The Energy Function of the Skyrme Model

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1 Introduction

About 1960 the British physicist Tony Hilton Royle Skyrme (1922 - 1987) introduced a field theory, which is now known as the **Skyrme model** [1,2]. This is a non-linear field theory of low energy hadron physics, which identifies baryons as topological solitons of a field theory for pions.

A **Skyrme field** is a smooth map

$$U: \mathbb{R}^3 \times \mathbb{R} \longrightarrow SU(2)$$

with the boundary condition $U(x) \to 1$ for $|x|_{\mathbb{R}^3} \to \infty$. In the next chapter we will see that the assumption of finite energy already implies this boundary condition. Here $\mathbb{R}^3 \times \mathbb{R}$ is the physical space-time and $SU(2)$ is the Lie-group of special unitary 2×2 matrices. The space of all these maps U is called **configuration space** $\mathcal{C}$. For the following equations we need the Maurer-Cartan form

$$L := U^{-1}dU: \mathbb{R}^3 \times \mathbb{R} \to su(2) \ ,$$

where $su(2)$ is the Lie-algebra of $SU(2)$, i. e. the algebra of skrew-symmetric 2×2 matrices. The model is characterized by its **Lagrangian**

$$\mathcal{L} = -F_\pi |L|^2 - e|L \times L|^2 \ , \tag{1}$$

where the first term is the Lagrangian of the σ-model and the second is the so-called Skyrme term, which ensures the existence of soliton solutions. The norm $|.|^2$ is given by the Killing form $\frac{1}{2}Tr(.^2)$. The constants are the pion decay constant F_π and a unit-free parameter e introduced by Skyrme to stabilize the solitons [3]. Without loss of generality, we will fix $F_\pi \cdot e^{-1} = 1$. Because of the Skyrme term the model is no longer renormalizable.

We will only look for time-independent critical points of the energy function. Therefore, we consider the **energy function** $\mathcal{E}$ given by:

$$\mathcal{E} = -\int_{\mathbb{R}^3} \mathcal{L}\ dx^3 = \int_{\mathbb{R}^3} |L|^2 + |L \times L|^2\ dx^3 \tag{2}$$

This function is invariant under action of the chiral group $SU_L(2) \times SU_R(2)$, i. e. invariant by transformations of the form

$$U \rightsquigarrow A\ U\ D$$

for fixed $A, D \in SU(2)$. This chiral symmetry of the Skyrme model is obvious when we note

$$L_{AUD} = D^{-1} L_U D\ .$$

Hence $|L_{AUD}|^2 = |L_U|^2$ and in the same way $|L_{AUD} \times L_{AUD}|^2 = |L_U \times L_U|^2$. Therefore, the energy is invariant under chiral transformations.

A variation of the energy function (2) yields the **field equation** for the Skyrme model:

$$\delta\left(L + \frac{1}{4}\big[L, [L, L]\big]\right) = 0 \tag{3}$$

Here $\delta := *d*$, where $*$ is the Hodge-operator. Solutions of this differential equation, called **Skyrmions**, are characterized by a topological invariant, called **baryon number** B of the particle. If U is extended to the compactification S^3 of $\mathbb{R}^3$, which is possible due to the boundary condition, the invariant B is given by the degree of the field $U\colon S^3 \to S^3$. It has the form

$$B(U) := \deg(U) = \int_{S^3} U^*\omega \Big/ \int_{S^3} \omega = \frac{1}{12\pi^2} \int_{S^3} Tr(L \wedge L \wedge L)\ \in \mathbb{Z} \tag{4}$$

for the canonical volume form ω on S^3. So, this baryon number B classifies the configuration space $\mathcal{C}$ and for every $B \in \mathbb{Z}$ there will be at least a local minimum of the energy function [4].

In the next chapter we will introduce a general ansatz, will prove that the boundary condition of the field U comes out of finite energy, and will present the classical ansatz, the so-called **hedgehog-ansatz**. In the last chapter we will prove two inequalities, which show that the energy has a lower boundary and that there is binding-energy between two Skyrmions.

2 A General Ansatz

For solving the differential equation (3) and for getting more information of the energy (2) we need an ansatz. Here we will use the most general one. Every field $U\colon \mathbb{R}^3 \to SU(2)$ in $\mathcal{C}$ can be written in the form

$$U = \cos(f) + \sin(f) \cdot \vec{n} \tag{5}$$

for a smooth function $f\colon \mathbb{R}^3 \to \mathbb{R}$ and a unit vector $\vec{n}$ in $\mathbb{R}^3$. Note that $\vec{n}$ is perpendicular to $d\vec{n}$. So we get by easy calculation:

$$\begin{aligned}
L &= df \cdot \vec{n} + \cos(f)\sin(f)d\vec{n} - \sin^2(f)(\vec{n} \times d\vec{n}) \\
L \times L &= \tfrac{1}{2}df \sin(2f)(\vec{n} \times d\vec{n}) + df \sin^2(f)d\vec{n} + \sin^2(f)(d\vec{n} \times d\vec{n}) \\
|L|^2 &= (df)^2 + \sin^2(f)|d\vec{n}|^2 \\
|L \times L|^2 &= df \sin^2(f)|d\vec{n}|^2 + \sin^4(f)|d\vec{n} \times d\vec{n}|^2
\end{aligned}$$

With this results we obtain the

Lemma 1 *Using the general ansatz (5) we get for the Lagrangian and the energy:*

$$\begin{aligned}
\mathcal{L} &= -(df)^2(1 + \sin^2(f)|d\vec{n}|^2) - \sin^2(f)|d\vec{n}|^2 - \sin^4(f)|d\vec{n} \times d\vec{n}|^2 \\
\mathcal{E} &= \int_{\mathbb{R}^3} (df)^2(1 + \sin^2(f)|d\vec{n}|^2) + \sin^2(f)|d\vec{n}|^2 + \sin^4(f)|d\vec{n} \times d\vec{n}|^2 \; dx^3
\end{aligned}$$

When we look more carefully at the energy we receive the

Corollary 1 *If the energy of a Skyrmion is finite, the field of this particle fulfils the boundary condition*

$$U(x) \longrightarrow 1 \quad \textit{for} \quad |x|_{\mathbb{R}^3} \to \infty \ .$$

Proof: Since every term in the integrand of the energy function (see Lemma 1) is positive we get from $\mathcal{E} < C$, for a constant C, especially

$$\int_{\mathbb{R}^3} (df(x))^2 \; dx^3 < C \quad \text{and} \quad \int_{\mathbb{R}^3} \sin^2(f)|d\vec{n}|^2 \; dx^3 < C \ .$$

So $(df(x))^2$ tends to 0 faster than x^{-3} for great $|x|$. Hence $f(x) \to \mathit{const}$ and because of the second inequality we get $f(x) \to 2\pi \cdot n$ ($n \in \mathbb{Z}$) for $|x|_{\mathbb{R}^3} \to \infty$, i. e. $U(x) \to 1$. $\square$

Now, we want to have a look at the baryon number B defined in (4).

Lemma 2 *The baryon number B is given by $\pi^{-1} \cdot f(0)$.*

Proof: When we use ansatz (5) we get:

$$
\begin{aligned}
B(U) &= \frac{1}{24\pi^2} \int_{S^3} \langle L \times L, L \rangle \; dx^3 = \\
&= \frac{1}{12\pi^2} \int_{S^3} df \sin^2(f) |\vec{n} \times d\vec{n} \times d\vec{n}| vol(S^3) = \\
&= \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} df \sin^2(f) \; dr \; d\phi \; d\theta = \\
&= \frac{1}{2\pi^2} \int_0^{2\pi} \int_0^{\pi} f(0,\phi,\theta) - \tfrac{1}{2} \sin 2f(0,\phi,\theta) \; d\phi \; d\theta = \\
&= \frac{1}{\pi} \Big(f(0) - \tfrac{1}{2} \sin 2f(0) \Big) \; ,
\end{aligned}
$$

since $f(0,\phi,\theta)$ is independent of ϕ and θ. Hence we receive $B(U) = \pi^{-1} \cdot f(0)$. □

Example: A special ansatz is the so-called **hedgehog-ansatz**, which has rotational symmetry in all three space directions. It is the most common ansatz in this model (e. g. [5]). Such a Skyrme field has the form

$$
U(x) = \exp\Big(f(r) \frac{x}{r} \Big) = \Big(\cos f(r), \frac{x}{r} \sin f(r) \Big) \tag{6}
$$

with $x = (x_1, x_2, x_3)$, $r = |x|$ and a smooth function $f \colon [0,\infty] \to \mathbb{R}$. Finite energy implies, of course, again $f(r) \overset{r\to\infty}{\longrightarrow} 0$, c. f. [4].
For this hedgehog ansatz there is $\vec{n} = \frac{\mathrm{x}}{r}$, hence, $|\vec{n}| = 1$, $|d\vec{n}| = \frac{2}{r^2}$ and $|d\vec{n} \times d\vec{n}| = \frac{1}{r^4}$ and we obtain the well known results for Lagrangian, energy, baryon number and field equation:

$$
\begin{aligned}
\mathcal{L} &= -\frac{1}{2} \left(\frac{df(r)}{dr} \right)^2 - \frac{1}{r^2} \sin^2 f(r) - \frac{1}{r^2} \left(\frac{df(r)}{dr} \right)^2 \sin^2 f(r) - \frac{1}{2r^4} \sin^4 f(r) \\
\mathcal{E} &= 2\pi \int_0^{\infty} \left(\frac{df(r)}{dr} \right)^2 \cdot \Big(r^2 + 2\sin^2 f(r) \Big) + \sin^2 f(r) \Big(2 + \frac{1}{r^2} \sin^2 f(r) \Big) \; dr \\
B &= -\frac{2}{\pi} \int_0^{\infty} df(r) \sin^2(f(r)) \; dr = \pi^{-1} \cdot f(0) \\
0 &= 2r \frac{df(r)}{dr} + \frac{d^2 f(r)}{dr^2} \Big(r^2 + 2\sin^2 f(r) \Big) + \left(\frac{df(r)}{dr} \right)^2 \sin 2f(r) - \\
&\quad - \sin 2f(r) - \frac{1}{r^2} \sin^2 f(r) \sin 2f(r)
\end{aligned}
$$

3 Boundaries of the energy

First, we want to show that the energy $\mathcal{E}$ of a Skyrmion has a lower boundary depending on the baryon number B. We also prove that this boundary is not sharp and therefore the Skyrme model is not of Bogomol′nyi-typ, i. e. there is no reduction of the field equation by the Bogomol′nyi-trick to a differential equation of degree 1.

Lemma 3 *The energy of a Skyrmion of baryon number $B > 0$ fulfils*

$$\mathcal{E} > 12\pi^2 B \ .$$

Proof: By quadratic completion, c. f. [6], we obtain

$$\begin{aligned} \mathcal{E} &= \int_{\mathbb{R}^3} |L|^2 + |L \times L|^2 \ dx^3 \\ &= \int_{\mathbb{R}^3} |L + \frac{1}{4} L \times L|^2 \ dx^3 + 12\pi^2 B \end{aligned}$$

Hence, $\mathcal{E} \geq 12\pi^2 B$ and equality holds, iff $L = -\frac{1}{4} L \times L$. With the help of the Maurer-Cartan structure equation $*dL + L \times L = 0$ we get

$$L = \tfrac{1}{4} * dL \ .$$

Therefore,

$$|L|^2 = *(L \wedge *L) = \tfrac{1}{8} * d(L \wedge L)$$

and we get by Stokes theorem

$$\int_{\mathbb{R}^3} |L|^2 \ dx^3 = \tfrac{1}{8} \int_{\mathbb{R}^3} d(L \wedge L) = 0 \ ,$$

i. e. $|L|^2 = 0 \Leftrightarrow L \equiv 0$. So, equality holds just for $U \equiv 1$, i. e. $B(U) = 0$. □

Now we want to go on with a proof for binding-energy between two Skyrmions.

Lemma 4 *For any two natural numbers $n, k \in \mathbb{N}$ the energies $\mathcal{E}_n$ and $\mathcal{E}_k$ of Skyrmions of baryon number n and k fulfil the inequality*

$$\mathcal{E}_{n+k} < \mathcal{E}_n + \mathcal{E}_k \ .$$

Proof: The most common way to construct a Skyrmion of baryon number $n+k$ out of two others of baryon number n and k is to use the so-called **product-ansatz**, i. e. the field is of the form UV with fields U and V of Skyrmions with baryon number n and k. Here we will use this product-ansatz that results in a field of baryon number $n+k$, c. f. [7,p.88], and show that the energy difference

$$\Delta\mathcal{E} := \mathcal{E}_{n+k} - \mathcal{E}_n - \mathcal{E}_k$$

may be negative. This is sufficient, since the energy of a Skyrmion of baryon number $n+k$ has, at most, the lowest value of $\mathcal{E}_{n+k}$ that we can construct here. The fields of the two Skyrmions are given by

$$U = \cos(f) + \sin(f)\cdot\vec{n} \quad \text{and} \quad V = \cos(g) + \sin(g)\cdot\vec{m}\ ,$$

with $f(0) = n\cdot\pi$ and $g(0) = k\cdot\pi$. Now, we place the two Skyrmions so far apart that at least one of the functions $f, g \in C^\infty(\mathbb{R}^3)$ is almost zero at each point of the space $\mathbb{R}^3$. Therefore, we can concentrate on the leading terms of the energy difference $\Delta\mathcal{E}$; i. e., we will ignore terms that are at least quadratic in both functions f and g. We obtain:

$$\Delta\mathcal{E} \sim \int_{\mathbb{R}^3} df \cos^2(f)\cos(g)\sin(g)\langle\vec{n}, d\vec{m}\rangle + dg\cos(f)\sin(f)\cos^2(g)\langle\vec{m}, d\vec{n}\rangle\ dx^3$$

Both terms of the integrand are highest in the middle, between the two Skyrmions, since everywhere else one of the functions and its derivative is almost zero. In this area, we can approximate $\cos(f)$ and $\cos(g)$ by 1 and $\sin(f)$ resp. $\sin(g)$ by f resp. g. Hence, a first approximation of the energy difference is given by

$$\Delta\mathcal{E} \sim \int_{\mathbb{R}^3} g df\langle\vec{n}, d\vec{m}\rangle + f dg\langle\vec{m}, d\vec{n}\rangle\ dx^3\ .$$

Now, we will make use of the chiral symmetry of the Skyrme model. By a chiral transformation $U \rightsquigarrow AUD$ we obtain

$$\Delta\mathcal{E} \sim \int_{\mathbb{R}^3} A\ g df\langle\vec{n}, d\vec{m}\rangle\ D + A\ f dg\langle\vec{m}, d\vec{n}\rangle\ D\ dx^3\ ,$$

since the integrand is linear in $\vec{n}$. So we can choose $A, D \in SU(2)$ in such a way that $\Delta\mathcal{E}$ is negative. Hence, there is binding-energy. $\square$

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