On the eta–invariant of certain non–local boundary value problems

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Contents

1. Introduction **1**

2. Generalities **2**

3. Expansion theorems and the gluing law **12**

4. Proofs **23**

1. Introduction

A very intriguing feature of elliptic operators on compact manifolds is the locality of their indices. Specifically, if M denotes a compact Riemannian spin manifold, $S \to M$ a spinor bundle, $E \to M$ a hermitian coefficient bundle with unitary connection, and D^E the Dirac operator on M with coefficients in E then, by the Atiyah–Singer theorem,

$$\mathrm{ind} D_+^E = \int_M \hat{A}(M) \wedge \mathrm{ch}\, E. \tag{1.1}$$

Here D_+^E arises from splitting $S \otimes E$ under the involution induced by the complex volume element on M.

If M decomposes along a compact hypersurface, N, as $M = M_1 \cup M_2$, with $\partial M_i = N$ for $i = 1, 2$, then one is lead to ask whether the obvious decomposition of the right hand side in (1.1) corresponds to a decomposition of the (essentially) self–adjoint operator D^E into self–adjoint operators D_i^E, defined in M_i by suitable boundary conditions on N, such that

$$\mathrm{ind} D_{1,+}^E + \mathrm{ind} D_{2,+}^E = \mathrm{ind} D_+^E. \tag{1.2}$$

This question was answered in the affirmative by Atiyah, Patodi, and Singer [APS] who formulated the correct boundary conditions (cf. Sec. 2 for details). More importantly, the resulting index formula (2.6) displayed a new spectral invariant of self–adjoint elliptic operators (defined on N) which they called the η–invariant. It is not locally computable by a formula as in (1.1) as can be seen from its behaviour under coverings. Nevertheless, one can ask how the η–invariant behaves under splitting N as $N_1 \cup N_2$, and this is the problem we address in this work.

One motivation for posing this question may be seen in trying to understand the signature theorem on manifolds with corners. From a systematical point of view, splitting formulas for spectral invariants should also be very useful for computational purposes – as illustrated nicely by the analytic torsion, cf. [Ch, M1] – and as a possible source of new invariants. Another recent motivation is provided by topological quantum field theory.

The "gluing law" for η–invariants we prove here (Thm. 3.9) is not new; cf. Sec.2 for an account of previous work. Our proof, however, attacks the problem directly on the cut manifold, M^{cut}, by analizing families of "generalized Atiyah–Patodi–Singer boundary value problems." These new abstract boundary conditions are defined by three simple axioms ((3.23)–(3.25) below) which are designed in such a way that the heat kernel of the model operator is explicitly computable. Incidentally, our formula generalizes a result of Sommerfeld in the scalar case. Moreover, under this class we find the spectral boundary conditions introduced by Atiyah, Patodi, and Singer as well as the (local) absolute and relative boundary conditions for the Gauß–Bonnet operator. Thus, our method gives a uniform way to derive the asymptotic expansion of the heat trace in both cases, generalizing in particular recent work by Grubb and Seeley [GrSe] (cf. Thm. 3.4). The family we define interpolates between the "uncut manifold" (the case of smooth transmission) and actual Atiyah–Patodi–Singer boundary value problems; this is similar to Vishik's approach to the splitting behavior of the analytic torsion, and we hope to exploit this further in another publication. The special structure of our family, on the other hand, resembles closely the finite–dimensional variations constructed by Lesch and Wojciechowski [LW]. This allows us to produce explicit variation formulas

(Thm. 3.5). We evaluate them using the vanishing of the noncommutative residue on pseudodifferential idempotents and a special symmetry of the cutting problem.

The plan of the paper is as follows: In Section 2, we review some abstract facts on η–invariants and previous work on the gluing law. All results are presented in Section 3 while the details of most proofs are carried out in Section 4.

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2. Generalities

In this section we briefly review some more or less well known properties of η–invariants which are needed below, together with some of the previous work leading to the gluing law.

The η–invariant was introduced in the seminal work [APS] by Atiyah, Patodi, and Singer. They considered the signature operator, $D = d + \delta$, on a smooth oriented Riemannian manifold, M, with compact boundary $\partial M = N, \dim M = m = 4k$. Assuming that the metric is a product in a neighborhood

$$U \simeq [0, 1) \times N \tag{2.1a}$$

of the boundary, separation of variables leads to the representation

$$D = \gamma(\frac{\partial}{\partial x} + A). \tag{2.1b}$$

Here, we use the decomposition of a smooth form, α, as $\alpha = dx \wedge \alpha_1(x) + \alpha_2(x)$. Thus, the operator on the right acts on $C_0^\infty((0,1), \Omega(N) \oplus \Omega(N))$, $\Omega(N)$ the smooth forms on N, and one has

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes I, \quad A = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \otimes (d_N + \delta_N). \tag{2.1c}$$

Thus A is symmetric, and we have the relations

$$\gamma^2 = -I, \quad \gamma^* = -\gamma, \quad \gamma A + A\gamma = 0. \tag{2.2}$$

A symmetric operator of type (2.1b) does not in general admit local boundary conditions which define a self–adjoint extension (cf., however, [GSm] and [Si]), even though local boundary conditions do exist in the special case (2.1c) i.e. the absolute and relative boundary conditions. But there is always a nonlocal boundary condition given (essentially) by the Calderón projector [C]. Thus we introduce the boundary condition

$$P_{>0}(A)u(0) = 0, \tag{2.3a}$$

where $P_{>0}(A)$ is the orthogonal projection onto the subspace spanned by eigenvectors of A with positive eigenvalues. To define a symmetric operator, this needs to be supplemented by

$$P_\sigma u(0) = 0, \tag{2.3b}$$

where P_σ projects onto a Lagrangian subspace of $\ker A$ with respect to the symplectic form (note that $\dim\ker A$ is even)

$$\omega(u,v) := <\gamma u, v>, \quad u, v \in \ker A,$$

and such a space can always be viewed as the $+1$–eigenspace of an involution, σ, on $\ker A$ satisfying

$$\sigma\gamma + \gamma\sigma = 0; \tag{2.4a}$$

then

$$P_\sigma = \frac{1}{2}(I + \sigma). \tag{2.4b}$$

In the case at hand, a convenient choice of σ is (Clifford multiplication by) the complex volume element, ω_M, i.e. we put

$$\sigma_0 := \omega_M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \omega_N,$$

where ω_N denotes the complex volume element on N.

It is not hard to see that these data define a self–adjoint extension of D, D_{σ_0}, which anticommutes with ω_M. Then the signature operator, D_S, for a manifold with boundary is the closure of

$$D_{\sigma_0}|\mathcal{D}(D_{\sigma_0}) \cap \left\{ u \in \Omega(\overline{M}) \mid \omega_M u = u \right\},$$

and [APS, Thm. (I.3.10)] asserts that D_S is a Fredholm operator with

$$\mathrm{ind} D_S = \int_M L(M) - \frac{1}{2}(\eta(B) + \dim\ker B). \tag{2.5}$$

Here, $L(M)$ denotes the Hirzebruch L–form and the operator B is defined by a representation of D_S in U analogous to (2.1b). In fact, near ∂M we have

$$\begin{aligned} D_S &= \omega_N\left(\partial_x + \omega_N(d_N + \delta_N)\right) \\ &=: \omega_N\left(\partial_x + B\right), \end{aligned}$$

and a core is given by the space (with obvious notation)

$$\mathcal{D}(D_S) = \left\{ u \in \Omega(\overline{M}) \mid P_{\geq 0}(B)u(0) = 0 \right\}.$$

Rewriting (2.5) in terms of the signature of M (as a manifold with boundary) gives [APS, Thm. (I.4.14)]

$$\mathrm{sign}\, M = \int_M L(M) - \frac{1}{2}\eta(B), \tag{2.6}$$

and thus an analytic interpretation of the additivity of the signature under cutting along a separating hypersurface.

The η–invariant figuring in (2.5) and (2.6) is derived from a meromorphic function generalizing the ζ–function of an elliptic operator. It is convenient to derive the main properties of these functions in an abstract functional analytic setting. Thus consider a self–adjoint operator, A, with dense domain, $\mathcal{D}(A)$, in some Hilbert space, H. If we assume that

$$(A + i)^{-1} \in C_p(H), \quad \text{for some } p > 0, \tag{2.7}$$

(where C_p denotes the Schatten–von Neumann class of order p) then the function

$$\eta(A;s) := \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{(s-1)/2}\mathrm{tr}_H(Ae^{-tA^2})dt = \sum_{\lambda\in\operatorname{spec}A\setminus\{0\}}(\operatorname{sgn}\lambda)|\lambda|^{-s} \tag{2.8}$$

is holomorphic for large $\operatorname{Re}s$. More generally, if $B:\mathcal{D}(A)\to H$ is any bounded operator satisfying

$$P_0(A)BP_0(A)=0, \tag{2.9}$$

$P_0(A)$ the orthogonal projection onto $\ker A$, then the same is true of

$$\begin{aligned}\eta(A,B;s) &:= \frac{1}{\Gamma(\frac{s+1}{2})}\int_0^\infty t^{(s-1)/2}\mathrm{tr}_H(Be^{-tA^2})dt\\ &= \sum_{\lambda\in\operatorname{spec}A\setminus\{0\}}(\mathrm{tr}_{\ker(A-\lambda)}B)|\lambda|^{-s-1}.\end{aligned} \tag{2.10}$$

It is very important to determine conditions on A and B which guarantee the existence of a meromorphic extension of (2.10) to the whole complex plane. The standard source of such an extension is an asymptotic expansion

$$\mathrm{tr}_H(Be^{-tA^2}) \sim_{t\to 0+} \sum_{\substack{\operatorname{Re}\alpha\to\infty\\ 0\le k\le k(\alpha)}} a_{\alpha k}(A,B)\,t^\alpha\log^k t. \tag{2.11}$$

The notation used means, of course, that $\{\alpha\in\mathbf{C}\mid a_{\alpha k}(A,B)\neq 0 \text{ for some } k\in\mathbf{Z}_+, k\le k(\alpha)\}$ is a countable subset of $\mathbf{C}$ whose real parts accumulate at most at ∞.

Using the notation $f(s) =: \sum_k \operatorname{Res}_k f(s_0)(s-s_0)^{-k}$, introduced in [BS2] for Laurent expansions, one has

Lemma 2.1 *Under the conditions* (2.7), (2.9), *and* (2.11), η *extends to a meromorphic function on* $\mathbf{C}$.

The poles are situated at the points $s_\alpha = -2\alpha-1$ *and the principal part of* η *at* s_α *is given by*

$$\frac{1}{\Gamma(\frac{s+1}{2})}\sum_{k=0}^{k(\alpha)} a_{\alpha,k}(A,B)(-1)^k k!2^{k+1}(s-s_\alpha)^{-k-1}.$$

In particular, the poles are of order

(1) $k(\alpha)+1$, *if* $\alpha\notin\mathbf{Z}_+$, *and*

$$\operatorname{Res}_{k(\alpha)+1}\eta(A,B;s_\alpha) = \frac{(-1)^{k(\alpha)}k(\alpha)!2^{k(\alpha)+1}}{\Gamma(-\alpha)}a_{\alpha,k(\alpha)}(A,B), \tag{2.12a}$$

and

(2) $k(\alpha)$, *if* $\alpha\in\mathbf{Z}_+$, *and*

$$\operatorname{Res}_{k(\alpha)}\eta(A,B;s_\alpha) = (-1)^{k(\alpha)+\alpha}\alpha!k(\alpha)!2^{k(\alpha)}a_{\alpha,k(\alpha)}(A,B). \tag{2.12b}$$

Lemma 2.2 *Under the conditions* (2.7) *and* (2.9) *the following statements are equivalent:*

(i) $\mathrm{tr}_H(Be^{-tA^2})$ *has an asymptotic expansion of type* (2.11) *which can be differentiated, i.e. for* $N, K > 0$ *we have*

$$\Big|\partial_t^N\Big(\mathrm{tr}_H(Be^{-tA^2}) - \sum_{\substack{\mathrm{Re}\,\alpha\le N+K\\ 0\le k\le k(\alpha)}} a_{\alpha k}(A,B)\, t^\alpha \log^k t\Big)\Big| \le C_{N,K}t^K, \quad t\to 0. \quad (2.13)$$

(ii) $\Gamma(\frac{s+1}{2})\eta(A,B;s)$ *is holomorphic in the half plane* $\{s\in\mathbf{C}\,|\,\mathrm{Re}\,s > p\}$ *and extends meromorphically to* $\mathbf{C}$*. Moreover, for* $a,b\in\mathbf{R}$ *there exists* $s_0 = s_0(a,b) > 0$ *such that* $\Gamma(\frac{s+1}{2})\eta(A,B;s)$ *is holomorphic for* $a\le \mathrm{Re}\,s\le b, |s|\ge s_0$ *with estimate*

$$\Big|\Gamma(\tfrac{s+1}{2})\eta(A,B;s)\Big| \le C(a,b,N)|s|^{-N}, \quad a\le \mathrm{Re}\,s\le b, |s|\ge s_0, \quad (2.14)$$

for any $N > 0$.

Proof (i)⇒(ii): In view of (2.7) and (2.9) $\Gamma(\frac{s+1}{2})\eta(A,B;s)$ is holomorphic in the half plane $\{s\in\mathbf{C}\,|\,\mathrm{Re}\,s>p\}$ and extends meromorphically to $\mathbf{C}$, by Lemma 2.1. Integration by parts gives

$$\Gamma(\tfrac{s+1}{2})\eta(A,B;s) = \frac{(-1)^N 2^N}{(s+1)(s+3)\cdot\ldots\cdot(s+2N-1)}\int_0^\infty t^{(s-1)/2+N}\partial_t^N \mathrm{tr}_H(Be^{-tA^2})dt. \quad (2.15)$$

In view of (2.9) we have for $a\le\mathrm{Re}\,s\le b$

$$\Big|\int_1^\infty t^{(s-1)/2+N}\partial_t^N\mathrm{tr}_H(Be^{-tA^2})dt\Big| \le C\int_1^\infty t^{(b-1)/2+N}e^{-ct}dt =: C_{N,b}. \quad (2.16)$$

Furthermore, choosing K such that $(a-1)/2+K+N>-1$, we may write

$$\begin{aligned}&\int_0^1 t^{(s-1)/2+N}\partial_t^N\mathrm{tr}_H(Be^{-tA^2})dt\\ =:&\int_0^1 t^{(s-1)/2+N}\varphi_{K,N}(t)dt + \sum_{\substack{\mathrm{Re}\,\alpha\le N+K\\ 0\le k\le k(\alpha)}} a_{\alpha k}(A,B)\int_0^1 t^{(s-1)/2+N}\partial_t^N t^\alpha\log^k t\,dt \quad (2.17)\end{aligned}$$

with $|\varphi_{K,N}(t)|\le C_{K,N}t^K$. Hence, we have for $a\le\mathrm{Re}\,s\le b$

$$\Big|\int_0^1 t^{(s-1)/2+N}\varphi_{K,N}(t)dt\Big| \le C_{N,K}. \quad (2.18)$$

Using $\partial_t^N t^\alpha\log^k t = \sum\limits_{i=0}^k c_i t^{\alpha-N}\log^i t$ we get

$$\int_0^1 t^{(s-1)/2+N}\partial_t^N t^\alpha\log^k t = \sum_{i=0}^k c_i(-1)^i i!((s+1)/2+\alpha)^{-i-1}. \quad (2.19)$$

Combining (2.15) through (2.19) we reach the conclusion.

(ii)$\Rightarrow$(i): In view of the estimate (2.14) we can apply the inverse Mellin transform to find, for $c > p$,

$$\operatorname{tr}_H(Be^{-tA^2}) = \frac{1}{4\pi i}\int_{\operatorname{Re} s=c} t^{-(s+1)/2}\,\Gamma(\tfrac{s+1}{2})\eta(A,B;s)ds.$$

Moreover, we can shift the contour of integration to the left and apply the Residue Theorem to get

$$\operatorname{tr}_H(Be^{-tA^2}) \sim_{t\to 0+} \frac{1}{2}\sum_{s\in\mathbf{C}} \operatorname{Res}_1\Big(t^{-(s+1)/2}\,\Gamma(\tfrac{s+1}{2})\eta(A,B;s)\Big).$$

Clearly, this asymptotic expansion can be differentiated. □

Remarks 1) Of course, $B := I - P_0(A)$ gives the ζ–function of A^2,

$$\zeta_{A^2}(\tfrac{s+1}{2}) = \eta(A, I - P_0(A); s).$$

In particular, we can read off the regularity at 0 of ζ_{A^2} provided that the asymptotic expansion of $\operatorname{tr}_H(e^{-tA^2})$ exists and does not contain contributions to $\log^k t, k \in \mathbf{N}$.

2) If A and B are classical pseudodifferential operators on a compact manifold, M, $\dim M =: m$ and A is self–adjoint and elliptic, then (2.7) holds and we have an asymptotic expansion [GrSe, Theorem 2.7]

$$\operatorname{tr}_H(Be^{-tA^2}) \sim_{t\to 0+} \sum_{j=0}^{\infty} a_j(A,B)\,t^{(j-m-b)/2a} + \sum_{j=0}^{\infty} b_j(A,B)\,t^j\log t, \tag{2.20}$$

where $a := \operatorname{ord} A, b := \operatorname{ord} B$. Moreover, this asymptotic expansion can be differentiated in view of the identity

$$\partial_t^N \operatorname{tr}_H(Be^{-tA^2}) = (-1)^N \operatorname{tr}_H(BA^{2N}e^{-tA^2}).$$

If, in addition, (2.9) holds then we can apply Lemma 2.2 to conclude that (2.14) holds for A and B.

Note that in view of (2.20) and Lemma 2.1, in this case $\eta(A,B;s)$ has a meromorphic continuation to $\mathbf{C}$ with *simple* poles.

The estimate (2.14) suffices to shift the contour of integration and to deduce a short time asymptotic expansion. However, for some classical pseudodifferential operators A, B an even stronger result holds: Namely, if A has scalar principal symbol then it follows from [DG] that $\eta(A,B;s)$ is of polynomial growth on finite vertical strips. Since $\Gamma(\frac{s+1}{2})$ decays exponentially on finite vertical strips this implies the estimate (2.14). However, our method of proving (2.14) is completely elementary while [DG] uses the machinery of Fourier integral operators.

Given these preparations we define, under the assumptions of Lemma 2.1 (actually, a partial expansion in (2.11) would suffice), the *η–invariant of A* as

$$\eta(A) := \operatorname{Res}_0\eta(A;0), \tag{2.21a}$$

and, in view of the index formula (2.5), the *reduced η–invariant of* A as

$$\xi(A) := \frac{1}{2}\left(\eta(A) + \dim\ker A\right). \tag{2.21b}$$

Generally, $\eta(A)$ is difficult to compute. It is thus of great importance that suitable one–parameter variations turn out to be "locally computable" in the sense of asymptotic expansions of the type (2.11).

To deal with variations in the abstract framework above we now impose the following assumptions. Consider a connected open subset, J, of $\mathbf{R}$ and for $a \in J$ a family

$$A(a) : \mathcal{D} \longrightarrow H, \tag{2.22a}$$

of self–adjoint operators with fixed domain $\mathcal{D}$, satisfying (2.7).

Moreover, assume that this family has kernel of constant rank, i.e. for $P_0(a) := P_0(A(a))$ we have

$$\dim P_0(a) \quad \text{is constant in } J. \tag{2.22b}$$

Likewise, let

$$B(a) : \mathcal{D} \longrightarrow H, \tag{2.22c}$$

be another family of bounded operators satisfying (2.9) which, in addition, commutes with $A(a)^2$ in the sense that

$$[B(a), (A(a)^2 - \zeta)^{-1}] = 0, \quad a \in J, \quad \zeta \notin \operatorname{spec} A(a)^2. \tag{2.22d}$$

Note that these conditions imply that

$$B(a) = (I - P_0(a))B(a)(I - P_0(a)).$$

Finally, we assume that

the families $(A(a))_{a\in J}, (B(a))_{a\in J} \subset \mathcal{L}(\mathcal{D}, H)$ are strongly differentiable in J, with strongly continuous derivative. (2.22e)

Under these assumptions, the operator families $P_0(a)$ and

$$\widetilde{A}(a) := (I - P_0(a))A(a) + P_0(a) \tag{2.23}$$

are strongly differentiable, too. Using the representation

$$e^{-tA(a)^2} = \frac{(m-1)!\ t^{1-m}}{2\pi i}\int_\Gamma e^{-t\zeta}(A(a)^2 - \zeta)^{-m}d\zeta,$$

with Γ a suitable contour, one can easily derive the identity

$$\begin{aligned}\frac{\partial}{\partial a}\operatorname{tr}_H\left[B(a)e^{-tA(a)^2}\right] &= \operatorname{tr}_H\left[B'(a)e^{-tA(a)^2}\right]\\ &\quad + t\frac{\partial}{\partial t}\operatorname{tr}_H\left[B(a)\left(\frac{d}{da}A(a)^2\right)\widetilde{A}(a)^{-2}e^{-tA(a)^2}\right].\end{aligned}$$

Our assumptions imply the absolute and locally uniform convergence of the relevant t–integrals, and we arrive at

Lemma 2.3 *Under the assumptions* (2.7) *and* (2.22a-e) *we have the identity*

$$\begin{aligned}\frac{\partial}{\partial a}\eta(A(a),B(a);s) &= \eta(A(a),B'(a);s)\\ &\quad -\frac{s+1}{2}\eta(A(a),B(a)\left(\frac{d}{da}A(a)^2\right)\tilde{A}(a)^{-2};s). \end{aligned} \tag{2.24}$$

If we assume in addition that

$$[B(a),(A(a)-\zeta)^{-1}]=0 \quad \text{for} \quad a\in J, \zeta\notin \operatorname{spec} A(a), \tag{2.22d'}$$

then (2.24) simplifies to

$$\frac{\partial}{\partial a}\eta(A(a),B(a);s)=\eta(A(a),B'(a);s)-(s+1)\,\eta(A(a),A'BA\tilde{A}(a)^{-2};s). \tag{2.25}$$

So, if both sides extend meromorphically to $\mathbf{C}$ then (2.25) holds in $\mathbf{C}$, too. We note in particular that

$$\frac{\partial}{\partial a}\eta(A(a);s)=-s\,\eta(A(a),A'(a);s). \tag{2.26}$$

Thus we obtain the well known

Corollary 2.4 *Assume* (2.7), (2.22a,b,e), *and* (2.11) *with* $A(a)$ *and* $A'(a)$ *in place of* B. *Then, for* $k\in\mathbf{Z}_+$,

$$\begin{aligned}\frac{d}{da}\mathrm{Res}_k\eta(A(a);0) &= -\mathrm{Res}_{k+1}\eta(A(a),A'(a);0)\\ &= \frac{(-1)^{k+1}k!2^{k+1}}{\sqrt{\pi}}a_{-1/2,k}(A(a),A'(a)). \end{aligned}\tag{2.27}$$

The condition (2.22b) is not satisfied in interesting situations. One can get rid of it in choosing a real number $c>0$ so that $c\notin \operatorname{spec}(A(a))$ for a near $a_0\in J$. Then we put $\tilde{P}_{<c}(a):=P_{<c}(a)P_{>-c}(a)$, $\tilde{P}_{>c}(a):=I-\tilde{P}_{<c}(a)$ and replace $A(a)$ by $A^c(a):=\tilde{P}_{>c}(a)A(a)+\tilde{P}_{<c}(a)$ and $B(a)$ by $B^c(a):=\tilde{P}_{>c}(a)B(a)\tilde{P}_{>c}(a)+\tilde{P}_{<c}(a)$, obtaining the modified η–function $\eta^c(A(a),B(a);s):=\eta(A^c(a),B^c(a);s)$. η^c admits, near a_0, the same analysis as outlined for η with (2.22b), and from (2.10) we obtain

$$(\eta-\eta^c)(A(a),B(a);s)=\sum_{\substack{\lambda\in\operatorname{spec}A(a)\\ 0<|\lambda|<c}}|\lambda|^{-s-1}\mathrm{tr}_{\ker(A(a)-\lambda)}B(a)-\dim\tilde{P}_{<c}(a). \tag{2.28}$$

This is a smooth function of a and holomorphic in $s\in\mathbf{C}$; on the other hand, the negative t-powers in the expansion (2.11) are unaffected if we modify A and B by an operator of finite rank. Evaluating (2.28) with $B(a):=A(a)$ we obtain

$$\frac{1}{2}(\eta-\eta^c)(A(a);s)+\frac{1}{2}\dim\ker A(a)=\sum_{\substack{\lambda\in\operatorname{spec}A(a)\\ 0<|\lambda|<c}}\frac{1}{2}(\operatorname{sgn}\lambda|\lambda|^{-s}-1),$$

and consequently

Lemma 2.5 *Assume that the family $A(a)_{a\in J}$ satisfies* (2.7), (2.22a,e), *and* (2.11) *with $A(a)$ and $A'(a)$ in place of B. Then, for $a, a_0 \in J$,*

$$\xi(A(a)) - \xi(A(a_0)) + \frac{1}{\sqrt{\pi}}\int_{a_0}^{a} a_{-1/2,0}(A(a), A'(a))da \ \in \mathbf{Z}. \tag{2.29}$$

This implies that the function

$$\tau(A(a)) := e^{2\pi i \xi(A(a))} \tag{2.30}$$

is always smooth in $a \in J$ under our assumptions; the invariant τ was introduced in [DF].

If the asymptotic expansion of $\mathrm{tr}_H[A^j(a)e^{-tA(a)^2}]$ does not contain terms of the form $t^\alpha \log^k t$ with $\alpha < 0$ and $k \in \mathbf{N}$ for $j = 0, 1$ – as it is the case for (classical) elliptic pseudodifferential operators on compact manifolds, cf. the remarks after Lemma 2.2 – then it follows from Lemmas 2.1 and 2.3 that 0 is at most a simple pole of η and that the residue is a homotopy invariant. This is the basis for proving that $\eta(A; s)$ is, in fact, regular at $s = 0$ if A happens to be a (classical) pseudodifferential operator on a compact manifold, cf. [G, Sec. 3.8]. More generally, Wodzicki observed the remarkable fact that, in this class of operators,

$$\mathrm{res}\, B := (\mathrm{ord}\, A)\mathrm{Res}_1\eta(A, B; -1) = -2(\mathrm{ord}\, A)\, a_{0,1}(A, B) \tag{2.31}$$

defines the unique trace (up to a constant) on classical pseudodifferential operators if A is elliptic of positive order, $\mathrm{ord}\, A$. Wodzicki also observed the following result, which is stated without proof in his thesis (Steklov Institute 1984):

Lemma 2.6 *If B is a classical pseudodifferential operator on a compact manifold and an idempotent, then*

$$\mathrm{res}\, B = 0.$$

The only proof we know of shows that the statement of this lemma follows from the regularity at 0 of the η–function for general classical elliptic pseudodifferential operators on a compact manifold. For completeness we indicate that these facts are actually equivalent.

Lemma 2.7 *The assertion of Lemma* 2.6 *is equivalent to the following: Let P be a self–adjoint classical elliptic pseudodifferential operator of positive order on the compact manifold M. Then*

$$\mathrm{Res}_1\eta(P; 0) = 0.$$

Proof 1. First we assume Lemma 2.6. Let P be a self–adjoint classical elliptic pseudodifferential operator of order d on a compact manifold, M. We consider the pseudodifferential operator

$$\mathrm{sgn}\, P := P|P|^{-1} : x \mapsto \begin{cases} |P|^{-1}Px, & x \in \ker P^\perp, \\ 0, & x \in \ker P. \end{cases}$$

We find

$$\eta(P^2, \operatorname{sgn} P; s) = \sum_{\lambda \in \operatorname{spec} P} (\operatorname{sgn} \lambda)|\lambda|^{-s-1} = \eta(P; s+1)$$

an hence in view of (2.31)

$$0 = \operatorname{res} \operatorname{sgn} P = (\operatorname{ord} P)\operatorname{Res}_1 \eta(P^2, \operatorname{sgn} P; -1) = (\operatorname{ord} P)\operatorname{Res}_1 \eta(P; 0).$$

2. To prove the converse we consider a classical pseudodifferential idempotent, B, on a compact manifold, M. B is similar to a self–adjoint idempotent and it is not difficult to see that the similarity can be effected through a pseudodifferential operator. Since the residue is a trace, similar operators have the same residue. Hence we may assume B to be an orthogonal projection. We put $\tau := 2P - I$ and let $\sigma_\tau \in C^\infty(S^*M)$ be the principal symbol of τ. We can choose an invertible first order self–adjoint pseudodifferential operator, Q, with principal symbol σ_τ. Then we put

$$P := \frac{1}{2}(Q|Q|^{-1} + I).$$

This is an operator of order 0 and $P - B$ is of order -1. Then one shows that there exists a pseudodifferential projection P_1, a smoothing operator R, and a pseudodifferential operator K, $\|K\| < 1$, such that $P + R$ is a projection and

$$B = P + R + K.$$

Since $\|K\| < 1$ the projections B and $P + R$ are similar and since res is a trace which vanishes on smoothing operators we find

$$\operatorname{res} B = \operatorname{res}(P + R) = \operatorname{res} P.$$

Since $\operatorname{res} I = 0$ we end up with

$$\begin{aligned} \operatorname{res} P &= \frac{1}{2}\operatorname{res}(Q|Q|^{-1}) \\ &= \frac{1}{2}\operatorname{Res}_1 \eta(Q; 0) = 0. \end{aligned}$$

□

We emphasize, however, that neither for index theorems [BS1] nor for the gluing law to be proved below the regularity at 0 of the η–function is essential; the definition (2.21a) is perfectly sufficient.

If one wants to widen the class of operators which admit reasonable η–invariants then it is most natural to consider elliptic boundary value problems. As illustrated by the gluing question, one may also expect further insight in the compact case. The first work in this direction seems to be [GSm] which deals with local boundary conditions leading to (mildly) nonself–adjoint operators which do, however, admit reasonable η–invariants. This was used by Singer [Si] who showed (among other things) that the difference of η–invariants associated to two natural boundary value problems of this kind is an interesting spectral invariant of the boundary, at least asymptotically. More precisely, let M be an odd dimensional Riemannian spin manifold with spinor bundle $S(M)$ and assume again that the metric is a product near N (this assumption will be kept from now on). Thus, a neighborhood of N in M is isometric to the cylinder $N_R = [0, R) \times N$, for some $R > 0$. Then we have again a representation of type (2.1b) for the Dirac operator, D^M, on $S(M)$ where $A = D^N$ becomes the Dirac operator on $S(N) = S(M)|N$. Under γ, $S(N)$ splits into $S^+(N) \oplus S^-(N)$ with projections $Q_\pm : L^2(S(N)) \longrightarrow L^2(S^\pm(N))$. Then $D^M_\pm := (D^M, Q_\pm)$ are well–posed boundary value problems to which the analysis of [GSm] applies, and Singer proves that by stretching N_R the difference of η–invariants localizes i.e.

$$\lim_{R\to\infty} (\eta(D^M_+) - \eta(D^M_-)) = \frac{1}{4\pi i} \log\, \det(D^N)^2. \tag{2.32}$$

Singers investigation was motivated by Witten's identification of the covariant anomaly with the so–called adiabatic limit of an η–invariant [W] but his work, in turn, stimulated greatly the interest in η–invariants for manifolds with boundary.

Douglas and Wojciechowski [DW] then studied systematically the properties of η–invariants for generalized Dirac operators on odd–dimensional manifolds with boundary. They assumed (2.1b) with the additional hypothesis

$$\ker A = 0, \tag{2.33}$$

and chose the boundary condition (2.3a); in this situation, they established Lemmas 2.1 and 2.3, and for suitable families of such operators they proved (2.27) for $k = 0$. Moreover, they showed that stretching the cylinder N_R produces an "adiabatic limit" in the sense that

$$\lim_{R\to\infty} \eta(D_R) =: \eta_\infty \tag{2.34}$$

exists. Then the challenge was to identify η_∞ and to extend the results to $\ker A \neq 0$. In this case, there is considerable freedom of choice for the "supplementary" boundary condition (2.4a,b), and its variation ought to be allowed, too, in a suitable generalization of (2.27). Note that the analysis of Lemma 2.3 does not apply to this situation right away since the operators under consideration do not have constant domain, so one has to search for a suitable transformation of the family. This was done by Lesch and Wojciechowski [LW]. Since their method also served as a basic motivation for this paper, we will present a suitable version of their argument. Theorem 3.5 below generalizes considerably the original construction and is the main analytic tool of our present work.

The result of [LW] was obtained independently by Müller [M2]. In addition, Müller presented a thorough analysis of the operators D_σ in the general case. In particular, he showed that η_∞ exists and can be interpreted as the suitably defined η–invariant for an operator on the manifold $\widetilde{M} := M \cup N_\infty$. Moreover, he proved that

$$\eta_\infty = \eta(D_{\sigma_1}) \tag{2.35}$$

for a suitable σ_1, obtained from scattering theory on $\widetilde{M}$. He also obtained the regularity of the η–function of D_σ if D is assumed to be of Dirac type.

In the context of Melrose's "b–calculus", Hassell, Mazzeo and Melrose [MM, HMM] define an η–invariant on manifolds with boundary, and they prove a gluing law in this situation. This η–invariant coincides again with η_∞.

(2.35) can be taken as the starting point to prove the gluing law for η–invariants as done by Müller (unpublished). Bunke [B] gave a complete proof of the gluing law based on cutting the manifold in question thrice and reassembling the pieces into a cylinder (carrying both boundary conditions) and a compact manifold where one can do essentially only "interior" analysis, in view of the finite propagation speed enjoyed by all D_σ. This reduces the analysis to the explicit computation on the cylinder carried out in [LW]. Bunke's result is, at least theoretically, more precise than ours since he gives a formula for the unknown integer in (2.29). This is possible since his deformation induces a relatively compact perturbation. By contrast, our construction is more direct and more general but less rigid with regard to compactness.

Bunke's argument, in turn, was generalized and simplified in a substantial paper by Dai and Freed [DF]; they interpreted the invariant (2.30) as a section of the determinant line if one considers families of operators D_σ fibered over a compact Riemannian manifold. This allows a natural interpretation of Witten's anomaly formula, and also illustrates nicely the philosophy developed in Singer's paper [Si].

Our proof of the gluing law (Theorem 3.9 below) arises as a byproduct of an extension of the variation formula to a wider class of boundary conditions, thus furnishing a proof of a rather different nature than those described before.

3. Expansion theorems and the gluing law

Our approach to the proof of the gluing law was originally inspired by Vishik's proof of the Cheeger–Müller Theorem [V]. Working out the details we discovered, however, that we were lead to a very natural generalization of the approach in [LW], designed to determine the variation of $\eta(D_\sigma)$ under a change of σ.

At any rate, the analysis we are going to present deals with operators of type (2.1b) but with more general boundary conditions than (2.3). We will now explain how this class arises naturally from the gluing problem, define it in general, and outline the proof of the gluing law. Most details are deferred to Sec. 4.

Let now M be a *compact* Riemannian manifold, $\dim M = m$, and let

$$D_0 : C_0^\infty(S) \longrightarrow C_0^\infty(S) \tag{3.1}$$

be a first order symmetric elliptic differential operator on the hermitian vector bundle $S \to M$. The main examples are, of course, Dirac operators associated to a Dirac bundle (S, ∇), but we will work in a more general context, allowing for example Dirac operators with potential.

Let $N \subset M$ be a compact hypersurface. We assume that N has a tubular neighborhood U isometric to $(-1,1) \times N$ and such that the hermitian structure of S is a product, too. Moreover, we assume that on U the operator D_0 has the form

$$D_0 = \gamma(\frac{\partial}{\partial x} + A), \tag{3.2}$$

where $\gamma \in C^\infty(\mathrm{End}(S_N))$ is a unitary bundle automorphism and A is a first order self–adjoint elliptic differential operator on $S_N := S|N$. If D_0 is a compatible Dirac operator, then γ is Clifford multiplication by the inward normal vector and A is (essentially) a Dirac operator on N. We assume, furthermore, that γ and A satisfy (2.2).

Let D be the restriction of D_0 to $C_0^\infty(S|M \setminus N)$. This operator is no longer essentially self–adjoint; in order to obtain self–adjoint extensions one has to impose boundary conditions. The natural boundary condition inherited from M is the *continuous transmission* boundary condition. Interpreting sections of S with support in U as functions $[-1,1] \to L^2(S_N)$ in the obvious way, this boundary condition reads

$$f(0-) = f(0+). \tag{3.3}$$

It is fairly clear that the resulting self–adjoint operator is unitarily equivalent to the closure of D in $L^2(S)$. On the other hand, D lives naturally on

$$M^{\mathrm{cut}} := (M \setminus U) \cup_{\partial(M\setminus U)} ((-1,0] \times N \cup [0,1) \times N) \tag{3.4}$$

obtained by cutting M along N (we adopt here the notation from [DF, p. 5164 and Sec. 4]). Thus, M^{cut} is obtained from M by artificially introducing two copies of N as boundary.

On M^{cut} we can introduce spectral boundary conditions as in Sec. 2. The natural interpolation between the continuous transmission and the Atiyah–Patodi–Singer boundary condition is furnished by the boundary conditions

$$\begin{aligned} \cos\theta\, P_{>0}(A) f(0+) &= \sin\theta\, P_{>0}(A) f(0-), \\ \sin\theta\, P_{<0}(A) f(0+) &= \cos\theta\, P_{<0}(A) f(0-), \end{aligned} \tag{3.5a}$$

$$P_0(A) f(0+) = P_0(A) f(0-), \tag{3.5b}$$

where $|\theta| < \pi/2$.

To render this more transparent, we employ the isomorphism (with $H := L^2(S_N)$)

$$\Phi : L^2(S|U) \simeq L^2([-1,1], H) \longrightarrow L^2([0,1], H \oplus H), \tag{3.6a}$$

which sends $f \in L^2([-1,1], H)$ to Φf,

$$\Phi f(x) = f(x) \oplus f(-x), \quad x \in [0,1]. \tag{3.6b}$$

It is easy to see that, under Φ, D is transformed to

$$\widetilde{D} := \begin{pmatrix} \gamma & 0 \\ 0 & -\gamma \end{pmatrix} \left(\frac{\partial}{\partial x} + \begin{pmatrix} A & 0 \\ 0 & -A \end{pmatrix} \right) =: \tilde{\gamma}\Big(\frac{\partial}{\partial x} + \widetilde{A}\Big), \tag{3.7}$$

and the boundary condition to

$$\cos\theta\, P_{>0}(\widetilde{A}) u(0) = \sin\theta\, \tau P_{<0}(\widetilde{A}) u(0), \tag{3.8a}$$

where

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{3.8b}$$

supplemented on $\ker \widetilde{A}$ by

$$P_\sigma u(0) = 0, \tag{3.8c}$$

with

$$\sigma := -\begin{pmatrix} 0 & P_0(A) \\ P_0(A) & 0 \end{pmatrix}. \tag{3.8d}$$

Note that

$$\tau\tilde{\gamma}+\tilde{\gamma}\tau=0=\tau\tilde{A}+\tilde{A}\tau, \quad \tau^2=1, \quad \tau=\tau^*. \tag{3.9}$$

Next we observe that this boundary condition can be written as

$$\tilde{P}(\theta)u(0)=0, \tag{3.10a}$$

if we introduce the projection

$$\tilde{P}(\theta):=\cos^2\theta P_{>0}(\tilde{A})+\sin^2\theta P_{<0}(\tilde{A})-\frac{1}{2}(\sin 2\theta)\tau(P_{>0}(\tilde{A})+P_{<0}(\tilde{A}))+P_\sigma. \tag{3.10b}$$

It is useful to note the following properties of this family of projections, all of which are easily verified.

First, we see that

$$\tilde{\gamma}\tilde{P}(\theta)=(I-\tilde{P}(\theta))\tilde{\gamma}, \tag{3.11}$$

and that $\tilde{P}(\theta)$ commutes with $\tilde{A}^2$,

$$[\tilde{P}(\theta),\tilde{A}^2]=0. \tag{3.12}$$

We do not have commutativity with $\tilde{A}$, however. Instead we find

$$\tilde{P}(\theta)\tilde{A}\tilde{P}(\theta)=\cos 2\theta|\tilde{A}|\tilde{P}(\theta). \tag{3.13}$$

Remembering the argument of Lesch and Wojciechowski [LW] we are lead to ask for a natural "parametrization" of the family $(\tilde{P}(\theta))_{|\theta|<\pi/2}$. It is easy to verify that with

$$U(\theta):=\Big(\cos\theta(P_{>0}(\tilde{A})+P_{<0}(\tilde{A}))+\sin\theta(P_{>0}(\tilde{A})-P_{<0}(\tilde{A}))\tau\Big)\oplus I_{\ker\tilde{A}} \tag{3.14}$$

and

$$\operatorname{sgn}\tilde{A}:=P_{>0}(\tilde{A})-P_{<0}(\tilde{A}) \tag{3.15}$$

we have

$$\tilde{P}(\theta) = U(\theta)\tilde{P}(0)U(\theta)^*, \tag{3.16}$$

$$U(\theta) = e^{(\operatorname{sgn}\tilde{A}\tau)\theta}. \tag{3.17}$$

Thus we obtain a family of generalized Atiyah–Patodi–Singer boundary conditions, and the gluing law becomes just the variational formula for this class of operators in the sense of Sec. 2.

In fact, we will generalize the situation further. *Thus from now on we consider the following setting.*

M is a Riemannian manifold of dimension m, $S\to M$ is a smooth hermitian vector bundle over M, and D is a first order symmetric elliptic differential operator on $C_0^\infty(S)$. We assume that M can be decomposed as

$$M=U\cup M_1, \tag{3.18}$$

where M_1 is a compact manifold with boundary $N=\partial M_1=\partial U$ and U is open. Moreover, we assume an isometry of Hilbert spaces,

$$\Phi: L^2(S|U)\longrightarrow L^2([0,1],H), \tag{3.19}$$

where S_N is a smooth hermitian bundle over N and $H = L^2(S_N)$ as before. This isometry maps smooth sections to smooth sections in the sense that

$$\Phi(C^\infty(S|U) \cap L^2(S|U)) \subset C^\infty((0,1), C^\infty(S_N)) \cap L^2([0,1], H). \tag{3.20}$$

Thus we can transform D on U, and we require that

$$\Phi D \Phi^* = \gamma(\partial_x + A) =: \widetilde{D}, \tag{3.21}$$

with A a symmetric elliptic operator of first order on S_N which we identify with its self–adjoint closure, and γ a bounded operator on H. We assume, moreover, that γ and A satisfy the relations (2.2) and (2.7).

Finally, we require that for $\phi \in C_0^\infty(-1,1)$ there is $\psi_\phi \in C^\infty(M)$ such that $\psi_\phi = 0$ in a neighborhood of ∂M_1, and

$$\Phi(\psi_\phi u) = \phi \Phi u, \quad u \in L^2(S); \tag{3.22a}$$

and

$$\phi = 1 \text{ near } 0 \text{ implies } 1 - \psi_\phi \in C_0^\infty(M). \tag{3.22b}$$

As usual, we extend $\widetilde{D}$ to $L^2(\mathbf{R}_+, H) =: \mathcal{H}$ to obtain the model operator. To define a family of boundary conditions we proceed as in the above analysis of the cutting problem: we consider a family $P(\theta)_{|\theta|<\pi/2}$ of orthogonal projections with the following properties.

$$\gamma P(\theta) = (I - P(\theta))\gamma; \tag{3.23}$$

$$[P(\theta), A^2] = 0; \tag{3.24}$$

$$\begin{aligned} &A(\theta) := P(\theta) A P(\theta) = a(\theta)|A|P(\theta) \text{ for some} \\ &a \in C^\infty(-\pi/2, \pi/2) \text{ with } a > -1. \end{aligned} \tag{3.25}$$

These projections are again assumed to be conjugate to $P(0)$ under a family of unitaries, $U(\theta)$,

$$P(\theta) = U(\theta) P(0) U(\theta)^*. \tag{3.26}$$

We assume, moreover, a representation

$$U(\theta) = e^{iT(\theta)}, \tag{3.27}$$

with $T(\theta)$ bounded and self–adjoint in H, smooth in $(-\pi/2, \pi/2)$, and such that

$$[\gamma, T(\theta)] = 0, \tag{3.28a}$$

$$AT(\theta) + T(\theta)A = 0. \tag{3.28b}$$

With these data we define boundary conditions for D and $\widetilde{D}$ via

$$\widetilde{\mathcal{D}}_\theta := \left\{ u \in C(\mathbf{R}_+, H) \cap \mathcal{H} \,\middle|\, u \in \mathcal{D}(\widetilde{D}^*), P(\theta)u(0) = 0 \right\}, \tag{3.29a}$$

$$\mathcal{D}_\theta := \left\{ u \in L^2(S) \,\middle|\, \begin{matrix} u \in \mathcal{D}(D^*), \Phi(\psi_\phi u) \in \widetilde{\mathcal{D}}_\theta \quad \text{for some} \\ \phi \in C_0^\infty(-1,1) \quad \text{with} \quad \phi = 1 \quad \text{near} \quad 0 \end{matrix} \right\}, \tag{3.29b}$$

and

$$D_\theta := D|\mathcal{D}_\theta, \quad \widetilde{D}_\theta := \widetilde{D}|\widetilde{\mathcal{D}}_\theta. \tag{3.30}$$

A good part of the subsequent analysis rests on these assumptions. For the asymptotic expansions to exist it is convenient to require in addition that

$$\begin{aligned}&P(\theta), T(\theta) \text{ are classical pseudodifferential operators}\\ &\text{of order zero on } N, \text{ for } |\theta| < \pi/2.\end{aligned} \tag{3.31}$$

This assumption is clearly satisfied in the gluing case (3.10a,b).

We will refer to the family $(D_\theta)_{|\theta|<\pi/2}$ with the properties listed above as a *deformation of Atiyah–Patodi–Singer* (APS) type. Then we have seen that cutting along a compact hypersurface leads naturally to such a family. In this case, we do have a bit more structure since, in (3.25), we have $a(\theta) = \cos 2\theta$, in view of (3.13), and we have the additional symmetry, τ, with the properties (3.9).

We note that a single projection, P, with the properties (3.23), (3.24), (3.25) defines a self–adjoint extension of D, D_P, to which the analysis of Sec. 2 applies. This we call a *generalized APS operator* since, clearly, $P = P_{>0}(A) + P_\sigma$ falls in this class.

We proceed to the spectral analysis of D_θ, the proofs being given in Sec. 4.

Proposition 3.1 *The operators D_θ and $\widetilde{D}_\theta$ are essentially self–adjoint.*

We will identify D_θ and $\widetilde{D}_\theta$ with their respective closures in the sequel.

Proposition 3.2 *D_θ satisfies* (2.7) *i.e.*

$$(D_\theta + i)^{-1} \in C_p(L^2(S)) \quad \textit{for every} \quad p > m.$$

We want to apply Lemma 2.5 to the family $(D_\theta)_{|\theta|<\pi/2}$ which requires that we first apply a transformation to satisfy (2.22a,e). This we do as in [LW], and this is the motivation for the assumptions (3.26), (3.27), and (3.28a,b).

Thus we choose $\phi \in C_0^\infty(-1,1)$ with $\phi = 1$ near 0 and introduce the unitary transformation

$$\begin{aligned}\Psi_\theta : L^2([0,1],H) &\longrightarrow L^2([0,1],H),\\ \Psi_\theta u(x) &:= e^{i\phi(x)T(\theta)}(u(x)).\end{aligned} \tag{3.32}$$

Then $P(0)u(0) = 0$ implies $P(\theta)\Psi_\theta u(0) = 0$, in view of (3.26). Hence, extending Ψ_θ to $L^2([0,1],H) \oplus L^2(S|M_1)$ as the identity on $L^2(S|M_1)$ and similarly Φ in (3.19), we obtain an isometry

$$\Phi_\theta := \Phi^*\Psi_\theta\Phi$$

of $L^2(S)$ mapping $\mathcal{D}_0$ to $\mathcal{D}_\theta$. Consequently, the family

$$\check{D}_\theta := \Phi_\theta^* D_\theta \Phi_\theta \tag{3.33}$$

has constant domain, $\mathcal{D}_0$, and the same spectral invariants as D_θ. It is easy to see that $(\check{D}_\theta)_{|\theta|<\pi/2}$ satisfies (2.22a,e). It remains to establish the asymptotic expansions (2.11), with $\check{D}_\theta$, $\frac{d}{d\theta}\check{D}_\theta$ in place of B.

Our expansion results will be expressed in terms of the Mellin transform of a certain meromorphic function, F_a, which we have to introduce first.

Lemma 3.3 *Consider for $a \in (-1,1]$ and $x > 0$ the function*

$$F_a(x) := x \int_0^\infty \operatorname{erfc}(z) e^{-2axz - x^2} dz,$$

where

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-u^2} du$$

denotes the complementary error function.

Then the Mellin transform of F_a is, for $0 < |a| < 1$,

$$\begin{aligned} \mathcal{M}F_a(w) &= \frac{1}{4a}\Big[(1-(1-a^2)^{-w/2})\Gamma(w/2) \\ &\quad + \frac{2}{\sqrt{\pi}}(1-a^2)^{-w/2}\int_0^a (1-t^2)^{(w/2)-1} dt\, \Gamma((w+1)/2)\Big], \end{aligned} \tag{3.34}$$

whereas

$$\mathcal{M}F_0(w) = \frac{1}{2\sqrt{\pi}}\Gamma((w+1)/2), \tag{3.35}$$

and

$$\mathcal{M}F_1(w) = \frac{1}{4}\Big[\Gamma(w/2) - \frac{2}{w\sqrt{\pi}}\Gamma((w+1)/2)\Big]. \tag{3.36}$$

Hence $\mathcal{M}F_a(w)$ is meromorphic in $\mathbf{C}$ with simple poles at the points $-k, k \in \mathbf{Z}_+$. For $|a| < 1$, the residues are

$$\begin{aligned} \operatorname{Res}_1 \mathcal{M}F_a(-2l) &= \frac{(-1)^l}{l!2a}(1-(1-a^2)^l), \quad l \in \mathbf{Z}_+, \\ \operatorname{Res}_1 \mathcal{M}F_a(-2l-1) &= \frac{(-1)^l}{l!\sqrt{\pi}a}(1-a^2)^{l+1/2}\int_0^a (1-t^2)^{-l-3/2} dt, \quad l \in \mathbf{Z}_+. \end{aligned} \tag{3.37}$$

For $a = 0, 1$ one has to take the corresponding limit in (3.37). More precisely,

$$\begin{aligned} \operatorname{Res}_1 \mathcal{M}F_0(-2l) &= 0, \quad l \in \mathbf{Z}_+, \\ \operatorname{Res}_1 \mathcal{M}F_0(-2l-1) &= \frac{(-1)^l}{\sqrt{\pi}l!}, \quad l \in \mathbf{Z}_+, \\ \operatorname{Res}_1 \mathcal{M}F_1(-2l) &= \begin{cases} 0, & l = 0, \\ \dfrac{(-1)^l}{l!2}, & l \in \mathbf{N}, \end{cases} \\ \operatorname{Res}_1 \mathcal{M}F_1(-2l-1) &= \frac{(-1)^l}{l!\sqrt{\pi}(2l+1)}, \quad l \in \mathbf{Z}_+. \end{aligned} \tag{3.38}$$

Now we present our first expansion result.

Theorem 3.4 *Assume that* (3.18) *through* (3.31) *hold. For $l = 0, 1$ we have an asymptotic expansion of the form*

$$\begin{aligned}\operatorname{tr}_{L^2(S)}[D_\theta^l e^{-tD_\theta^2}] \quad \sim_{t\to 0+} \quad & \sum_{j=0}^\infty a_j(\theta,l)\, t^{j-m/2} + \sum_{j=0}^\infty b_j(\theta,l)\, t^{j/2}\log t \\ & + \sum_{j=0}^\infty c_j(\theta,l)\, t^{(j-n-l)/2} + \sum_{j=0}^\infty d_j(\theta,l)\, t^{j/2}. \end{aligned} \tag{3.39}$$

Here, the coefficients a_j are integrals of local densities on the metric double, $\widetilde{M}$, of M, b_j and c_j are integrals of local densities on N, and d_j are nonlocal invariants of N; they are given explicitly in the formulas (4.15), (4.21a), (4.21b), (4.30a), *and* (4.30b) *below.*

For $l = 0$, the leading term is

$$a_0(\theta, 0) = \Gamma(m/2+1)\operatorname{vol}(T_1^*M), \tag{3.40}$$

where $T_1^*M = \{\xi \in T^*M \mid \sigma_{D_\theta^2}(\xi) \le 1\}$.

The logarithmic terms vanish if $l = 0$ and m is odd. If $l = 0$ and m is even then $b_{2j}(\theta, 0) = 0$. However, the logarithmic terms are present in general.

For $l = 1$, the expansion (3.39) *implies that $\eta(D_\theta; s)$ has a meromorphic extension to $\mathbf{C}$ with at most double poles. 0 is a simple pole and for the residue at 0 we find*

$$\operatorname{Res}_1\eta(D_\theta;0) = \frac{2}{\sqrt{\pi}}a_{n/2}(\theta,1) + \Big(\frac{2a(\theta)}{\sqrt{\pi}}\mathcal{M}F_{a(\theta)}(1) - \frac{1}{2}\Big)\operatorname{res}\left(\gamma(\operatorname{sgn} A)P(\theta)\right). \tag{3.41}$$

For the APS boundary condition, this result has been obtained by Grubb and Seeley [GrSe]. Our approach differs from theirs by using the explicit heat kernel (4.1); this method seems to provide explicit formulas for the coefficients in (3.39) more directly. The same expansion result is sketched in Müller [M2, Lemma 1.17] overlooking, however, the coefficients which are not local in $\widetilde{M}$ in the case $l = 0$. In the case $l = 1$ and for APS boundary conditions, these nonlocal terms are actually not present.

To explain this let for the moment $\widetilde{D}_\sigma$ be the operator with APS boundary condition. Then a simple symmetry argument shows that for any cut–off function $\phi \in C_0^\infty(\mathbf{R})$

$$\operatorname{tr}_{L^2(S)}[\phi\widetilde{D}_\sigma e^{-t\widetilde{D}_\sigma^2}] = 0, \quad t > 0, \tag{3.42}$$

and hence $b_j(\sigma,1) = c_j(\sigma,1) = d_j(\sigma,1) = 0$ (cf. [L2, Lemma 5.2.4]). For general $P(\theta)$ we cannot expect (3.42) to hold.

In the next step, we evaluate the formula for the variation of the ξ–invariant in Lemma 2.5, via the asymptotic expansion of $\operatorname{tr}((\frac{d}{d\theta}\check{D}_\theta)e^{-t\check{D}_\theta^2})$.

Theorem 3.5 *Under the assumptions of Theorem* 3.4 *we have the following variation formulas:*

$$\begin{aligned}\frac{d}{d\theta}\operatorname{Res}_1\eta(D_\theta;0) &= \frac{1}{\sqrt{\pi}}\operatorname{res}\left(\gamma i T'(\theta)\right), & (3.43)\\ \frac{d}{d\theta}\xi(D_\theta) &= \frac{1}{2\pi}a_{00}(A,\gamma i T'(\theta)) \\ &\quad + \Big(\frac{2a(\theta)}{\sqrt{\pi}}\mathcal{M}F_{a(\theta)}(1) - \frac{1}{2}\Big)\operatorname{res}\left(\gamma i T'(\theta)(\operatorname{sgn} A)P(\theta)\right). & (3.44)\end{aligned}$$

Corollary 3.6 *In the situation of Theorem* 3.5*, assume in addition that*

$$T'(\theta)P(\theta) = (I - P(\theta))T'(\theta). \tag{3.45}$$

Then

$$\mathrm{res}\,(\gamma i T'(\theta)(\mathrm{sgn}\,A)P(\theta)) = \frac{a(\theta)}{2}\mathrm{res}\,(\gamma i T'(\theta)).$$

In particular, if $\mathrm{res}\,(\gamma i T'(\theta)) = 0$ *then* $\mathrm{Res}_1\eta(D_\theta; 0)$ *is independent of* θ *and*

$$\frac{d}{d\theta}\xi(D_\theta) = \frac{1}{2\pi}a_{0,0}(A, \gamma i T'(\theta)).$$

Proof We use (3.45), (3.23), (3.25), and the trace property of the noncommutative residue to compute

$$\begin{aligned}\mathrm{res}\,(\gamma i T'(\theta)(\mathrm{sgn}\,A)P(\theta)) &= \mathrm{res}\,(\gamma i T'(\theta)P(\theta)(\mathrm{sgn}\,A)P(\theta)) \\ &= a(\theta)\mathrm{res}\,(\gamma i T'(\theta)P(\theta)).\end{aligned}$$

Here we have used that res vanishes on smoothing operators. Furthermore, in view of (3.28a),

$$\begin{aligned}\mathrm{res}\,(\gamma i T'(\theta)P(\theta)) &= \mathrm{res}\,(\gamma i(I - P(\theta))T'(\theta)) \\ &= \mathrm{res}\,(i(I - P(\theta))T'(\theta)\gamma) \\ &= \mathrm{res}\,(i(I - P(\theta))\gamma T'(\theta)) \\ &= \mathrm{res}\,(\gamma i T'(\theta)(I - P(\theta))),\end{aligned}$$

and we reach the conclusion. □

Next we introduce a special class of deformations of APS type which is still slightly more general than the gluing situation (3.5a)–(3.17):

We consider again the framework (3.18)–(3.22b). Furthermore, let $\tau : C^\infty(S_N) \to C^\infty(S_N)$ be a unitary classical pseudodifferential operator satisfying (cf. (3.9))

$$\tau\gamma + \gamma\tau = 0 = \tau A + A\tau, \quad \tau^2 = I, \quad \tau = \tau^*. \tag{3.46}$$

We abbreviate

$$K^\pm := (\ker A) \cap \ker(\gamma \mp i). \tag{3.47}$$

The relations (3.46) immediately imply

$$\dim K^+ = \dim K^-. \tag{3.48}$$

However, the presence of τ is not really necessary for this equality. (3.48) follows already from (3.18)–(3.22b). If D is a Dirac operator, this is the well–known cobordism theorem for Dirac operators [P, Chapter XVII]. For general D, this is due to the second named author [L1, Theorem 6.2], [L2, Chapter IV]. It was also proved independently by W. Müller [M2, Prop. 4.26].

In view of (3.48) we can choose an isometry

$$U : K^+ \longrightarrow K^- \tag{3.49}$$

and put

$$\sigma = \begin{pmatrix} 0 & U^* \\ U & 0 \end{pmatrix} : \ker A \longrightarrow \ker A. \tag{3.50}$$

With these data we can introduce the projection (cf. (3.10b))

$$P(\theta) := \cos^2\theta P_{>0}(A) + \sin^2\theta P_{<0}(A) - \frac{1}{2}(\sin 2\theta)\tau(P_{>0}(A) + P_{<0}(A)) + P_\sigma \tag{3.51}$$

and the unitary family (cf. (3.14))

$$U(\theta) = (\cos\theta(P_{>0}(A) + P_{<0}(A)) + \sin\theta(\operatorname{sgn} A)\tau) \oplus I_{\ker A} = e^{(\operatorname{sgn} A)\tau\theta}. \tag{3.52}$$

One immediately checks the relations (3.11)–(3.13), (3.16), hence we are lead to a deformation of APS type. We denote the corresponding family of operators by $D_{\theta,\sigma}$, indicating explicitly the dependence on the choice of σ. If we fix θ and consider a one parameter family of reflections, σ_u, we obtain another deformation of APS type. In this way we recover the main result of Lesch and Wojciechowski [LW] as a special case of our present work:

Proposition 3.7 (cf. [LW, M2, DF]) *Let $\cos\theta \neq 0$ and $U_u : K^+ \to K^-$ be a smooth family of unitary operators. Put*

$$\sigma_u := \begin{pmatrix} 0 & U_u^* \\ U_u & 0 \end{pmatrix}.$$

Then $(D_{\theta,\sigma_u})_u$ is a deformation of APS *type,* $\operatorname{Res}_1\eta(D_{\theta,\sigma_u};0)$ *is independent of u and*

$$\frac{d}{du}\xi(D_{\theta,\sigma_u}) = \frac{1}{2\pi i}\operatorname{tr}_{K^+}[U_u^{-1}\frac{d}{du}U_u].$$

Proof We put

$$P_u(\theta) := \cos^2\theta P_{>0}(A) + \sin^2\theta P_{<0}(A) - \frac{1}{2}(\sin 2\theta)\tau(P_{>0}(A) + P_{<0}(A)) + P_{\sigma_u}.$$

Furthermore, we fix u_0 and define the unitary operator $V_u \in \mathcal{L}(H)$ by

$$V_u|K^+ := U_u^* U_{u_0}, \quad V_u|K^- \oplus (\ker A)^\perp := I. \tag{3.53}$$

Then we choose a smooth family of self–adjoint operators, T_u, such that

$$V_u = e^{iT_u}, \quad T_{u_0} = 0, \quad T_u|K^- \oplus (\ker A)^\perp = 0. \tag{3.54}$$

It follows that

$$V_u P_{u_0}(\theta) V_u^* = P_u(\theta)$$

and one checks that $(D_{\theta,\sigma_u})_u$ is a deformation of APS type. Since T_u' is an operator of finite rank, we have

$$\operatorname{res}(\gamma i T_u') = \operatorname{res}(\gamma i T_u'(\operatorname{sgn} A)P_u(\theta)) = 0.$$

We deduce from Theorem 3.5

$$\frac{d}{du}\mathrm{Res}_1\eta(D_{\theta,\sigma_u};0)=0,$$

and

$$\begin{aligned}\frac{d}{du}\xi(D_{\theta,\sigma_u}) &= \frac{1}{2\pi}a_{00}(A,\gamma iT'_u)\\ &= \frac{i}{2\pi}\lim_{u\to 0}\mathrm{tr}_H[\gamma T'_u e^{-tA^2}]\\ &= \frac{i}{2\pi}\mathrm{tr}_{K^+}[\gamma T'_u]\\ &= \frac{1}{2\pi i}\mathrm{tr}_{K^+}[U_u^{-1}\frac{d}{du}U_u].\end{aligned}$$

□

Next we deal with the deformation $(D_{\theta,\sigma})_{|\theta|<\pi/2}$:

Proposition 3.8 $\mathrm{Res}_1\eta(D_{\theta,\sigma};0)$ *is independent of* θ *and*

$$\begin{aligned}\frac{d}{d\theta}\xi(D_{\theta,\sigma}) &= \frac{1}{2\pi}a_{00}(A,\gamma(\mathrm{sgn}\,A)\tau)\\ &= \frac{1}{2\pi}\mathop{\mathrm{LIM}}_{t\to 0}\mathrm{tr}_H[\gamma(\mathrm{sgn}\,A)\tau e^{-tA^2}].\end{aligned}$$

Here $\mathrm{LIM}_{t\to 0}$ *is just another common notation for the constant term in the asymptotic expansion as* $t\to 0$.

Proof In view of (3.52) we put

$$T(\theta):=-i(\mathrm{sgn}\,A)\tau\theta.$$

Then one checks that (3.23)–(3.28b) and (3.45) are satisfied. We want to apply Corollary 3.6 to compute $\frac{d}{d\theta}\xi(D_{\theta,\sigma})$. Since res vanishes on operators of finite rank we may replace

$$\gamma iT'(\theta)=\gamma(\mathrm{sgn}\,A)\tau$$

by

$$\gamma((\mathrm{sgn}\,A)+\sigma)\tau$$

in the assumptions of Corollary 3.6. Since

$$(\gamma((\mathrm{sgn}\,A)+\sigma)\tau)^2=I$$

we infer from Lemma 2.6 that $\mathrm{res}\,(\gamma((\mathrm{sgn}\,A)+\sigma)\tau)=0$. Thus $\mathrm{Res}_1\eta(D_{\theta,\sigma};0)$ is independent of θ and

$$\begin{aligned}\frac{d}{d\theta}\xi(D_{\theta,\sigma}) &= \frac{1}{2\pi}a_{00}(A,\gamma iT'(\theta))\\ &= \frac{1}{2\pi}a_{00}(A,\gamma(\mathrm{sgn}\,A)\tau)\\ &= \frac{1}{2\pi}\mathop{\mathrm{LIM}}_{t\to 0}\mathrm{tr}_H[\gamma(\mathrm{sgn}\,A)\tau e^{-tA^2}].\end{aligned}$$

□

Finally, we present the gluing law. In this situation (3.5a)–(3.17) we have yet another structure: namely, introducing (with same notation as in (3.7), (3.8b))

$$\mu := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

we see that

$$\mu^2 = -I, \quad \mu\tau + \tau\mu = \tau\tilde{\gamma} + \tilde{\gamma}\mu = \mu\tilde{A} + \tilde{A}\mu = 0.$$

This observation leads to

Theorem 3.9 (Gluing Law) *Consider the deformation of* APS *type,* $(D_{\theta,\sigma})_{|\theta|<\pi/2}$*, introduced in* (3.46)–(3.52)*. If there exists a unitary classical pseudodifferential operator* $\mu : C^\infty(S_N) \to C^\infty(S_N)$ *satisfying*

$$\mu^2 = -I, \quad \mu\tau + \tau\mu = \tau\gamma + \gamma\mu = \mu A + A\mu = 0 \tag{3.55}$$

then

$$\frac{d}{d\theta}\xi(D_{\theta,\sigma}) = 0.$$

Proof In view of (3.55) we have

$$\mu\gamma(\operatorname{sgn} A)\tau + \gamma(\operatorname{sgn} A)\tau\mu = 0,$$

hence

$$\operatorname{tr}_H[\gamma(\operatorname{sgn} A)\tau e^{-tA^2}] = 0.$$

In particular $a_{00}(A, \gamma(\operatorname{sgn} A)\tau) = 0$ and, by Proposition 3.8, we reach the conclusion. □

Our last comment concerns the residue at 0 of the η–function. We expect that in general the residue in (3.41) will not vanish. In the cutting case, however, there is no pole:

Theorem 3.10 *If* $(D_\theta)_{|\theta|<\pi/2}$ *arises from cutting* M *along a compact hypersurface* (*as explained in* (3.5a)–(3.17)) *then* $\eta(D_\theta; s)$ *is regular at* $s = 0$*, for all* θ*.*

Proof By Proposition 3.8, $\operatorname{Res}_1\eta(D_\theta; 0)$ is independent of θ, hence

$$\operatorname{Res}_1\eta(D_\theta; 0) = \operatorname{Res}_1\eta(D_{\pi/4}; 0) = 0$$

since the η–function of a self–adjoint elliptic differential operator on a compact manifold is regular at 0 [G, Sec. 3.8]. □

4. Proofs

We now prove the statements used in the previous section.

Proof of Proposition 3.1 We consider $\widetilde{D}_\theta$ first. Let $u \in \mathcal{D}(\widetilde{D}_\theta^*)$ satisfy

$$\widetilde{D}_\theta^* u = \pm\sqrt{-1}u.$$

This implies, for $v \in \widetilde{\mathcal{D}}_\theta$, that

$$(\widetilde{D}_\theta v, u) = \mp\sqrt{-1}(v, u).$$

Then a standard regularity argument shows that $u \in C(\mathbf{R}_+, L^2(S_N))$ with

$$P(\theta)u(0) = 0,$$

by (3.23). Choosing $\phi \in C_0^\infty(\mathbf{R})$ with $\phi = 1$ near 0 we put $\phi_N(x) := \phi(x/N)$ and obtain $\phi_N^2 u \in \widetilde{\mathcal{D}}_\theta$. Consequently, we find that

$$\pm\sqrt{-1}\|u\|^2 = \lim_{N\to\infty}(\widetilde{D}_\theta\phi_N^2 u, u) = \lim_{N\to\infty}(u, \widetilde{D}_\theta\phi_N^2 u) \in \mathbf{R},$$

hence $u = 0$.

For D_θ, we appeal to the localization principle for deficiency indices derived in [L1, Thm. 2.1] (cf. also [L2, Chapter IV]). □

In what follows it will be crucial that we can give an explicit formula for the operator heat kernel of $\widetilde{D}_\theta$. It is the operator analogue of a formula derived by Sommerfeld [So, p. 61].

Theorem 4.1 *We have for $t, x, y > 0$*

$$\begin{aligned} e^{-t\widetilde{D}_\theta^2}(x,y) &= (4\pi t)^{-1/2}\left(e^{-(x-y)^2/4t} + (I - 2P(\theta))e^{-(x+y)^2/4t}\right)e^{-tA^2} \\ &\quad + (\pi t)^{-1/2}(I - P(\theta))\int_0^\infty e^{-(x+y+z)^2/4t}\widetilde{A}(\theta)e^{\widetilde{A}(\theta)z - tA^2}\,dz, \end{aligned} \tag{4.1}$$

where $\widetilde{A}(\theta) := (I - P(\theta))A(I - P(\theta))$.

Proof The point is the convergence of the integral in (4.1). Note that $P(\theta)$ commutes with $|A|$ by (3.24) and the discreteness of A. Thus from (3.23), (2.2), and (3.25)

$$\begin{aligned} \widetilde{A}(\theta) &= \gamma P(\theta)\gamma^* A\gamma P(\theta)\gamma^* = -\gamma P(\theta)AP(\theta)\gamma^* \\ &= -a(\theta)\gamma|A|P(\theta)\gamma^* \\ &= -a(\theta)|A|(I - P(\theta)). \end{aligned}$$

In particular, $\widetilde{A}(\theta)$ commutes with $(I - P(\theta))$ so

$$\widetilde{A}(\theta)e^{\widetilde{A}(\theta)z - tA^2} = -a(\theta)|A|(I - P(\theta))e^{-a(\theta)|A|z - tA^2}.$$

Introducing $a_-(\theta) := -\min\{0, a(\theta)\} \in [0,1)$ we find

$$-a(\theta)|A|z \le a_-(\theta)\left(\frac{z^2}{4t} + A^2 t\right),$$

and

$$0 \le |A|(I - P(\theta))e^{\widetilde{A}(\theta)z - tA^2} \le |A|(I - P(\theta))e^{a_-(\theta)z^2/4t}e^{-(1-a_-(\theta))tA^2}. \tag{4.2}$$

This implies that the integral converges in the trace norm of $L^2(S_N)$.

Now pick $u \in C_0^\infty((0,\infty), L^2(S_N))$ and form

$$Q_t u(x) := \int_0^\infty Q_t(x,y)u(y)dy$$

where Q_t denotes the right hand side of (4.1). Then it is a routine matter to check that we have

$$\begin{aligned} &Q_t u \in C^1((0,\infty), \mathcal{D}(\widetilde{D}^*)) \cap C(\mathbf{R}_+, \mathcal{H}),\\ &(\partial_t + (\widetilde{D}^*)^2 Q_t u(x) = 0, \quad t, x > 0,\\ &\lim_{t\to 0+} Q_t u(x) = u(x). \end{aligned} \tag{4.3}$$

Hence it remains to verify the boundary conditions. Clearly,

$$P(\theta)Q_t(x,y) = (4\pi t)^{-1/2}\left(e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right)P(\theta)e^{-tA^2} \underset{x\to 0+}{\longrightarrow} 0,$$

and the same holds for $P(\theta)Q_t u(x)$ and $AP(\theta)Q_t u(x)$, by dominated convergence. This implies

$$Q_t u \in \mathcal{D}(\widetilde{D}_\theta).$$

We finally have to show that

$$\begin{aligned} 0 &= \lim_{x\to 0+} P(\theta)\gamma(\partial_x + A)Q_t u(x)\\ &= \lim_{x\to 0+} \gamma(I - P(\theta))(\partial_x + A)Q_t u(x)\\ &= \lim_{x\to 0+} \left\{\gamma(\partial_x + \widetilde{A}(\theta))(I - P(\theta))Q_t u(x) + \gamma(I - P(\theta))AP(\theta)Q_t u(x)\right\}\\ &= \lim_{x\to 0+} \gamma(\partial_x + \widetilde{A}(\theta))(I - P(\theta))Q_t u(x). \end{aligned}$$

An easy calculation shows that

$$\begin{aligned} &(\partial_x + \widetilde{A}(\theta))(I - P(\theta))Q_t(x,y)\\ =\; &(4\pi t)^{-1/2}\left\{e^{-(x-y)^2/4t}\left(\tfrac{y-x}{2t} + \widetilde{A}(\theta)\right) + e^{-(x+y)^2/4t}\left(-\tfrac{y+x}{2t} + \widetilde{A}(\theta)\right)\right\}(I - P(\theta))e^{-tA^2}\\ &-(\pi t)^{-1/2}e^{-(x+y)^2/4t}\widetilde{A}(\theta)(I - P(\theta))e^{-tA^2} \underset{x\to 0+}{\longrightarrow} 0. \end{aligned}$$

Then the proof is completed as above. □

Proof of Proposition 3.2 We propose to show that, for $u \in \mathcal{D}(D_\theta^k)$ with $k > m/2$, we have the estimate

$$|u(p)| \leq C(1-a_-(\theta))^{-k-1/2}(\|u\|_{L^2(S)} + \|D_\theta^k u\|_{L^2(S)}). \tag{4.4}$$

As explained in [L3] (cf. also [L2, Sec. 1.4]), this estimate implies the Hilbert–Schmidt property of suitable functions of D_θ and, in particular, the assertion of the proposition.

To prove (4.4), it is clearly enough to assume that $\operatorname{supp} u \subset U$, and we are reduced to proving the analogue of (4.4) for $\widetilde{D}_\theta$ if $\operatorname{supp} u \subset [0,1)$. To do so, we write for $q \in N$

$$\begin{aligned} u(x)(q) &= (\widetilde{D}_\theta^2+1)^{-j}(\widetilde{D}_\theta^2+1)^j u(x)(q) \\ &= \frac{1}{\Gamma(j)}\int_0^\infty e^{-t}t^{j-1}\int_0^\infty e^{-t\widetilde{D}_\theta^2}(x,y)(\widetilde{D}_\theta^2+1)^j u(y)\,dy\,dt(q). \end{aligned} \tag{4.5}$$

From the ellipticity of A we get for $k > (m-1)/2$

$$|u(x)(q)| \leq C_k\|(A^2+1)^k u(x)\|_{L^2(S_N)},$$

hence, with $j = k + 1/2 + \varepsilon$, $\varepsilon > 0$,

$$\begin{aligned} |u(x)(q)| \;\leq\; & C_k\int_0^\infty e^{-t}t^{k-1/2+\varepsilon}\int_0^\infty \|(A^2+1)^k e^{-t\widetilde{D}_\theta^2}(x,y)\,\cdot \\ & \cdot((\widetilde{D}_\theta^2+1)^{k+1/2+\varepsilon}u(y))\|_{L^2(S_N)}\,dy\,dt \end{aligned} \tag{4.6}$$

From (4.1) and (4.2) we derive the norm estimate

$$\begin{aligned} &\|(A^2+1)^k e^{-t\widetilde{D}_\theta^2}(x,y)\|_{\mathcal{L}(L^2(S_N))} \\ \leq\;& C_k(1-a_-(\theta))^{-k-1}\,t^{-k-1/2}\left(e^{-(x-y)^2/4t}+e^{-(x+y)^2/4t}\right). \end{aligned} \tag{4.7}$$

Using (4.7) and the Cauchy–Schwarz inequality in (4.6) we obtain the result. □

Proof of Lemma 3.3 An integration by parts gives

$$\begin{aligned} F_a(x) &= -\frac{1}{2a}\int_0^\infty \operatorname{erfc}(z)\frac{\partial}{\partial z}(e^{-2axz-x^2})\,dz \\ &= \frac{1}{2a}e^{-x^2} - \frac{1}{a\sqrt{\pi}}\int_0^\infty e^{-(2axz+x^2+z^2)}\,dz \\ &=: \frac{1}{2a}\Big(G(x)-\widetilde{F}_a(x)\Big). \end{aligned} \tag{4.8}$$

Clearly,

$$\mathcal{M}G(w) = \frac{1}{2}\Gamma(w/2). \tag{4.9}$$

To determine $\mathcal{M}\widetilde{F}_a$, we observe that

$$\widetilde{F}_a(x) = e^{-(1-a^2)x^2}\operatorname{erfc}(ax)$$

and derive a differential equation in a. In fact, for $\operatorname{Re} w > 0$, $0 < |a| < 1$,

$$\begin{aligned}
\frac{\partial}{\partial a}(1-a^2)^{w/2}\mathcal{M}\widetilde{F}_a(w) &= \frac{\partial}{\partial a}\int_0^\infty x^{w-1}e^{-x^2}\operatorname{erfc}(\frac{a}{\sqrt{1-a^2}}x)dx \\
&= -\frac{2}{\sqrt{\pi}}\int_0^\infty x^w e^{-x^2/(1-a^2)}dx\ (1-a^2)^{-3/2} \\
&= -\frac{1}{\sqrt{\pi}}\Gamma((w+1)/2)(1-a^2)^{w/2-1} \qquad (4.10)
\end{aligned}$$

The initial condition at $a = 0$ is

$$\mathcal{M}\widetilde{F}_0(w) = \frac{1}{2}\Gamma(w/2). \qquad (4.11)$$

The solution of this initial value problem is, for $|a| < 1$,

$$\mathcal{M}\widetilde{F}_a(w) = (1-a^2)^{-w/2}\Big(\frac{1}{2}\Gamma(w/2) - \frac{1}{\sqrt{\pi}}\Gamma((w+1)/2)\int_0^a (1-t^2)^{w/2-1}dt\Big),$$

hence

$$\begin{aligned}
\mathcal{M}F_a(w) &= \frac{1}{2a}\Big[(1-(1-a^2)^{-w/2})\frac{1}{2}\Gamma(w/2) \\
&\quad + \frac{1}{\sqrt{\pi}}(1-a^2)^{-w/2}\int_0^a (1-t^2)^{w/2-1}dt\ \Gamma((w+1)/2)\Big]. \qquad (4.12)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathcal{M}\widetilde{F}_1(w) &= \int_0^\infty x^{w-1}\operatorname{erfc}(x)dx \\
&= \frac{2}{w\sqrt{\pi}}\int_0^\infty x^w e^{-x^2}dx = \frac{1}{w\sqrt{\pi}}\Gamma((w+1)/2), \qquad (4.13)
\end{aligned}$$

thus

$$\mathcal{M}F_1(w) = \frac{1}{4}\Big[\Gamma(w/2) - \frac{2}{w\sqrt{\pi}}\Gamma((w+1)/2)\Big]. \qquad (4.14)$$

The poles and residues of $\mathcal{M}F_a$ can now easily be calculated in terms of the poles and residues of the Γ–function. □

We turn to the

Proof of Theorem 3.4 We choose $\phi \in C_0^\infty(-1,1)$ with $\phi = 1$ near 0. Then, from [G, Lemma 1.9.1] (cf. Remark 2) after Lemma 2.2) we obtain the asymptotic expansion, for $l = 0, 1$,

$$\operatorname{tr}_{L^2(S)}[(1-\psi_\phi)D_\theta^l e^{-tD_\theta^2}] \sim_{t\to 0+} \sum_{j=0}^\infty a_j(\phi;\theta,l)\ t^{j-m/2}. \qquad (4.15)$$

The coefficients can be computed locally in terms of the natural extension of D to the metric double, $\widetilde{M}$, of M, and ψ_ϕ.

Thus, since $e^{-t\widetilde{D}^2}$ can serve as a parametrix for D_θ^2 we obtain from [L3, Theorem 2.10 and Prop. 3.4] (note that what is called there the "singular elliptic estimate" was proved in (4.4)) that

$$\mathrm{tr}_{L^2(S)}[\psi_\phi D_\theta^l e^{-tD_\theta^2}] \sim_{t\to 0+} \mathrm{tr}_{\mathcal{H}}[\phi \widetilde{D}_\theta^l e^{-t\widetilde{D}_\theta^2}], \tag{4.16}$$

and it is enough to expand the right hand side of (4.16) for $l = 0, 1$.

Consider $l = 0$ first. We obtain from the explicit formula (4.1) and the Trace Lemma [BS1, Appendix] that

$$\begin{aligned}
\mathrm{tr}_{\mathcal{H}}[\phi e^{-t\widetilde{D}_\theta^2}] &= \int_0^\infty \phi(x)(4\pi t)^{-1/2}\mathrm{tr}_H[e^{-tA^2}]dx \\
&\quad + \int_0^\infty \phi(x) e^{-x^2/t}(4\pi t)^{-1/2}\mathrm{tr}_H[(I-2P(\theta))e^{-tA^2}]dx \\
&\quad -a(\theta)\int_0^\infty\int_0^\infty \phi(x) e^{-(2x+z)^2/4t}(\pi t)^{-1/2}\mathrm{tr}_H[P(\theta)|A|e^{-a(\theta)|A|z-tA^2}]dzdx \\
&=: \mathrm{I}(t)+\mathrm{II}(t)+\mathrm{III}(t).
\end{aligned} \tag{4.17}$$

Since A is elliptic on S_N we have for the first term

$$\mathrm{I}(t) \sim_{t\to 0+} (4\pi t)^{-1/2}\int_0^\infty \phi(x)dx \sum_{j=0}^\infty b_j(A^2)t^{j-(m-1)/2}. \tag{4.18}$$

Next, as an easy consequence of (3.23) we see that

$$\mathrm{II}(t) \equiv 0. \tag{4.19}$$

For $\mathrm{III}(t)$, we write, with

$$c(\lambda) := \dim\ker(|A|-\lambda) = 2\,\mathrm{tr}_{\ker(|A|-\lambda)}(P(\theta)),$$

$$\begin{aligned}
\mathrm{III}(t) &= -a(\theta)\int_0^\infty\int_0^\infty \phi(x\sqrt{t})\frac{1}{\sqrt{\pi}}e^{-(x+z)^2}\sum_{\lambda\in\mathrm{spec}\,|A|\setminus\{0\}} c(\lambda)\sqrt{t}\lambda e^{-2a(\theta)\sqrt{t}\lambda z - t\lambda^2}dzdx \\
&\sim_{t\to 0+} -\frac{a(\theta)}{2}\int_0^\infty \mathrm{erfc}(z)\sum_{\lambda\in\mathrm{spec}\,|A|\setminus\{0\}} c(\lambda)\sqrt{t}\lambda e^{-2a(\theta)\sqrt{t}\lambda z-t\lambda^2}dz \\
&= -\frac{a(\theta)}{2}\sum_{\lambda\in\mathrm{spec}\,|A|\setminus\{0\}} c(\lambda)F_{a(\theta)}(\sqrt{t}\lambda) \\
&= -\frac{a(\theta)}{2}\sum_{\lambda\in\mathrm{spec}\,|A|\setminus\{0\}} c(\lambda)\frac{1}{2\pi i}\int_{\mathrm{Re}\,w=c>>0} t^{-w/2}\lambda^{-w}\mathcal{M}F_{a(\theta)}(w)dw \\
&= -\frac{a(\theta)}{4\pi i}\int_{\mathrm{Re}\,w=c} t^{-w/2}\zeta_{A^2}(w/2)\mathcal{M}F_{a(\theta)}(w)dw.
\end{aligned} \tag{4.20}$$

We now collect the various contributions. First, replacing ϕ by ϕ_ε, $\phi_\varepsilon(x) := \phi(x/\varepsilon)$, and letting $\varepsilon \to 0$ we obtain from (4.15) and (4.18) a contribution

$$\widetilde{\mathrm{I}}(t) \sim_{t\to 0+} \sum_{j=0}^\infty a_j(\theta,0)\, t^{j-m/2}, \tag{4.21a}$$

where

$$a_j(\theta,0)=\int_M \tilde u_j(\theta,0),$$

with $\tilde u_j$ a local density computed for the natural extension of D to the double, $\widetilde M$, of M.

The remaining contribution, $\mathrm{III}(t)$, can be evaluated by the Residue Theorem since the integrand decays in vertical strips with bounded real part (by Lemma 3.3, Lemma 2.2, and (2.20)). Thus we find (using e.g. the description of the singularities of ζ_{A^2} in [BL, Lemma 2.1])

$$\begin{aligned}\mathrm{III}(t) \quad &= \quad -\frac{a(\theta)}{2}\sum_{w\in\mathbf{C}}\mathrm{Res}_1(t^{-w/2}\zeta_{A^2}(w/2)\mathcal{M}F_{a(\theta)}(w))\\ \sim_{t\to 0+} \quad & \frac{a(\theta)}{2}\sum_{j=0}^{\infty}t^{j-n/2}\Big\{\log t\,\mathrm{Res}_1\zeta_{A^2}(n/2-j)\mathrm{Res}_1\mathcal{M}F_{a(\theta)}(n-2j)\\ &-2\mathrm{Res}_1\zeta_{A^2}(n/2-j)\mathrm{Res}_0\mathcal{M}F_{a(\theta)}(n-2j)\Big\}\\ &-\frac{a(\theta)}{2}\sum_{j=0}^{\infty}t^{j/2}\mathrm{Res}_0\zeta_{A^2}(-j/2)\mathrm{Res}_1\mathcal{M}F_{a(\theta)}(-j).\end{aligned}\tag{4.21b}$$

From this, we can read off our assertions on the structure of the coefficients. First of all, the leading contribution comes from (4.21a) only, as $a_0t^{-m/2}$, and so is computed as in the compact case. Next, we observe that ζ_{A^2} has no poles at the points $n/2-j$ for $j\ge n/2$ if n is even. If n is odd, however, the log–terms occur as can be seen from Lemma 3.3. The coefficients of the terms in the first sum in (4.21b) are computed from local densities on N, whereas those in the second sum are, in general, nonlocal.

Next we consider the case $l=1$. In view of (4.15) and the previous analysis it is enough to expand

$$\int_0^\infty \phi(x)\mathrm{tr}_H[\gamma(\partial_x+A)e^{-t\widetilde D_\theta^2}(x,x)]dx =: \tilde{\mathrm{I}}(t)+\tilde{\mathrm{II}}(t)+\widetilde{\mathrm{III}}(t),\tag{4.22}$$

numbering again the contributions according to the three terms in (4.1). In view of (3.23), (3.24), (2.2), and (3.48) we find

$$\begin{aligned}&\mathrm{tr}_H[\gamma e^{-tA^2}]=\mathrm{tr}_H[\gamma P(\theta)e^{-tA^2}]=0\\ &\mathrm{tr}_H[\gamma P(\theta)|A|e^{-a(\theta)|A|z-tA^2}]=0,\end{aligned}\tag{4.23}$$

and thus

$$\mathrm{tr}_H[\gamma\partial_x e^{-t\widetilde D_\theta^2}(x,x)]=0.\tag{4.24}$$

Again from (2.2) we conclude

$$\mathrm{tr}_H[\gamma A e^{-tA^2}]=0,\tag{4.25}$$

which implies

$$\tilde{\mathrm{I}}(t)\equiv 0.\tag{4.26}$$

Furthermore,

$$\tilde{\mathrm{II}}(t) \quad = \quad (4\pi t)^{-1/2}\int_0^\infty \phi(x)e^{-x^2/t}\mathrm{tr}_H[\gamma A(I-2P(\theta))e^{-tA^2}]dx$$

$$
\begin{aligned}
&= (4\pi)^{-1/2}\int_0^\infty \phi(x\sqrt{t})e^{-x^2}\,dx\,\mathrm{tr}_H[\gamma A(I-2P(\theta))e^{-tA^2}]\\
&\sim_{t\to 0+} \frac{1}{4}\mathrm{tr}_H[\gamma A(I-2P(\theta))e^{-tA^2}]\\
&= -\frac{1}{2}\mathrm{tr}_H[\gamma AP(\theta)e^{-tA^2}]. \qquad (4.27)
\end{aligned}
$$

Finally, we note that, using again (3.23) and (2.2),

$$
\mathrm{tr}_H[\gamma A(I-P(\theta))\widetilde{A}(\theta)e^{\widetilde{A}(\theta)z-tA^2}] = a(\theta)\mathrm{tr}_H[\gamma AP(\theta)|A|e^{-a(\theta)|A|z-tA^2}], \quad (4.28)
$$

and so, as in (4.20), with $d(\lambda) := \mathrm{tr}_{\ker(|A|-\lambda)}[\gamma AP(\theta)]$,

$$
\begin{aligned}
\widetilde{\mathrm{III}}(t) &= a(\theta)\int_0^\infty\int_0^\infty \phi(x)e^{-(2x+z)^2/4t}(\pi t)^{-1/2}\mathrm{tr}_H[\gamma AP(\theta)|A|e^{-a(\theta)|A|z-tA^2}]dzdx\\
&= a(\theta)\int_0^\infty\int_0^\infty \phi(x\sqrt{t})\frac{2}{\sqrt{\pi}}e^{-(x+z)^2}\sum_{\lambda\in\mathrm{spec}\,|A|\setminus\{0\}} d(\lambda)\sqrt{t}\lambda e^{-2a(\theta)\sqrt{t}\lambda z-t\lambda^2}dzdx\\
&\sim_{t\to 0+} a(\theta)\sum_{\lambda\in\mathrm{spec}\,|A|\setminus\{0\}} d(\lambda)F_{a(\theta)}(\sqrt{t}\lambda)\\
&= \frac{a(\theta)}{2\pi i}\int_{\mathrm{Re}\,w=c} t^{-w/2}\eta(A,\gamma AP(\theta);w-1)\mathcal{M}F_{a(\theta)}(w)dw. \qquad (4.29)
\end{aligned}
$$

Combining our computations, we see that the terms local on $\widetilde{M}$ protrude from (4.15) as before.

We obtain the second contribution from (4.27). However, since $P(\theta)$ is a pseudodifferential operator we now have to employ the general expansion theorem for pseudodifferential operators (2.20) [GrSe, Theorem 2.7]. Namely

$$
\begin{aligned}
\widetilde{\mathrm{II}}(t) &\sim_{t\to 0+} -\frac{1}{2}\mathrm{tr}_H[\gamma AP(\theta)e^{-tA^2}]\\
&\sim_{t\to 0+} \sum_{j=0}^\infty c_j^1(\theta,1)t^{(j-m)/2} + \sum_{j=0}^\infty\Big(b_j^1(\theta,1)t^j\log t + d_j^1(\theta,1)t^j\Big). \qquad (4.30\mathrm{a})
\end{aligned}
$$

Here, b_j^1, c_j^1 are integrals of local densities over N whereas the $d_j^1(\theta,1)$ are, in general, nonlocal spectral invariants on N.

For the third contribution, we use again the estimate (2.14) with $B = \gamma AP(\theta)$ (stemming from the fact that $P(\theta)$ is a pseudodifferential operator) to obtain

$$
\widetilde{\mathrm{III}}(t) \sim_{t\to 0+} a(\theta)\sum_{w\in\mathbf{C}}\mathrm{Res}_1\Big(t^{-w/2}\eta(A,\gamma AP(\theta);w-1)\mathcal{M}F_{a(\theta)}(w)\Big).
$$

From the expansion (4.30a) and Lemma 2.1 one derives that $\eta(A,\gamma AP(\theta);w)$ is meromorphic in $\mathbf{C}$ with simple poles at the points $n-k, k\in\mathbf{Z}_+$. Furthermore, the residues of the poles are integrals of local densities over N. Thus

$$
\begin{aligned}
\widetilde{\mathrm{III}}(t) \sim_{t\to 0+} & -\frac{a(\theta)}{2}\sum_{j=0}^\infty t^{j/2}\log t\ \mathrm{Res}_1(\eta(A,\gamma AP(\theta);-j-1)\mathrm{Res}_1\mathcal{M}F_{a(\theta)}(-j)\\
&+a(\theta)\sum_{j=0}^\infty t^{(j-m)/2}\mathrm{Res}_1(\eta(A,\gamma AP(\theta);m-j-1)\mathrm{Res}_0\mathcal{M}F_{a(\theta)}(m-j)\\
&+a(\theta)\sum_{j=0}^\infty t^{j/2}\mathrm{Res}_0(\eta(A,\gamma AP(\theta);-j-1)\mathrm{Res}_1\mathcal{M}F_{a(\theta)}(-j). \qquad (4.30\mathrm{b})
\end{aligned}
$$

The coefficients in the first and second sum are again local, like c_j^1 in (4.30a), whereas those in the second sum are not.

It remains to compute the contribution to $t^{-1/2}$ from (4.30a,b). Using Lemma 2.1, it turns out to be equal to

$$\begin{aligned}
&-\frac{1}{2}a_{-1/2,0}(A,\gamma AP(\theta)) + a(\theta)\mathrm{Res}_0\mathcal{M}F_{a(\theta)}(1)\mathrm{Res}_1\eta(A,\gamma AP(\theta);0)\\
=\;& (-\frac{\sqrt{\pi}}{4} + a(\theta)\mathcal{M}F_{a(\theta)}(1))\mathrm{Res}_1\eta(A,\gamma AP(\theta);0)\\
=\;& (-\frac{\sqrt{\pi}}{4} + a(\theta)\mathcal{M}F_{a(\theta)}(1))\mathrm{Res}_1\eta(A,\gamma(\operatorname{sgn} A)P(\theta);-1)\\
=\;& (-\frac{\sqrt{\pi}}{4} + a(\theta)\mathcal{M}F_{a(\theta)}(1))\mathrm{res}\,(\gamma(\operatorname{sgn} A)P(\theta)),
\end{aligned}$$

using (2.31) in the last step. □

Proof of Theorem 3.5 We choose $\tilde\phi \in C_0^\infty(-1,1)$ with $\tilde\phi = 1$ in a neighborhood of $\operatorname{supp}\phi$, with ϕ from (3.32). Then, for $u \in \mathcal{D}_0$, one easily computes (writing $\tilde\psi = \psi_{\tilde\phi}$)

$$\begin{aligned}
\check D_\theta\tilde\psi u &= \Phi_\theta^*\Phi^*\gamma(\partial_x + A)\Phi\Phi_\theta\tilde\psi u &(4.31)\\
&=: \Phi^* i\gamma\phi' T(\theta)\Phi\tilde\psi u + \Phi^*\Phi_\theta^*\gamma A\Phi_\theta\Phi\tilde\psi u + v, &(4.32)
\end{aligned}$$

with v independent of θ, hence

$$\frac{d}{d\theta}\check D_\theta\tilde\psi u = \Phi_\theta^*\Phi^* i\gamma(\phi' T'(\theta) - 2\phi T'(\theta)A)\Phi\Phi_\theta\tilde\psi u$$

and

$$\mathrm{tr}_{L^2(S)}[\frac{d}{d\theta}\check D_\theta e^{-t\check D_\theta^2}] = i\mathrm{tr}_{L^2(S)}[\gamma(\phi' T'(\theta) - 2\phi T'(\theta)A)e^{-tD_\theta^2}\tilde\psi]. \tag{4.33}$$

We can argue as in the proof of Theorem 3.4 to replace $e^{-tD_\theta^2}$ by $e^{-t\tilde D_\theta^2}$, i.e.

$$\begin{aligned}
&i\mathrm{tr}_{L^2(S)}[\gamma(\phi' T'(\theta) - 2\phi T'(\theta)A)e^{-tD_\theta^2}\tilde\psi]\\
\sim_{t\to 0+}\;& i\mathrm{tr}_{L^2(S)}[\gamma(\phi' T'(\theta) - 2\phi T'(\theta)A)e^{-t\tilde D_\theta^2}\tilde\psi]. &(4.34)
\end{aligned}$$

Again as in the proof of Theorem 3.4, we obtain twice three terms from plugging the kernel (4.1) in (4.34).

We start with

$$\begin{aligned}
&i\mathrm{tr}_{L^2(S)}[\gamma\phi' T'(\theta)e^{-t\tilde D_\theta^2}]\\
=\;& i\int_0^\infty \phi'(x)\mathrm{tr}_H[\gamma T'(\theta)e^{-t\tilde D_\theta^2}(x,x)]dx\\
=:\;& \mathrm{I}(t) + \mathrm{II}(t) + \mathrm{III}(t). &(4.35)
\end{aligned}$$

We find

$$\begin{aligned}
\mathrm{I}(t) &= i(4\pi t)^{-1/2}\int_0^\infty \phi'(x)dx\ \mathrm{tr}_H[\gamma T'(\theta)e^{-tA^2}]\\
&= -i(4\pi t)^{-1/2}\mathrm{tr}_H[\gamma T'(\theta)e^{-tA^2}]. &(4.36)
\end{aligned}$$

Since ϕ' is supported away from zero, it is easy to see that

$$\mathrm{II}(t) \sim_{t\to 0+} \mathrm{III}(t) \sim_{t\to 0+} 0. \tag{4.37}$$

The second contribution is

$$\begin{aligned} & -2i\mathrm{tr}_{L^2(S)}[\gamma\phi T'(\theta)Ae^{-t\widetilde{D}_\theta^2}] \\ = \;& -2i\int_0^\infty \phi(x)\mathrm{tr}_H[\gamma T'(\theta)Ae^{-t\widetilde{D}_\theta^2}(x,x)]dx \\ =: \;& \widetilde{\mathrm{I}}(t) + \widetilde{\mathrm{II}}(t) + \widetilde{\mathrm{III}}(t). \end{aligned} \tag{4.38}$$

We compute

$$\widetilde{\mathrm{I}}(t) = -2i(4\pi t)^{-1/2}\int_0^\infty \phi(x)\mathrm{tr}_H[\gamma T'(\theta)Ae^{-tA^2}]dx = 0, \tag{4.39}$$

since γ commutes with $T'(\theta)$ but anticommutes with A. Next we see that

$$\begin{aligned} \widetilde{\mathrm{II}}(t) \quad &= \quad -2i(4\pi t)^{-1/2}\int_0^\infty \phi(x)e^{-x^2/t}\,\mathrm{tr}\left[\gamma T'(\theta)A(I-2P(\theta))e^{-tA^2}\right]dx \\ &\sim_{t\to 0+} \quad i\,\mathrm{tr}\left[\gamma T'(\theta)AP(\theta)e^{-tA^2}\right]. \end{aligned} \tag{4.40}$$

Finally, with $d(\lambda) = \mathrm{tr}_{\ker(|A|-\lambda)}[\gamma T'(\theta)AP(\theta)e^{-tA^2}]$,

$$\begin{aligned} \widetilde{\mathrm{III}}(t) \quad &\sim_{t\to 0+} \quad -2ia(\theta)\sum_{\lambda\in\mathrm{spec}\,|A|\setminus\{0\}} d(\lambda)\sqrt{t}\lambda\int_0^\infty e^{-2a(\theta)\lambda\sqrt{t}z-t\lambda^2}\mathrm{erfc}(z)dz \\ &= \quad -2ia(\theta)\sum_{\lambda\in\mathrm{spec}\,|A|\setminus\{0\}} d(\lambda)F_{a(\theta)}(\sqrt{t}\lambda) \\ &= \quad -\frac{a(\theta)}{\pi}\int_{\mathrm{Re}\,w=c} t^{-w/2}\eta(A,\gamma T'(\theta)AP(\theta);w-1)\mathcal{M}F_{a(\theta)}(w)dw. \end{aligned} \tag{4.41}$$

The existence of the asymptotic expansion hence follows from our assumptions, Lemma 3.3, and (4.36), (4.40), (4.41). Consequently, we obtain with (2.12a), (2.10), (2.31), and (2.20):

$$\begin{aligned} a_{-1/2,1}(\check{D}_\theta, \frac{d}{d\theta}\check{D}_\theta) \;&=\; -(4\pi)^{-1/2}a_{0,1}(A,\gamma iT'(\theta) \\ &=\; \frac{1}{4\sqrt{\pi}}\mathrm{res}\,(\gamma iT'(\theta)), \\ a_{-1/2,0}(\check{D}_\theta, \frac{d}{d\theta}\check{D}_\theta) \;&=\; -(4\pi)^{-1/2}a_{00}(A,\gamma iT'(\theta)) \\ &\quad +a_{-1/2,0}(A,\gamma iT'(\theta)AP(\theta)) \\ &\quad -2a(\theta)\mathcal{M}F_{a(\theta)}(1)\mathrm{Res}_1\eta(A,\gamma iT'(\theta)AP(\theta);0) \\ &=\; -(4\pi)^{-1/2}a_{00}(A,\gamma iT'(\theta)) \\ &\quad +\Big(\frac{\sqrt{\pi}}{2} - 2a(\theta)\mathcal{M}F_{a(\theta)}(1)\Big)\mathrm{res}\,(\gamma iT'(\theta)(\mathrm{sgn}\,A)P(\theta)). \end{aligned}$$

In view of (2.27) we reach the conclusion. □

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