_{132}\,\overset{23}{\mathcal{R}}\,\Phi_{123}\;,\;\; (id \otimes \Delta)\,\mathcal{R} = \Phi^{-1}_{231}\,\overset{13}{\mathcal{R}}\,\Phi_{213}\,\overset{12}{\mathcal{R}}\,\Phi^{-1}_{123}\;, \tag{2.8}$$

$$(\epsilon \otimes id)\,\mathcal{R} \;=\; (id \otimes \epsilon)\,\mathcal{R} \;=\; e\;. \tag{2.9}$$

The analogue of Yang-Baxter equation for $\mathcal{R}$ follows from (2.7) and (2.8) and looks like

$$\overset{12}{\mathcal{R}}\;\Phi_{312}\;\overset{13}{\mathcal{R}}\;\Phi^{-1}_{132}\;\overset{23}{\mathcal{R}}\;\Phi_{123} = \Phi_{321}\;\overset{23}{\mathcal{R}}\;\Phi^{-1}_{231}\;\overset{13}{\mathcal{R}}\;\Phi_{213}\;\overset{12}{\mathcal{R}}\;. \tag{2.10}$$

A crucial observation [6, 21, 7] is that the construction of exact generating matrices, which we described in § 1.3, involves the quasi-Hopf algebra $\mathcal{G}_{\mathcal{F}}$ (where $\mathcal{R}_+(\vec{p})$ plays the role of the element $\mathcal{R}$) obtained as a twist of the symmetry algebra $\mathcal{G}$. More precisely, there exists an invertible element $\mathcal{F}(\vec{p}) \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{C}$ such that one can construct with its help from standard co-multiplication and R-matrices (which obey axioms of Hopf algebra) the following objects which obey all the axioms of quasi-triangular quasi-Hopf algebra:[9]

$$\Delta_{\mathcal{F}}(\xi) \;=\; \mathcal{F}^{-1}(\vec{p})\,\Delta(\xi)\,\mathcal{F}(\vec{p}) \quad \text{for all } \xi \in \mathcal{G}\;, \tag{2.11}$$

$$\mathcal{R}_{\pm}(\vec{p}) \;=\; (\mathcal{F}'(\vec{p}))^{-1}\,R_{\pm}\,\mathcal{F}(\vec{p}) \quad \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{C}\;, \tag{2.12}$$

$$\Phi(\vec{p})_{123} \;=\; \overset{12}{\mathcal{F}}{}^{-1}(\vec{p}_3)\,\overset{12}{\mathcal{F}}(\vec{p}) \quad \in \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{G} \otimes \mathcal{C}\;. \tag{2.13}$$

[9] In fact, here we deal with some generalization of the Drinfeld's scheme, since $\mathcal{F}(\vec{p})$, $\mathcal{R}_{\pm}(\vec{p})$ and $\Phi(\vec{p})$ possess additional $\mathcal{C}$-valued tensor component. But all Hopf-algebra operations are applied only to $\mathcal{G}$-parts of these objects.

Here $\vec{p}_i$ denotes, as before, the shift (1.19) with the $\vec{H}$-term appearing in i^{th} tensor component. These formulae show that equation (1.20) introduced above is a particular realization of the abstract form (2.10) of the twisted Yang-Baxter equation.

The fact that $\mathcal{R}_{\pm}(\vec{p})$ introduced in (1.20)-(1.22) admit decomposition of type (2.12) is crucial in the context of the fusion problem for exact generating matrices. Indeed, suppose we are given two exact generating matrices, U^I and U^J, which obey (1.24)-(1.27) with certain $\mathcal{R}^{IJ}_{\pm}(\vec{p})$. Applying the formula (1.11) with some matrix $F^{IJ}(\vec{p})$ (it may be $\mathcal{C}$-valued) to these U^I and U^J, we get new matrix U^{IJ}_K. It automatically obeys (1.9). Moreover, it is easy to verify that exchange relations between U^{IJ}_K and any exact generating matrix U^L have the form (1.24) but contain matrices $\overset{1,32}{R}{}^{LK}_{\pm}$ given by (1.13) in the r.h.s. and some p-dependent R-matrices in the l.h.s. The latter look like following

$$\overset{1,32}{\mathcal{R}}{}^{LK}_{\pm}(\vec{p}) \;=\; \overset{23}{P}{}^{IJ}_{K}\;\overset{23}{F}{}^{IJ}(\vec{p}_1)\;\overset{13}{\mathcal{R}}{}^{LJ}_{\pm}(\vec{p})\;\overset{12}{\mathcal{R}}{}^{LI}_{\pm}(\vec{p}_3)\,(\overset{23}{F}{}^{IJ}(\vec{p}))^{-1}\;\overset{23}{P}{}^{IJ}_{K}\;, \tag{2.14}$$

where, similarly as in (1.13), the basis of $V^L \otimes V^I \otimes V^J$ is used. This is an analogue of the fusion formula (1.13) for standard R-matrices. The demand that the new generating matrix U^{IJ}_K is exact, i.e., in particular, it obeys (1.24), implies that expression (2.14) rewritten in the basis of $V^I \otimes V^J$ must coincide with $\mathcal{R}^{LK}_{\pm}(\vec{p})$. Taking into account that $\mathcal{R}_{\pm}(\vec{p})$ satisfies (2.12) with some element $\mathcal{F} \in \mathcal{G}\otimes\mathcal{G}\otimes\mathcal{C}$, we get the equality

$$\begin{aligned}\mathcal{R}^{LK}_{\pm}(\vec{p}) \;=& \overset{23}{\mathrm{C}}\,[IJK]\;\overset{23}{F}{}^{IJ}(\vec{p}_1)\;\overset{13}{\mathcal{R}}{}^{LJ}_{\pm}(\vec{p})\,(\overset{21}{\mathcal{F}}{}^{IL}(\vec{p}_3))^{-1}\;\overset{21}{\mathcal{F}}{}^{IL}(\vec{p})\;\times \\ \times\;& \overset{12}{\mathcal{R}}{}^{LI}_{\pm}(\vec{p})\,(\overset{12}{\mathcal{F}}{}^{LI}(\vec{p}))^{-1}\;\overset{12}{\mathcal{F}}{}^{LI}(\vec{p}_3)\,(\overset{23}{F}{}^{IJ}(\vec{p}))^{-1}\;\overset{23}{\mathrm{C}}{}'[IJK]\;.\end{aligned} \tag{2.15}$$

The latter is equivalent due to (1.15) and (2.11) to the identity

$$\begin{aligned}\Big(\rho^L\otimes\rho^I\otimes\rho^J\Big)\;(id\otimes\Delta_{\mathcal{F}})\,\mathcal{R} \;=\; (\overset{23}{F}{}^{IJ}(\vec{p}))^{-1}\;\overset{23}{F}{}^{IJ}(\vec{p}_1)\;\overset{13}{\mathcal{R}}{}^{LJ}_{\pm}(\vec{p})\;\times \\ \times\,(\overset{21}{\mathcal{F}}{}^{IL}(\vec{p}_3))^{-1}\;\overset{21}{\mathcal{F}}{}^{IL}(\vec{p})\;\overset{12}{\mathcal{R}}{}^{LI}_{\pm}(\vec{p})\,(\overset{12}{\mathcal{F}}{}^{LI}(\vec{p}))^{-1}\;\overset{12}{\mathcal{F}}{}^{LI}(\vec{p}_3)\;,\end{aligned}$$

which, as we see from (2.8) and (2.13), takes place only if $F^{IJ}(\vec{p}) = \mathcal{F}^{IJ}(\vec{p})$.

§ 2.3. Properties of the twisting element

Let us give a résumé of the previous paragraph. Let U be a universal generating matrix for a given symmetry algebra $\mathcal{G}$ (in the sense of §1.3) and let $\mathcal{R}_{\pm}(p)$ be the corresponding (twisted) R-matrix. Let $\mathcal{F}(\vec{p})$ be the twisting element which transforms the Hopf algebra $\mathcal{G}$ into the quasi-Hopf algebra, $\mathcal{G}_{\mathcal{F}}$, for which $\mathcal{R}_{\pm}(\vec{p})$ is an R-matrix (in the sense of eqs. (2.6)-(2.10)). If we are given two concrete representations of the exact generating matrix, U^I and U^J [and, hence, we know $\mathcal{R}^{IJ}_{\pm}(\vec{p})$], then to construct its another representation U^K we must substitute the matrix $\mathcal{F}^{IJ}(\vec{p}) = (\rho^I\otimes\rho^J)\mathcal{F}(\vec{p})$ into the fusion formula (1.11). An obstacle to application of this prescription is that explicit universal expressions for $\mathcal{F}(\vec{p})$ and $\mathcal{R}(\vec{p})$ are usually unknown. However, one can formulate some conditions which $\mathcal{F}^{IJ}(\vec{p})$ has to obey:

1. $\mathcal{F}^{IJ}(\vec{p})$ is a solution of the following equation (for given $\mathcal{R}^{IJ}_{\pm}(\vec{p})$):

$$R^{IJ}_{\pm}\,\mathcal{F}^{IJ}(\vec{p}) \;=\; (\mathcal{F}^{IJ}(\vec{p}))'\,\mathcal{R}^{IJ}_{\pm}(\vec{p})\;; \tag{2.16}$$

2. $\mathcal{F}^{IJ}(\vec{p})$ obeys the symmetry condition

$$[\,\mathcal{F}^{IJ}(\vec{p}),\, q^{H^I_i}\otimes q^{H^J_i}\otimes e\,] \;=\; 0 \quad \text{for} \;\; i=1,..,n\;; \tag{2.17}$$

3. $\mathcal{F}^{IJ}(\vec{p})$ is such that all the entries of the matrix

$$U_0^{IJ} \;=\; P_0^{IJ}\,\mathcal{F}^{IJ}(\vec{p})\;\overset{2}{U}{}^{J}\,\overset{1}{U}{}^{I}\,P_0^{IJ} \quad \in \mathrm{End}(V^I\otimes V^J)\otimes \mathrm{End}\mathcal{H}\ , \tag{2.18}$$

commute with all entries of generating matrix U^M for any weight M;

4. $\chi^{IJ} = \mathcal{F}^{IJ}(\vec{p})(\mathcal{F}^{IJ}(\vec{p}))^*$ is a p-independent object.

Let us comment these conditions. First of them is equivalent to eq. (2.12); its necessity has been explained in the previous paragraph. However, this is not a sufficient condition since, in general, (2.16) possesses a family of solutions. In principle, we could select the right solution in this family verifying whether a substitution of this solution in fusion formulae (2.14) or (2.15) yields matrices $\mathcal{R}(\vec{p})$ obeying (1.20)-(1.23). But such a verification would be quite cumbrous in practice.

The second condition ensures that the descendant matrix U^K obeys eq. (1.27) and, as a consequence, eq. (1.25). This can be easily checked applying (1.27) to (1.11). In fact, eq. (2.17) implies that for our specific example of quasi-Hopf algebra the co-multiplication on the Cartan subalgebra is not deformed, i.e., it is the same as for $U_q(\mathcal{J})$ and $U(\mathcal{J})$.

The third condition is derived from eq. (2.9) and the property $(\epsilon\otimes id)R = (id\otimes\epsilon)R = e$ of the standard R-matrices (recall that ϵ stands for the trivial one-dimensional representation of $\mathcal{G}$). Indeed, applying $(\epsilon\otimes id)$ or $(id\otimes\epsilon)$ to (1.24), we conclude that $U^0 = (\epsilon\otimes id)U \in \mathrm{End}\mathcal{H}$ commutes with all entries of U^J for any J. For U_0^{IJ} in eq. (2.18) we have: $U_0^{IJ} = U_0 P_0^{IJ}$.[10] Therefore, if the trivial representation $\rho^0 \equiv \epsilon$ appears in the decomposition of the product $\rho^I\otimes\rho^J$, then the condition *3* is non-trivial.

The fourth condition claims, in fact, that the element $\chi = \mathcal{F}(\vec{p})(\mathcal{F}(\vec{p}))^*$ belongs to $\mathcal{G}\otimes\mathcal{G}$ (or, more precisely, that the last tensor component of χ in $\mathcal{G}\otimes\mathcal{G}\otimes\mathcal{C}$ is trivial). To clarify this, let us first recall that for $|q| = 1$ the standard co-multiplication has the following property with respect to the conjugation in $\mathcal{G}$: $\Delta^*(\xi) = \Delta'(\xi^*)$. On the other hand, relations (1.23) imply that (see also [7])

$$(\Delta_{\mathcal{F}}(\xi))^* \;=\; \Delta_{\mathcal{F}}(\xi^*) \quad \text{for all } \xi\in\mathcal{G}\ . \tag{2.19}$$

Next, we observe that eqs. (1.23), (2.8) and (2.19) ensure a unitarity of the co-associator:

$$\Phi^*(\vec{p})_{123} \;=\; \Phi^{-1}(\vec{p})_{123}\ . \tag{2.20}$$

According to (2.13), the latter equation leads to the condition $\overset{12}{\chi}(\vec{p}) = \overset{12}{\chi}(\vec{p}_3)$ or, equivalently, to $\overset{3}{U}(\overset{12}{\chi}(\vec{p}))(\overset{3}{U})^{-1} = \overset{12}{\chi}(\vec{p})$. This implies p-independence of χ.

Actually, the element χ plays an important role in the theory of exact generating matrices. Let us mention here that due to eq. (1.23) it satisfies the following relation:

$$R_{\pm}\,\chi \;=\; \chi'\,R_{\mp}^{-1}\ . \tag{2.21}$$

We shall discuss some other properties of χ below, in §3.4.

[10] Hence $tr(U_0^{IJ})$ coincides (up to a constant) with U^0. The latter can be regarded [8] as a generalization of the quantum determinant [11, 12]. Notice also that it can be written with the help of the CG-maps as follows: $U^0 = \mathrm{C}[IJ0]\,\mathcal{F}^{IJ}(\vec{p})\;\overset{2}{U}{}^{J}\,\overset{1}{U}{}^{I}\,\mathrm{C}'[IJ0]$ (assuming that $\rho^0\equiv\epsilon$ appears in the decomposition of $\rho^I\otimes\rho^J$). See §3.2 for the case of $U_q(sl(2))$.

III. $U_q(sl(2))$ CASE

In this section the preceding discussion will be illustrated on some explicit calculations. Although solutions for the twisted Yang-Baxter equation (1.20) are known [6, 14, 16, 18] for the fundamental representations of $\mathcal{G} = U_q(sl(n))$, we shall consider here only the case of $U_q(sl(2))$. But let us stress that in the more general case of $U_q(sl(n))$ the computations would be essentially the same.

§ 3.1. $\mathcal{R}(p)$ and U in the fundamental representation

As we mentioned above, the twisted Yang-Baxter equation possesses a family of solutions even in the simplest case of the fundamental representation of $U_q(sl(2))$. But imposing additional conditions (1.21)-(1.23), we get the unique solution $\mathcal{R}_{\pm}(p)$ [5, 6, 14] which depends on the single variable $p = 2J+1$ with J being the spin. In fact, $\mathcal{R}_{-}(p)$ coincides with $\mathcal{R}_{+}(p)$ if q is replaced with q^{-1}. The entries of $\mathcal{R}_{\pm}^{IK}(p)$ coincide with values of the $6j$-symbols involving spins I, K and J (see [15] for details). Moreover, the asymptotics of $\mathcal{R}_{\pm}(p)$ in the formal limits $q^p \to +\infty$ and $q^{-p} \to +\infty$ are given by:

$$\begin{array}{rcll} \mathcal{R}_{\pm}(p) & \to & R_{\pm} & \text{when } q^p \to +\infty \ , \\ \mathcal{R}_{\pm}(p) & \to & R_{\mp}^{-1} & \text{when } q^{-p} \to +\infty \ . \end{array} \tag{3.1}$$

That is, in these limits we return to the case of Hopf algebra; in particular, the co-associator becomes trivial: $\Phi_{123} \to e \otimes e \otimes e$. Furthermore, relations (3.1) together with eqs. (2.12), (2.21) allow to add the following condition to the properties of $\mathcal{F}(\vec{p})$ listed in §2.3:

5. (Asymptotic behaviour)

$$\mathcal{F}(q^p \to +\infty) \ = \ e \otimes e \ , \quad \mathcal{F}(q^{-p} \to +\infty) \ = \ \chi \ , \tag{3.2}$$

where the element $\chi \in \mathcal{G} \otimes \mathcal{G}$ was described in §2.3. Let us stress that we have derived this additional condition, in its present form, only for $\mathcal{J} = sl(2)$. It would be interesting to find analogues of (3.2) in the case when $\vec{p}$ has several components.

In the simplest non-trivial case, $I = K = \frac{1}{2}$, the solution $\mathcal{R}_{\pm}^{IK}(p)$ is given by (all non-specified entries are zeros)

$$\mathcal{R}_{+}^{\frac{1}{2}\frac{1}{2}}(p) \ = \ (\mathcal{R}_{-}^{\frac{1}{2}\frac{1}{2}})' \ = \ q^{-1/2} \begin{pmatrix} q & & & \\ & \frac{\sqrt{[p+1][p-1]}}{[p]} & \frac{q^p}{[p]} & \\ & -\frac{q^{-p}}{[p]} & \frac{\sqrt{[p+1][p-1]}}{[p]} & \\ & & & q \end{pmatrix} . \tag{3.3}$$

Here $[x]$ stands, as usually, for q-number.

Now let us turn to the solution $U^{\frac{1}{2}}$ of eqs. (1.24)-(1.26) for $\mathcal{R}_{\pm}^{\frac{1}{2}\frac{1}{2}}(p)$ given by (3.3). It was considered in different contexts in [5, 6, 13] and has been shown to be unique up to the transformations (1.28)-(1.29). In agreement with the general description in §1.3, the entries of $U^{\frac{1}{2}} = \begin{pmatrix} U_1 & U_2 \\ U_3 & U_4 \end{pmatrix}$ act on the model space $\mathcal{M} = \oplus_{J=0}^{\infty} \mathcal{H}_J$ as the basic shifts (see Fig.1).

A particular realization of $U^{\frac{1}{2}}$ can be written (see, e.g., [13]) in terms of multiplication and difference derivative operators for two complex variables:

$$U^{\frac{1}{2}} \ = \ \begin{pmatrix} z_1 \, q^{\frac{1}{2} z_2 \partial_2} & z_2 \, q^{-\frac{1}{2} z_1 \partial_1} \\ -z_2^{-1} [z_2 \partial_2] \, q^{-\frac{1}{2}(z_1 \partial_1 + 1)} & z_1^{-1} [z_1 \partial_1] \, q^{\frac{1}{2}(z_2 \partial_2 + 1)} \end{pmatrix} \frac{1}{\sqrt{[p]}} \ , \tag{3.4}$$

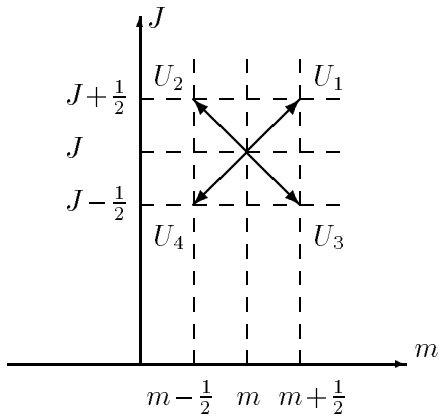

Fig.1 Action of the operators U_i on the model space.

where $p = z_1\partial_1 + z_2\partial_2 + 1$. Entries of (3.4) are operators of basic shifts on the model space realized as $D_q(z_1, z_2)$ – the space of holomorphic functions of two complex variables equipped with such a scalar product (a deformation of the standard one) that the monomials $|J, m\rangle := \frac{z_1^{J+m} z_2^{J-m}}{\sqrt{[J+m]![J-m]!}}$ form an orthonormal basis.

Moreover, the specific realization (3.4) of the exact generating matrix of spin 1/2 may be called *precise* in the following sense. Matrix elements of its entries $\langle J', m'|U_i|J'', m''\rangle$ evaluated on $D_q(z_1, z_2)$ exactly coincide with the CG coefficients $\left\{ \begin{smallmatrix} J' & 1/2 & J'' \\ m' & \pm 1/2 & m'' \end{smallmatrix} \right\}_q$ (four of them are non-vanishing); see [13] for more detailed comments.

Now we encounter the simplest version of the fusion problem – to build up the exact generating matrix U^1 (of spin 1) from $U^{\frac{1}{2}}$. For this purpose we have to find an explicit form of the corresponding twisting element $\mathcal{F}(\vec{p})$ in the fundamental representation.

Before going into the computations let us mention that a universal formula (i.e., applicable for representations of any spin) for solution $\widetilde{\mathcal{R}}(p)$ of eq. (1.20) and a universal expression for $\widetilde{\mathcal{F}}(p)$ obeying (2.12) with this $\widetilde{\mathcal{R}}(p)$ have been obtained in [17]. But this solution $\widetilde{\mathcal{R}}(p)$ does not satisfy (1.22)-(1.23) and, therefore, being evaluated, say, in the fundamental representation it differs from (3.3). Thus, solution $\widetilde{U}$ of (1.24) for such $\widetilde{\mathcal{R}}(p)$ would not be an exact generating matrices in our sense. In particular, the solution for spin 1/2 would differ from the one given by (3.4) and, therefore, would not have the remarkable properties mentioned above.

Although, let us stress that such $\widetilde{U}$ still would be a generating matrix in the sense of definition (1.9). Therefore, one could examine whether it can be converted into an exact generating matrix by means of the transformation $\widetilde{U} = M(p)U$ with $M(p) \in \mathcal{G} \otimes \mathcal{C}$. If such $M(p)$ exists, then the following relations hold:

$$\begin{aligned} \widetilde{\mathcal{F}}(p) &= \Big((\Delta \otimes id)M(p)\Big)\, \mathcal{F}(p)\, \Big(\overset{1}{M}(p_2)\, \overset{2}{M}(p)\Big)^{-1}\,, \\ \widetilde{\mathcal{R}}_{\pm}(p) &= \overset{2}{M}(p_1)\, \overset{1}{M}(p)\, \mathcal{R}_{\pm}(p)\, (\overset{1}{M}(p_2)\, \overset{2}{M}(p))^{-1}\,, \end{aligned}$$

and we can construct our $\mathcal{F}(p)$ from $\widetilde{\mathcal{F}}(p)$ and $M(p)$. However, bearing in mind possible applications in the cases where no universal formulae for $\mathcal{R}(\vec{p})$ are known, it is more instructive to give a direct computation of $\mathcal{F}(p)$.

§ 3.2. Computation of $\mathcal{F}^{\frac{1}{2}\frac{1}{2}}(p)$ and U^1

The matrix $\mathcal{F}^{\frac{1}{2}\frac{1}{2}}(p)$ must satisfy the conditions listed in §2.3. First of all, it must be a solution of the equation (2.16), where $\mathcal{R}_\pm(p)$ in the r.h.s. are given by (3.3) and the standard R-matrices in the l.h.s. are

$$R_+^{\frac{1}{2}\frac{1}{2}} = q^{-1/2}\begin{pmatrix} q & & & \\ & 1 & \omega & \\ & & 1 & \\ & & & q \end{pmatrix}, \quad R_-^{\frac{1}{2}\frac{1}{2}} = q^{1/2}\begin{pmatrix} q^{-1} & & & \\ & 1 & & \\ & -\omega & 1 & \\ & & & q^{-1} \end{pmatrix}, \tag{3.5}$$

with $\omega = q - q^{-1}$. The symmetry condition *2* dictates to look for the solution of eq. (2.16) in the following form:

$$\mathcal{F}(p) = \begin{pmatrix} 1 & & & \\ & \alpha(p) & \beta(p) & \\ & \gamma(p) & \delta(p) & \\ & & & 1 \end{pmatrix}. \tag{3.6}$$

The straightforward check shows that only two of the functions $\alpha(p)$, $\beta(p)$, $\gamma(p)$, $\delta(p)$ are independent, and we can express, say, the entries in the third row in (3.6) via the entries of the second row. The result reads as follows:

$$\gamma(p) = -\tfrac{q^{-p}}{[p]}\,\alpha(p) + \tfrac{\sqrt{[p+1][p-1]}}{[p]}\,\beta(p)\ , \quad \delta(p) = \tfrac{\sqrt{[p+1][p-1]}}{[p]}\,\alpha(p) + \tfrac{q^{p}}{[p]}\,\beta(p)\ . \tag{3.7}$$

Now we shall use the condition *3*. For this purpose, we can employ the the following formulae for the fundamental R-matrices of $U_q(sl(n))$ (see [11] for details):

$$P_\pm = \frac{q^{\frac{1}{n}\pm 1}\,\widehat{R}_+ - q^{-\frac{1}{n}\mp 1}\,\widehat{R}_-}{q^2 - q^{-2}}\ , \tag{3.8}$$

where $\widehat{R}_\pm = PR_\pm$ [P is the permutation matrix, i.e., $PaP = a'$ for $a \in \mathcal{G}\otimes\mathcal{G}$] and P_+, P_- are the projectors in $\mathbb{C}^n \otimes \mathbb{C}^n$ (q-symmetrizer and q-antisymmetrizer) of ranks $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$, respectively. In the case of $U_q(sl(2))$ these projectors are given by

$$P_+ = \begin{pmatrix} 1 & & & \\ & q^{-1}\lambda & \lambda & \\ & \lambda & q\lambda & \\ & & & 1 \end{pmatrix}, \quad P_- = \begin{pmatrix} 0 & & & \\ & q\lambda & -\lambda & \\ & -\lambda & q^{-1}\lambda & \\ & & & 0 \end{pmatrix}, \tag{3.9}$$

where $\lambda = \frac{1}{[2]} = (q+q^{-1})^{-1}$. It is easy to find their eigenvectors $\vec{x}_i$ such that $\vec{x}_i^{\,t}\vec{x}_j = \delta_{ij}$:

$$P_+ = \sum_{i=1}^{3} \vec{x}_i \otimes \vec{x}_i^{\,t}\ , \quad \vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x}_2 = \sqrt{\lambda}\begin{pmatrix} 0 \\ q^{-1/2} \\ q^{1/2} \\ 0 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; \tag{3.10}$$

$$P_- = \vec{x}_0 \otimes \vec{x}_0^{\,t}\ , \quad \vec{x}_0 = \sqrt{\lambda}\begin{pmatrix} 0 \\ q^{1/2} \\ -q^{-1/2} \\ 0 \end{pmatrix}. \tag{3.11}$$

According to (1.14) we can construct from these vectors the following CG maps

$$\mathrm{C}[\tfrac{1}{2}\tfrac{1}{2}0] = \sqrt{\lambda}\,(0,\, q^{1/2},\, -q^{-1/2},\, 0)\ , \quad \mathrm{C}[\tfrac{1}{2}\tfrac{1}{2}1] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\lambda}\,q^{-1/2} & \sqrt{\lambda}\,q^{1/2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\ . \qquad (3.12)$$

Now, substituting (3.9) into (2.18), we can compute U^0. To be able to use the condition *3* we have to compare U^0 with the central element of the algebra $\mathcal{U}$ generated by the entries U_i of the matrix $U^{\frac{1}{2}}$ and the spin operator p. It has been shown in [13] that the only nontrivial central element of the algebra $\mathcal{U}$ is given by the following "p-deformed" determinant of $U^{\frac{1}{2}}$:

$$\mathrm{Det}\, U^{\frac{1}{2}} \;=\; (U_1U_4 - q\,U_2U_3)\sqrt{\tfrac{[p]}{[p-1]}} \;=\; (q\,U_4U_1 - U_3U_2)\sqrt{\tfrac{[p]}{[p+1]}}\ . \qquad (3.13)$$

Omitting simple calculations, we give the result: U^0 coincides (up to a numerical factor) with (3.13) only if the constraint

$$\alpha(p)\sqrt{[p+1]} - \beta(p)\sqrt{[p-1]} \;=\; \varepsilon\sqrt{[p]}. \qquad (3.14)$$

holds. Thus, (3.6) contains only one independent function. Moreover, from the condition *5* we infer that the numerical constant ε in the r.h.s. of (3.14) is fixed: $\varepsilon = q^{1/2}$.

Finally, we can use the conditions *4* and *5*. To apply the former in practice, we can first consider the non-deformed case ($q = 1$) when entries of $\mathcal{F}(p)$ are self-conjugated and then extend the solution to generic q in such a way that the condition *5* would be satisfied. After simple calculations we get

$$\begin{aligned} \alpha(p) \;&=\; \delta(p) \;=\; \tfrac{1}{[2]}\Big(\, q^{\frac{1}{2}}\sqrt{\tfrac{[p+1]}{[p]}} + q^{-\frac{1}{2}}\sqrt{\tfrac{[p-1]}{[p]}}\,\Big)\ , \\ \beta(p) \;&=\; -\gamma(p) \;=\; \tfrac{1}{[2]}\Big(\, q^{-\frac{1}{2}}\sqrt{\tfrac{[p+1]}{[p]}} - q^{\frac{1}{2}}\sqrt{\tfrac{[p-1]}{[p]}}\,\Big)\ . \end{aligned} \qquad (3.15)$$

Thus $\mathcal{F}^{\frac{1}{2}\frac{1}{2}}(p)$ is found. Observe that $\det \mathcal{F}^{\frac{1}{2}\frac{1}{2}}(p) = 1$.

Now, substituting (3.12) into (1.17) and exploiting the explicit form of $\mathcal{F}^{\frac{1}{2}\frac{1}{2}}(p)$, we can build up the exact generating matrix of spin 1. It looks like following:

$$U^1 \;=\; \begin{pmatrix} U_1^2 & q^{-\frac{1}{2}}\sqrt{[2]}\,U_1U_2 & U_2^2 \\ \sqrt{\frac{[2][p]}{[p+1]}}\,U_1U_3 & \sqrt{\frac{[p]}{[p+1]}}\,(q^{\frac{1}{2}}\,U_1U_4 + q^{-\frac{1}{2}}\,U_2U_3) & \sqrt{\frac{[2][p]}{[p+1]}}\,U_2U_4 \\ U_3^2 & q^{-\frac{1}{2}}\sqrt{[2]}\,U_3U_4 & U_4^2 \end{pmatrix}\ . \qquad (3.16)$$

Let us briefly comment this formula. First, as we could expect [due to eqs. (3.2)], in the formal limit $q^p \to +\infty$ the structure of U^1 becomes identical to that of g^1 given in (2.3). Also, it is easy to see that the second row of (3.16) can be identified (up to rescaling by $\sqrt{\frac{[2]}{[p+1][p-1]}}$) with the spin 1 tensor operator (1.7) constructed from the generators of $U_q(sl(2))$. Finally, the entries U^1_{ij}, $i,j = 1,2,3$ act on the model space $\mathcal{M}$ as shifts from the state $|J,m\rangle$ to the states $|J+(2-i), m+(2-j)\rangle$, which is natural because we have applied the fusion scheme to the matrix $U^{\frac{1}{2}}$ whose entries are basic shifts on $\mathcal{M}$.

Furthermore, if we substitute in (3.16) the realization (3.4) of the operators U_i, then U^1 also will be precise in the described sense. Namely, it can be checked then that matrix elements $\langle J', m'|U^1_{ij}|J'', m''\rangle$ evaluated on $D_q(z_1, z_2)$ precisely coincide with the CG coefficients $\left\{ \begin{smallmatrix} J' & 1 & J'' \\ m' & 2-j & m'' \end{smallmatrix} \right\}_q$ (nine of them are non-vanishing). Thus, the fusion procedure preserves the "preciseness" of exact generating matrices, which is useful in practical applications.

§ 3.3. Another construction for $\mathcal{F}$

The computations performed in the previous paragraph inspire us to introduce p-dependent counterparts of the projectors $P_\pm$ used above. Indeed, we can consider the following analogue of the decomposition formula (3.8):

$$\mathcal{P}_\pm \;=\; \frac{q^{\frac{1}{n}\pm 1}\,\widehat{\mathcal{R}}_+(\vec{p}) - q^{-\frac{1}{n}\mp 1}\,\widehat{\mathcal{R}}_-(\vec{p})}{q^2 - q^{-2}}\;. \tag{3.17}$$

It is obvious from the formula $\widehat{\mathcal{R}}_\pm(\vec{p}) = (\mathcal{F}(\vec{p}))^{-1}\widehat{R}_\pm\mathcal{F}(\vec{p})$ that the objects $\mathcal{P}_+$ and $\mathcal{P}_-$ are also projectors of ranks $\frac{n(n+1)}{2}$ and $\frac{n(n-1)}{2}$, respectively. [11] In the case of $U_q(sl(2))$ we find (cf. formulae (3.9))

$$\mathcal{P}_\pm \;=\; \begin{pmatrix} \frac{1\pm1}{2} & & & \\ & \lambda\frac{[p\mp1]}{[p]} & \pm\lambda\frac{\sqrt{[p+1][p-1]}}{[p]} & \\ & \pm\lambda\frac{\sqrt{[p+1][p-1]}}{[p]} & \lambda\frac{[p\pm1]}{[p]} & \\ & & & \frac{1\pm1}{2} \end{pmatrix}\;. \tag{3.18}$$

Repeating the procedure described in the previous paragraph, we can find the eigenvectors $\vec{\mathrm{x}}_i$ such that $\vec{\mathrm{x}}_i^t\vec{\mathrm{x}}_j = \delta_{ij}$ and $\mathcal{P}_- = \vec{\mathrm{x}}_0\otimes\vec{\mathrm{x}}_0^t$, $\mathcal{P}_+ = \sum_{i=1}^3 \vec{\mathrm{x}}_i\otimes\vec{\mathrm{x}}_i^t$. Next, using the same formulae (1.14), we can construct p-dependent counterparts of the CG-maps. They look like following:

$$\begin{aligned} \mathrm{C}_p[\tfrac12\tfrac12 0] \;&=\; \sqrt{\lambda}\,(0,\,\sqrt{\tfrac{[p+1]}{[p]}},\,-\sqrt{\tfrac{[p-1]}{[p]}},\,0)\;, \\ \mathrm{C}_p[\tfrac12\tfrac12 1] \;&=\; \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{\lambda}\sqrt{\frac{[p-1]}{[p]}} & \sqrt{\lambda}\sqrt{\frac{[p+1]}{[p]}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\;. \end{aligned} \tag{3.19}$$

Now, straightforward check shows that the matrix $\mathcal{F}^{\frac12\frac12}(p)$ found before can be obtained as follows: $\mathcal{F}^{\frac12\frac12}(p) = \mathrm{C}'[\frac12\frac12 0]\mathrm{C}_p[\frac12\frac12 0] + \mathrm{C}'[\frac12\frac12 1]\mathrm{C}_p[\frac12\frac12 1]$. This suggests to consider the more general formula:

$$\mathcal{F}^{IJ}(\vec{p}) \;=\; \sum_K \mathrm{C}'[IJK]\,\mathrm{C}_{\vec{p}}[IJK]\;. \tag{3.20}$$

Eq. (3.20) may be regarded as an alternative definition of the twisting element and it has already been considered in [7] (similar expressions also appeared in [21]) and proven to obey all axioms for the twisting element provided $\mathrm{C}[IJK]$ and $\mathrm{C}_{\vec{p}}[IJK]$ are properly defined. In this approach, however, the entries of the matrices $\mathrm{C}[IJK]$ and $\mathrm{C}_{\vec{p}}[IJK]$ are supposed to be a-priory identified with some specific values of the CG coefficients and the $6j$-symbols (explicit formulae for them might be quite cumbersome). Moreover, these values (in general defined not uniquely) have to be chosen, to be compatible, in particular, with the choice of the matrices $R_\pm$ and $\mathcal{R}_\pm(\vec{p})$. All this explains why we had not chosen eq. (3.20) as a starting point for constructing $\mathcal{F}^{\frac12\frac12}(p)$.

To sum up, we have demonstrated that eq. (3.20) with $\mathrm{C}[IJK]$ and $\mathrm{C}_{\vec{p}}[IJK]$, built up from eigenvectors of the projectors P_K^{IJ} and $\mathcal{P}_K^{IJ}$ according to given above prescriptions, gives correct expression for the twisting element. With this clarification the practical application of (3.20) becomes more straightforward.

[11] Notice that $P_-\mathcal{F}(p)\mathcal{P}_+ = P_+\mathcal{F}(p)\mathcal{P}_- = 0$. This is an alternative form of eq. (2.16) [for the case of fundamental representation].

Additionally, our approach allows to make the algebraic sense of (3.20) more transparent. Indeed, since $P_K^{IJ} = \mathrm{C}'[IJK]\mathrm{C}[IJK]$, we can rewrite the formula for decomposition of R-matrices over the projectors (we used its simplest case (3.8) above) in the following form:

$$\widehat{R}_{\pm}^{IJ} \;=\; \sum_K \mathrm{C}'[IJK]\, r_{K,\pm}^{IJ}\, \mathrm{C}[IJK]\;, \tag{3.21}$$

where $r_{K,\pm}^{IJ}$ are the corresponding eigenvalues [see [11] for the fundamental representations of $U_q(sl(n))$ and [12] for the highest irreps of $U_q(sl(2))$]. Now, bearing in mind the properties (1.16) of the CG maps and employing (3.20), we can transform, according to (2.12), eq. (3.21) into similar one for the twisted R-matrices:

$$\widehat{\mathcal{R}}_{\pm}^{IJ}(\vec{p}) \;=\; \sum_K \mathrm{C}'_{\vec{p}}[IJK]\, r_{K,\pm}^{IJ}\, \mathrm{C}_{\vec{p}}[IJK]\;. \tag{3.22}$$

Thus, being written in the language of the projectors, the objects belonging to Hopf and quasi-Hopf structures look quite identical.

§ 3.4. On properties of the element χ

In conclusion, we wish to discuss in more detail properties of the element χ which, as we have seen in §2.3 and §3.1, plays an essential role in the theory of exact generating matrices. First, we infer from (2.19) that the self-conjugated element $\chi = \mathcal{F}(\vec{p})\mathcal{F}^*(\vec{p})$ obeys the relation

$$\chi\,\Delta' \;=\; \Delta\,\chi\;. \tag{3.23}$$

Additionally, eq. (2.21) also implies that $[R_{\pm}, \chi\chi'] = 0$. Together with (3.23) this allows us to assume that

$$\chi' \;=\; \chi^{-1}\;. \tag{3.24}$$

Indeed, as has been demonstrated in [7], there exists the following universal expression for the element $\chi \in \mathcal{G}\otimes\mathcal{G}$:

$$\chi \;=\; \Delta(\kappa^{-1})\,(\kappa\otimes\kappa)\,R_+^{-1} \;=\; \Delta(\kappa)\,(\kappa\otimes\kappa)^{-1}\,R_-^{-1}\;, \tag{3.25}$$

where $\kappa^2 = v$ with v being a certain invertible central element of $\mathcal{G}$ (the *ribbon element*, see [22]), such that

$$R_-^{-1}\,R_+ \;=\; \Delta(v^{-1})\,(v\otimes v)\;,\quad S(v) \;=\; v\;,\quad \epsilon(v) \;=\; 1,\quad v^* \;=\; v^{-1}\;. \tag{3.26}$$

It is interesting to mention that, since the first relation in eq. (3.26) can be rewritten as $\Delta(v^{-1})\,(v\otimes v) = \widehat{R}_+^2 = \widehat{R}_-^{-2}$ [recall that $\widehat{R}_{\pm} \equiv PR_{\pm}$], eq. (3.25) admits the following form:

$$\widehat{R}_+ \;=\; \widehat{\chi}^{-1}\,\{\widehat{R}_+^2\}^{\frac{1}{2}}\;,\quad \widehat{R}_- \;=\; \{\widehat{R}_-^2\}^{\frac{1}{2}}\,\widehat{\chi}\;,$$

with $\widehat{\chi} \equiv \chi P$. In other words, $\widehat{\chi}^{\mp 1}$ appears as a (matrix) phase which fixes the choice of square root of $\widehat{R}_{\pm}^2$. And, although, $\widehat{\chi}^2 = e\otimes e$ [according to (3.24)], this phase turns out to be quite nontrivial, as we shall see below.

The properties (2.21) and (3.23)-(3.24) [as well as $\chi^* = \chi$] become quite obvious for χ being defined as in (3.25). However, eq. (3.25) is not convenient if we want to get an explicit form of χ^{IJ}. Therefore, below we shall discuss, exploiting the language of quantum projectors, an alternative way of constructing χ.

First, let us find with the help (3.15) and (3.2) an explicit expression for the element χ in the fundamental representation of $U_q(sl(2))$:

$$\chi^{\frac{1}{2}\frac{1}{2}} = \begin{pmatrix} 1 & & & \\ & 2\lambda & -\omega\lambda & \\ & \omega\lambda & 2\lambda & \\ & & & 1 \end{pmatrix} . \tag{3.27}$$

In the non-deformed limit, $q \to 1$, we have $\chi^{\frac{1}{2}\frac{1}{2}} \to e \otimes e$, as expected. Now we notice that (3.27) looks very simply in terms of projectors (3.9). Namely, for $\widehat{\chi}^{\frac{1}{2}\frac{1}{2}} = \chi^{\frac{1}{2}\frac{1}{2}} P^{\frac{1}{2}\frac{1}{2}}$ we have

$$\widehat{\chi}^{\frac{1}{2}\frac{1}{2}} = P_+ - P_- , \tag{3.28}$$

where $P_+ \equiv P_1^{\frac{1}{2}\frac{1}{2}}$, $P_- \equiv P_0^{\frac{1}{2}\frac{1}{2}}$. To explain this formula, we need the relation (3.20). Substituting the latter into the definition of χ, we obtain:

$$\widehat{\chi}^{IJ} = \mathcal{F}^{IJ} (\mathcal{F}^{IJ})^* P^{IJ} = \sum\nolimits_K C'[IJK] \overline{C[JIK]} , \tag{3.29}$$

where $\overline{C[\ldots]} =$ denotes a matrix complex conjugated to $C[\ldots]$ and we have used the identity $\mathrm{C}_{\vec{p}}[IJK](\mathrm{C}_{\vec{p}}[IJL])^* = \delta_{KL}$ which follows from (2.19). Now, taking into account the well-known symmetry of q-deformed CG coefficients (see, e.g., [4])

$$C[JIK]_{q^{-1}} = (-1)^{I+J-K} C[IJK]_q , \tag{3.30}$$

we rewrite (3.29) as follows

$$\widehat{\chi}^{IJ} = \sum\nolimits_{K=|I-J|}^{I+J} (-1)^{I+J-K} P_K^{IJ} . \tag{3.31}$$

Thus, for $U_q(sl(2))$ the element $\widehat{\chi}$ is an altered sum of the quantum projectors. For $U_q(sl(n))$ the symmetries of the CG coefficients are of more sophisticated form (see, e.g., [23]). Therefore, in general, we should expect more complicated formula for χ, but, presumably, still in terms of the quantum projectors. In particular, formula (3.28) remains, probably, true for any fundamental representation of $U_q(sl(n))$ [since in that case there are only two projectors and the corresponding coefficients are uniquely fixed by the properties of χ].

Conclusion

In the present paper we have demonstrated that the theory of (deformed) tensor operators and, in particular, the fusion procedure can be most naturally described employing the R-matrix language and revealing the underlying quasi-Hopf-algebraic structure. We clarified the role in this context of the projectors and their p-dependent counterparts which appear, respectively, in decompositions of R-matrices and twisted R-matrices. From the practical point of view, the suggested prescription for constructing exact generating matrices can be used, e.g., for explicit computations and studies of (deformed) CG coefficients for quantum Lie algebras of higher ranks. On the other hand, the specific quasi-Hopf algebra [defined by the pair $R_\pm$ and $\mathcal{F}(\vec{p})$] appearing in this context should certainly be studied in more detail since it provides non-trivial (and presumably somewhat simplified) realization of the abstract general scheme. Explicit formulae like that we have derived for $\mathcal{F}^{\frac{1}{2}\frac{1}{2}}(p)$ may be useful here.

Although the present paper dealt mainly with the mathematical side of the theory of tensor operators, we are going to discuss some physical applications in future. Finally, we

would like to note that it would be interesting to extend the developed technique to the case of q being a root of unity, which would involve truncated quasi-Hopf algebras.

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References

[1] E.P.Wigner, *Group theory and its applications to the quantum mechanics of atomic spectra* (Academic Press, 1959);
L.C.Biedenharn, J.D.Louck, *Angular momentum in quantum physics* (Addison-Wesley, 1981).

[2] V.G.Drinfeld, *Quantum groups*. In: Proc. of ICM, Berkeley 1986 (AMS Publ., 1987).

[3] L.C.Biedenharn, M.Tarlini, Lett. Math. Phys. **20**, 271 (1990);
F.Pan, J. Phys. **A 24**, L803 (1991);
V.Rittenberg, M.Scheunert, J. Math. Phys. **33**, 436 (1992);
A.U.Klimyk, J. Phys. **A25**, 2919 (1992).

[4] L.C.Biedenharn, M.Lohe, *Quantum group symmetry and q-tensor algebras* (World Scientific, 1995).

[5] L.D.Faddeev, Comm. Math. Phys. **132**, 131 (1990);
A.Yu.Alekseev, L.D.Faddeev, Comm. Math. Phys. **141**, 413 (1991);
L.D.Faddeev, *Quantum symmetry in conformal field theory by Hamiltonian methods*. In: *New symmetry principles in quantum field theories* (Plenum Press, 1992);
F.Falceto, K.Gawedzki, J. Geom. Phys. **11**, 251 (1993).

[6] M.Chu, P.Goddard, I.Halliday, D.Olive, A.Schwimmer, Phys. Lett. **B 266**, 71 (1991);
M.Chu, P.Goddard, Phys. Lett. **B 337**, 285 (1994);
M.Chu, P.Goddard, Nucl. Phys. **B 445**, 145 (1995).

[7] A.G.Bytsko, V.Schomerus, *Vertex operators – from toy model to lattice algebras*, preprint Berlin Sfb288 No.215 and q-alg/9611010 (1996).

[8] A.G.Bytsko, *Tensor operators in R-matrix approach*, preprint DESY 95-254, q-alg/9512030 (1995).

[9] V.G.Drinfeld, Algebra and analysis **1**, 1419 (1990).

[10] E.Abe, *Hopf algebras* (Cambridge Univ. Press, 1980);
M.E.Sweedler, *Hopf algebras* (Benjamin, 1969).

[11] L.D.Faddeev, N.Yu.Reshetikhin, L.A.Takhtajan, Algebra and analysis **1**, 193 (1990).

[12] P.P.Kulish, E.K.Sklyanin, Lect. Notes Phys. **151**, 61 (1982);
P.P.Kulish, N.Yu.Reshetikhin, E.K.Sklyanin, Lett. Math. Phys. **5**, 393 (1981).

[13] A.G.Bytsko, L.D.Faddeev, J. Math. Phys. **37**, 6324 (1996).

[14] A.P.Isaev, J. Phys. **A 29**, 6903 (1996).

[15] A.N.Kirillov, N.Yu.Reshetikhin, Adv. Series in Math. Phys. **11**, 202 (1990).

[16] J-L.Gervais, A.Neveu, Nucl. Phys. **B 238**, 125 (1984);
E.Cremmer, J-L.Gervais, Comm. Math. Phys. **134**, 619 (1990).

[17] O.Babelon, Comm. Math. Phys. **139**, 619 (1991).

[18] O.Babelon, Phys. Lett **B 215**, 523 (1988);
A.Bilal, J-L.Gervais, Nucl. Phys. **318**, 579 (1989);
L.Bonora, V.Bonservizi, Nucl. Phys. **B 390**, 205 (1993).

[19] J.Avan, O.Babelon, E.Billey, Comm. Math. Phys. **178**, 281 (1996).

[20] G.Mack, V.Schomerus, Nucl. Phys. **B 370**, 185 (1992).

[21] O.Babelon, D.Bernard, E.Billey, Phys. Lett. **B 375**, 89 (1996).

[22] N.Yu.Reshetikhin, V.G.Turaev, Comm. Math. Phys. **127**, 1 (1990);
A.N.Kirillov, N.Yu.Reshetikhin, Comm. Math. Phys. **134**, 421 (1991).

[23] A.U.Klimyk, J. Math. Phys. **36**, 508 (1995).