Schlesinger Transformations for Bonnet Surfaces

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Abstract

Bonnet surfaces, i.e. surfaces in Euclidean 3-space, which admits a one-parameter family of isometries preserving the mean curvature function, can be described in terms of solutions of some special Painlevé equations. The goal of this work is to use the well-known Schlesinger transformations for solutions of Painlevé VI equations to create new Bonnet surfaces from a known one.

Introduction

Let us consider surfaces in an Euclidean 3-space which admits a family of isometric surfaces having the same mean curvature function. It what the french mathematician O. Bonnet who first discussed whether there exists such surfaces or not. He stated that excluding surfaces of revolution, helicoids and constant mean curvature surfaces there is a class of such surfaces, which can be described by ordinary differential equations. We call these surfaces Bonnet surfaces.
A. Hazzidakis formulated the problems in terms of an ordinary differential equation of third order for the mean curvature function on a Bonnet surface and integrated this equation once. He also discussed the first (non-geometrical) solution of it. Fourty years later it was E. Cartan, who gave a classification of all Bonnet-surfaces, which is up to now the state of the art. He showed that Bonnet-surfaces depends - up to rigid motion - on at most 3 real parameters.
In [1], it is shown how to describe Bonnet-surfaces in terms of solutions of some special Painlevé V(III) and VI equations. The first integral found by Hazzidakis is thereby a parameter in the Painlevé equations. The main tool hereby is the so-called Ψ-function, which is a matrix-valued solution of the corresponding linear systems (the Painlevé equations are the zero-curvature conditions for integration of the corresponding linear systems). This Ψ-function generates a moving frame on the surfaces and have so a geometrical meaning.
As it is known for specialists in Painlevé equations, there are so called Schlesinger-transformation, i.e. left-matrix-multiplication of the Ψ-functions by which from one solution of a Painlevé equation one can generate solutions of discrete many different Painlevé equation. These transformations are discussed in [3],[4] and especially for Painlevé VI in [5]. Using the formal ansatz from [5] I produce a Schlesinger transformation for Bonnet surfaces by shifting of the first integral

parameter by an integer number. Up to now this recept can be geometrically interpretated only in the Cartan **B**-case[6], since for the Cartan **A**-case I can not characterize geometrical solutions of the corresponding Painlevé VI equation. In the Cartan **C**-case the first integral is not involved in the corresponding linear system for the Ψ-function. This comes from the fact, that this parameter is not an essential parameter of the surfaces. It is only a scaling factor and could be fixed - unless it is not zero (all Bonnet where this parameter vanishs are described explicitly in hypergeometric functions in [1]).
I should mention that I do not find a complete geometrical meaning of the Schlesinger transformation up to now.

Basics

Let $su(2)$ be the Lie-algebra of $SU(2)$ and

$$\mathbb{i} = \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}, \quad \mathbb{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbb{k} = \begin{pmatrix} -\mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix},$$

be a basis of it. To find a proper description of Bonnet-surfaces in terms of Painlevé functions it is usefull to identify R^3 with $su(2)$ and use the 2 by 2 representation of the Painlevé equations introduced in [3]. Here this identicifation is realized by the isomorphism

$$(e_1\,, e_2\,, e_3) \in \mathbb{R}^3 \;\rightarrow\; (\mathbb{i}\,, \mathbb{j}\,, \mathbb{k}) \in su(2). \tag{1}$$

How by this identification a local theory of surfaces in R^3 can be formulated is stated in the following

Lemma 1 *Let $\mathcal{F} \subset \mathbb{R}^3$ be a smooth surface and $F = F(x, y)$ be a smooth local conformal parametrization with the induced metric*

$$ds^2 = e^{u(x,y)}(d\,x^2 + d\,y^2),$$

Then there exists a smooth $SU(2)$-valued function $\Psi(x, y)$ such that

$$\begin{aligned} F_x(x,y) &= e^{u(x,y)/2}\,\Psi^{-1}(x,y)\,\mathbb{i}\,\Psi(x,y), \\ F_y(x,y) &= e^{u(x,y)/2}\,\Psi^{-1}(x,y)\,\mathbb{j}\,\Psi(x,y), \\ N(x,y) &= \Psi^{-1}(x,y)\,\mathbb{k}\,\Psi(x,y), \end{aligned} \tag{2}$$

where F_x and F_y denotes the partial derivatives of $F(x, y)$ and $N(x, y)$ is the corresponding Gauss-map, each identified with a $su(2)$-valued function by the isomorphism given above.

Proof : The proof is given in [7]. It is stated there, that Ψ have to solve the following linear system

$$\begin{aligned}
\Psi_x\,\Psi^{-1} &= \begin{pmatrix} -\mathrm{i}\,\dfrac{u_y}{4} & -Q\,e^{-u/2} - \dfrac{H}{2}\,e^{u/2} \\ \bar{Q}\,e^{-u/2} + \dfrac{H}{2}\,e^{u/2} & \mathrm{i}\,\dfrac{u_y}{4} \end{pmatrix} \\
\Psi_y\,\Psi^{-1} &= \mathrm{i}\begin{pmatrix} \dfrac{u_x}{4} & -Q\,e^{-u/2} + \dfrac{H}{2}\,e^{u/2} \\ -\bar{Q}\,e^{-u/2} + \dfrac{H}{2}\,e^{u/2} & -\dfrac{u_x}{4} \end{pmatrix},
\end{aligned} \tag{3}$$

where $u = u(x,y)$ is the metric function as above, $H = H(x,y)$ is the mean curvature function and $Q = Q(x,y)$ is the Hopf-differential defined by

$$Q(x,y) = \frac{1}{4}\,(< F_{xx}(x,y) - F_{yy}(x,y), N(x,y) > -2\,\mathrm{i}\, < F_{xy}(x,y), N(x,y) >)\,.$$

$\bar{Q}(x,y)$ is the conjugated function of this.
As in the classical theory there are integrability condition for moving frames, here the same conditions are the compatibility condition for (3). This is what is stated in the following

Lemma 2 *Let $u(x,y)$, $H(x,y)$ and $Q(x,y)$ be a solution of*

$$\begin{aligned}
\Delta\,u + 2\,H^2\,e^u - 8|Q|^2\,e^{-u} &= 0, \\
(Q+\bar{Q})_x + \mathrm{i}\,(Q-\bar{Q})_y &= H_x\,e^u, \\
\mathrm{i}\,(Q-\bar{Q})_x - (Q+\bar{Q})_y &= H_y\,e^u.
\end{aligned} \tag{4}$$

Then there exists (up to left-multiplication with a constant $SU(2)$-matrix) a unique $SU(2)$-valued function Ψ, which solve (3). Moreover, by (2) a (up to a constant rotation) unique moving frame on a surface is defined. The corresponding surface is (up to a translation) unique integrale of this moving frame.

Proof : One observe that (4) are the compatibility condition for (3). The solution of this equation is unique up to left-multiplication with a constant $SU(2)$ matrix (if the initial data matrix is in $SU(2)$). Now, defining functions F_x, F_y, N by (2) it holds $(F_x)_y = (F_y)_x$. Therefore they can be integrated to one function $F = F(x,y)$. This integation is unique up to an additional term in $su(2)$. Finally using the isomorphism (1) we get a surface in $\mathbb{R}^3$ unique up to rigid motion.

General results on Bonnet A and B surfaces

Following the general results formulated in [1] we find

Theorem 1 *Let $\mathcal{F}$ be a Bonnet-surface of Cartan-type* **A** *or* **B**. *Then there exists a (local) conformal coordinate $z = x + \mathrm{i}\, y$, such that the fundamental functions reads as follows:*

$$\begin{aligned} Q_A(x,y) &= -2\,\frac{\sin(2\,(x-\mathrm{i}\,y))}{\sin(4\,x)\,\sin(2\,(x+\mathrm{i}\,y)} & e^{u_A(x)} &= -\frac{16}{\sinh^2(4\,x)\,H_A'(x)} \\ Q_B(x,y) &= -2\,\frac{\sinh(2\,(x-\mathrm{i}\,y))}{\sinh(4\,x)\,\sinh(2\,(x+\mathrm{i}\,y)} & e^{u_B(x)} &= -\frac{16}{\sin^2(4\,x)\,H_B'(x)} \end{aligned} \tag{5}$$

and $H_\nu(x)$, $\nu = A, B$, the mean curvature function on the corresponding surface, solves the differential equation

$$\left(\frac{H_\nu''(x)}{H_\nu'(x)}\right)' - 2\,H_\nu'(x) = \frac{8\,\alpha_\nu^2}{\sinh^2(\alpha_\nu\,x)}\left(1 - \frac{H_\nu{}^2(x)}{H_\nu'(x)}\right), \tag{6}$$

where $\alpha_A = 4\,\mathrm{i}$ and $\alpha_B = 4$.

Proof : The proof is stated in [1].
The parameter α_ν is not an essential parameter of the differential equation. This will be shown in the following

Lemma 3 *Let $H(x) = H_\nu(x)$ be an arbitrary solution of (6) with arbitrary $\alpha = \alpha_\nu \in \mathbb{C}\setminus\{0\}$. Then the function $H_\theta(x)$ defined by*

$$H(x) = \frac{1}{\theta}\,H_\theta\left(\frac{\theta}{x}\right)$$

solve (6) but with $\alpha_\nu = \alpha\,\theta$.

Proof : This follows from the chain rule.

Remark 1 *Lemma* **3** *stated, that one can express the general solutions of (6) in the Bonnet* **A**-*case by the general solution in the Bonnet* **B**-*case and vice versa. Of course by general solution I mean general complex-valued solutions. So Lemma* **3** *has only an analytical meaning but not a geometrical one.*

A Schlesinger Transformation for Bonnet B surfaces

Hazzidakis[8] found a first integral of (6). It was shown in [1] that there is a 1:1-relation between the solutions of (6) and the solutions of a Painlevé VI equation. For that it was shown that the moving frame equation (3) for Bonnet **A** and **B** surfaces are the zero-curvature representation for such a Painlevé equation, described in [3]. The solution of this linear system gives the Ψ-function.
For the rest of this text I fix my view on the Bonnet **B**-case. By Lemma **3** it is clear that all analytical properties found here holds as well in the **A**-case.

Lemma 4 *Equation (6) for $\alpha_B = 4$ has the first integral*

$$\left(\frac{H''(x)}{8\,H'(x)} + \coth(4\,x)\right)^2 - \frac{H^2(x)}{H'(x)\,\sinh^2(4\,x)} - \frac{H'(x)}{16} - \frac{H(x)}{2}\,\coth(4\,x) = \mu^2. \qquad (7)$$

Proof : This is a simple calculation.

Remark 2 *If $H(x)$ is a solution of (6) with the first integral (7) then $-H(-x)$ is also a solution of (6) with the same first integral.*

Let us use another way to prove Lemma 4, namely via the Lax-representation of (6). By introducing the similarity variables

$$s = e^{8\,x}, \qquad \lambda = e^{4\,(x+\mathrm{i}\,y)}, \qquad \Psi(s,\lambda) = \Psi(x(s,\lambda), y(s,\lambda)) = \Psi(x,y). \qquad (8)$$

By inserting them into the moving frame equation (3) together with (5) we arrive at:

$$\begin{aligned}
\Psi_\lambda\,\Psi^{-1} &= \frac{1}{\lambda}\begin{pmatrix} a(s) & \varphi_1(s) \\ \varphi_2(s) & -a(s)\end{pmatrix} + \frac{\varphi_2(s)-\varphi_1(s)}{\lambda-1}\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} + \\
&\qquad \frac{\varphi_1(s)-\varphi_2(s)}{\lambda-s}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}, \\
\Psi_s\,\Psi^{-1} &= \frac{\varphi_2(s)-\varphi_1(s)}{\lambda-s}\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} + \begin{pmatrix} -\dfrac{a(s)}{2\,s} & \dfrac{\varphi_1(s)-s\,\varphi_2(s)}{s\,(s-1)} \\ \dfrac{\varphi_2(s)-\varphi_1(s)}{s-1} & \dfrac{a}{2\,s}\end{pmatrix},
\end{aligned} \qquad (9)$$

where

$$\begin{aligned}
a(s) &= -\frac{1}{2}\left(\frac{H''(x)}{8\,H'(x)} + \coth(4\,x)\right) \\
\varphi_1(s) &= \frac{H(x)}{2\,\sinh(4\,x)\,\sqrt{-H'(x)}} - e^{4\,x}\,\frac{\sqrt{-H'(x)}}{8} \\
\varphi_2(s) &= \frac{H(x)}{2\,\sinh(4\,x)\,\sqrt{-H'(x)}} - e^{-4\,x}\,\frac{\sqrt{-H'(x)}}{8}.
\end{aligned} \qquad (10)$$

Here $'$ denotes the differentiation with respect to x. Now the compatibility conditions for the sytems (9) leads to

$$a^2(s) + \varphi_1(s)\,\varphi_2(s) = \left(\frac{\mu}{2}\right)^2 \qquad (11)$$

which is (7).

Theorem 2 *Let $H(x)$ be a solution of (6) in the case $\alpha_B = 4$ with the first integral μ given by (7). Then*

$$y(\tau) = \frac{1}{4\,H'(x)}\left(\frac{8\,H'(x)\,(\mu\,\sinh(4\,x)+\cosh(4\,x))+\sinh(4\,x)\,H''(x)}{4\,H(x)+e^{4\,x}\,\sinh(4\,x)\,H'(x)}\right)^2, \qquad \tau = e^{-8x}, \tag{12}$$

solves the Painlevé VI equation

$$\begin{aligned} y'' \quad &= \quad \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-\tau}\right)y'^2 - \left(\frac{1}{\tau}+\frac{1}{\tau-1}+\frac{1}{y-\tau}\right)y' + \\ & \frac{y(y-1)(y-\tau)}{\tau^2(\tau-1)^2}\left(\alpha+\beta\,\frac{\tau}{y^2}+\gamma\,\frac{\tau-1}{(y-1)^2}+\delta\,\frac{\tau(\tau-1)}{(y-\tau)^2}\right) \end{aligned} \tag{13}$$

with

$$\alpha = 0, \quad \beta = 0, \quad \gamma = \frac{\mu^2}{2}, \quad \delta = -\frac{\mu(\mu+2)}{2}. \tag{14}$$

Inversely let $y(\tau)$ be an arbitrary solution of (13) with (14), then

$$H(x) = -\frac{2\,(\tau-1)(\tau^2\,y'^2(\tau)-\mu^2\,y^2(\tau))}{y(\tau)\,(y(\tau)-1)(y(\tau)-\tau)}, \qquad \tau = e^{-8\,x} \tag{15}$$

solves (6). Moreover if $\mu \in I\!R$ and $y(\tau) < 0$ then there exists a parametrization of a Bonnet **B** *type surface, with the mean curvature $H(x)$ as in (15) and the metric (5) given by*

$$e^{u(x)} = -\frac{4\,y(\tau)\,(y(\tau)-\tau)^2}{(\tau-1)^2(\mu\,y(\tau)+\tau\,y'(\tau))^2}. \tag{16}$$

Remark 3 *It is simple to prove, that in the Bonnet* **B** *case μ could be real as well as pour imaginary. The above theorem holds in general. This is not explicitly formulated, since I am not able to characterize geometrical solutions of (13) in the general case.*

Proof : For the proof let us introduce the auxilary function

$$\xi(s) = \left(\frac{a(s)-\mu/2}{\varphi_1(s)}\right)^2, \tag{17}$$

which is according (10) for given solution of (6) expressable in terms of $H(x)$, $H'(x)$ and $H''(x)$. The the compatibility conditions of (9) reads as follows:

$$\begin{aligned} a'(s) \quad &= \quad \frac{(\mu^2+4\,a^2(s))(1-s\,\xi^2(s))-4\,\mu\,a(s)(1+s\,\xi^2(s))}{4\,s(s-1)\,\xi(s)}, \\ \xi'(s) \quad &= \quad \frac{2\,a(s)(1+s\,\xi(s))(1+\xi(s))+\mu\,(s\xi^2(s)-1)}{s(s-1)}. \end{aligned} \tag{18}$$

This can be formulated as second order equation for $\xi(s)$ only:

$$\begin{aligned} \xi'' \;=\; & \frac{1}{2}\left(\frac{1}{\xi}+\frac{1}{\xi+1}+\frac{s}{1+s\,\xi}\right)\xi'^2-\left(\frac{1}{s}+\frac{1}{s-1}-\frac{\xi}{1+s\,\xi}\right)\xi'+ \\ & \frac{\xi(\xi+1)(1+s\,\xi)}{s^2(s-1)^2}\left(\frac{\mu\,(\mu+2)}{2}\,\frac{s(s-1)}{(1+s\,\xi)^2}-\frac{\mu^2}{2}\,\frac{s-1}{(\xi+1)^2}\right). \end{aligned} \tag{19}$$

In view that

$$y(\tau)=-\xi(s), \quad \tau=\frac{1}{s}, \tag{20}$$

we find the first part of the theorem.
Inversely, starting with an arbitray solution $y(\tau)$ of (13) with the parameters (14) given by an arbitrary $\mu \in \mathbb{C}$. Define $\xi(s)$ and $a(s)$ by (20) and (18). Now set $\varphi_1(s)$ and $\varphi_2(s)$ as

$$\varphi_1(s)=\frac{a(s)-\mu/2}{\sqrt{\xi(s)}} \quad \varphi_2(s)=-\sqrt{\xi(s)}\,(a(s)+\mu/2). \tag{21}$$

By (10) we can express $H(x)$, $H'(x)$ and $H''(x)$ in terms of $a(s)$, $\varphi_1(s)$ and $\varphi_2(s)$. The ambiguity of the definition of $\varphi_1(s)$ and $\varphi_2(s)$ (coming form the square root of ξ) is cancelled in this expressions. Using (21) we find (15) and an expression for $H'(x)$, which is compatible to the derivation of (15) by use of (13), (14). It is simple to check that (15) solves (6) if $y(\tau)$ solves (13), (14). Finally by taking (formally) (5) for the metric we reach at (16). For $\mu \in \mathbb{R}$ this is strictly positive, iff $y(\tau)<0$.
The last part of the proof may motivate the following

Definition 1 *A solution $y(\tau)$ of (13),(14) for $\mu \in \mathbb{R}$ is said to be geometrical on a non-empty interval $\mathcal{I} \subset \mathbb{R}$, if $y(\tau) \in \mathbb{R}^-$, $\forall \tau \in \mathcal{I}$.*

Remark 4 *Lemma* **4** *and Theorem* **2** *shows, that for any solution $H(x)$ of (6) there exist two first integrals, namely $\pm\mu$. Consequently two functions $y(\tau)$ solving two different Painlevé VI equations (13), (14) are related to each solution of (6).*

Remark 5 *Formulas (12) and (16) shows, that for $\mu \in \mathbb{R}$ in the Bonnet* **B** *case a solution $y(\tau)$ can be characterized to be geometrical, which means that the corresponding function $H(x)$ given by (15) is realized on a Bonnet* **B** *surface. This is obvious that a solution $y(\tau)$ of (13), (14) is geometrical if and only if $y(\tau)<0$ for all τ in an interval in $\mathbb{R}^+$.*

For the next theorem it will be convenient for me to use the auxilary function $\xi(s)$ instead of $y(\tau)$. From (20) the relation to $y(\tau)$ is clear.

Theorem 3 *Let $\mu \in \mathbb{C}$, $\xi_\mu = \xi_\mu(s)$ be an arbitray solution of (19), $a(s)$, $\varphi_1(s)$ and $\varphi_2(s)$ be defined by (18) and (21), respectively, and $\Psi_\mu(\lambda, s)$ the solution of the corresponding linear system (9). Then*

$$\Psi_{\mu+1}(\lambda, s) = L(\lambda, s)\, \Psi_\mu(\lambda, s), \tag{22}$$

with

$$\begin{aligned} L(\lambda, s) = \frac{\mathrm{i}\sqrt{\lambda}}{\sqrt{(1+\xi_\mu)(1+s\,\xi_\mu)}} \Bigg\{ & \begin{pmatrix} -s^{-1/4} & \sqrt{\xi_\mu}\, s^{-1/4} \\ -\sqrt{\xi_\mu}\, s^{1/4} & \xi_\mu\, s^{1/4} \end{pmatrix} + \\ & \frac{1}{\lambda} \begin{pmatrix} -\xi_\mu\, s^{3/4} & -\sqrt{\xi_\mu}\, s^{3/4} \\ \sqrt{\xi_\mu}\, s^{1/4} & s^{1/4} \end{pmatrix} \Bigg\}, \end{aligned} \tag{23}$$

solves the linear system (9) corresponding to the solutions

$$\xi_{\mu+1}(s) = \frac{1}{s\,\xi_\mu} \left(\frac{\mu\,\xi_\mu\,(1+s+2\,s\,\xi_\mu) + 2\,s\,\xi_\mu(\xi_\mu+1) + s\,\xi_\mu'\,(s-1)}{2\,(\mu+1) + \mu\,\xi_\mu\,(s+1) + 2\,\xi_\mu - s\,\xi_\mu'\,(s-1)} \right)^2 \tag{24}$$

of (19) with $\mu+1$ instead of μ.

Proof: This is a direct calculation.

Remark 6 *I should remark that one can get the transformation-law for the ξ-functions directly. Therefore one has to see that*

$$\xi_{-(\mu+1)}(s) = \frac{1}{s\,\xi_\mu(s)} \tag{25}$$

solves (19) for $-(\mu+1)$ instead of μ. One the other hand

$$\xi_{-\mu}(s) = \xi_\mu(s) \left(\frac{2\,a_\mu(s) + \mu}{2\,a_\mu(s) - \mu} \right)^2$$

solves (19) but with $-\mu$ instead of μ. Both together gives a simples relation between ξ_μ and $\xi_{\mu+1}$ which is (24).

Example 1 *For $\mu = 0$ one finds $\xi_0(s) = \alpha \neq 0$ as a 1-parameter family of solutions of (18). By use of (24), one gets for any integer number $\mu = n \in \mathbb{Z}$*

$$\xi_n(s) = \alpha\, s^n, \quad y_n(\tau) = -\frac{\alpha}{\tau^n}. \tag{26}$$

Finally one can directly prove that this is a solution of (18) for any $\mu \in \mathbb{C}$ (instead of n).

Remark 7 *From the viewpoint of the equation (19), we found, that any solution of the linear differential equation*

$$\tau\, y'(\tau) = -\mu\, y(\tau) \tag{27}$$

is at the same time a solution of the corresponding Painlevé (13), (14). This behaviour is similar to that in the Painlevé V case discussed for the Bonnet **C** *case in [1]. All solution of (27) corresponds to the solution* $H(x) \equiv 0$ *of (6).*

Corollary 1 *Let* $H(x)$ *be a real solution of (6),* $H'(x) < 0$ *and* $\mu \in I\!R$ *be defined (up to the sign) by (7). Then* $L \in SU(2)$.

Proof: First one observes that $L \in SL(2)$. Moreover since $a(s)\,, \varphi_1^2(s), \mu \in I\!R$ ξ_μ must be positive (e.g. see (17)). With $s > 1$ we get $(1+\xi_\mu)(1+s\,\xi_\mu) > 0$. Now it is simple so see, that L is of the form

$$L = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

Remark 8 *Corollary* **1** *together with (24) means that the transformation* $\Psi \to L\,\Psi$ *transforms geometrical solutions of (19) into geometrical.*

Lemma 5 *Let* $H(x) = H_\mu(x)$ *be a solution of (6)* ($\alpha_B = 4$) *with the first integral* μ *given by (7). Then*

$$H_{\mu+1}(x) = \left(1 + \frac{2\,y_\mu(\tau)}{\mu\,y_\mu(\tau) - \tau\,y'_\mu(\tau)}\right) H_\mu(x), \quad \tau = e^{-8\,x}, \tag{28}$$

with $y_\mu(\tau)$ *as in (12) solves (6), too. The first integral (7) for this solution is* $\pm(\mu+1)$.
Moreover if $H_\mu(x)$ *satisfies the conditions of Corollary* **1**, *then* H_μ *and* $H_{\mu+1}$ *are both geometrical. The metric (5) of the corresponding surfaces are related by:*

$$e^{u(x)}_{\mu+1} = \frac{\tau\,(y_\mu(\tau)-1)^2}{(y_\mu(\tau)-\tau)^2}\; e^{u(x)}_{\mu}. \tag{29}$$

Proof : Since $H_\mu(x)$ does not depend on the sign of μ (see (7)), we can take (25) to define $y_{-(\mu+1)}(\tau)$ via (20) . Now using (15) but with $-(\mu+1)$ instead of μ leads to an expression for $H_{\mu+1}(x)$. Cancelling the part $H_\mu(x)$ (see (15)) gives (28).
For the second part of the lemma one should take Corollary **1** and (16), where $y_{-(\mu+1)}(\tau)$ is inserted and the term $e^{u_\mu(x)}$ is cancelled. This means (29).
To illustrate, that one can use Lemma **5** to create new solutions of (6) (as well as of (13), (14)), we gave the following (non-geometrical)

Example 2 *Taking the one-parameter family of solutions of (6), ($\alpha_B = 4$)*

$$H_{\pm 1}(x) = \frac{1}{\alpha - x}$$

which have the first integral $\mu = \pm 1$. The corresponding solution family of the Painlevé VI (13),(14) with $\mu = 1$ is

$$y_1(\tau) = \tau\, \left(\frac{1-\tau+8\,\alpha+\log(\tau)}{1-\tau+\tau(8\,\alpha+\log(\tau))}\right)^2.$$

Now (28) leads the one-parameter family of solution for $\mu = 2$

$$H_{\pm 2}(x) = 8\,\frac{1-e^{16\,x}-16\,e^{8\,x}(\alpha-x)}{(e^{8\,x}-1)^2-64\,e^{8\,x}\,(\alpha-x)^2}.$$

By view of (25) we find directly

$$y_{-2}(\tau) = \left(\frac{1-\tau+\tau(8\,\alpha+\log(\tau))}{1-\tau+8\,\alpha+\log(\tau)}\right)^2.$$

Recursively one can create a discrete set of one-parameter families of solutions of (6) rational in $e^{8\,x}$ and x and simultaniously solutions of (13),(14) for any $\mu \in \mathbb{Z}$.

Finally let me describe the Schlesinger transformation on the level of the moving frame. This will be handled in the following

Lemma 6 *Let $y(\tau) = y_\mu(\tau)$ be a geometrical solution of (13), (14) with arbitray $\mu \in \mathbb{R}$. $\Psi(x,y) = \Psi^\mu(x,y)$ denote the solution of the corresponding lienar system (9). Define a moving frame $F_x^\mu(x,y)$, $F_y^\mu(x,y)$ and $N^\mu(x,y)$ by (2). Then for the Schlesinger-transformed solution $y_{\mu+1}(\tau)$ the moving frame defined by the similar procedure $F_x^{\mu+1}(x,y)$, $F_y^{\mu+1}(x,y)$ and $N^{\mu+1}(x,y)$ is related to initial frame as follows:*

$$\begin{pmatrix} F_y^{\mu+1}(x,y) \\ F_y^{\mu+1}(x,y) \\ N^{\mu+1}(x,y) \end{pmatrix} = S(x,y) \begin{pmatrix} F_y^{\mu}(x,y) \\ F_y^{\mu}(x,y) \\ N^{\mu}(x,y) \end{pmatrix}, \tag{30}$$

where

$$S(x,y) = \begin{pmatrix} -\omega_1\,\omega_2\,\cos(4\,y) & \omega_2\,\sin(4\,y) & \omega_2\,\sqrt{1-\omega_1^2}\,\cos(4\,y) \\ \omega_1\,\sin(4\,y) & \cos(4\,y) & -\sqrt{1-\omega_1^2}\,\sin(4\,y) \\ \omega_1\,\sqrt{1-\omega_2^2}\,\cos(4\,y) & -\sqrt{1-\omega_2^2}\,\sin(4\,y) & -\sqrt{(1-\omega_1^2)(1-\omega_2^2)}\,\cos(4\,y) \end{pmatrix} + \begin{pmatrix} -\sqrt{(1-\omega_1^2)(1-\omega_2^2)} & 0 & -\omega_1\,\sqrt{1-\omega_2^2} \\ 0 & 0 & 0 \\ -\omega_2\,\sqrt{1-\omega_1^2} & 0 & -\omega_1\,\omega_2 \end{pmatrix}, \tag{31}$$

with

$$\omega_1 = \omega_1(\tau) = \frac{1 + y(\tau)}{1 - y(\tau)}, \qquad \omega_2 = \omega_2(\tau) = \frac{\tau + y(\tau)}{\tau - y(\tau)}$$

and $y(\tau)$ a geometrical solution of (13), (14).

Proof : The proof follows directly from (23) with (2) and the isomorphism (1) between $su(2)$ and $\mathbb{R}^3$.
Of course the moving frames are given in a natural way by the partial derivatives (and there cross product) along the parametrization $F^\mu(x, y)$ and $F^{\mu+1}(x, y)$ respectively.

Remark 9 *In general the matrix $S(x, y)$ for arbitrary $y(\tau)$ has determinate* 1 *and the columns (rows) are orthogonal to each other. But they are real only if and only the function $y(\tau)$ is geomtrical.*

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