Hyperelliptic Riemann surfaces of infinite genus and solutions of the KdV equation

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Abstract

In this article we construct (connected) hyperelliptic Riemann surfaces of infinite genus which have period matrices close to the diagonal matrix $\sqrt{-1}\mathrm{diag}(\tau_1, \tau_2, \cdots)$ with arbitrary $\tau_i > 0$. By previous results of the authors, one may therefore associate (renormalized) theta functions to these surfaces. Via an Its–Matveev formula this gives, in particular, rise to new solutions of the KdV equations

*Supported by DFG SFB 288 "Differentialgeometrie und Quantenphysik"

1 Introduction

In this paper, we construct new classes of quasi-periodic solutions of the Korteweg-de Vries equation

$$u_t = 6uu_x - u_{xxx}, \quad u|_{t=0} = q, \tag{1.1}$$

where q is a function of $x \in \mathbb{R}$. To explain our main result, we recall some basic facts about the Cauchy problem for the KdV equation with periodic initial conditions. The investigation of this problem was started independently by Novikov [17], Lax [10], Matveev [12], and McKean and van Moerbeke [13], and it is now well known that for initial conditions q that are smooth and periodic in $x \in \mathbb{R}$, the integration of (1.1) is closely related to the inverse problem for the Schrödinger operator

$$H = -\frac{d^2}{dx^2} + q(x) \tag{1.2}$$

with the periodic potential q. An important part in the study of the periodic (or, more generally, quasi-periodic) problem for the KdV equation is the investigation of the class of potentials q for which the Schrödinger operator (1.2) has only finitely many gaps (zones of instability) in the spectrum on the whole real line. The associated spectral surface $\mathcal{M}$ is a hyperelliptic Riemann surface of genus g (=number of gaps), and the theory of abelian functions can be used to reconstruct the potential from the spectrum of the periodic Schrödinger operator and to solve the KdV equation with these initial conditions. More precisely, for finite-zone periodic potentials, the solution of the KdV equation can be expressed explicitly by the Its-Matveev formula in terms of the Riemann theta functions associated to the corresponding spectral surface [8], [5], [4].

This approach can be generalized, eliminating all reference to the Schrödinger operator. The boundaries of the finitely many gaps can be prescribed arbitrarily, and all results extend to this more general setting. In particular, the solutions of the KdV equation are given by an extension of the Its-Matveev formula which we describe now. Let $\mathcal{M}$ be the hyperelliptic Riemann surface associated with the fixed gap structure. Let Ω be the period matrix of $\mathcal{M}$, and let $\vartheta_g(z|\Omega)$ be the corresponding Riemann theta function. Then there exist $U, V \in \mathbb{R}^g$ and a constant C such that for every $W \in \mathbb{R}^g$,

$$u(x,t) = -2\frac{d^2}{dx^2} \log \vartheta_g \left(Ut + Vx + W|\Omega \right) + C. \tag{1.3}$$

is a solution of the KdV equation (see, e.g., [2], [8], [4]). The vectors U and V are given by periods of certain normalized meromorphic 1-forms on $\mathcal{M}$ with a unique pole at ∞ of order 2 and 4, respectively, and C is also determined by $\mathcal{M}$. The solutions obtained in this way are quasi-periodic.

McKean and Trubowitz [14], [15] extended this theory to all smooth periodic potentials. The spectrum of the Schrödinger operator with an arbitrary periodic potential has, in general, infinitely many gaps and the corresponding spectral surface $\mathcal{M}$ is an infinite-genus hyperelliptic Riemann surface. Some of the function theory that exists for finite-genus surfaces can also be developed for such infinite-genus Riemann surfaces. In particular, there exist Riemann theta functions whose definition is originally due to Tahara [19]. Using these theta functions, the Its-Matveev formula (1.4) can be generalized, i.e. the solution of the KdV equation with initial data a periodic potential can be expressed in terms of these theta functions by a formula that is analogous to (1.3). The study of Riemann surfaces of infinite genus associated with infinite–gap solutions of the KdV equation has been continued by Feldman, Knörrer, Trubowitz [7] and one of the authors [18]. In all these cases, the period matrix Ω of the corresponding surface satisfies

$$\langle z, \operatorname{Im}\Omega z\rangle \geq \frac{1}{2\pi}\sum_{j\geq 1} |\log t_j|\, |z_j|^2\,,\ z\in\mathbb{C}^\infty,$$

where the $t_j \in (0,1)$ are such that

$$\sum_{j\geq 1} t_j^\beta < \infty$$

for some $0<\beta<1/2$. Again, the reference to the Schrödinger operator can be eliminated.

In the present paper, we generalize this approach to the Cauchy problem for the KdV equation, using other classes of hyperelliptic Riemann surfaces of infinite genus. The surfaces that we are going to consider are determined by a sequence of ramification points of the form

$$\Lambda:\ \lambda_0 = 0 < \lambda_1 < \lambda_2 < \cdots \to \infty, \tag{1.4}$$

where we assume that Λ has no finite points of accumulation. Such a surface has a canonical homology basis $A_1, B_1, A_2, B_2, \ldots.$ We construct explicitly a basis $\psi_1, \psi_2, \ldots$ for the space of holomorphic square integrable 1–forms on $\mathcal{M}$ which is normalized by

$$\int_{A_i}\psi_j = \delta_{ij}.$$

Such a basis is uniquely determined and, in analogy with the finite–genus case, the period matrix of $\mathcal{M}$ is defined by

$$\Omega = \left(\int_{B_i} \psi_j \right).$$

Then one of our main results can be stated as

Theorem 1.1 *Given a diagonal matrix $T = diag(\tau_1, \tau_2, \dots)$ with $\tau_j > 0$, $j \geq 1$, and real numbers $p > 1, C > 0$, there exists a sequence Λ such that the period matrix Ω of $\mathcal{M}(\Lambda)$ is purely imaginary and satisfies*

$$\left| \left(\Omega - \sqrt{-1} T \right)_{ij} \right| \leq \frac{C}{i^p j^p} \, \tau_i^{1/2} \, \tau_j^{1/2} \tag{1.5}$$

for all $i, j \geq 1$.

We add some remarks to illustrate our theorem. Let $\mathfrak{M}_g$ be the moduli space of genus g Riemann surfaces. Recall that the moduli space has a natural compactification $\overline{\mathfrak{M}}_g$ such that the boundary consists of surfaces with ordinary double points. If we approach the boundary along some path then the period matrix degenerates into a block diagonal matrix (see Corollary 3.2 in [6]). In particular, a hyperelliptic surface may degenerate along a path $\sigma(t) = X_t$, $t \in (0, 1]$, such that the limit surface X_0 has $g - 1$ double points $p_1, ..., p_{g-1}$ and $X_0 - \{p_1, ..., p_{g-1}\}$ is the disjoint union of g punctured torii with periods $\tau_1, ..., \tau_g$. Let $\Omega(t)$ be the period matrix of X_t. Then $\Omega(t)$ has an expansion

$$\Omega(t) = \begin{pmatrix} \tau_1 & & 0 \\ & \ddots & \\ 0 & & \tau_g \end{pmatrix} + O(t), \tag{1.6}$$

where $O(t)$ is a matrix whose entries tend to zero as $t \to 0$. This follows by repeated use of Corollary 3.2 in [6]. Theorem 1.1 may be regarded as a generalization of (1.6) to infinite–genus hyperelliptic surfaces. Geometrically, we take a sequence of torii Y_i, $i \in \mathbb{N}$, such that Y_i has period $\sqrt{-1}\ \tau_i$ and connect the Y_i's successively by hyperbolic cylinders. If we make the length of the deviding geodesics sufficiently small, we can arrange the surface so that the period matrix becomes arbitrarily close to $\sqrt{-1}\, T$. This is essentially how Theorem 1.1 is proved.

In [16], we introduced an infinite–dimensional analogue H_T^p of the Siegel upper half–space and Theorem 1.1 means simply that $\Omega \in H_T^p$. Let

$$\ell_{p,T} = \Big\{ (z_i)_{i\in\mathbb{N}} \in \mathbb{C}^\infty \;\Big|\; \| z \|_{p,T}^2 = \textstyle\sum_{i\geq 1} |z_i|^2 \tau_i^{-1} i^p < \infty \Big\}. \tag{1.7}$$

In [16], we also defined renormalized theta functions $\vartheta_T(z|\Omega)$ which depend on $z \in \ell_{p,T}$ and $\Omega \in H_T^p$. Using these theta functions, we extend the Its–Matveev formula (1.3) to our situation, and construct in this way new solutions of the KdV equation. In section 5 we prove

Theorem 1.2 *Let $p > 3$ and $T = \mathrm{diag}(\tau_1, \tau_2, \dots)$ be given. Let Ω be the period matrix of a hyperelliptic surface $\mathcal{M}(\Lambda)$ such that (1.5) holds. Then $\Omega \in H_T^p$ and there exist $U, V \in \ell_{p,T}$ and $C \in \mathbb{C}$ such that for every real $W \in \ell_{p,T}$,*

$$u(x,t) = -2\,\frac{d^2}{dx^2} \log \vartheta_T(Ux + Vt + W|\Omega) + C, \quad x, t \in \mathbb{R}, \tag{1.8}$$

is a nontrivial solution of the KdV equation (1.1).

The vectors U and V are again given by periods of certain meromorphic 1-forms. The spectral surface of the corresponding Schrödinger operator is obtained from $\mathcal{M}(\Lambda)$ by applying the lift of the map $\lambda \mapsto 1/\lambda$ of $\mathbf{P}^1$ to $\mathcal{M}(\Lambda)$. In particular, the gaps of the potentials decrease exponentially and accumulate at zero. Our solutions u of the KdV equation are limits of solutions which are associated to Riemann surfaces of finite genus. As Novikov [17] has shown, finite–gap potentials are quasi–periodic. If T is such that $\tau_i \geq C\, i^{2p}$, $i \geq 1$, for some $p > 1$, our potentials are uniformly approximated by finite–gap potentials and therefore, Novikov's result implies that in this case our potentials are quasi–periodic. Moreover, the corresponding solutions of the KdV equation are quasi–periodic in x and t. These solutions are different from the ones constructed in [11]. We don't know yet if, in general, our solutions are quasi–periodic.

Finally, we briefly describe the content of the paper. In section 2 we summarize some facts about hyperelliptic Riemann surfaces of infinite genus and we introduce a canonical family of holomorphic 1–forms called elementary 1–forms. In section 2 we study the periods of the elementary 1–forms w.r.t. the canonical homology basis $A_1, B_1, A_2, B_2, \dots$. The main purpose in this section is to prove Theorem 3.1 which states that there exist surfaces

$\mathcal{M}(\Lambda)$ such that the matrices of the A– and B–periods are close to Id and $\sqrt{-1}T$, respectively. In section 4 we prove Theorem 1.1. The final section 5 is devoted to the proof of Theorem 1.2.

2 Hyperelliptic surfaces of infinite genus

In this section we shall collect some basic facts about the infinite–genus hyperelliptic surfaces that we will consider in this paper . Let Λ be a strictly increasing sequence (1.4). We assume that $\lambda_1 = 1$. We take two copies of $\mathbb{C}$, cut each of them from $-\infty$ to λ_0 and from λ_{2i-1} to λ_{2i}, $i \geq 1$. Attach all the lower banks on one copy of $\mathbb{C}$ to the corresponding upper banks on the other one and vice versa. The resulting noncompact hyperelliptic Riemann surface of infinite genus will be denoted by $\mathcal{M} = \mathcal{M}(\Lambda)$.

Throughout this paper we shall assume that the λ_i satisfy the estimate

$$\lambda_{2i-1}, \lambda_{2i} \geq Ci^2, \quad i \geq 1,$$

for some constant $C > 0$ independent of i. Then the analytic function

$$R(\lambda) = \left(-\lambda \prod_{i=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_{2i-1}} \right) \left(1 - \frac{\lambda}{\lambda_{2i}} \right) \right)^{1/2} \tag{2.1}$$

is well defined and entire on $\mathcal{M}$. In fact, we may regard $\mathcal{M}$ as the Riemann surface of the transcendental hyperelliptic irrationality $R(\lambda)$. Note that $\mathcal{M}$ is a ramified double covering $\mathcal{M} \xrightarrow{\pi} \mathbb{C}$ with ramification at the points λ_i.

Remark: Let $q \in C^{\infty}(\mathbb{R})$ and assume that $q(x+1) = q(x)$, $x \in \mathbb{R}$. Then $H = -d^2/dx^2 + q(x)$ is Hill's operator associated to q. The periodic and antiperiodic eigenfunctions of H determine an infinite sequence $\lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots \longrightarrow \infty$ of simple or double eigenvalues which satisfy

$$\lambda_{2i-1}, \lambda_{2i} = \pi^2 i^2 + O(1/i^2)$$

and the widths $\Delta_i = \lambda_{2i} - \lambda_{2i-1}$, $i \geq 1$, are rapidly decreasing. Suppose that the periodic spectrum is simple, i.e., $\lambda_{2i-1} < \lambda_{2i}$, $i \geq 1$. Then $[\lambda_{2i-1}, \lambda_{2i}]$ are the gaps (or the intervals of instability) and the surface $\mathcal{M}$ constructed above with respect to this sequence of eigenvalues is Hill's surface for the potential q [15].

The surface $\mathcal{M}$ has a canonical homology basis

$$A_1, B_1, A_2, B_2, \ldots, A_n, B_n, \ldots,$$

i.e., $\{A_i, B_i\}_{i \in \mathbb{N}}$ is a basis of $H_1(\mathcal{M}, \mathbb{Z})$ which satisfies

$$A_i \cdot A_j = B_i \cdot B_j = 0 \quad \text{and} \quad A_i \cdot B_j = \delta_{ij}.$$

The cycle A_i is homologous to the boundary of the cut from λ_{2i-1} to λ_{2i}, $i \geq 1$, oriented clockwise. The cycle B_i is the union of two semi–circles B_i^+ and B_i^- where B_i^+ is contained in the upper half–plane of the first sheet with endpoints at -1 and $(\lambda_{2i} + \lambda_{2i-1})/2$ and B_i^- is the semi–circle in the lower half–plane of the second sheet with the same endpoints. The resulting simple closed curve on $\mathcal{M}$ is oriented counter clockwise (see Fig.1).

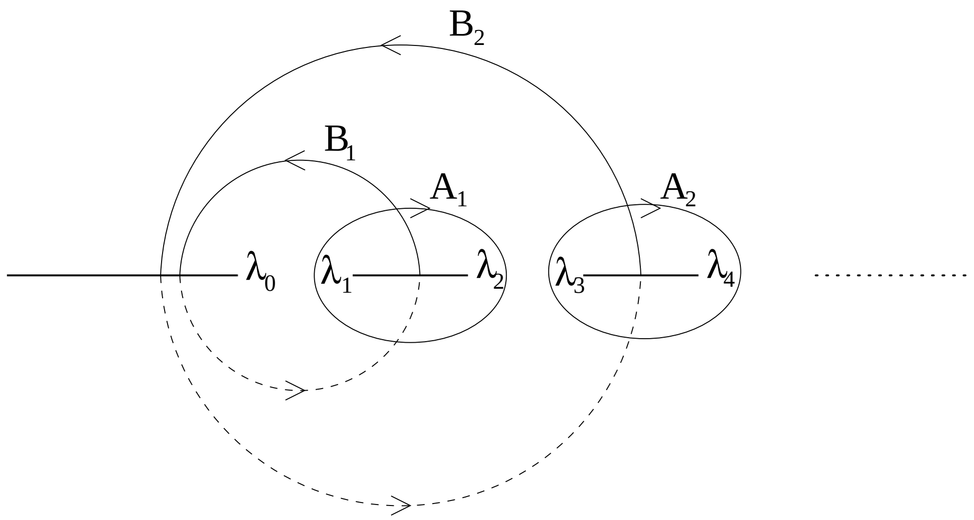

Fig.1 The canonical homology basis of $\mathcal{M}$.

We shall also use the elementary cycles S_j, $j \geq 1$, which are simple closed curves around the j–th hole of $\mathcal{M}$ (see Fig.2).

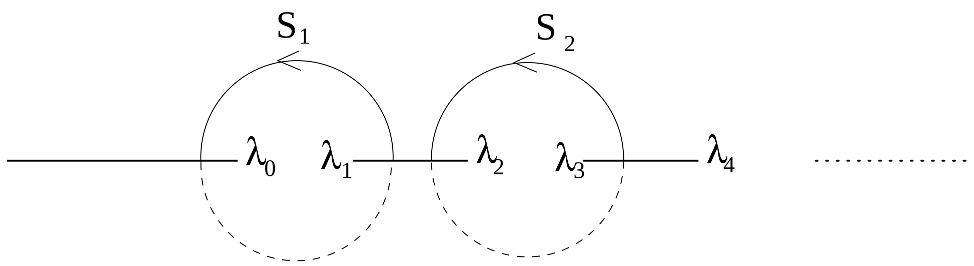

Fig.2 The elementary cycles S_j.

Note that B_i is homologeous to $\sum_{j=1}^{i} S_j$.

Furthermore, we recall that the complex stucture of $\mathcal{M}$ determines an inner product in the space $\Lambda^1(\mathcal{M})$ of differential 1–forms on $\mathcal{M}$. Let $\phi \in \Lambda^1(\mathcal{M})$ and let $z = x + iy$ be a coordinate on $U \subset \mathcal{M}$. Write

$$\phi\mid_U = fdx + gdy.$$

Then the Hodge star operator is given locally by

$$*\phi\mid_U = -gdx + fdy,$$

and the L^2–norm in $\Lambda^1(\mathcal{M})$ is defined by

$$\| \phi \|^2 = \int_{\mathcal{M}} \phi \wedge \overline{*\phi}.$$

We denote by $L^2\Lambda^1(\mathcal{M})$ the Hilbert space of all measurable square integrable 1–forms on $\mathcal{M}$. Let

$$\mathcal{H}^1(\mathcal{M}) = \{\phi \in \Lambda^1(\mathcal{M}) \mid \ d\phi = d * \phi = 0, \ \| \phi \| < \infty\}$$

be the space of L^2–harmonic 1–forms on $\mathcal{M}$. Then $\mathcal{H}^1(\mathcal{M})$ is a closed subspace of $L^2\Lambda^1(\mathcal{M})$ and therefore, $\mathcal{H}^1(\mathcal{M})$ is a Hilbert space.

Recall that a 1–form ω on $\mathcal{M}$ is called holomorphic, if in each coordinate system (U, z), $\phi|_U = f(z)dz$ and f is holomorphic on U. It is easy to see that ω is a holomorphic 1–form if and only if $d\omega = 0$ and $*\omega = -i\omega$. In particular, each holomorphic 1–form is harmonic, i.e., $d\omega = d * \omega = 0$. Let $\Omega^1(\mathcal{M})$ denote the space of square integrable holomorphic 1–forms on $\mathcal{M}$. Then we have

$$\Omega^1(\mathcal{M}) = \{\omega \in H^1(\mathcal{M}) \mid *\omega = -i\omega\}.$$

Since $*$ is a bounded linear operator in $\mathcal{H}^1(\mathcal{M})$, it follows that $\Omega^1(\mathcal{M})$ is a closed subspace of $\mathcal{H}^1(\mathcal{M})$ and therefore, $\Omega^1(\mathcal{M})$ is a Hilbert space with inner product

$$\langle \omega_1, \omega_2 \rangle = \int_{\mathcal{M}} \omega_1 \wedge \overline{*\omega_2} = i \int_{\mathcal{M}} \omega_1 \wedge \overline{\omega_2}.$$

Since $\mathcal{M}$ is hyperelliptic and defined by $R(\lambda)$, it is easy to see that every holomorhic 1–form ϕ on $\mathcal{M}$ can be written as

$$\phi = \frac{\varphi(\lambda)}{R(\lambda)}\, d\lambda$$

where φ is entire on $\mathbb{C}$. Moreover, the L^2–norm of φ is given by

$$H(\varphi) = \| \phi \|^2 = \frac{i}{2} \int_{\mathbb{C}} \left| \frac{\varphi(\lambda)}{R(\lambda)} \right|^2 d\lambda \wedge d\overline{\lambda}. \tag{2.2}$$

Thus, we may identify $\Omega^1(\mathcal{M})$ with the Hilbert space of all entire functions φ for which $H(\varphi) < \infty$.

Next observe that $\mathcal{M}$ admits an obvious exhaustion by compact surfaces with boundary. Let $f(z) = |\pi(z)|$, where $\pi : \mathcal{M} \to \mathbb{C}$ is the covering map. Then for each $t \neq \lambda_i$, $i \geq 0$, $\mathcal{M}_t = f^{-1}([0,t])$ is a compact Riemann surface with boundary and $\mathcal{M} = \cup_{t>0}\mathcal{M}_t$. This implies that the Riemann bilinear relations hold for $\mathcal{M}$ in the following sense [1]

Theorem 2.1 *Let ϕ and ψ be smooth, closed square integrable 1–forms on $\mathcal{M}$ such that*

$$\int_{A_i} \phi = \int_{A_i} \psi = 0$$

for all but finitely many indices i. Then we have

$$\int_{\mathcal{M}} \phi \wedge \psi = \sum_{i=1}^{\infty} \Big(\int_{A_i} \phi \int_{B_i} \psi \; - \; \int_{A_i} \psi \int_{B_i} \phi \Big).$$

Corollary 2.2 *Let ω be a square integrable holomorphic 1-form on $\mathcal{M}$ such that $\int_{A_i} \omega = 0$ for all $i \geq 1$, or $\int_{B_i} \omega = 0$ for all $i \geq 1$. Then $\omega = 0$.*

Proof. Since ω is holomorphic, both ω and $\overline{\omega}$ are closed and satisfy the assumption of Theorem 2.1 Hence

$$\| \omega \|^2 = i \int_{\mathcal{M}} \omega \wedge \overline{\omega} = i \sum_{i=1}^{\infty} \Big(\int_{A_i} \omega \int_{B_i} \overline{\omega} \; - \; \int_{A_i} \overline{\omega} \int_{B_i} \omega \Big) = 0.$$

□

As a consequence of Theorem 3.8 in [7] we have

Theorem 2.3 *There exists a unique basis $\{\omega_k\}_{k\in\mathbb{N}}$ of $\Omega^1(\mathcal{M})$ which satisfies*

$$\int_{A_i} \omega_j = \delta_{ij}, \quad i,j \in \mathbb{N}.$$

Our goal is to construct the forms ω_i explicitely. For this purpose, we introduce a sequence of auxiliary forms ϕ_i, $i \in \mathbb{N}$. Set

$$\mu_i = \sqrt{\lambda_{2i}\lambda_{2i-1}}, \quad i \geq 1. \tag{2.3}$$

Furthermore, let

$$\begin{aligned} a_1 &= 2\int_{\lambda_1}^{\lambda_2} \frac{d\lambda}{\left(\lambda\left(\frac{\lambda}{\lambda_1}-1\right)\left(1-\frac{\lambda}{\lambda_2}\right)\right)^{1/2}}, \\ a_i &= \frac{2}{\left(\lambda_{2i-2}\right)^{1/2}}\int_{\lambda_{2i-1}}^{\lambda_{2i}} \frac{d\lambda}{\left(\left(\frac{\lambda}{\lambda_{2i-2}}-1\right)\left(\frac{\lambda}{\lambda_{2i-1}}-1\right)\left(1-\frac{\lambda}{\lambda_{2i}}\right)\right)^{1/2}}, \\ & \qquad i \geq 2. \end{aligned} \tag{2.4}$$

Using these definitions, we set

$$\varphi_i(\lambda) = a_i^{-1} \prod_{\substack{1\leq k<\infty \\ k\neq i}} \left(1-\frac{\lambda}{\mu_k}\right),\ i \geq 1. \tag{2.5}$$

Then $\varphi_i(\lambda)$ is entire on $\mathcal{M}$ and we may define the following holomorhic 1–forms

$$\phi_i = \frac{\varphi_i(\lambda)}{R(\lambda)}\, d\lambda, \quad i \geq 1. \tag{2.6}$$

We call the forms ϕ_i elementary holomorphic 1–forms of our hyperelliptic surface. The following lemma is easy to prove

Lemma 2.4 *The 1–forms ϕ_i are square integrable.*

3 The periods of elementary forms

In this section we study the periods of the elementary 1–forms ϕ_i introduced in the last section. Let $\{\tau_i\}_{i\in\mathbb{N}}$ be a sequence of positive real numbers and let

$$T = \mathrm{diag}(\tau_1, \tau_2, \cdots) = \begin{pmatrix} \tau_1 & & & 0 & \\ & \tau_2 & & & \\ & & \ddots & & \\ & 0 & & \ddots & \\ & & & & \ddots \end{pmatrix}. \tag{3.1}$$

be the diagonal matrix with diagonal entries $\tau_1, \tau_2, \ldots$. Then we shall show that there exist hyperelliptic surfaces $\mathcal{M}$ whose period matrix is arbitrarily close to $\sqrt{-1}T$.
Let

$$\begin{aligned} A_{ij} &= \int_{A_i} \phi_j \\ B_{ij} &= -\sqrt{-1}\int_{B_i} \phi_j, \quad 1 \le i, j < \infty, \end{aligned} \tag{3.2}$$

where ϕ_j is the 1–form defined by (2.6). ¿From the definition, it follows that the periods A_{ij} and B_{ij} are both real. Let

$$\rho(i,j) = \mathrm{Min}(1, \tau_i^{\pm 1}, \tau_j^{\pm 1}, \tau_i^{\pm 1}\tau_j^{\pm 1}),\ i, j \in \mathbb{N}. \tag{3.3}$$

Theorem 3.1 *Given a sequence $\{\tau_i\}_{1\le i\le\infty}$ of positive real numbers and $p > 1$, there exists a sequence $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$, which has the following properties:*

1) $\lambda_0 = 0$, $\lambda_1 = 1$, and $\lambda_{2i-1}, \lambda_{2i} \ge i^2$,

2) There exists a constant $S < (\sum_{i\ge 1} i^{-2p})^{-1}$ such that

$$\begin{aligned} |A_{ij} - \delta_{ij}| &\le S\,\frac{\rho(i,j)}{i^p j^p}, \\ |B_{ij} - \tau_i\delta_{ij}| &\le S\,\frac{\rho(i,j)}{i^p j^p}, \quad i, j \ge 1. \end{aligned} \tag{3.4}$$

The proof of Theorem 3.1 needs some preparation. Let $v_1 < v_2 < v_3$ be given real numbers and set $P(z) = (z - v_1)(z - v_2)(z - v_3)$. Let E be the elliptic curve defined by the equation $w^2 = P(z)$. Then E is isomorphic to a complex torus $\mathbb{C}/L$ where $L = \mathbb{Z} \bigoplus \mathbb{Z}\tau$ and $\operatorname{Im}\tau > 0$. Let $A, B \in H_1(E, \mathbb{Z})$ be the canonical homology basis (see Fig. 3).

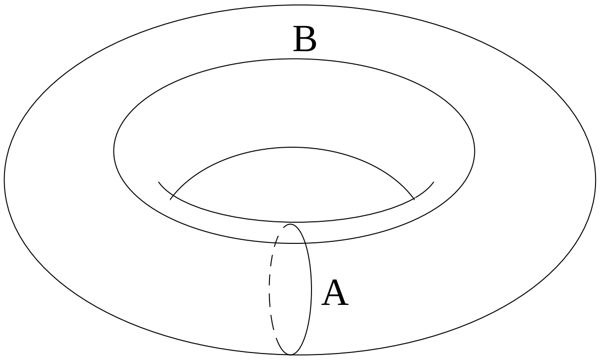

Fig.3

A holomorphic 1–form on E is given by

$$\omega = \frac{dz}{\sqrt{P(z)}}$$

and for the corresponding periods we obtain

$$\begin{aligned} \alpha &= \int_A \omega = 2\int_{v_2}^{v_3} \frac{dz}{\sqrt{P(z)}}, \\ \beta &= \int_B \omega = 2\int_{v_1}^{v_2} \frac{dz}{\sqrt{P(z)}}. \end{aligned} \tag{3.5}$$

Note that β is real, α is pure imaginary and $\tau = \beta/\alpha \in H = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$ is the period matrix of E.

Now we introduce real valued functions $f(u), g(u)$ and $h(u)$ of the real variable $u > 0$ by

$$f(u) = \int_0^1 \frac{dy}{(y(1-y)(1+u-y))^{1/2}}, \tag{3.6}$$

$$g(u) = \int_0^1 \frac{dy}{(y(1-y)(uy+1))^{1/2}}, \tag{3.7}$$

$$h(u) = \frac{f(u)}{g(u)}. \tag{3.8}$$

Using (3.5) and changing variables, it follows that

$$\tau = \frac{\beta}{\alpha} = ih\Big(\frac{v_3 - v_2}{v_2 - v_1}\Big). \tag{3.9}$$

Lemma 3.2 *The function h is a diffeomorphism of $(0,\infty)$ onto $(0,\infty)$.*

Proof. If $u \to 0$, then $f(u)$ diverges and $g(u)$ tends to a finite limit $\neq 0$. Hence, $h(u) \to \infty$ as $u \to 0$. On the other hand, we have

$$f(u) \le \frac{1}{\sqrt{u}} \int_0^1 \frac{dy}{(y(1-y))^{1/2}}$$

and

$$\begin{aligned} g(u) &\ge \frac{1}{\sqrt{u}} \int_0^{1/2} \frac{dy}{(y(1-y)(y+\frac{1}{u}))^{1/2}} \\ &\ge \frac{1}{\sqrt{u}} \int_0^{1/2} \frac{dy}{(y(y+\frac{1}{u}))^{1/2}} \ge \frac{1}{\sqrt{u}} \int_0^{1/2} \frac{dy}{y+\frac{1}{u}} \\ &= \frac{1}{\sqrt{u}} \log(1+\frac{u}{2}). \end{aligned} \tag{3.10}$$

Hence $h(u) \to 0$ as $u \to \infty$. This proves that h is surjective. Furthermore, the derivative of h is given by

$$h'(u) = \frac{1}{2\, g(u)^2} \int_0^1 \int_0^1 \frac{(y(1-x)-1)\, dxdy}{(x(1-x)y(1-y))^{1/2}(1+u-x)^{3/2}(uy+1)^{3/2}}$$

which shows that $h'(u) < 0$ for $u \in (0,\infty)$.

□

Thus, by Lemma 3.2, h is $1-1$ and therefore, the period τ determines the ratio $(v_3 - v_2)/(v_2 - v_1)$ of the roots of the polynomial $P(z)$ which defines the elliptic curve. We note that the inverse h^{-1} can be described in terms of the Weierstraß $\wp$–function. First observe that by a linear transformation $z = aw+b$, $P(z)$ can be brought into Weierstraß normal form $4w^3 - g_2 w - g_3 = 4(w-e_1)(w-e_2)(w-e_3)$, $e_1 < e_2 < e_3$. Let $\wp(z;\tau)$ be the Weierstraß $\wp$–function. Then the roots e_i are given by

$$e_1 = \wp\left(\frac{1}{2};\tau\right),\ e_2 = \wp\left(\frac{\tau}{2};\tau\right),\ e_3 = \wp\left(\frac{1+\tau}{2};\tau\right)$$

[9]. Thus, we have

$$\frac{v_3 - v_2}{v_2 - v_1} = \frac{e_3 - e_2}{e_2 - e_1} = \frac{\wp\left(\frac{1+\tau}{2};\tau\right) - \wp\left(\frac{\tau}{2};\tau\right)}{\wp\left(\frac{\tau}{2};\tau\right) - \wp\left(\frac{1}{2};\tau\right)} = h^{-1}(-i\tau).$$

Now we return to the hyperelliptic surfaces of infinite genus defined by a sequence $\{\lambda_i\}_{i=0}^{\infty}$. We introduce new parameters α_i and Δ_i by

$$\begin{aligned} \alpha_i &= \lambda_{2i-1} - \lambda_{2i-2} \\ \Delta_i &= \lambda_{2i} - \lambda_{2i-1}, \quad i \geq 1, \end{aligned} \tag{3.11}$$

where $\lambda_0 = 1$, $\lambda_1 = 1$ such that $\alpha_1 = 1$. The λ_i's are then given in terms of these parameters by

$$\begin{aligned} \lambda_{2i-1} &= \alpha_1 + \Delta_1 + \alpha_2 + \ldots + \alpha_i \\ \lambda_{2i} &= \alpha_1 + \Delta_1 + \alpha_2 + \ldots + \alpha_i + \Delta_i. \end{aligned} \tag{3.12}$$

Given a matrix T as in (3.1), the Δ_i will always be fixed in terms of the τ_i and α_i by the condition

$$\frac{\Delta_i}{\alpha_i} = h^{-1}(\tau_i), \quad i \geq 1. \tag{3.13}$$

In view of our previous discussion, this means the following. Each $\sqrt{-1}\tau_i$ determines a complex torus $E_i = \mathbb{C}/L_i$, $L_i = \mathbb{Z} \bigoplus \sqrt{-1}\tau_i\mathbb{Z}$ and, as elliptic

curve, E_i is given by an equation of the form $w^2 = P_i(z)$ where $P_i(z)$ has roots $\lambda_{2i}, \lambda_{2i-1}$ and λ_{2i-2}.
Note also that the definition (2.4) for the a_i may be rewritten as

$$a_i = 2\left(\frac{\lambda_{2i-1}\lambda_{2i}}{\alpha_i}\right)^{1/2} g\left(\frac{\Delta_i}{\alpha_i}\right),\ i \geq 1. \tag{3.14}$$

For later use we introduce the analogous numbers

$$\begin{aligned} b_1 &= 2\int_0^1 \frac{dy}{\left(\lambda\left(1-\lambda\right)\left(1-\frac{\lambda}{\lambda_2}\right)\right)^{1/2}} \\ b_i &= \frac{2}{\left(\lambda_{2i-2}\right)^{1/2}} \int_{\lambda_{2i-2}}^{\lambda_{2i-1}} \frac{d\lambda}{\left(\left(\frac{\lambda}{\lambda_{2i-2}}-1\right)\left(1-\frac{\lambda}{\lambda_{2i-1}}\right)\left(1-\frac{\lambda}{\lambda_{2i}}\right)\right)^{1/2}}, \\ &\qquad i \geq 2. \end{aligned} \tag{3.15}$$

Then we have

$$b_i = 2\left(\frac{\lambda_{2i-1}\lambda_{2i}}{\alpha_i}\right)^{1/2} f\left(\frac{\Delta_i}{\alpha_i}\right) \tag{3.16}$$

and therefore

$$\frac{b_i}{a_i} = h\left(\frac{\Delta_i}{\alpha_i}\right) = \tau_i, \quad i \geq 1. \tag{3.17}$$

We now briefly explain the main strategy for proving Theorem 3.1. For a given period matrix $\sqrt{-1}T$, the ratios Δ_i/α_i are fixed by (3.13) and by (3.12), the sequence $\{\lambda_j\}_{j=0}^{\infty}$ with $\lambda_0 = 0, \lambda_1 = 1$ is then uniquely determined in terms of the sequence $\alpha = \{\alpha_i\}_{i\in\mathbb{N}}$. We want to make the α_k's so large that intuitively $\mathcal{M}$ is obtained by gluing together tori Y_i with period matrices $\sqrt{-1}\tau_i$ and "separating distances" Δ_i between Y_i and Y_{i+1}. This will be achieved by induction, where at the l–th step we construct α^l for which the estimates (3.4) hold for all $i, j \leq l-1$. To perform the induction step we introduce the following

Definition 3.3 A sequence $\alpha = \{\alpha_k\}_{k\in\mathbb{N}}$ is called admissible if the following three conditions are satisfied

$$\begin{aligned} \alpha_k &\geq k^2 \lambda_{2k-2} \\ \alpha_{k+1} &\geq (\alpha_k)^2, \quad k \geq 1 \end{aligned} \tag{3.18}$$

and

$$\sum_{j=1}^{\infty} \left(\frac{\alpha_j}{\Delta_j} \right)^{1/2} \alpha_j^{-1/4} < \infty. \tag{3.19}$$

The λ_k's in (3.18) are given in terms of the α_k's by (3.12) and (3.13) with fixed τ_k's.

Remark. In view of (3.13), condition (3.19) depends on the growth of the periods τ_i. Suppose, for example, that there exist constants $C_1 > 0$ and $0 \leq c < \log 2$ such that

$$\tau_k \leq C_1 e^{ck}, \quad k \geq 1. \tag{3.20}$$

It is easy to see that for $u < 1$, we have

$$h(u) \geq C_1 |\log u|$$

for some constant $C_2 > 0$ and therefore,

$$h(e^{-x}) \geq C_1 x, \quad x > 0.$$

Let $C = C_1/C_2$. Then, by (3.20), we get

$$\tau_k \leq h(e^{-Ce^{ck}}).$$

Since h is monotonically decreasing, it follows from (3.13) that

$$\frac{\alpha_k}{\Delta_k} \leq e^{Ce^{ck}}. \tag{3.21}$$

Furhermore, (3.18) implies that $\alpha_k \geq (\alpha_2)^{2^{(k-2)}}$ and $\alpha_2 \geq 2$. Hence, we get

$$\sum_{k \geq 2} \left(\frac{\alpha_k}{\Delta_k} \right)^{1/2} \alpha_k^{-1/4} \leq \sum_{k \geq 2} e^{(C/2)e^{ck}} 2^{-2^{(k-3)}} < \infty.$$

Thus, if the periods τ_k satisfy (3.20), then (3.19) is a consequence of (3.18). It is easy to construct admissible sequences. We also introduce the notation

$$\alpha \leq_l \beta$$

for two admisseble sequences α and β to mean that $\alpha_k \leq \beta_k$ for all k with equality for $1 \leq k \leq l$. Our induction step will be such that $\alpha^l \leq_l \alpha^{l+1}$.

Remark. Note that if α is admissible and $\gamma_k,\ 1 \leq k < k_0 \leq \infty$, are numbers with $\alpha_k \leq \gamma_k$ with equality for $k \leq l$, then there exists an admissible sequence β with $\alpha \leq_l \beta$ and $\gamma_k \leq \beta_k$ for all $1 \leq k < k_0$.

For each admissible sequence there are some a priori estimates we will frequently make use of. First we claim that

$$\left(\frac{\lambda}{\lambda_{2k-1}} - 1\right)^{-1/2} \left(\frac{\lambda}{\lambda_{2k}} - 1\right)^{-1/2} \leq 2\frac{(\lambda_{2k-1}\lambda_{2k})^{1/2}}{\lambda} \tag{3.22}$$

for all $\lambda \geq \lambda_{2k+1}$. Indeed, by (3.18) we have $\lambda_{2k+1} = \lambda_{2k} + \alpha_{k+1} \geq (k+2)\lambda_{2k}$ which implies $\lambda/\lambda_{2k-1} \geq \lambda/\lambda_{2k} \geq k+2$. Hence $\lambda/\lambda_{2k-1} - 1 \geq \lambda/(2\lambda_{2k-1})$ and $\lambda/\lambda_{2k} - 1 \geq \lambda/(2\lambda_{2k})$, and (3.22) follows. Similarly one obtains

$$\left(1 - \frac{\lambda}{\lambda_{2k-1}}\right)^{-1/2} \left(1 - \frac{\lambda}{\lambda_{2k}}\right)^{-1/2} \leq 2 \tag{3.23}$$

for $\lambda \leq \lambda_{2k-2},\ k \geq 2$.
Next we introduce the functions

$$F_k(\lambda) = \frac{1 - \frac{\lambda}{\mu_k}}{\left(\left(1 - \frac{\lambda}{\lambda_{2k-1}}\right)\left(1 - \frac{\lambda}{\lambda_{2k}}\right)\right)^{1/2}} \tag{3.24}$$

for $\lambda \geq 0,\ k \geq 1$. F_k depends on the sequence α considered as a parameter. If is easy to check that $F_k(\lambda)$ is monotonically increasing in λ for $\lambda \leq \lambda_{2k-1}$. Hence, we get

$$0 \leq F_k(\lambda) \leq \frac{1}{\left(1 - \frac{\lambda_{2j}}{\lambda_{2k-1}}\right)^{1/2}\left(1 - \frac{\lambda_{2j}}{\lambda_{2k}}\right)^{1/2}}$$

for $\lambda \le \lambda_{2j}$ and $j < k$. Using (3.18), an easy calculation shows that

$$\begin{aligned}\lambda_{2k-1} &= \lambda_{2j} + \alpha_{j+1} + \Delta_{j+1} + \ldots + \alpha_k \\ &\ge j^2\lambda_{2j} + k^2\lambda_{2k-2} \ge (j^2+k^2)\lambda_{2j}.\end{aligned} \tag{3.25}$$

Hence, it follows that

$$0 \le F_k(\lambda) \le \left(1 - \frac{1}{1+k^2}\right)^{-1} \tag{3.26}$$

and therefore, we get

$$0 \le \prod_{k>j} F_k(\lambda) \le \prod_{m=1}^{\infty} \left(1 - \frac{1}{1+m^2}\right)^{-1} = q < \infty \tag{3.27}$$

for $\lambda \le \lambda_{2j}$, uniformly for all j and all admissible sequences α.

For $\lambda > \lambda_{2k}$ we have

$$|F_k(\lambda)| = \frac{\frac{\lambda}{\mu_k} - 1}{\left(\frac{\lambda}{\lambda_{2k-1}} - 1\right)^{1/2}\left(\frac{\lambda}{\lambda_{2k}} - 1\right)^{1/2}} \tag{3.28}$$

and this function is monotonically decreasing there. Hence, we obtain the estimate

$$|F_k(\lambda)| \le \frac{1}{\left(1 - \frac{\lambda_{2k-1}}{\lambda_{2j-1}}\right)^{1/2}\left(1 - \frac{\lambda_{2k}}{\lambda_{2j-1}}\right)^{1/2}} \tag{3.29}$$

which holds for $\lambda \ge \lambda_{2j-1}$, $k < j$, and where we used (2.3) and (3.23). Furthermore, since

$$\begin{aligned}\lambda_{2j-1} &= \lambda_{2k} + \alpha_{k+1} + \Delta_{k+1} + \ldots + \alpha_j \\ &\ge \lambda_{2k}(1+k^2)\end{aligned} \tag{3.30}$$

for $k < j$, we also have

$$\prod_{1\le k<j} |F_k(\lambda)| \le q \tag{3.31}$$

for all $\lambda \ge \lambda_{2j-1}$, uniformly for all j and all admissible sequences α.

Lemma 3.3 *There exists a sequence α^l, $l \in \mathbb{N}$, of admissible sequences with the following properties:*

(i) *For each $l \geq 2$, the relation $\alpha^{l-1} \leq_{l-1} \alpha^l$ holds.*

(ii) *For each admissible sequence α with $\alpha^l \leq_l \alpha$ the estimates (3.4) hold for all $i, j \leq l-1$.*

By our previous discussion, this proves Theorem 3.1.

Proof of Lemma 3.3. For $l = 1$ we only have to construct an admissible sequence α^1. Define $\alpha_1^1 = 1$ and set $\alpha_k^1 = \mathrm{Max}((\alpha_{k-1}^1)^2, k^2\lambda_{2k-2}^1)$, $k \geq 2$, inductively using (3.12) and (3.13). Now suppose that α^{l-1}, $l \geq 2$, has been constructed. In what follows, α will always be an admissible sequence with $\alpha^{l-1} \leq_{l-1} \alpha$. By the induction assumption we already have the desired estimates for A_{ij}, B_{ij}, $1 \leq i, j \leq l-1$, for any $\mathcal{M}(\Lambda)$ with the λ_i's given in terms of the α_i's. Hence, it suffices to consider $A_{il}, A_{li}, B_{il}, B_{li}$ for $i \leq l$.

We start with an estimate for A_{il}, $1 \leq i \leq l-1$. First we note that $R(\lambda)$, as given by (2.1), is real for $\lambda_{2k-1} \leq \lambda \leq \lambda_{2k}$, $k \geq 1$, and pure imaginary for all other real λ. Hence, we have

$$|A_{il}| \leq \frac{2}{a_l} \int_{\lambda_{2i-1}}^{\lambda_{2i}} \prod_{k \neq l} |F_k(\lambda)| \frac{d\lambda}{\left(\lambda\left(1 - \frac{\lambda}{\lambda_{2l-1}}\right)\left(1 - \frac{\lambda}{\lambda_{2l}}\right)\right)^{1/2}} \tag{3.32}$$

Now, for $\lambda \in [\lambda_{2i-1}, \lambda_{2i}]$ we write

$$\begin{aligned} \prod_{k \neq l} |F_k(\lambda)| =& \frac{\left(\frac{\lambda}{\mu_{i-1}} - 1\right)}{\left(\frac{\lambda}{\lambda_{2i-3}} - 1\right)^{1/2}\left(\frac{\lambda}{\lambda_{2i-2}} - 1\right)^{1/2}} \\ &\cdot \frac{\left(1 - \frac{\lambda}{\mu_i}\right)}{\left(\frac{\lambda}{\lambda_{2i-1}} - 1\right)^{1/2}\left(1 - \frac{\lambda}{\lambda_{2i}}\right)^{1/2}} \prod_{1 \leq k < i-1} |F_k(\lambda)| \prod_{\substack{i<k \\ k \neq l}} F_k(\lambda). \end{aligned} \tag{3.33}$$

In case $i = 1$, the first factor is replaced by 1. The last two factors in (3.33) are controlled by (3.27) and (3.31). Also by (3.23) we have

$$\left(1 - \frac{\lambda}{\lambda_{2l-1}}\right)^{-1/2}\left(1 - \frac{\lambda}{\lambda_{2l}}\right)^{-1/2} \leq 2. \tag{3.34}$$

This leads to the bound

$$|A_{il}| \leq a_l^{-1} K_i, \tag{3.35}$$

where K_i depends only on $\alpha_1, \ldots, \alpha_{l-1}$. Using (3.12) and (3.13), it follows that $\lambda_{2l-1} = \alpha_l + \beta_l$ and $\lambda_{2l} = \alpha_l + \gamma_l$, where β_l and γ_l depend only on $\alpha_1, \ldots, \alpha_{l-1}$. Hence, by (3.14) we obtain

$$a_l^{-1} \leq C_l \left(1 + \frac{\beta_l}{\alpha_l}\right)^{-1/2} \left(1 + \frac{\gamma_l}{\alpha_l}\right)^{-1/2} \alpha_l^{-1/2} \leq C_l \alpha_l^{-1/2}$$

where C_l depends only on τ_l which is fixed. Thus, we have proved that

$$|A_{il}| \leq C_i(\alpha_1, \ldots, \alpha_{l-1}) \alpha_l^{-1/2} \tag{3.36}$$

and C_i depends only on $\alpha_1, \ldots, \alpha_{l-1}$. Next consider A_{li}, $1 \leq i \leq l-1$. As above, we get

$$|A_{li}| \leq \frac{2}{a_i} \int_{\lambda_{2l-1}}^{\lambda_{2l}} \prod_{k \neq i} |F_k(\lambda)| \cdot \frac{1}{\lambda^{1/2} \left(\frac{\lambda}{\lambda_{2i-1}} - 1\right)^{1/2} \left(\frac{\lambda}{\lambda_{2i}} - 1\right)^{1/2}} \, d\lambda. \tag{3.37}$$

By (3.22) we have

$$\left(\frac{\lambda}{\lambda_{2i-1}} - 1\right)^{-1/2} \left(\frac{\lambda}{\lambda_{2i}} - 1\right)^{-1/2} \leq 2 \frac{(\lambda_{2i-1}\lambda_{2i})^{1/2}}{\lambda}$$

in the region of integration. We write

$$\begin{aligned} \prod_{k \neq i} |F_k(\lambda)| = & \frac{\left|1 - \frac{\lambda}{\mu_l}\right|}{\left(\frac{\lambda}{\lambda_{2l-1}} - 1\right)^{1/2} \left(1 - \frac{\lambda}{\lambda_{2l}}\right)^{1/2}} \\ & \cdot \prod_{\substack{1 \leq k \leq l-1 \\ k \neq i}} |F_k(\lambda)| \cdot \prod_{k > l} F_k(\lambda). \end{aligned} \tag{3.38}$$

Now observe that

$$\left|1-\frac{\lambda}{\mu_l}\right| \leq \frac{\Delta_l}{\mu_l} \leq \frac{\Delta_l}{\alpha_l}$$

in the region of integraion, since $\lambda_{2l-1} \leq \mu_l \leq \lambda_{2l}$. Again, the last two factors in (3.38) are controlled by (3.27) and (3.31). Hence we obtain an estimate of the form

$$|A_{li}| \leq 4\left(\lambda_{2i-1}\lambda_{2i}\right)^{1/2} a_i^{-1}\frac{\Delta_l}{\alpha_l} q^2 \cdot \int_{\lambda_{2l-1}}^{\lambda_{2l}} \frac{d\lambda}{\lambda^{3/2}\left(\frac{\lambda}{\lambda_{2l-1}}-1\right)^{1/2}\left(1-\frac{\lambda}{\lambda_{2l}}\right)}. \tag{3.39}$$

The integral on the r.h.s. can be estimated by

$$\left(\lambda_{2l-1}\right)^{-1}\left(\lambda_{2l}\right)^{1/2}\int_0^{\Delta_l}\frac{d\lambda}{\lambda^{1/2}\left(\Delta_l-\lambda\right)^{1/2}} = \frac{\lambda_{2l}^{1/2}}{\lambda_{2l-1}}\int_0^1\frac{d\lambda}{(\lambda(1-\lambda))^{1/2}}.$$

Furthermore, by (3.12) we have $\lambda_{2l-1} > \alpha_l$ and $\lambda_{2l} = \Delta_l + \alpha_l + \lambda_{2l-2}$. Hence, we get

$$\frac{\lambda_{2l}^{1/2}}{\lambda_{2l-1}} \leq c_l \alpha_l^{-1/2}.$$

Here c_l depends only on $\alpha_1, \ldots, \alpha_{l-1}$ and we recall that $\frac{\Delta_l}{\alpha_l}$ is fixed by τ_l. Combined with (3.39), it follows that

$$|A_{li}| \leq C_l(\alpha_1, \ldots, \alpha_{l-1})\alpha_l^{-1/2}, \quad i \leq l-1, \tag{3.40}$$

and $C_l(\alpha_1, \ldots, \alpha_{l-1})$ depends only on $\alpha_1, \ldots, \alpha_{l-1}$. Again we have used that Δ_l/α_l is fixed in terms of τ_l.

Now we come to A_{ll}. By definition we have

$$A_{ll} = 2a_l^{-1}\int_{\lambda_{2l-1}}^{\lambda_{2l}} \frac{\prod_{k\neq l} F_k(\lambda)}{\left(\lambda\left(\frac{\lambda}{\lambda_{2l-1}}-1\right)\left(1-\frac{\lambda}{\lambda_{2l}}\right)\right)^{1/2}} \quad d\lambda. \tag{3.41}$$

To analyze this expression we introduce the functions $L_k(\lambda),\ k \geq 1$, defined by

$$
\begin{aligned}
L_1(\lambda) &= \frac{\frac{\lambda}{\mu_1} - 1}{\left(\lambda\left(\frac{\lambda}{\lambda_1} - 1\right)\right)^{1/2}} \\
L_k(\lambda) &= \frac{\frac{\lambda}{\mu_k} - 1}{\left(\left(\frac{\lambda}{\lambda_{2k-1}} - 1\right)\left(\frac{\lambda}{\lambda_{2k-2}} - 1\right)\right)^{1/2}}, \quad k \geq 2.
\end{aligned}
\tag{3.42}
$$

Then we split the numerator in (3.41) as

$$
\prod_{k \neq l} F_k(\lambda) = \lambda^{1/2} \left(\frac{\lambda}{\lambda_{2l-2}} - 1\right)^{-1/2} \prod_{1 \leq k \leq l-1} L_k(\lambda) \cdot \prod_{k > l} F_k(\lambda).
\tag{3.43}
$$

Now observe that $L_k(\lambda)$ is increasing for $\lambda > \mu_k$. Moreover, since $\mu_k > \lambda_{2k-1} > \lambda_{2k-2}$, we have $L_k(\lambda) < 1$ for $\lambda > \mu_k$. This implies that

$$
\begin{aligned}
0 \leq (\lambda_{2l-2})^{-1/2} - &\prod_{1 \leq k \leq l-1} L_k(\lambda) \\
&\leq (\lambda_{2l-2})^{-1/2} - \prod_{1 \leq k \leq l-1} L_k(\lambda_{2l-1}) \\
&\leq 1 - \prod_{1 \leq k \leq l-1} \frac{\lambda_{2l-1} - \mu_k}{\left((\lambda_{2l-1} - \lambda_{2k-1})(\lambda_{2l-1} - \lambda_{2k-2})\right)^{1/2}}
\end{aligned}
\tag{3.44}
$$

for $\lambda \geq \lambda_{2l-1}$, where we used that $\lambda_{2l-1} \geq 1$. It remains to estimate $\prod_{k>l} F_k(\lambda)$. Recall that $F_k(\lambda),\ k > l$, is increasing on $[\lambda_{2l-1}, \lambda_{2l}]$ and satisfies $F_k(\lambda) > 1$ on this interval. Hence, we obtain

$$
0 \leq \prod_{k>l} F_k(\lambda) - 1 \leq \prod_{k>l} F_k(\lambda_{2l}) - 1
\tag{3.45}
$$

for $\lambda \in [\lambda_{2l-1}, \lambda_{2l}]$. Let $k \geq l+2$. Using (3.18), we get

$$
\begin{aligned}
\lambda_{2k} > \lambda_{2k-1} > \alpha_k &\geq (\alpha_{k-1})^2 \geq \alpha_{k-1}(k-1)^2 \lambda_{2k-4} \\
&\geq (k-1)^2 \lambda_{2l} (\alpha_l)^{2(k-l-1)} \geq (k-1)^2 \lambda_{2l} \alpha_l^2
\end{aligned}
$$

for $\alpha_l \geq 1$. From this estimate we obtain

$$\begin{aligned}\prod_{k\geq l+2} F_k(\lambda_{2l}) &\leq \prod_{k\geq l+2}\left(1-\frac{1}{(k-1)^2\alpha_l^2}\right)^{-1} \\ &\leq 1+\frac{1}{\alpha_l^2}\prod_{k\geq l+2}\left(1-\frac{1}{(k-1)^2}\right)^{-1}.\end{aligned} \tag{3.46}$$

The function $F_{l+1}(\lambda)$ can be estimated in the same way. Combining (3.45) and (3.46), we get

$$0 \leq \prod_{k>l} F_k(\lambda) - 1 \leq \frac{c_l(\alpha_1,\dots,\alpha_{l-1})}{\alpha_l} \tag{3.47}$$

for $\lambda \in [\lambda_{2l-1},\lambda_{2l}]$. Here $c_l(\alpha_1,\dots,\alpha_{l-1})$ is a constant that depends on $\alpha_1,\dots,\alpha_{l-1}$. Thus, if we replace in (3.41) the product of the F_k's by $\lambda^{1/2}(\lambda_{2l-2})^{-1/2}(\lambda/\lambda_{2l-2}-1)^{-1/2}$, then it follows from (2.4) together with (3.43), (3.44) and (3.47) that

$$|A_{ll}-1| \leq \frac{C_l(\alpha_1,\dots,\alpha_{l-1})}{\alpha_l} \tag{3.48}$$

for some constant $C_l(\alpha_1,\dots,\alpha_{l-1})$ that depends on $\alpha_1,\dots,\alpha_{l-1}$.

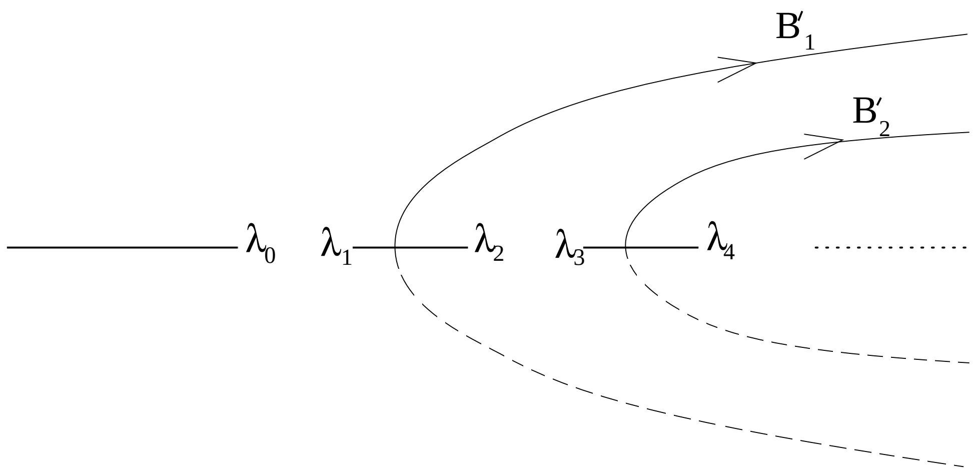

Fig.4 The deformed countours B_l'.

The periods B_{il}, $1 \leq i \leq l-1$, can be treated in essentially the same way as the periods A_{il}. By definition, we have

$$B_{il} = 2a_l^{-1} \sum_{j=1}^{i} \int_{\lambda_{2j-2}}^{\lambda_{2j-1}} \frac{\prod_{k\neq l} F_k(\lambda)}{\left(\lambda\left(1-\frac{\lambda}{\lambda_{2l-1}}\right)\left(1-\frac{\lambda}{\lambda_{2l}}\right)\right)^{1/2}} \quad d\lambda. \tag{3.49}$$

Each integral can be estimated as above which gives

$$|B_{il}| \leq \frac{C_l(\alpha_1, \dots, \alpha_{l-1})}{\alpha_l^{1/2}}, \quad i < l. \tag{3.50}$$

To handle B_{ll}, we replace the cycles B_l by the cycles B_l' which are as shown in figure 4.
Using a contour deformation, it is easy to see that we may replace the cycles B_l by the B_l'. This leads to

$$\begin{aligned} |B_{ll} - \tau_l| \leq & \left| \tau_l - 2a_l^{-1} \int_{\lambda_{2l}}^{\lambda_{2l+1}} \frac{\prod_{k\neq l} F_k(\lambda)}{\left(\lambda\left(\frac{\lambda}{\lambda_{2l-1}}-1\right)\left(\frac{\lambda}{\lambda_{2l}}-1\right)\right)^{1/2}}\, d\lambda \right| \\ & + \sum_{j\geq l+1} 2a_l^{-1} \int_{\lambda_{2j}}^{\lambda_{2j+1}} \frac{\prod_{k\neq l} |F_k(\lambda)|}{\left(\lambda\left(\frac{\lambda}{\lambda_{2l-1}}-1\right)\left(\frac{\lambda}{\lambda_{2l}}-1\right)\right)^{1/2}}\, d\lambda. \end{aligned} \tag{3.51}$$

We start with the discussion of the first term on the r.h.s. and split the numerator as

$$\lambda^{-1/2} \prod_{k\neq l} F_k(\lambda) = \prod_{1\leq k\leq l-1} L_k(\lambda) \cdot \prod_{k>l+1} F_k(\lambda) \cdot \frac{F_{l+1}(\lambda)}{\left(\frac{\lambda}{\lambda_{2l-2}}-1\right)^{1/2}}.$$

The first and the second factor can be treated using (3.44) and (3.47). Thus, it suffices to study the integral

$$\frac{2}{(\lambda_{2l-2})^{1/2}} a_l^{-1} \int_{\lambda_{2l}}^{\lambda_{2l+1}} \frac{F_{l+1}(\lambda)}{\left(\left(\frac{\lambda}{\lambda_{2l-2}}-1\right)\left(\frac{\lambda}{\lambda_{2l-1}}-1\right)\left(\frac{\lambda}{\lambda_{2l}}-1\right)\right)^{1/2}}\, d\lambda. \tag{3.52}$$

Now consider the hyperelliptic Riemann surface X which is obtained by cutting two copies of the complex plane along the three intervals $(-\infty, \lambda_{2l-2})$, $(\lambda_{2l-1}, \lambda_{2l})$ and $(\lambda_{2l+1}, \lambda_{2l+2})$ and gluing boundaries correspondingly. Then (3.52) is just the integral of a holomorphic 1–form on X over an elementary cycle S_l which runs around $(\lambda_{2l}, \lambda_{2l+1})$ (see Fig.2). By Cauchy's theorem, we may replace S_l by the complementary elementary cycles on X. Hence, (3.52) equals

$$\begin{aligned} \frac{1}{a_l}\,\frac{2}{(\lambda_{2l-2})^{1/2}} \Bigg\{ & \int_{\lambda_{2l-2}}^{\lambda_{2l-1}} \frac{F_{l+1}(\lambda)}{\left(\left(\frac{\lambda}{\lambda_{2l-2}}-1\right)\left(1-\frac{\lambda}{\lambda_{2l-1}}\right)\left(1-\frac{\lambda}{\lambda_{2l}}\right)\right)^{1/2}}\,d\lambda \\ & + \int_{\lambda_{2l+2}}^{\infty} \frac{F_{l+1}(\lambda)}{\left(\left(\frac{\lambda}{\lambda_{2l-2}}-1\right)\left(\frac{\lambda}{\lambda_{2l-1}}-1\right)\left(\frac{\lambda}{\lambda_{2l}}-1\right)\right)^{1/2}}\,d\lambda \Bigg\}. \end{aligned} \tag{3.53}$$

The second integral can be estimated by

$$\begin{aligned} \left(\frac{\lambda_{2l+2}}{\alpha_{l+1}}\right)^{3/2} & \int_{\lambda_{2l+2}}^{\infty} \lambda^{-3/2} F_{l+1}(\lambda)\,d\lambda \\ & = \left(\frac{\lambda_{2l+2}}{\alpha_{l+1}}\right)^{3/2} \int_{\lambda_{2l+2}}^{\infty} \frac{\lambda-\mu_{l+1}}{\lambda^{3/2}(\lambda-\lambda_{2l+1})^{1/2}(\lambda-\lambda_{2l+2})^{1/2}}\,d\lambda \\ & \le \left(\frac{\lambda_{2l+2}}{\alpha_{l+1}}\right)^{3/2} \int_{\lambda_{2l+2}}^{\infty} \frac{d\lambda}{\lambda(\lambda-\lambda_{2l+2})^{1/2}} \\ & = \frac{\lambda_{2l+2}}{(\alpha_{l+1})^{3/2}} \int_{1}^{\infty} \frac{1}{\lambda(\lambda-1)^{1/2}}\,d\lambda. \end{aligned} \tag{3.54}$$

By (3.12) and (3.18), this number is bounded by $C_l \alpha_l^{-1}$, where C_l depends on $\alpha_1, \ldots, \alpha_{l-1}$. Furthermore, using (3.14) it follows that $\mu_l / a_l \le C \alpha_l^{1/2}$. Thus, the second term in (3.53) is bounded by $\widetilde{C}_l \alpha_l^{-1/2}$.
To handle the first integral, we note that for $\lambda \in [\lambda_{2l-2}, \lambda_{2l-1}]$, we have

$$\frac{\lambda}{\lambda_{2l+2}} \le \frac{\lambda}{\mu_{l+1}} \le \frac{\lambda}{\lambda_{2l+1}} \le \frac{\lambda_{2l-1}}{\lambda_{2l+1}} = \frac{\alpha_l + \lambda_{2l-2}}{\alpha_{l+1} + \alpha_l\left(1+\frac{\Delta_l}{\alpha_l}\right) + \lambda_{2l-2}},$$

and by (3.18), this expression can be estimated by $C_l \alpha_l^{-1}$. Thus, it follows that $|F_{l+1}(\lambda) - 1| \le C_l \alpha_l^{-1}$, uniformly for $\lambda \in [\lambda_{2l-2}, \lambda_{2l-1}]$. Furthermore, if we replace in the first integral in (3.53) $F_{l+1}(\lambda)$ by 1, then by (3.15) and (3.17), we get $b_l/a_l = \tau_l$. Putting our estimates together, it follows that the first term on the r.h.s. of the inequality (3.51) is bounded by $C_l \alpha_l^{-1/2}$.

It remains to investigate the infinite sum on the r.h.s. of (3.51). Given $j \geq l+1$, we split the numerator of the j–th integral as follows

$$\prod_{k \neq l} |F_k(\lambda)| = \prod_{\substack{1 \leq k < j-1 \\ k \neq l}} |F_k(\lambda)| \cdot \prod_{k > j+1} F_k(\lambda) \cdot |F_j(\lambda)| \, F_{j+1}(\lambda). \tag{3.55}$$

The first two factors on the r.h.s. can be estimated by (3.27) and (3.31). As for the remaining two factors, for $\lambda \in [\lambda_{2j}, \lambda_{2j+1}]$ we have

$$\begin{aligned} |F_j(\lambda)| \, F_{j+1}(\lambda) &= \frac{(\lambda - \mu_j)(\mu_{j+1} - \lambda)}{((\lambda - \lambda_{2j-1})(\lambda - \lambda_{2j})(\lambda_{2j+1} - \lambda)(\lambda_{2j+2} - \lambda))^{1/2}} \\ &\leq 2(\alpha_{j+1} + \Delta_{j+1})^{1/2} \min\left\{1, \left(\frac{\alpha_{j+1}}{\Delta_{j+1}}\right)^{1/2}\right\} \\ &\qquad \cdot \frac{\lambda}{((\lambda - \lambda_{2j-1})(\lambda - \lambda_{2j})(\lambda_{2j+1} - \lambda))^{1/2}}. \end{aligned}$$

One case of the inequality is obtained by using that $\mu_{j+1} - \lambda < \lambda_{2j+2} - \lambda$, together with

$$\mu_{j+1} - \lambda < \lambda_{2j+2} - \lambda_{2j} = \alpha_{j+1} + \Delta_{j+1}$$

for $\lambda \in [\lambda_{2j}, \lambda_{2j+1}]$. For the other case we observe that by (3.18), one has $\lambda_{2j} < \alpha_{j+1}$ and hence, for $\lambda \in [\lambda_{2j}, \lambda_{2j+1}]$, we get

$$(\lambda_{2j+2} - \lambda)^{-1/2} \leq (\lambda_{2j+2} - \lambda_{2j+1})^{-1/2} = (\Delta_{j+1})^{-1/2}.$$

Furthermore, the numerator is estimated by $\lambda \mu_{j+1}$. Using (2.3) and $\lambda_{2j+2} = \Delta_{j+1} + \alpha_{j+1} + \lambda_{2j}$ together with $\lambda_{2j} < \alpha_{j+1}$, the inequality follows.

Next observe that by (3.22) the denominator of the j–th integral can be estimated by $2\mu_l \lambda^{-3/2}$. Furthermore, it is easy to see that

$$\begin{aligned} &\int_{\lambda_{2j}}^{\lambda_{2j+1}} \frac{d\lambda}{(\lambda(\lambda - \lambda_{2j-1})(\lambda - \lambda_{2j})(\lambda_{2j+1} - \lambda))^{1/2}} \\ &\qquad \leq \frac{1}{\alpha_{j+1}} \int_0^1 \frac{d\lambda}{\left(\lambda + \frac{\Delta_j}{\alpha_{j+1}}\right) (\lambda(1 - \lambda))^{1/2}} \leq C(\alpha_{j+1}\Delta_j)^{-1/2}. \end{aligned}$$

Using these estimates and (3.18), it follows that the infinite sum on the r.h.s. of (3.51) is bounded by a constant times

$$\sum_{j\geq l+1}\left(\frac{\alpha_j}{\Delta_j}\right)^{1/2}\left(\frac{\alpha_l}{\alpha_j}\right)^{1/2}. \tag{3.56}$$

Now note that by (3.18), we have

$$\left(\frac{\alpha_l}{\alpha_j}\right)^{1/2}\leq \alpha_j^{-1/4} \text{ for } j\geq l+1.$$

Hence (3.56) is bounded by

$$\sum_{j\geq l+1}\left(\frac{\alpha_j}{\Delta_j}\right)^{1/2}\alpha_j^{-1/4}, \tag{3.57}$$

which converges by condition (3.19) for an admissible sequence α . Combining our estimates, we have proved that

$$|B_{ll}-\tau_l|\leq \frac{C_l(\alpha_1,\ldots,\alpha_{l-1})}{\alpha_l^{1/8}}. \tag{3.58}$$

By the same method, we can also treat the periods B_{li}, $1\leq i\leq l-1$. Again, using the cycles B_l', we get

$$|B_{li}|\leq 2a_i^{-1}\sum_{j\geq l}\int_{\lambda_{2j}}^{\lambda_{2j+1}}\frac{\prod_{k\neq l}|F_k(\lambda)|}{\left(\lambda\left(\frac{\lambda}{\lambda_{2i-1}}-1\right)\left(\frac{\lambda}{\lambda_{2i}}-1\right)\right)^{1/2}}\,d\lambda.$$

Arguing as above, we obtain

$$|B_{li}|\leq \frac{C_l(\alpha_1,\ldots,\alpha_{l-1})}{\alpha_l^{1/8}}. \tag{3.59}$$

By the estimates that we derived above, it is clear that we can proceed by induction to prove Lemma 3.5. $\square$

4 The period matrix

In this section we study the period matrix of the surfaces $\mathcal{M}(\Lambda)$ and we prove Theorem 1.1. Let $T = \mathrm{diag}(\tau_1, \tau_2, \dots)$ be a given diagonal matrix with positive entries τ_i. Let $\Lambda = \{\lambda_i\}_{i=0}^{\infty}$ be a sequence which, with respect to T, satisfies (3.4) and the additional conditions $\lambda_0 = 0,\ \lambda_1 = 1$ and $\lambda_{2i},\ \lambda_{2i-1} \geq i^2,\ i \geq 1$. Let

$$A = (A_{ij})_{i,j=1}^{\infty} \quad \text{and} \quad B = (B_{ij})_{i,j=1}^{\infty},$$

where A_{ij} and B_{ij} are the periods (3.2). Let ℓ^2 denote the Hilbert space of square summable sequences. Then it follows from (3.4) that $A - \mathrm{Id}$ and $T^{-1/2}BT^{-1/2} - \mathrm{Id}$ are both Hilbert Schmidt operators with Hilbert Schmidt norm less than $S_0 = S(\sum_{i\geq 1} i^{-2p}) < 1$ (see (2.3) in [16]). From this fact if follows that A has a bounded inverse A^{-1} on ℓ^2 and $A^{-1} - \mathrm{Id}$ is Hilbert Schmidt. In fact, A^{-1} is given by a convergent Neumann series in the form

$$\begin{aligned} A^{-1} &= (\mathrm{Id} - (\mathrm{Id} - A))^{-1} \\ &= \sum_{n=0}^{\infty} (\mathrm{Id} - A)^n. \end{aligned} \tag{4.1}$$

Let $(A^{-1})_{ij},\ i,j \in \mathbb{N}$, denote the entries of A^{-1}. To estimate $(A^{-1})_{ij}$, we may use (4.1). First observe that the function $\rho(i,j)$, defined by (3.3), satisfies

$$\rho(i,j)\rho(j,k) \leq \rho(i,k), \quad i,j,k \in \mathbb{N}.$$

Using (3.4), it follows by induction that

$$\left|(\mathrm{Id} - A)^n)_{ij}\right| \leq \frac{S \cdot S_0^{n-1}}{i^p j^p}\, \rho(i,j), \quad i,j \in \mathbb{N}, \tag{4.2}$$

where $S_0 = S(\sum i^{-2p}) < 1$ and therefore, we get

$$\left|(A^{-1})_{ij}\right| \leq \delta_{ij} + \frac{C}{i^p j^p}\, \rho(i,j) \tag{4.3}$$

Now we introduce the normalized forms

$$\psi_j = \sum_{k=1}^{\infty} (A^{-1})_{kj} \phi_k, \qquad j \in \mathbb{N}. \tag{4.4}$$

Recall that the space $\Omega^1(\mathcal{M})$ of square integrable holomorphic 1–forms on $\mathcal{M}$ is a Hilbert space. Therefore, in order to show that the ψ_j's are holomorphic 1–forms on $\mathcal{M}$, it is sufficient to prove that the series (4.4) converges in L^2. This will be established by

Theorem 4.1 *For each $j \in \mathbb{N}$, the series (4.4) converges in L^2 to a square integrable holomorphic 1–form ψ_j.*

Proof. First, we claim that there exists a constant $C > 0$ such that

$$\| \phi_k \| \leq C \tau_k^{1/2}, \quad k \geq 1. \tag{4.5}$$

To verify (4.5), we use the definition (2.6) of ϕ_k and introduce the function

$$F(r) = \int_{|\lambda| \leq r} \left| \frac{\varphi_k(\lambda)}{R(\lambda)} \right|^2 d\mathrm{vol}$$

Then $F(r)$ is obviously monotonically increasing. Let C_r be the circle in $\mathbb{C}$ of radius r with center at the origin and set

$$L_k(r) = \int_{C_r} |\phi_k| \,.$$

Furthermore, let

$$r_n = \frac{1}{2}(\lambda_{2n} + \lambda_{2n+1}), \quad n \geq 1.$$

Following the proof of Theorem 1 in section 5 of [15], we obtain

$$\left| 2F(r_n) + \sum_{i=1}^{n} A_{ik} B_{ik} \right| \leq \quad L_k^2(r_n) \tag{4.6}$$

Next we shall estimate $L_k(r_n)$. Set $C_n = C_{r_n}$. By definition, we have

$$L_k(r_n) \leq 2\pi r_n \sup_{\lambda \in C_n} \left| \frac{\varphi_k(\lambda)}{R(\lambda)} \right| . \tag{4.7}$$

We write

$$\left|\frac{\varphi_k(\lambda)}{R(\lambda)}\right| = |\lambda|^{-1/2}\prod_{l\neq k}|F_l(\lambda)|\,\frac{1}{\left|\left(1-\frac{\lambda}{\lambda_{2k-1}}\right)^{1/2}\left(1-\frac{\lambda}{\lambda_{2k}}\right)^{1/2}\right|} \tag{4.8}$$

and start with the discussion of $|F_l(\lambda)|$ for $l \leq n$. For $\lambda \in C_n$ we have

$$\begin{aligned} |F_l(\lambda)| &\leq \frac{\left(\frac{r_n}{\mu_l}+1\right)}{\left(\frac{r_n}{\lambda_{2l-1}}-1\right)^{1/2}\left(\frac{r_n}{\lambda_{2l}}-1\right)^{1/2}} \\ &= \frac{\left(1+\frac{\mu_l}{r_n}\right)}{\left(1-\frac{\lambda_{2l-1}}{r_n}\right)^{1/2}\left(1-\frac{\lambda_{2l}}{r_n}\right)^{1/2}}. \end{aligned} \tag{4.9}$$

Suppose that $l < n$. Since $r_n \geq \lambda_{2n} \geq \lambda_{2n-1}$, it follows from (3.25) that

$$\frac{\lambda_{2l-1}}{r_n} \leq \frac{\mu_l}{r_n} \leq \frac{\lambda_{2l}}{r_n} \leq \frac{\lambda_{2l}}{\lambda_{2n-1}} \leq (n^2+l^2)^{-1}.$$

Hence, for $\lambda \in C_n$ we get

$$\prod_{\substack{l\neq k\\ l<n}}|F_l(\lambda)| \leq \prod_{1\leq i<\infty}\frac{\left(1+\frac{1}{i^2}\right)}{\left(1-\frac{1}{i^2}\right)} = q' < \infty,$$

uniformly in n. Let $l = n$. Then it follows from (3.18) that

$$\frac{\lambda_{2n-1}}{r_n} \leq \frac{\lambda_{2n}}{r_n} = \frac{2\lambda_{2n}}{\lambda_{2n}+\lambda_{2n+1}} \leq \frac{2}{n+2} \leq \frac{2}{3}$$

which implies

$$|F_n(\lambda)| \leq 5,$$

uniformly for $\lambda \in C_n$. For $l \geq n+1$ we use the estimate

$$|F_l(\lambda)| \leq \frac{1+\frac{r_n}{\mu_l}}{\left(1-\frac{r_n}{\lambda_{2l-1}}\right)^{1/2}\left(1-\frac{r_n}{\lambda_{2l}}\right)^{1/2}}. \tag{4.10}$$

Let $l > n+1$. Since $\lambda_{2l} \geq \mu_l \geq \lambda_{2l-1}$ and $r_n \leq \lambda_{2n+1}$, it follows again from (3.25) that

$$\frac{r_n}{\lambda_{2l}} \leq \frac{r_n}{\mu_l} \leq \frac{\lambda_{2n+1}}{\lambda_{2l-1}} \leq (n^2 + l^2)^{-1}.$$

This implies

$$\prod_{l>n+1} |F_l(\lambda)| \leq q'$$

for all $\lambda \in C_n$, uniformly in n. Finally, for $l = n+1$, we proceed as in the case $l = n$, using (4.10) instead of (4.9) and we get $|F_{n+1}(\lambda)| \leq 5,\ \lambda \in C_n$.

Now we turn to the last factor in (4.8). Let $n \geq k+4$. Using (3.18), we get $r_n > (n-3)^8 \lambda_{2k}^4 \lambda_{2k-1}^4$, which implies that

$$\left| \left(1 - \frac{\lambda}{\lambda_{2k-1}}\right)^{-1/2} \left(1 - \frac{\lambda}{\lambda_{2k}}\right)^{-1/2} \right| \leq r_n^{-7/8}, \quad \lambda \in C_n.$$

Combining our estimates with (4.7) and (4.8), we get

$$L_k(r_n) \leq C r_n^{-3/8}. \tag{4.11}$$

By (3.4), we have

$$\sum_{i \geq 1} |A_{ik}|\ |B_{ik}| \leq \sum_i \left(\delta_{ik} + \frac{3S}{i^p k^p}\right) \tau_k \leq C\tau_k. \tag{4.12}$$

Together with (4.6) and (4.11), this proves our claim (4.5). Hence, using the definition (4.4) of ψ_j and (4.3), we get

$$\| \psi_j \| \leq C_1 \sum_k \left(\delta_{kj} + \frac{C\tau_k^{-1/2}\tau_j^{1/2}}{k^p j^p}\right) \tau_k^{1/2} \leq C_2 \tau_j^{1/2}, \quad j \in \mathbb{N}. \tag{4.13}$$

This completes the proof of our theorem. □

Observe that by (4.4), the 1–forms ψ_i satisfy

$$\int_{A_i} \psi_j = \delta_{ij}, \quad i, j \geq 1. \tag{4.14}$$

Now let $\{\omega_j\}_{j\in\mathbb{N}}$ be the unique normalized basis of $\Omega^1(\mathcal{M})$ which exists by Theorem 2.3. Then, for each $j \in \mathbb{N}$, we have

$$\int_{A_i} (\omega_j - \psi_j) = 0 \quad \text{for all } i \geq 1.$$

Thus, by Corollary 2.2 it follows that $\omega_j = \psi_j, \quad j \geq 1$. This gives

Corollary 4.2 *The set of 1–forms $\{\psi_j\}_{j\in\mathbb{N}}$ is a basis of $\Omega^1(\mathcal{M})$ which is normalized by (4.14).*

In analogy with the finite genus case we can now introduce the period matrix of the marked Riemann surface $(\mathcal{M}; A_1, B_1, \dots)$ by

$$\Omega = (\omega_{ij}) = \left(\int_{B_i} \psi_j \right). \tag{4.15}$$

Then the period relations Theorem (2.1) imply

Proposition 4.3 *The period matrix Ω is symmetric and* $\operatorname{Im}\Omega$ *is positive definite.*

Proof. Since ψ_i and ψ_j are holomorphic, it follows from Theorem 4.14 and Theorem 2.1 that

$$0 = \int_{\mathcal{M}} \psi_i \wedge \psi_j = \int_{B_i} \psi_j - \int_{B_j} \psi_i, \tag{4.16}$$

since $\psi_i \wedge \psi_j = 0$. In the same way, using Theorem 2.1 and (4.16), we get

$$\langle \psi_k, \psi_l \rangle = \int_{\mathcal{M}} \psi_k \wedge *\overline{\psi}_l = \sqrt{-1} \int_{\mathcal{M}} \psi_k \wedge \overline{\psi}_l = \sqrt{-1} \left(\int_{B_k} \overline{\psi}_l - \int_{B_l} \psi_k \right) =$$

$$\sqrt{-1} \left(\overline{\int_{B_l} \psi_k} - \int_{B_l} \psi_k \right) = 2 \operatorname{Im} \Omega_{kl},$$

which shows that $\operatorname{Im}\Omega$ is positive definite. $\square$

We are now ready to prove Theorem 1.1. By definition, we have
$\Omega = \sqrt{-1}BA^{-1}$ and by (4.1), we get

$$BA^{-1} - T = B - T + \sum_{n=1}^{\infty} B(\mathrm{Id} - A)^n.$$

Using (3.4) and (4.2) we obtain the estimate

$$\begin{aligned} \left| \left(BA^{-1} - T\right)_{ij} \right| &\leq \frac{S\left(1 + (1 - S_0)^{-1}\right)}{i^p j^p}\,\rho(i,j)(\tau_i + 1) \\ &\leq C\frac{\tau_j}{i^p j^p}. \end{aligned} \tag{4.17}$$

Since $BA^{-1} - T$ is symmetric, the proof of Theorem 1.1 follows by symmetrizing the above inequality. $\square$

5 Solutions of the KdV equation

The purpose of this section is to prove Theorem 1.2. Our construction will be based on an extension of the Its–Matveev formula (1.3), where the theta functions are replaced by the renormalized theta functions introduced in [16]. The construction is such that our solutions are approximated by solutions of the KdV equation that are associated to Riemann surfaces of finite genus.

So, in order to establish notation, we will briefly recall the standard construction for the case of hyperelliptic Riemann surfaces of finite genus (see [2], [4], [5], and references quoted there). Consider a sequence of nonnegative real numbers

$$0 = E_0 < E_1 < E_2 < \cdots < E_{2g}$$

and let $\mathcal{M}_g$ be the corresponding hyperelliptic surface of genus g, obtained by gluing two copies of $\mathbb{C}$, each cut along along the intervals $(-\infty, 0)$, (E_1, E_2), $\ldots$, (E_{2g-1}, E_{2g}), where the upper banks of one copy are glued to the corresponding lower banks of the other copy and vice versa. Then $\mathcal{M}_g$ has a canonical homology basis $A_1, B_1, \ldots, A_g, B_g \in H_1(\mathcal{M}_g, \mathbb{Z})$ which is chosen similarly as in Fig.1.

Let Ω_g be the corresponding period matrix which is pure imaginary in the present case. Let Φ_1 and Φ_2 be the standard normalized meromorphic 1–forms on $\mathcal{M}_g$ with a unique pole at $P_0 = \infty$ of order two and four, respectively (see [4], p. 232). Normalization means that

$$\int_{A_j} \Phi_1 = \int_{A_j} \Phi_2 = 0, \quad 1 \leq j \leq g.$$

Let $U, V \in \mathbb{R}^g$ be the vectors with components given by

$$\begin{aligned} U_j &= \frac{1}{2\pi i} \int_{B_j} \Phi_1, \\ V_j &= \frac{1}{2\pi i} \int_{B_j} \Phi_2, \quad j = 1, \ldots, g. \end{aligned} \tag{5.1}$$

Let $\vartheta_g(z|\Omega_g)$, $z \in \mathbb{C}^g$, be the Riemann zeta function of the surface $\mathcal{M}_g$. Then there exists $C \in \mathbb{C}$ such that for every real $W \in \mathbb{R}^g$,

$$u(x,t) = -2\frac{d^2}{dx^2} \log \vartheta_g(Ux + Vt + W \mid \Omega_g) + C = \tilde{u}(x,t) + C \tag{5.2}$$

is a solution of the KdV equation (1.1) (see, e.g., §3 of [2]). Note that by (1.1) and (5.2) the constant C is given by

$$C = -\frac{1}{6\tilde{u}_x(x,t)} \{6\tilde{u}(x,t)\tilde{u}_x(x,t) - \tilde{u}_{xxx}(x,t) - \tilde{u}_t(x,t)\} \tag{5.3}$$

for any (x,t) such that $\tilde{u}(x,t) \neq 0$.

This constuction can not be extended immediately to the Riemann surfaces of infinite genus considered above, mainly because the cuts accumulate at $P_0 = \infty$. Instead, we will chose the pole of the meromorphic 1–forms to be at $\lambda = 0$, which is well behaved. Geometrically, this corresponds to considering the Riemann surface obtained from the original Riemann surface by applying the lift of the map $\lambda \longmapsto \frac{1}{\lambda}$ of $\mathbf{P}^1$ to it. This turns our Riemann surface into a hyperelliptic Riemann surface with cuts accumulating at zero, such that the new Riemann surface is well behaved at $P_0 = \infty$.

For fixed T and $p > 3$, consider a Riemann surface $\mathcal{M} = \mathcal{M}(\Lambda)$ defined by a sequence Λ of the form (1.4) and suppose that its period matrix Ω is contained in H^p_T. The existence of such surfaces is guaranteed by Theorem 1.1. Now we introduce two meromorphic 1–forms by

$$\begin{aligned} \Phi_1 &= \frac{a_1}{\lambda}\phi_1 - \sum_{j\geq 1}\left(\int_{A_j} \frac{a_1}{\lambda}\phi_1\right)\psi_j, \\ \Phi_2 &= \frac{a_1}{\lambda^2}\phi_1 - \sum_{j\geq 1}\left(\int_{A_j} \frac{a_1}{\lambda}\phi_1\right)\psi_j. \end{aligned} \tag{5.4}$$

Here a_1 is given by (2.4), ϕ_1 by (2.6) and ψ_j, $j \geq 1$, are the normalized holomorphic 1–forms defined by (4.4).

Remark: The overall factor a_1 in (5.4) is introduced such that Φ_1 and Φ_2 behave locally like $\frac{d\lambda}{\lambda\sqrt{\lambda}}$ and $\frac{d\lambda}{\lambda^2\sqrt{\lambda}}$, respectively, near $\lambda = 0$ (compare [4], p.232), in particular, note that Φ_1 has a pole of second order and Φ_2 has a pole of fourth order at $P_0 = 0$.

Using the bounds for A_{j1} provided by Theorem 3.1, we obtain

$$\begin{aligned} \left|\int_{A_j} \frac{a_1}{\lambda}\phi_1\right| &\leq \frac{1}{\lambda_{2j-1}}\left(\delta_{j1} + C_2 j^{-p}\rho(j,1)\right) \\ &\leq \frac{C_1}{j^2}\left(\delta_{j1} + C_2 j^{-p}\rho(j,1)\right) \end{aligned} \tag{5.5}$$

and similarly,

$$\left| \int_{A_j} \frac{a_1}{\lambda^2} \, \phi_1 \right| \le \frac{C_1}{j^4} \Big(\delta_{j1} + C_2 j^{-p} \rho(j,1) \Big). \tag{5.6}$$

Here and in what follows, $C_1, C_2, \ldots$ are suitable constants. This implies that the second summands on the r.h.s. of (5.4) are each of the form $\frac{\psi(\lambda)}{R(\lambda)} \, d\lambda$, where $\psi(\lambda)$ is analytic on $\mathcal{M}$. In particular, Φ_1 and Φ_2 are well defined. By definition, we have

$$\int_{A_j} \Phi_1 = \int_{A_j} \Phi_2 = 0, \quad j \ge 1. \tag{5.7}$$

Now define $U_j, V_j \in \mathbb{R}$ by

$$\begin{aligned} U_j &= \frac{1}{2\pi i} \int_{B_j} \Phi_1, \\ V_j &= \frac{1}{2\pi i} \int_{B_j} \Phi_2. \end{aligned} \tag{5.8}$$

Using (3.4) and the mean value theorem, we get

$$\begin{aligned} \left| \int_{B_j} \frac{a_1}{\lambda} \phi_1 \right| &\le \frac{a_1}{\lambda_{2j}} \, |B_{j1}| \le \frac{C}{j^{2+p}} \rho(1,j), \\ \left| \int_{B_j} \frac{a_1}{\lambda^2} \phi_1 \right| &\le \frac{a_1}{\lambda_{2j}^2} \, |B_{j1}| \le \frac{C}{j^{4+p}} \rho(1,j). \end{aligned} \tag{5.9}$$

Combined with (1.5) and (5.5), we obtain

$$\begin{aligned} |U_j| &\le \frac{C}{j^{2+p}} \rho(1,j) \\ &\quad + C_1 \sum_{k \ge 1} \frac{1}{k^2} \left(\delta_{k1} + C_2 k^{-p} \rho(1,k) \right) \left(\delta_{jk} \tau_j + \frac{C_3}{j^p k^p} \tau_j^{1/2} \tau_k^{1/2} \right) \\ &\le \frac{C_4}{j^{2+p}} \tau_j^{1/2}. \end{aligned} \tag{5.10}$$

and, using (5.6) in place of (5.5),

$$|V_j| \leq \frac{C_4}{j^{p+4}} \tau_j^{1/2}. \tag{5.11}$$

Let $U = (U_1, U_2, \dots), V = (V_1, V_2 \dots)$. Then (5.10) and (5.11) imply that $U, V \in \ell_{p,T}$ (see (1.7)). Now for $z \in \ell_{p,T}$, let $\vartheta_T(z \mid \Omega)$ be the theta function introduced in [16], where Ω is the period matrix of $\mathcal{M}$. Let $x, t \in \mathbb{R}$ and $C \in \mathbb{C}$. Since Ω is purely imaginary and belongs to H_T^p, it follows from Theorem 4.4 in [16] that for each real $W \in \ell_{p,T}$, the function

$$u(x,t) = -2 \frac{d^2}{dx^2} \log \vartheta_T(Ux + Vt + W|\Omega) + C \tag{5.12}$$

is well defined. Moreover, by Corollary 4.2 in [16] $u(x,t)$ is real analytic in $(x,t) \in \mathbb{R}^2$. To prove Theorem 1.2, it suffices to prove

Theorem 5.1 *There exists $C \in \mathbb{C}$ such that $u(x,t)$, defined by (5.12), is a nontrivial solution of the KdV equation (1.1).*

Proof. We shall construct solutions $u_g(x,t)$ of the KdV equations obtained from suitable surfaces of genus g such that as $g \to \infty$, u_g together with its derivatives converges to u, uniformly on compact subsets.
For a given sequence Λ, let $\mathcal{M}_g$ be the hyperelliptic Riemann surface defined by

$$R_g(\lambda) = \left(-\lambda \prod_{i=1}^{g} \left(1 - \frac{\lambda}{\lambda_{2i-1}}\right) \left(1 - \frac{\lambda}{\lambda_{2i}}\right) \right)^{1/2}. \tag{5.13}$$

The cycles $A_1, B_1, \dots, A_g, B_g$ form a canonical homology basis for $\mathcal{M}_g$. A basis of the space of holomorphic 1–forms on $\mathcal{M}_g$ is given by

$$\phi_i^g = \frac{1}{a_i} \prod_{\substack{1 \leq k \leq g \\ k \neq i}} \left(1 - \frac{\lambda}{\mu_j}\right) R_g^{-1}(\lambda)\, d\lambda, \quad i = 1, \dots, g, \tag{5.14}$$

where μ_k is defined by (2.3) and a_i by (2.4). Let $\psi_1^g, \dots, \psi_g^g$ be the basis of $\Omega^1(\mathcal{M}_g)$ which is normalized by

$$\int_{A_j} \psi_i^g = \delta_{ij}, \quad i,j = 1,\dots,g,$$

and let Ω_g be the corresponding period matrix. Furthermore, we introduce meromorphic 1–forms Φ_1^g and Φ_2^g by

$$\Phi_1^g = \frac{a_1}{\lambda}\phi_1^g - \sum_{j=1}^{g}\left(\int_{A_j}\frac{a_1}{\lambda}\phi_1^g\right)\psi_j^g, \tag{5.15}$$

$$\Phi_2^g = \frac{a_1}{\lambda^2}\phi_1^g - \sum_{j=1}^{g}\left(\int_{A_j}\frac{a_1}{\lambda^2}\phi_1^g\right)\psi_j^g. \tag{5.16}$$

Then Φ_1^g has a pole of second order and Φ_2^g has pole of fourth order at $P_0 = 0$. Moreover, by construction we have

$$\int_{A_j}\Phi_1^g = \int_{A_j}\Phi_2^g = 0, \quad 1 \le j \le g.$$

Let $U^g, V^g \in \mathbb{R}^g$ be the vectors with components given by

$$U_j^g = \frac{1}{2\pi i}\int_{B_j}\Phi_1^g$$

$$V_j^g = \frac{1}{2\pi i}\int_{B_j}\Phi_2^g.$$

Let $W \in \ell_{p,T}$ be a real vector, and let $W^g \in \mathbb{R}^g$ be the vector with components $W_j^g = W_j$, $1 \le j \le g$. Let $\ell_{fn} \subset \ell_{p,T}$ be the subspace consisting of all sequences $(z_1, z_2, \dots) \in \mathbb{C}^\infty$ such that $z_j = 0$ for amost all j. Let $i(U^g), i(V^g), i(W^g) \in \ell_{fn}$ be the vectors obtained by the obvious inclusion $\mathbb{C}^g \subset \ell_{fn}$. Furthermore, let $i_T(\Omega_g) \in H_T^p$ be defined by

$$i_T(\Omega_g)_{jk} = \begin{cases} (\Omega_g)_{jk}, & 1 \le j,k \le g, \\ \sqrt{-1}\delta_{jk}\tau_j, & \text{otherwise.} \end{cases}$$

By the construction of the theta function ϑ_T in [16], we have

$$\begin{aligned}\vartheta_T\Big(i(U^g)x + i(V^g)t{+}i(W^g) \mid i_T(\Omega_g)\Big)&\\ = \vartheta_g\left(U^g x + V^g t + W^g \mid \Omega_g\right)/\vartheta_g(0 \mid \sqrt{-1}T_g),&\end{aligned}$$

where T_g is the diagonal matrix with diagonal entries $\tau_1, \dots, \tau_g$ and ϑ_g is the standard theta function in g variables. Therefore, we have the relation

$$\begin{aligned}u_g(x,t) &= -2\frac{d^2}{dx^2}\log\vartheta_g\left(U^g x + V^g t + W^g \mid \Omega_g\right) + C_g\\ &= -2\frac{d^2}{dx^2}\log\vartheta_T\left(i(U^g)x + i(V^g)t + i(W^g) \mid i_T(\Omega_g)\right)\\ &\quad + C_g,\end{aligned} \tag{5.17}$$

and for a suitable constant C_g, this is a solution of the KdV equation (1.1). By definition, it is clear that

$$i(W^g) \xrightarrow[g\to\infty]{} W \quad \text{in} \quad \ell_{p,T}. \tag{5.18}$$

Therefore, the main ingredient of the proof is

Lemma 5.2 *Let $3 < p' < p$. As $g \to \infty$, $i(U^g)$ and $i(V^g)$ converge in $\ell_{p,T}$ to U and V, respectively. Similarly, $i_T(\Omega_g)$ converges in $H_T^{p'}$ to Ω as $g \to \infty$.*

Proof of Lemma 5.2 We start with estimates for the difference $i(U^g) - U$. Let $j > g$. By (5.10) we have

$$|i(U^g)_j - U_j| \le \frac{C_4}{j^{p+2}}\tau_j^{1/2} \le \frac{C_4}{gj^{p+1}}\tau_j^{1/2}. \tag{5.19}$$

On the other hand, if $j \le g$, it follows from the definitions that

$$\begin{aligned}i(U^g)_j - U_j =&\frac{1}{2\pi}\left\{\int_{B_j}\frac{a_1}{\lambda}\phi_1^g - \int_{B_j}\frac{a_1}{\lambda}\phi_1\right\}\\ &- \frac{1}{2\pi}\left\{\int_{A_j}\frac{a_1}{\lambda}\phi_1^g - \int_{A_j}\frac{a_1}{\lambda}\phi_1\right\}.\end{aligned} \tag{5.20}$$

The second term on the r.h.s. can be estimated as follows

$$\left|\frac{1}{2\pi}\int_{A_j}\frac{a_1}{\lambda}\phi_1^g - \frac{1}{2\pi}\int_{A_j}\frac{a_1}{\lambda}\phi_1\right|$$
$$\leq \frac{1}{\pi}\int_{\lambda_{2j-1}}^{\lambda_{2j}} \lambda^{-1}\frac{|\varphi_1(\lambda)|}{|R(\lambda)|}\left|\prod_{k>g} F_k^{-1}(\lambda) - 1\right|\, d\lambda, \qquad (5.21)$$

where $F_k(\lambda)$ is given by (3.24). To handle the last term in the integrand, we proceed as follows. For $0 \leq x \leq 1/2$ one has $1 \leq (1-x)e^{2x}$. Hence, for real $0 \leq x_i, y_i, z_i \ \leq 1/2, \ i \in \mathbb{N}$, we obtain

$$e^{-\sum_i (y_i+z_i)} \leq \prod_{i=1}^{\infty}\frac{(1-y_i)^{1/2}(1-z_i)^{1/2}}{1-x_i} \leq e^{2\sum_i x_i}$$

Now we use that for $x > 0$, one has $e^x - 1 \leq xe^x$ and $1 - e^{-x} \leq xe^x$. This implies the following apriori estimate

$$\left|1 - \prod_{i=1}^{\infty}\frac{(1-y_i)^{1/2}(1-z_i)^{1/2}}{1-x_i}\right| \leq 2\sum_{i=1}^{\infty}(x_i + y_i + z_i)e^{2\sum_i(x_i+y_i+z_i)}.$$

Therefore, for $\lambda \in [\lambda_{2j-1}, \lambda_{2j}]$ we get

$$\left|\prod_{k>g} F_k(\lambda)^{-1} - 1\right| \leq 6\lambda_{2j}\left(\sum_{k>g}\frac{1}{\lambda_{2k-1}}\right)\cdot e^{6\lambda_{2j}\sum_{k>g}(\lambda_{2k-1})^{-1}}.$$

Inserting this into (5.21), using the estimates that bound $|A_{j1}|$ and (3.25), we obtain

$$\begin{aligned}\left|\int_{A_j}\frac{a_1}{\lambda}\Phi_1^g - \int_{A_j}\frac{a_1}{\lambda}\Phi_1\right| &\leq \frac{C_5}{\lambda_{2j-1}}\sum_{k\geq 1}\frac{1}{k^2+g^2}\cdot\int_{\lambda_{2j-1}}^{\lambda_{2j}}\left|\frac{\varphi_1(\lambda)}{R(\lambda)}\right|\, d\lambda \\ &\leq C_6\frac{1}{g}\cdot\frac{1}{j^{2+p}}\rho(j+1) \\ &\leq C_6\frac{1}{g}\cdot\frac{1}{j^{2+p}}\tau_j^{1/2}.\end{aligned}$$

By similar arguments, the first term on the r.h.s. of (5.20), satisfies the same bound. Combined with (5.19), it follows that $i(U_g)$ tends to U in $\ell_{p,T}$ as $g \to \infty$.
The convergence of $i(V_g)$ to V in $\ell_{p,T}$ follows by similar arguments.
Next consider $i_T(\Omega_g) - \Omega$. Let $A(g)$ and $B(g)$ be the $g \times g$ matrices given by

$$\begin{aligned} A(g)_{ij} &= \int_{A_i} \phi_j^g \\ B(g)_{ij} &= -\sqrt{-1} \int_{B_j} \phi_i^g, \qquad 1 \leq i, j \leq g. \end{aligned}$$

Then one has

$$\Omega_g = \sqrt{-1} B(g) \cdot A(g)^{-1}.$$

Set

$$i(A(g))_{kl} = \begin{cases} (A(g))_{kl}, & 1 \leq k, l \leq g \\ \delta_{kl}, & \text{otherwise.} \end{cases}$$

and similarly for $A(g)^{-1}$. Then one has $i(A(g)^{-1}) = i(A(g))^{-1}$ and

$$i_T(\Omega_g) = \sqrt{-1}(B(g)) \cdot i(A(g))^{-1}.$$

Now, by arguments similar to those used above, we get

$$\begin{aligned} |(i(A(g)) - A)_{kl}| &\leq \frac{C_8}{g^{q'} k^q j^q} \rho(i,j) \\ |(i_T(B(g)) - B)_{kl}| &\leq \frac{C_8}{g^{q'} k^q j^q} \rho(i,j), \end{aligned}$$

for any $3 < q < p$ and $q' = \mathrm{Min}(\frac{1}{2}, p-q) > 0$. If we proceed as in the proof of (4.17), it follows that as $g \to \infty$, $i_T(\Omega_g)$ converges to Ω in H_T^q. This concludes the proof of the lemma. □
Now put

$$\widetilde{u}_g(x,t) = -2\frac{d^2}{dx^2}\log\vartheta_g(U^g x + V^g t + W^g \mid \Omega_g),$$
$$\widetilde{u}(x,t) = -2\frac{d^2}{dx^2}\log\vartheta_T(Ux + Vt + W \mid \Omega).$$

Then (5.17) together with (5.18), Lemma 5.2 and Theorem 3.1 in [16] imply that as $g \to \infty$, $\widetilde{u}_g$ converges to $\widetilde{u}$, uniformly on compact subsets. By Corollary 4.2 in [16], we get

$$\frac{\partial^{k+l}\widetilde{u}_g(x,t)}{\partial x^k \partial t^l} \underset{g\to\infty}{\longrightarrow} \frac{\partial^{k+l}\widetilde{u}(x,t)}{\partial x^k \partial t^l}, \quad k, l \in \mathbb{N}, \tag{5.22}$$

uniformly on compact subsets.

To finish the proof of the theorem, it remains to show that C_g converges to some constant $C \in \mathbb{C}$. We use (5.3) to express C_g by $\widetilde{u}_g(x,t)$. By (5.22) it suffices to show that there exists (x,t) such that $\widetilde{u}(x,t) \neq 0$. To find such (x,t), suppose that the function of x, $\partial^2/\partial x^2 \log\vartheta_T(Ux+W|\Omega)$, is identically zero.
Then there exist $C, c \in \mathbb{C}$ such that

$$\vartheta_T(Ux + W|\Omega) = Ce^{cx}.$$

By Theorem 4.4 of [16], it follows that $C, c \in \mathbb{R}$, $C > 0$ and for any $x_i, x_j \in \mathbb{R}$, $1 \leq i, j \leq n$,

$$\sum_{i,j=1}^{n} e^{c(x_i - x_j)}\lambda_i\overline{\lambda}_j \geq 0$$

for all $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$. This is only possible if $c = 0$. But by Corollary 4.2 in [16], $\vartheta_T(Ux + W|\Omega)$ is nonconstant, unless $U = 0$. Thus there exists $x \in \mathbb{R}$ such that $\widetilde{u}(x,0) \neq 0$ and $C_g \to C \in \mathbb{C}$. Put $u(x,t) = \widetilde{u}(x,t) + C$. Then u is a nontrivial solution of the KdV equation which concludes the proof of the theorem. $\square$

By the proof of Theorem 5.1, the solutions $u(x,t)$ of the Korteweg–de Vries equation are limits of the solutions $u_g(x,t)$ which are associated to a Riemann surface of genus g. The initial conditions $u_g(x,0)$ are finite–gap potentials and therefore, $u_g(x,t)$ is quasi–periodic in x and t [17], [11]. Thus, $u(x,t)$ is the limit of quasi–periodic functions. But, in general, we only know that the functions converge uniformly on compact subsets. This is not sufficient to conclude that u is itself quasi–periodic. In special cases, however, one can prove that u_g converges uniformly on $\mathbb{R}\times\mathbb{R}$ to u. Suppose, for example, that the τ_i's are such that

$$\tau_i \geq C i^{2p}, \quad i \geq 1, \tag{5.23}$$

for some constant $C > 0$. Note that in this case

$$\vartheta(0|\sqrt{-1}T) = \prod_{i=1}^{\infty} \sum_{n\in\mathbb{Z}} e^{-\pi n^2 \tau_i} < \infty,$$

and it follows from (3.2) in [16] that

$$\vartheta_T(z|\Omega) = \frac{\vartheta(z|\Omega)}{\vartheta(0|\sqrt{-1}T)}$$

where

$$\vartheta(z|\Omega) = \lim_{n\to\infty} \vartheta_n\Big(r_n(z)|r_n(\Omega)\Big). \tag{5.24}$$

Here $r_n(z) \in \mathbb{C}^n$ is given by $r_n(z)_i = z_i$, $1 \leq i \leq n$, and $r_n(\Omega) \in M(n,\mathbb{C})$ is defined by $r_n(\Omega) = \{\Omega_{ij}\}_{1\leq i,j\leq n}$. Thus, if (5.23) holds, then the theta functions need not be renormalized. Next observe that by (1.7), $\ell_{p,T}$ contains the infinite unit cube

$$Q = \Big\{(z_i)_{i\in\mathbb{N}} \in \mathbb{C}^\infty \;\Big|\; |z_i| < 1,\ i \in \mathbb{N}\Big\},$$

and it follows from (3.26) in [16] that (5.24) converges uniformly for $z \in Q$. Now given $x, t \in \mathbb{R}$, let $N = (N_i)_{i\in\mathbb{N}} \in \mathbb{Z}^\infty$ be such that

$$|U_i x + V_i t + W_i - N_i| < 1, \quad i \in \mathbb{N},$$

i.e., $Ux + Vt + W - N \in Q$. Since $Q \subset \ell_{p,T}$, it follows that $N \in \ell_{p,T}$. Hence, using Theorem 3.1 in [16] and (4.1) in [16], we get

$$\vartheta(Ux + Vt + W|\Omega) = \vartheta(Ux + Vt + W - N|\Omega).$$

By (3.26) in [16], this implies that as $g \to \infty$, $\vartheta_g(U^g x + V^g t + W^g|\Omega_g)$ converges to $\vartheta(Ux+Vt+W|\Omega)$, uniformly for $(x,t) \in \mathbb{R}\times\mathbb{R}$. Since $\vartheta_g(U^g x + V^g t + W^g|\Omega_g)$ is quasi–periodic in x and t, Theorem V on p. 38 in [3] implies

Proposition 5.3 *Suppose that $T = diag(\tau_1, \tau_2, \dots)$ satisfies (5.23). Then for each real $W \in \ell_{p,T}$, the solution $u(x,t)$ of the KdV equation, given by (5.12), is quasi–periodic in x and t.*

We note that the case of smooth periodic solutions is covered by the discussion in [15].

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