Finite-size corrections of an integrable chain with alternating spins

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Abstract. In this paper we calculate the finite-size corrections of an anisotropic integrable spin chain, consisting of spins $s = 1$ and $s = \frac{1}{2}$. The calculations are done in two regions of the phase diagram with respect to the two couplings $\bar{c}$ and $\tilde{c}$. In case of conformal invariance we obtain the final answer for the ground state and its lowest excitations, which generalizes earlier results.

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1. Introduction

In 1992 de Vega and Woynarovich constructed the first example of a spin chain with alternating spins of the values $s = \frac{1}{2}$ and $s = 1$ [1] on the basis of the well-known XXZ($\frac{1}{2}$) model. We call this model XXZ($\frac{1}{2}, 1$). Later on a lot of interesting generalizations have been presented [2, 3, 4]. After de Vega *et al* [5, 6] we have studied the XXZ($\frac{1}{2}, 1$) model in two subsequent publications [7, 8]. In our last paper [8] we determined the ground state for different values of the two couplings $\bar{c}$ and $\tilde{c}$ (for the details see section 3 of that paper). Disregarding two singular lines we have found four regions in the $(\bar{c}, \tilde{c})$-plane which can be divided into two classes. The division is made with respect to the occurance of finite Fermi zones for Bethe ansatz roots. The two regions with infinite fermi zones only are well studied [1, 7] in the framework of Bethe ansatz. On that basis we consider the finite-size corrections for the ground state and its lowest excitations using standard techniques [9, 10, 11]. It is remarkable that they allow to obtain an explicit answer only in the conformally invariant cases, which are contained in the two regions considered. The results can easily be compared with the predictions of conformal invariance.

The paper is organized as follows. Definitions are reviewed in section 2. In section 3 we calculate the finite-size corrections for both couplings negative. The same is done in section 4 for positive couplings. Here it was necessary to set $\bar{c} = \tilde{c}$ to obtain explicit answers. Section 5 contains the interpretation of the results and our conclusions.

2. Description of the model

We consider the Hamiltonian of a spin chain of length $2N$ with N even

$$\mathcal{H}(\gamma) = \bar{c}\bar{\mathcal{H}}(\gamma) + \tilde{c}\tilde{\mathcal{H}}(\gamma). \tag{2.1}$$

The two Hamiltonians can (implicitly) be found in paper [1], they both contain a two-site and a three-site coupling part. Their explicit expressions are rather lengthy and do not provide any further insights. They include a XXZ-type anisotropy parametrized by $e^{i\gamma}$, we restrict ourselves to $0 < \gamma < \pi/2$. The isotropic limit XXX($\frac{1}{2}, 1$) is contained in [2]. The two real coupling constants $\bar{c}$ and $\tilde{c}$ dominate the qualitative behaviour of the model.

The Bethe ansatz equations (BAE) determining the solution of the model are

$$\left(\frac{\sinh(\lambda_j + i\frac{\gamma}{2})}{\sinh(\lambda_j - i\frac{\gamma}{2})}\frac{\sinh(\lambda_j + i\gamma)}{\sinh(\lambda_j - i\gamma)}\right)^N = -\prod_{k=1}^{M}\frac{\sinh(\lambda_j - \lambda_k + i\gamma)}{\sinh(\lambda_j - \lambda_k - i\gamma)}, \qquad j = 1\ldots M. \tag{2.2}$$

One can express energy, momentum and spin projection in terms of BAE roots λ_j:

$$E = \bar{c}\bar{E} + \tilde{c}\tilde{E},$$
$$\bar{E} = -\sum_{j=1}^{M}\frac{2\sin\gamma}{\cosh 2\lambda_j - \cos\gamma},$$

$$\tilde{E} = -\sum_{j=1}^{M} \frac{2\sin 2\gamma}{\cosh 2\lambda_j - \cos 2\gamma}, \tag{2.3}$$

$$P = \frac{i}{2}\sum_{j=1}^{M} \left\{ \ln\left(\frac{\sinh(\lambda_j + i\frac{\gamma}{2})}{\sinh(\lambda_j - i\frac{\gamma}{2})}\right) + \ln\left(\frac{\sinh(\lambda_j + i\gamma)}{\sinh(\lambda_j - i\gamma)}\right) \right\}, \tag{2.4}$$

$$S_z = \frac{3N}{2} - M. \tag{2.5}$$

We have defined energy and momentum to vanish for the ferromagnetic state.

3. Calculation of finite-size corrections for negative couplings

In section 3 of paper [8] we have carried out a detailed analysis of the thermodynamic Bethe ansatz equations (TBAE) at zero temperature and obtained the ground state.

We found a large antiferromagnetic region in the $(\bar{c}, \tilde{c})$-plane (depending on γ) where the ground state is formed by roots with imaginary parts $\frac{\pi}{2}$, the so-called $(1, -)$ strings. The Fourier transform of their density is given by [7]

$$\hat{\rho}_0(p) = \frac{1 + 2\cosh(p\gamma/2)}{2\cosh(p(\pi - \gamma)/2)}. \tag{3.1}$$

Depending on the signs of $\tilde{c}$ and $\bar{c}$ the region is described by the connection of three parts:

a)

$$\tilde{c} \le 0, \bar{c} \le 0;$$

b)

$$\begin{array}{lll} \tilde{c} < 0, \bar{c} > 0 & & \\ \dfrac{\bar{c}}{|\tilde{c}|} \le \dfrac{1}{2\cos\tilde{\gamma}} & \text{for} & 0 < \gamma \le \dfrac{2\pi}{5} \\ \dfrac{\bar{c}}{|\tilde{c}|} \le 2\cos\tilde{\gamma} & \text{for} & \dfrac{2\pi}{5} \le \gamma < \dfrac{\pi}{2}; \end{array}$$

c)

$$\begin{array}{lll} \tilde{c} > 0, \bar{c} < 0 & & \\ \dfrac{|\bar{c}|}{\tilde{c}} \ge \dfrac{8\cos^3\tilde{\gamma}}{4\cos^2\tilde{\gamma} - 1} & \text{for} & 0 < \gamma \le \dfrac{\pi}{3} \\ \dfrac{|\bar{c}|}{\tilde{c}} \ge \dfrac{2}{\cos\tilde{\gamma}} & \text{for} & \dfrac{\pi}{3} \le \gamma < \dfrac{\pi}{2}. \end{array} \tag{3.2}$$

Here for shortness we have introduced

$$\tilde{\gamma} = \frac{\pi\gamma}{2(\pi-\gamma)}. \tag{3.3}$$

We shall now calculate the finite-size corrections for the ground state and its excitations. In paper [7] the structure of excitations in the framework of BAE roots was obtained for $\tilde{c} < 0, \bar{c} < 0$. Our results immediately apply to the whole region (3.2), because we had to ensure only that the ground state consists of $(1,-)$ strings which follows from TBAE.

Because we are interested in the lowest excitations only, we disregard the bound states [7] and consider those excitations given by holes in the ground-state distribution which are located right (or left) from the real parts of all roots. The number of those holes we call H^+ (H^-). We follow the standard techniques developed in [9] and [10].

For transparency we employ the notations of [11] as much as possible. We decompose

$$\sigma_N = \rho_0^{(1)} + \rho_0^{(2)} + \Delta\sigma_N \tag{3.4}$$

where the upper index describes the two terms on the RHS of formula (3.1). The basic equations are then

$$\begin{aligned}\frac{\Delta E_N}{2N} \equiv e_N = \bar{c}\pi \int_{-\infty}^{\infty} d\lambda \rho_0^{(1)}(\lambda) \left\{ \frac{1}{N} \sum_k \delta(\lambda-\lambda_k) - \sigma_N(\lambda) \right\} \\ +\tilde{c}\pi \int_{-\infty}^{\infty} d\lambda \rho_0^{(2)}(\lambda) \left\{ \frac{1}{N} \sum_k \delta(\lambda-\lambda_k) - \sigma_N(\lambda) \right\}\end{aligned} \tag{3.5}$$

and

$$\Delta\sigma_N(\lambda) = -\int_{-\infty}^{\infty} d\mu \bar{p}(\lambda-\mu) \left\{ \frac{1}{N} \sum_k \delta(\lambda-\lambda_k) - \sigma_N(\lambda) \right\} \tag{3.6}$$

where the Fourier transform of the kernel $\bar{p}(\lambda)$ is given by

$$\bar{P}(\omega) = -\frac{\sinh((\pi/2-\gamma)\omega)}{2\sinh(\omega\gamma/2)\cosh(\omega(\pi/2-\gamma/2))}. \tag{3.7}$$

The summation on the RHS of (3.5) and (3.6) is carried out over the real parts of all the roots (without the holes). Using Euler-MacLaurin formula equations (3.5) and (3.6) are rewritten as usual, e.g.

$$\begin{aligned}&\sigma_N(\lambda) - \rho_0^{(1)}(\lambda) - \rho_0^{(2)}(\lambda) = \\ &\quad \int_{\Lambda^+}^{\infty} d\mu \sigma_N(\mu)\bar{p}(\lambda-\mu) - \frac{1}{2N}\bar{p}(\lambda-\Lambda^+) + \frac{1}{12N^2}\frac{1}{\sigma_N(\Lambda^+)}\bar{p}(\lambda-\Lambda^+) \\ &\quad \left(+ \int_{-\infty}^{\Lambda^-} d\mu \sigma_N(\mu)\bar{p}(\lambda-\mu) - \frac{1}{2N}\bar{p}(\lambda-\Lambda^-) - \frac{1}{12N^2}\frac{1}{\sigma_N(\Lambda^-)}\bar{p}(\lambda-\Lambda^-) \right)\end{aligned} \tag{3.8}$$

Here Λ^+ (Λ^-) is the real part of the largest (smallest) root. For $\lambda \geq \Lambda^+$ the part in round brackets can be omitted and equation (3.8) after a shift converts into a standard

Wiener-Hopf problem to be solved. In the expression for ΔE_N both parts have to be kept, so we need the solution for $\lambda \geq \Lambda^+$ and $\lambda \leq \Lambda^-$, which are simply related (but not equal) by symmetry.

Now we can use the factorization in paper [11] after observing

$$\bar{K}(\omega, \gamma) = K(\omega, \pi - \gamma) \tag{3.9}$$

where we have marked all values of our problem by a bar. That implies

$$\bar{G}_+(\omega, \gamma) = G_+(\omega, \pi - \gamma). \tag{3.10}$$

Further

$$\bar{F}_+(\omega) = \frac{e^{-\pi\Lambda^+/(\pi-\gamma)}}{\pi - i\omega(\pi-\gamma)}(1 + 2\cos\tilde{\gamma}) \tag{3.11}$$

and hence

$$\bar{Q}_+(\omega) = \frac{\bar{G}_+(i\pi/(\pi-\gamma))e^{-\pi\Lambda^+/(\pi-\gamma)}(1+2\cos\tilde{\gamma})}{\pi - i\omega(\pi-\gamma)}. \tag{3.12}$$

Next we must determine by normalization the value of the integral

$$\int_{\Lambda^+}^{\infty} \sigma_N(\lambda) d\lambda = z_N(\infty) - z_N(\Lambda^+).$$

After a thorough analysis we found for our case that the relation

$$\frac{H}{2} = \frac{H^+ + H^-}{2} = \left[\nu S + \frac{1}{2}\right] \tag{3.13}$$

holds where $\nu = \frac{\gamma}{\pi}$. Nevertheless we shall not claim formula (3.13) for all possible states [10], especially we expect effects like those described in [12] for higher excitations. Then we have

$$\begin{aligned} z_N(\pm\infty) - z_N(\Lambda^\pm) = & \pm \frac{1}{N}\left(\frac{1}{2} + \nu S_z \pm \frac{H^+ - H^-}{2}\right) \\ \equiv & \pm \frac{1}{N}\left(\frac{1}{2} \pm \Delta^\pm\right) \end{aligned} \tag{3.14}$$

yielding the important equation

$$\frac{\bar{G}_+(i\pi/(\pi-\gamma))e^{-\pi\Lambda^+/(\pi-\gamma)}(1+2\cos\tilde{\gamma})}{\pi} = \frac{1}{2N} - \frac{i\bar{g}_1}{12N^2\sigma_N(\Lambda^+)} + \frac{1}{N}\frac{1}{\sqrt{2\nu}}\Delta^+. \tag{3.15}$$

The other normalization equation is obviously

$$\sigma_N(\Lambda^+) = \frac{\bar{g}_1^2}{24N^2\sigma_N(\Lambda^+)} + \frac{i\bar{g}_1}{2N} + \frac{\bar{G}_+(i\pi/(\pi-\gamma))}{\pi-\gamma}e^{-\pi\Lambda^+/(\pi-\gamma)}(1+2\cos\tilde{\gamma}). \tag{3.16}$$

Now we can proceed in the usual way keeping in mind the changes arising especially from the last two equations.

$$\begin{aligned} \frac{\Delta E_N}{2N} = & -\frac{\pi}{\pi-\gamma}(\bar{c} + 2\tilde{c}\cos\tilde{\gamma})\bar{G}_+\left(\frac{i\pi}{\pi-\gamma}\right)\left[\bar{P}\left(\frac{i\pi}{\pi-\gamma}\right) + \bar{Q}_+\left(\frac{i\pi}{\pi-\gamma}\right)\right]e^{-\pi\Lambda^+/(\pi-\gamma)} \\ & + \left(\Lambda^+ \leftrightarrow \Lambda^-\right). \end{aligned} \tag{3.17}$$

After some algebra using equations (3.15) and (3.16) this turns out as

$$\frac{\Delta E_N}{2N} = -\frac{\pi^2}{\pi-\gamma}\frac{(\bar{c}+2\tilde{c}\cos\tilde{\gamma})}{1+2\cos\tilde{\gamma}} \times \\ \times \left\{ \left(-\frac{1}{24N^2}+\frac{1}{4N^2\nu}(\Delta^+)^2\right) + \left(-\frac{1}{24N^2}+\frac{1}{4N^2\nu}(\Delta^-)^2\right) \right\}. \tag{3.18}$$

Finally, for further interpretation we put it in the form

$$\frac{\Delta E_N}{2N} = -\frac{2\bar{c}+4\tilde{c}\cos\tilde{\gamma}}{1+2\cos\tilde{\gamma}}\frac{\pi}{\pi-\gamma}\left\{-\frac{\pi}{6}\frac{1}{4N^2}+\frac{2\pi}{4N^2}\left(\frac{S_z^2\nu}{2}+\frac{\Delta^2}{2\nu}\right)\right\}. \tag{3.19}$$

with $\Delta = (H^+ - H^-)/2$ as an integer number.

The interpretation of our result will be given in section 5. We stress once more, that to obtain equation (3.19) it was not necessary to put $\bar{c} = \tilde{c}$. The coupling constants are only constrained to stay in region (3.2).

4. Calculation of finite-size corrections for positive couplings

Now we consider region $\bar{c} > 0, \tilde{c} > 0$ and rely on the analysis of paper [1].

The ground state is given by two densities, $\sigma_N^{(1/2)}(\lambda)$ for the real roots and $\sigma_N^{(1)}(\lambda)$ for the real parts of the $(2,+)$ strings. One has

$$\sigma_\infty^{(1/2)}(\lambda) = \sigma_\infty^{(1)}(\lambda) = \frac{1}{2\gamma\cosh(\pi\lambda/\gamma)} \equiv s(\lambda). \tag{4.1}$$

The physical excitations are holes in those distributions. As in section 3 we consider only holes situated right (or left) from all roots. With the usual technique and the results of [1] we have obtained after some lengthy but straightforward calculations the basic system for the density corrections

$$\begin{aligned} \Delta\sigma_N^{(1/2)}(\lambda) = &- \int_{-\infty}^{\infty} d\mu\, s(\lambda-\mu)\left\{\frac{1}{N}\sum_{j=1}^{M_1}\delta(\mu-\xi_j) - \sigma_N^{(1)}(\mu)\right\} \\ \Delta\sigma_N^{(1)}(\lambda) = &- \int_{-\infty}^{\infty} d\mu\, s(\lambda-\mu)\left\{\frac{1}{N}\sum_{i=1}^{M_{1/2}}\delta(\mu-\lambda_i) - \sigma_N^{(1/2)}(\mu)\right\} \\ &- \int_{-\infty}^{\infty} d\mu\, r(\lambda-\mu)\left\{\frac{1}{N}\sum_{j=1}^{M_1}\delta(\mu-\xi_j) - \sigma_N^{(1)}(\mu)\right\}. \end{aligned} \tag{4.2}$$

We have denoted the real roots by λ_i (their number is $M_{1/2}$) and the real parts of the strings by ξ_j (number M_1). The function $r(\lambda)$ is given via its Fourier transform

$$R(\omega) = \frac{\sinh(\omega(\pi-3\gamma)/2)}{2\sinh(\omega(\pi-2\gamma)/2)\cosh(\omega\gamma/2)}. \tag{4.3}$$

The energy correction takes the form

$$\frac{\Delta E_N}{2N} = -\pi\bar{c}\int_{-\infty}^{\infty} d\lambda s(\lambda)\left\{\frac{1}{N}\sum_{i=1}^{M_{1/2}}\delta(\lambda-\lambda_i)-\sigma_N^{(1/2)}(\lambda)\right\}$$
$$-\pi\tilde{c}\int_{-\infty}^{\infty} d\lambda s(\lambda)\left\{\frac{1}{N}\sum_{j=1}^{M_1}\delta(\lambda-\xi_j)-\sigma_N^{(1)}(\lambda)\right\}. \tag{4.4}$$

Once more we shall follow [11]. The maximal (minimal) real roots we call $\Lambda_{1/2}^{\pm}$ and for the strings we use $\Lambda_1^{\pm}$ respectively. Instead of one $C(\omega)$ we have now $C_1(\omega)$ and $C_{1/2}(\omega)$ generalized in an obvious way. The main mathematical problem is the factorization of a matrix kernel

$$(1-K(\omega))^{-1} = G_+(\omega)G_-(\omega) \qquad \text{with}$$
$$G_-(\omega) = G_+(-\omega)^T \tag{4.5}$$

(see [5]) and

$$K(\omega) = \begin{pmatrix} 0 & S(\omega)e^{-i\omega\left(\Lambda_1^+-\Lambda_{1/2}^+\right)} \\ S(\omega)e^{i\omega\left(\Lambda_1^+-\Lambda_{1/2}^+\right)} & R(\omega)\end{pmatrix}. \tag{4.6}$$

G_+ is now a matrix function and G_+^T stands for its transposition. The two component vector $Q_+(\omega)$ is

$$Q_+(\omega) = \frac{G_+(i\pi/\gamma)^T}{\pi-i\omega\gamma}\begin{pmatrix} e^{-\pi\Lambda_{1/2}^+/\gamma} \\ e^{-\pi\Lambda_1^+/\gamma}\end{pmatrix}. \tag{4.7}$$

As usual we define the constant matrices G_1 and G_2 by

$$G_+(\omega) \overset{|\omega|\to\infty}{\longrightarrow} 1 + G_1\frac{1}{\omega} + G_2\frac{1}{\omega^2} + \mathcal{O}\left(\frac{1}{\omega^3}\right) \tag{4.8}$$

and as before one has

$$G_2 = \frac{1}{2}G_1^2. \tag{4.9}$$

The two component vector $P(\omega)$ is then

$$P(\omega) = \begin{pmatrix} -\frac{1}{2N} - \frac{i\omega}{12N^2\sigma_N^{(1/2)}(\Lambda_{1/2}^+)} \\ -\frac{1}{2N} - \frac{i\omega}{12N^2\sigma_N^{(1)}(\Lambda_1^+)}\end{pmatrix} + G_1\begin{pmatrix} \frac{i}{12N^2\sigma_N^{(1/2)}(\Lambda_{1/2}^+)} \\ \frac{i}{12N^2\sigma_N^{(1)}(\Lambda_1^+)}\end{pmatrix} \tag{4.10}$$

and therefore the shifted densities are expressed in the form

$$\begin{pmatrix} X_{1/2}^+(\omega) \\ X_1^+(\omega)\end{pmatrix} = \begin{pmatrix} C_{1/2}(\omega) \\ C_1(\omega)\end{pmatrix} + G_+(\omega)\left[P(\omega)+Q_+(\omega)\right]. \tag{4.11}$$

Now it is necessary to find the analogue of equation (3.14) for the two counting functions. Here it would be necessary to consider different cases depending on the fractions of

$\nu S_z/N$ or $2\nu S_z/N$. From our experience we know that the result of the finite-size corrections does not depend on those fractions, while relations like (3.13) obviously do. Being interested only in the former we shall proceed as easy as possible and consider only the case with vanishing fractions.

$$z_N^{(1/2)}(\pm\infty) - z_N^{(1/2)}(\Lambda_{1/2}^{\pm}) = \pm\frac{1}{N}\left(\frac{1}{2} - \nu S_z \pm H_{1/2}^{\pm}\right), \tag{4.12}$$

$$z_N^{(1)}(\pm\infty) - z_N^{(1)}(\Lambda_1^{\pm}) = \pm\frac{1}{N}\left(\frac{1}{2} - 2\nu S_z \pm H_1^{\pm}\right). \tag{4.13}$$

Easy counting leads to expressions for the numbers of the holes

$$\begin{aligned} H_1 &= 2S_z \\ H_{1/2} &= 2S_z + 2M_1 - N. \end{aligned} \tag{4.14}$$

We expect their modifications for non-vanishing fractions. We stress that both numbers are even.

Equation (3.15) is now more complicated

$$\frac{G_+(i\pi/\gamma)^T}{\pi}\begin{pmatrix} e^{-\pi\Lambda_{1/2}^+/\gamma} \\ e^{-\pi\Lambda_1^+/\gamma} \end{pmatrix} = G_+^{-1}(0)B^+ + \begin{pmatrix} \frac{1}{2N} \\ \frac{1}{2N} \end{pmatrix} - iG_1 \begin{pmatrix} \frac{1}{12N^2\sigma_N^{(1/2)}(\Lambda_{1/2}^+)} \\ \frac{1}{12N^2\sigma_N^{(1)}(\Lambda_1^+)} \end{pmatrix} \tag{4.15}$$

with the definitions

$$B^{\pm} = \begin{pmatrix} B_1^{\pm} \\ B_2^{\pm} \end{pmatrix} = \frac{1}{N}\begin{pmatrix} -\nu S_z \pm H_{1/2}^{\pm} \\ -2\nu S_z \pm H_1^{\pm} \end{pmatrix} = \frac{1}{N}\begin{pmatrix} S_z - \nu S_z + M_1 - \frac{1}{2} \pm \Delta^{(1/2)} \\ S_z - 2\nu S_z \pm \Delta^{(1)} \end{pmatrix} \tag{4.16}$$

and

$$\Delta^{(i)} = \frac{H_i^+ - H_i^-}{2}. \tag{4.17}$$

The other normalization condition is obviously

$$\begin{pmatrix} \sigma_N^{(1/2)}(\Lambda_{1/2}^+) \\ \sigma_N^{(1)}(\Lambda_1^+) \end{pmatrix} = \frac{G_1^2}{2}\begin{pmatrix} \frac{1}{12N^2\sigma_N^{(1/2)}(\Lambda_{1/2}^+)} \\ \frac{1}{12N^2\sigma_N^{(1)}(\Lambda_1^+)} \end{pmatrix} + iG_1\begin{pmatrix} \frac{1}{2N} \\ \frac{1}{2N} \end{pmatrix} + \frac{G_+(i\pi/\gamma)^T}{\gamma}\begin{pmatrix} e^{-\pi\Lambda_{1/2}^+/\gamma} \\ e^{-\pi\Lambda_1^+/\gamma} \end{pmatrix}. \tag{4.18}$$

After combining equations (4.18) and (4.15) we obtain from equation (4.4)

$$\begin{aligned} \frac{\Delta E_N}{2N} = \frac{\pi}{\gamma}\begin{pmatrix} \bar{c}e^{\pi\Lambda_{1/2}^+/\gamma} \\ \tilde{c}e^{\pi\Lambda_1^+/\gamma} \end{pmatrix}^T G_+\left(\frac{i\pi}{\gamma}\right)\Bigg[-\frac{1}{2}\begin{pmatrix} \frac{1}{2N} \\ \frac{1}{2N} \end{pmatrix} + \left(\frac{1}{2}iG_1 + \frac{\pi}{\gamma}\right)\begin{pmatrix} \frac{1}{12N^2\sigma_N^{(1/2)}(\Lambda_{1/2}^+)} \\ \frac{1}{12N^2\sigma_N^{(1)}(\Lambda_1^+)} \end{pmatrix} \\ +\frac{1}{2}G_+^{-1}(0)B^+\Bigg] + (\Lambda^+ \leftrightarrow \Lambda^-, B^+ \leftrightarrow B^-). \end{aligned} \tag{4.19}$$

This result is valid for any positive $\bar{c}$ and $\tilde{c}$. As in paper [5] no further progress can be made unless the factorization is explicitly known which is not the case up to now.

For $\bar{c} = \tilde{c} = c$ (conformal invariance) the problem simplifies and the final answer can be obtained.

After some lengthy calculations one arrives at

$$\frac{\Delta E_N}{2N} = \frac{\pi^2 c}{\gamma}\left[-\frac{1}{12N^2} + \frac{1}{2}\left(\left(B_1^+ - B_2^+\right)^2 + B_1^+ B_2^+ - (B_2^+)^2 \frac{\pi - 3\gamma}{2(\pi - \gamma)}\right)\right] + (B^+ \leftrightarrow B^-) \tag{4.20}$$

which can be brought into the form

$$\frac{\Delta E_N}{2N} = \frac{2\pi c}{\gamma}\left\{-\frac{\pi}{6}\frac{2}{4N^2} + \frac{2\pi}{4N^2}\left[\frac{1}{4}(1-2\nu)S_z^2 + \frac{1}{4}\left(H_{1/2} - \frac{H_1}{2}\right)^2 \right.\right.$$
$$\left.\left. +(\Delta^{(1/2)})^2 - \Delta^{(1/2)}\Delta^{(1)} + \frac{1}{2}(\Delta^{(1)})^2 \frac{1-\nu}{1-2\nu}\right]\right\}. \tag{4.21}$$

For $\bar{c} \neq \tilde{c}$ we expect the result to be much more complicated but of the same order $1/N^2$.

5. Conclusions

In sections 3 and 4 we have determined the finite-size corrections of our model for two different cases. Equation (3.19) gives the result for region (3.2), while equation (4.21) is valid for $\bar{c} = \tilde{c} = c > 0$. In both cases we have conformal invariance. That seems to be the reason why during the calculations a lot of terms cancel each other and the result is considerably simple.

From equation (3.19) we can read off

$$v_s = -\frac{2\pi}{\pi - \gamma}\frac{\bar{c} + 2\tilde{c}\cos\tilde{\gamma}}{1 + 2\cos\tilde{\gamma}} > 0 \tag{5.1}$$

for the speed of sound and the value of 1 for the central charge. Formula (5.1) generalizes the former result obtained for equal (but negative) couplings [8]. From equation (4.21) it follows

$$v_s = \frac{2c\pi}{\gamma} \tag{5.2}$$

and the central charge equals 2 [1].

For completeness we mention the heat capacities per site at low temperature given by

$$C = \frac{c_v T \pi}{3 v_s} \tag{5.3}$$

where we have denoted the central charge by c_v to avoid confusions. Therefore

$$C = -\frac{1 + 2\cos\tilde{\gamma}}{\bar{c} + 2\tilde{c}\cos\tilde{\gamma}}\frac{(\pi - \gamma)T}{6} \tag{5.4}$$

and

$$C = \frac{\gamma T}{3c} \tag{5.5}$$

in agreement with former results [8]. Formula (5.4) generalizes our calculations for $\bar{c} = \tilde{c}$.

The dimensions of the primary operators follow from (3.19)

$$x_n = \frac{S_z^2 \nu}{2} + \frac{\Delta^2}{2\nu} \tag{5.6}$$

for negative couplings. It is interesting to compare this with the result for the XXZ($\frac{1}{2}$) model, where ν is simply replaced by $1 - \nu$. Formula (5.6) has to be understood in the way that for general excited states (arbitrary holes and complex roots) S_z and Δ are replaced by more complicated integer numbers [10].

For positive couplings

$$x_n = \frac{1}{4}(1-2\nu)S_z^2 + \frac{1}{4}\left(H_{1/2} - \frac{H_1}{2}\right)^2 + \left(\Delta^{(1/2)}\right)^2 - \Delta^{(1/2)}\Delta^{(1)} + \frac{1}{2}\left(\Delta^{(1)}\right)^2 \frac{1-\nu}{1-2\nu}. \tag{5.7}$$

When $\nu \to 0$ the first two terms agree with paper [4] while the other terms are connected with the asymmetry of the state which was not considered there. The dimension of a general primary operator depends on four integer numbers. The second of them measures an asymmetry between the number of holes among real roots or strings respectively. Once more, for more complicated states the integers in equation (5.7) are replaced by other ones depending on the concrete structure of the state. We mention that relation (5.7) can be "diagonalized" to resemble the expression of two models both of central charge 1.

$$x_n = \frac{1}{2}\frac{(1-2\nu)}{2}S_z^2 + \frac{1}{2}\frac{\left(H_{1/2} - H_1/2\right)^2}{2} + \frac{1}{2}2\Delta_1^2 + \frac{1}{2}\frac{2}{1-2\nu}\Delta_2^2 \tag{5.8}$$

with

$$\begin{aligned} \Delta_1 &= \Delta^{(1/2)} - \frac{\Delta^{(1)}}{2}, \\ \Delta_2 &= \frac{\Delta^{(1)}}{2}. \end{aligned} \tag{5.9}$$

Expression (5.8) becomes even more symmetric if one remembers the first equation of (4.14) and the definition (4.17). Then twice a certain number of holes is linked with its appropriate asymmetry.

Finally, we shall determine the magnetic suszeptibilities per site at zero temperature and vanishing field from our finite-size results. That can be done, because the states we have considered include those with minimal energy for a given S_z (magnetization) [13]. Differentiating twice the energy with respect to S_z gives the inverse susceptibility.

Hence

$$\chi = \frac{\pi - \gamma}{4\pi\gamma} \left(-\frac{1 + 2\cos\tilde{\gamma}}{\bar{c} + 2\tilde{c}\cos\tilde{\gamma}} \right) = \frac{1}{v_s}\frac{1}{2\gamma} \tag{5.10}$$

and

$$\chi = \frac{1}{2c\pi}\frac{1}{\pi - 2\gamma} = \frac{1}{v_s}\frac{1}{\pi - 2\gamma} \tag{5.11}$$

for the two cases respectively in agreement with earlier results [4, 5, 8]. Expression (5.10) for general couplings in region (3.2) had not been derived before.

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