Free massive fermions inside the quantum discrete sine-Gordon model

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Abstract

We extend the notion of space shifts introduced in [FV3] for certain quantum light cone lattice equations of sine-Gordon type at root of unity (e.g. [FV1][FV2][BKP][BBR]). As a result we obtain a compatibility equation for the roots of central elements within the algebra of observables (also called current algebra). The equation which is obtained by exponentiating these roots is exactly the evolution equation for the "classical background" as described in [BBR].

As an application for the introduced constructions, a one to one correspondence between a special case of the quantum light cone lattice equations of sine-Gordon type and free massive fermions on a lattice as constructed in [DV] is derived.

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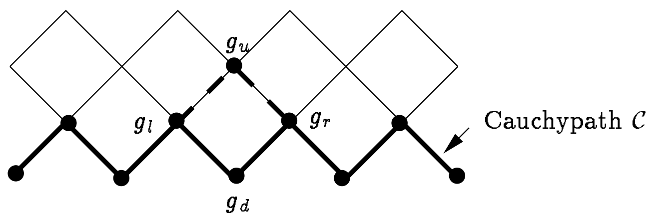
1 Introduction

Let us consider a class of integrable lattice systems defined by an evolution equation of the following type:

$$g_u = V'(g_l - g_r) + g_d = 0, \tag{1}$$

where u, d, l and r denote up, down, left and right, respectively and $V'(x)$ is the derivative of $V(x) : \mathbb{R} \to \mathbb{R}$. If we start with quasiperiodic initial data g_i along a Cauchy path $\mathcal{C}$ on a light cone lattice the local evolution given by (1) will determine the function g on the whole lattice.



Integrable lattice systems of the above type had been thoroughly discussed e.g. in [BRST, NCP, S, V, FV2, CN, EK])
In [EK] it was shown that it is possible to derive the above equation as an equation of motion from an explicitly given action. Moreover - using covariant phase space techniques - it is possible to derive the symplectic structure belonging to the above model via variation of this action. As a consequence one obtains unique Poisson commutation rules for the above variables which were first stated in [FV2]. The so obtained Poisson commutation rules are not local in space, while the induced commutation rules for the difference variables $g_l - g_r$ (also called current variables) are. Therefore the difference variables were looked at as referring to physical observables and consequently a lot of attention was drawn to them. (see e.g. [BKP, FV1, BBR, CN]).

Nevertheless it may be worthwile, also for physical reasons, to study the above variables despite their unpleasant nonultralocal commutation relations. An important step in this direction was made in the work of Faddeev and Volkov [FV3], were quantized models of the above variables were studied . Moreover in this work shifts in space direction as automorphisms on the quantized variables were constructed under the assumption of special monodromies.

We will extend their definition in the root of unity case to more general monodromies and derive as a consequence of this extension in connection with the corresponding quantum evolution a compatibility equation for the roots of central elements within the algebra of the current variables. The equation which is obtained by exponentiating these roots is exactly the evolution equation for the "classical background" as described in [BBR].

As a concrete example for the above construction of space and time shifts we will finally restrict ourselves to a subalgebra of the algebra generated by the quantized quasiperiodic variables g. The generators of this subalgebra satisfy canonical anticommutation rules (CAR). The quantum analog of the evolution in (1) reduced to this subalgebra for the case of $V(x)$ being the potential of the sine-Gordon model will be isomorphic to the evolution of free massive fermions on the lattice viewed as a special case of the massive Thirring model constructed by Destri and de Vega [DV] (see also [TS] and others cited there).

As a result we will may be understand more about the true nature of the famous sine-Gordon - massive Thirring model equivalence (see e.g. [C, KM]).

2 The phase space

2.1 Vertex variables

A (spatially) periodic light cone lattice L_{2p} (see also fig. 1) with period $2p$ may be viewed as $L/\mathbb{Z}$, where $\mathbb{Z}$ acts on the infinite light cone lattice L by shifts by $2p$ in space-like direction (cf. fig. 1). A quasi-periodic field is a mapping

$$g : L \to \mathbb{R}$$

with

$$g_{t,i+2p} - g_{t,i} = g_{t,i+2p+2} - g_{t,i+2} \quad \forall i \in \mathbb{Z},$$

i.e., there are two (space independent) monodromies $m_t^{(1)}, m_t^{(2)}$ defined by

$$m_t^{(i)} = g_{t,2p+2k+1-i} - g_{t,2k+1-i} \tag{2}$$

for an arbitrary $k \in \mathbb{Z}$.

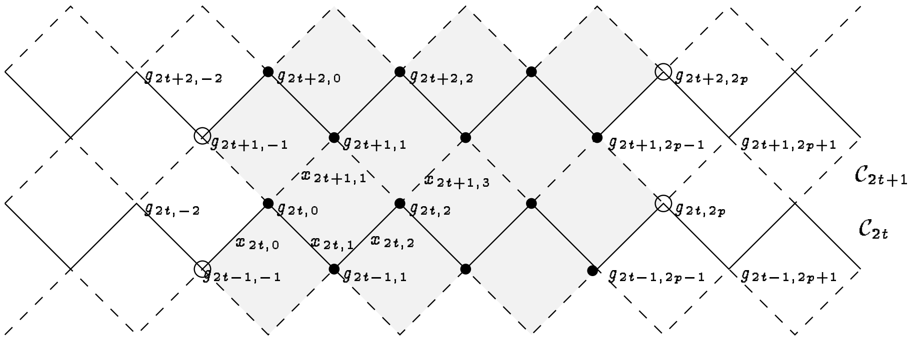


Figure 1

We define an evolution of the following form, where the indices t, k will be adapted to Figure 1, i.e. $k - t$ is chosen to be even:

$$g_{t+1,k+1} = V'(g_{t,k} - g_{t,k+2}) + g_{t-1,k+1} \qquad m_{t+1}^{(i)} = const \quad m_t^{(i)} = const \tag{3}$$

For a set of initial values $\mathcal{I}_{2t}^g = \left\{(g_{2t-1,k})_{k\in\{-1,\dots 2p-1\}}, (g_{2t,k})_{k\in\{0,\dots 2p\}}\right\}$ or $\mathcal{I}_{2t+1}^g = \left\{(g_{2t,k})_{k\in\{0,\dots 2p\}}, (g_{2t+1,k})_{k\in\{-1,\dots 2p-1\}}\right\}$, ($t \in \mathbb{Z}$ fixed) along an elementary Cauchy zig zag (see fig. 1) the above evolution equations define quasiperiodic field values at all other times. A quasiperiodic field g obtained in such a way will be called a solution to the above evolution (3). Hence the quasiperiodic initial values $\mathcal{I}_t$ can be interpreted as a (global) coordinate chart $g_t : \mathcal{P} \to \mathbb{R}^{2p+2}$ on the set of all solutions $\mathcal{P} = \{g|_{solution}\}$ via the identification:

$$g_{t,i}(g) = g_{t,i} \in \mathbb{R}.$$

The set of all solutions to a given evolution shall be called covariant phase space $\mathcal{P}$. It is now possible to define a translational invariant action on the set of quasiperiodic fields whose variation gives the above field equations as well as a time independent, translational invariant symplectic structure [EK] which in the turn defines the following Poisson relations:

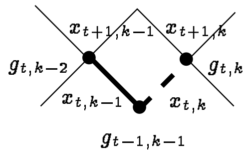
$$\begin{array}{rcll} \{g_{t,i}, g_{t,k}\} & = & 0, & \text{if } i-k \text{ even} \\ \{g_{t,i}, g_{\tilde t,k}\} & = & 1, & \text{if } i-k \text{ odd, } i<k,\ |i-k|<2p, \\ \{g_{t,i}, m_t^{(k)}\} & = & 0 & (i-k \text{ odd}), \\ \{g_{t,i}, m_{\tilde t}^{(k)}\} & = & 2 & (i-k \text{ even}), \\ \{m_t^{(1)}, m_{\tilde t}^{(2)}\} & = & 0 & \end{array} \tag{4}$$

where $|t-\tilde{t}| = 1$.
Since the quasiperiodic variables $\{g_{t,k}\}$ are associated to the vertices of the Minkowski lattice L they will be shortly called *vertex* variables. Since the monodromies as dynamical variables are by virtue of the evolution equations time independent we will skip from now on their time index t and keep in mind that $m^{(1)} = m^{(1)}_{2t}$ and $m^{(2)} = m^{(2)}_{2t-1}$.

2.2 Edge variables

Define the following variables for $(k-t)$ even:

$$\begin{aligned} x_{t,k} &:= g_{t,k} + g_{t-1,k-1} \\ x_{t,k-1} &:= g_{t-1,k-1} + g_{t,k-2} \end{aligned} \tag{5}$$



The variables $\{x_{t,k}\}$ are associated to the edges of the Minkowski lattice L. They will be shortly called *edge* variables.
A set of initial vertex values $\mathcal{I}^g_{2t+1}$ or $\mathcal{I}^g_{2t}$, respectively defines the initial edge values $\mathcal{I}^x_{2t+1} = \{(x_{2t+1,k})_{k\in\{0,\dots 2p\}}\}$ or $\mathcal{I}^x_{2t} = (x_{2t,k})_{k\in\{0,\dots 2p\}}\}$.
The edge variables are still quasiperiodic. Their monodromy is the sum of the two monodromies of the vertex variables :

$$x_{t,k+2p} = x_{t,k} + \underbrace{m^{(1)} + m^{(2)}}_{:=m^x}$$

The induced Poisson commutation rules are

$$\begin{aligned} \{x_{t,k}, x_{t,k+n}\} &= 2 \qquad n \in \{1\dots 2p-1\} \quad mod\, 2p \qquad (6) \\ \{x_{t,k}, m^x_t\} &= 4 \qquad (7) \end{aligned}$$

One obtains evolution equations in the edge variables by virtue of definition (5) and the vertex evolution equations (3), they read as:

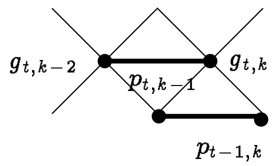
$$\begin{aligned} x_{t+1,k} &= V'(x_{t,k-1} - x_{t,k}) + x_{t,k} \qquad (8) \\ x_{t+1,k-1} &= V'(x_{t,k-1} - x_{t,k}) + x_{t,k-1} \qquad (9) \\ m^x &= const. \qquad (10) \end{aligned}$$

We will call these equations edge evolution equations.

2.3 Face variables

Define the following fields ($k-t$ even):

$$p_{t,k-1} := g_{t,k-2} - g_{t,k}$$



The difference variables $\{p_{t,k}\}$ are associated to the faces of the Minkowski lattice L. They will be shortly called *face* variables. A set of initial vertex values $\mathcal{I}^g_{2t+1}$ or $\mathcal{I}^g_{2t}$, respectively defines the initial face values $\mathcal{I}^p_{2t} = \{(p_{2t-1,2k})_{k\in\{0..p-1\}}, (p_{2t,2k+1})_{k\in\{0..p-1\}}\}$ or $\mathcal{I}^p_{2t+1} = \{(p_{2t,2k+1})_{k\in\{0..p-1\}}, (p_{2t+1,2k})_{k\in\{0..p-1\}}\}$.
Note that in particular $p_{t,k-1} = x_{t,k-1} - x_{t,k} = x_{t+1,k-1} - x_{t+1,k}$ ($k-t$ even). The above variables are periodic since the monodromies cancel out:

$$p_{t,2p+k-1} = p_{t,k-1} = g_{t,k-2} + m^{(i)} - (g_{t,k} + m^{(i)}).$$

The induced Poisson relations between these variables are ultralocal in space:

$$\begin{aligned} \{p_{t,j}, p_{\tilde{t},j+1}\} &= 2 & (11)\\ \{p_{t,j}, p_{t,j+1+k}\} &= 0 \quad \text{for} \quad k \in \{1\ldots 2p-3\} & (12) \end{aligned}$$

where $|t-\tilde{t}| = 1$.
The evolution equations in terms of the face variables read as

$$p_{t+1,k} = V'(p_{t,k-1}) - V'(p_{t,k+1}) + p_{t-1,k}.$$

We will call these equations face evolution equations.
The introduction of the above variables refers to a reduction of phase space as explained in [EK].

2.4 Relations to the sine-Gordon model

Let the halfperiod p of the lattice be even, i.e. $p = 2s$. In [EK] it was shown that the action, which describes the above dynamical systems is invariant under a redefinition of $g \rightsquigarrow -g$ along every second diagonal of the Minkowski lattice, i.e. all the above structure is preserved under such a redefinition.
Choosing

$$V'(x) = -i\, ln\left(\frac{1+ke^{ix}}{k+e^{ix}}\right) \tag{13}$$

and projecting the evolution to the torus S^{2p+2}:

$$e^{ig_{t+1,k+1}} = \frac{k + e^{i(g_{t,k}-g_{t,k+2})}}{1+ke^{i(g_{t,k}-g_{t,k+2})}} e^{ig_{t-1,k+1}}. \tag{14}$$

then with the above redefinition the vertex equation (3) turns into the famous Hirota equation [H], while the face equations are commonly known as doubly discrete sine-Gordon equations (see e.g.[BKP, BBR, FV2]). Without such a redefinition along the diagonals, but still with the special potential given in (13), the above model is related to the doubly discrete mKdV model [CN].

Due to its relation to the sine-Gordon model (see e.g. [IK]) , the to the torus projected vertex, edge and face equations with the potential (13) will be shortly called of sine-Gordon type. In the forthcoming we will study evolutions of sine-Gordon type.

3 Quantization of the models

3.1 General outline

Fix an initial Cauchy path as e.g. $\mathcal{C}_{2T}$ (see fig. 1).
Our quantization scheme follows the common procedure to substitute the canonical variables $I_{2T} := \{(e^{ig_{2T-1,2k+1}})_{k\in\{-1,\ldots p-1\}}\{(e^{ig_{2T,2k}})_{k\in\{0,\ldots p\}}\}$ as functions on phase space by unitary operators $\mathbf{I}_{2T+1} := \{(G_{2T+1,2k+1})_{k\in\{-1,\ldots p-1\}} \in U(\mathcal{H}),$

$(G_{2T,2k})_{k\in\{0,\dots p\}} \in U(\mathcal{H})\}$ obeying Weyl commutation relations, i.e. we search for a bijection $Q : \mathcal{F}(\mathcal{P}) \to U(\mathcal{H})$, $Q(e^{ig_{t,k}}) =: G_{t,k}$ with the properties

$$\begin{aligned} Q(const) &= const\, \mathbb{1} \\ Q(e^{ig_{t,k}} e^{ig_{\tilde{t},j}}) &= Q(e^{ig_{t,k}}) Q(e^{ig_{\tilde{t},j}})\, e^{i\, phase} \end{aligned}$$

such that

$$\begin{aligned} [G_{t,i}, G_{t,k}] &= 0, && \text{if } i-k \text{ even} \\ G_{t,i} G_{\tilde{t},k} &= q^{-\frac{m}{2}} G_{\tilde{t},k} G_{t,i}, && \text{if } i-k \text{ odd},\ i<k,\ |i-k|<2p, \\ [G_{t,i}, {M^{(k)}}_t] &= 0 && (i-k \text{ odd}), \\ G_{t,i} {M^{(k)}}_{\tilde{t}} &= q^{-m} M^{(k)}_t G_{t,i} && (i-k \text{ even}), \\ [{M^{(1)}}_t, {M^{(2)}}_{\tilde{t}}] &= 0 \end{aligned} \tag{15}$$

where $|t-\tilde{t}| = 1$, $t, \tilde{t} \in \{2T, 2T-1\}$, $q \in S^1 \subset \mathbb{C}$, $m \in \mathbb{N}$,

$$M^{(1)} := G_{2T,2p} G^{-1}_{2T,0} \qquad M^{(2)} := G_{2T-1,2p-1} G^{-1}_{2T-1,-1} \qquad G_{t,k+2p} := M^{(i)} G_{t,k},$$

$\mathbb{1} = (G_{t,k})^0$ is the identity in $U(\mathcal{H})$ and the product in $U(\mathcal{H})$ is given by the composition of operators. For our purpose q will always be chosen as a root of unity, i.e. $q = e^{\frac{2\pi i}{N}} \in S^1 \subset \mathbb{C}$.
Let $\mathcal{A}(G_{2T})$ be the algebra of Laurent polynomials in the generators $I_{2T} = \{(G_{2T,2k})_{k\in\{0,\dots p\}}, (G_{2T-1,2k+1})_{k\in\{-1,\dots p-1\}}\}$ ($T \in \mathbb{Z}$ fixed). Note that up to now the "quantization" map Q shall be only defined for the canonical variables $e^{ig_{t,k}}, t \in \{2T, 2T-1\}$ and, modulo a phase factor, on products of these. It shall not be specified for other functions on phase space $\mathcal{P}$ like for example on the time one evolved variables:

$$g_{2T+1,2k-1} = g_{2T+1,2k-1}(g_{2T,2k-2}, g_{2T,2k}, g_{2T-1,2k-1}).$$

The idea is that we will implicitly define a quantization for these functions by defining an automorphism $\mathbf{E}_{t,k-1} : \mathcal{A}(G_{2T}) \to \mathcal{A}(G_{2T})$, such that :

$$Q(e^{ig_{t+1,k-1}}) := \mathbf{E}_{t,k-1}(G_{t-1,k-1}) = \mathbf{E}_{t,k-1}(Q(e^{ig_{t-1,k-1}})),$$

where $e^{ig_{t+1,k}}$is given by the classical evolution.
The automorphism $\mathbf{E}_{t,k-1}$ will be very much adapted to our specific model.

We will neither discuss wether and how it would in general be possible to find such an automorphism nor will we be concerned with a discussion of the above quantization procedure with respect to completeness (when e.g. extending by linearity), uniqueness, connection to other quantizatons etc.
Let us proceed with an explicit construction of $\mathbf{E}_{t,k-1}$. In accordance with the classical definition (5) define edge operators ($k-t$ even):

$$X_{t,k} := G_{t,k} G_{t-1,k-1} \tag{16}$$

$$X_{t,k-1} := G_{t-1,k-1} G_{t,k-2} \tag{17}$$

$$\Rightarrow M_X := X_{t,2p} X^{-1}_{t,0} = q^{-m} M^{(1)} M^{(2)} \tag{18}$$

Let $\mathcal{A}(X_{2T})$ be the algebra of Laurent polynomials in the generators $I^X_{2T} = (X_{2T,k})_{k\in\{0..2p\}}$. Define face operators

$$P_{t,k-1} := q^{-\frac{m}{2}} G^{-1}_{t,k} G_{t,k-2} \tag{19}$$

Note that $P_{t,k-1} = X^{-1}_{t,k} X_{t,k-1} = X^{-1}_{t+1,k} X^{-1}_{t+1,k-1}$.
Let $\mathcal{A}(P_{2T})$ be the algebra of Laurent polynomials in the generators $I^P_{2T} = \{(P_{2T-1,2k})_{k\in\{0..p-1\}}, (P_{2T,2k+1})_{k\in\{0..p-1\}}\}$.
Clearly the above construction can also be done for the Cauchypath $\mathcal{C}_{2T-1}$, therefore the distinction between even and odd times will be sometimes omitted.

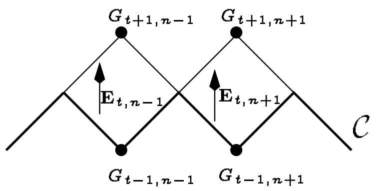
3.2 Almost hamiltonian quantum evolution

Let $R_k(P_{t,n-1})$ be a nonvanishing Laurent polynomial in the face operator $P_{t,n-1} := q^{-\frac{m}{2}} G_{t,n}^{-1} G_{t,n-2}$, i.e. $R_k(P_{t,n-1}) \in \mathcal{A}(G_t)$ which depends on a parameter $k \in [0,1)$. Let $e^{i\xi_k(P^B_{t,n-1})} \in S^1 \subset \mathbb{C}$ be a number which depends on a central element $P^B_{t,n-1} \in \mathcal{A}(G_t)$ (also called Casimir) and the same parameter $k \in [0,1)$, where $B \in \mathbb{N}$. Define recursively ($n-t$ even)

$$
\begin{aligned}
G_{t+1,n-1} := \mathbf{E}_{t,n-1}(G_{t-1,n-1}) \quad &:= \quad R_k(P_{t,n-1}) G_{t-1,n-1} R_k(P_{t,n-1})^{-1} e^{i\xi_k(P^B_{t,n-1})} \\
&= \quad \frac{R_k(P_{t,n-1})}{R_k(q^m P_{t,n-1})} G_{t-1,n-1} e^{i\xi_k(P^B_{t,n-1})} \qquad (20)
\end{aligned}
$$

$$
\mathbf{E}_{t,n-1}(G_{\tilde{t},j}) := G_{\tilde{t},j} \quad \text{for } j \neq n-1 \; mod\, 2p; \quad \tilde{t} \in \{t, t-1\} \qquad (21)
$$

i.e. the automorphisms $\mathbf{E}_{t,n-1}$ act nontrivially only on operators which are associated to the $(t,n-1)$'th face of the corresponding Cauchyzig zag $\mathcal{C}_t$.



Define

$$
\mathbf{E}_{2t} := \prod_{n=0}^{p-1} \mathbf{E}_{2t,2n+1} \qquad \mathbf{E}_{2t-1} := \prod_{n=0}^{p-1} \mathbf{E}_{2t-1,2n},
$$

which is well defined since the corresponding automorphisms $\mathbf{E}_{t,n-1}$ commute. $\mathbf{E}_t$ evolves all operators associated to a Cauchypath $\mathcal{C}_t$ one time step further and by the definition of the evolution automorphism:

$$
\mathbf{E}_t(G_{t-1,n-1}) = E_{t,n-1}(G_{t-1,n-1}).
$$

Using the commutation relations in (15) and the periodicity of the face operators, it follows immediately that the above automorphisms preserves the monodromies. We will call such automorphisms *almost hamiltonian evolution automorphisms.*

The induced evolution on the subalgebras $\mathcal{A}(P_T) \subset \mathcal{A}(X_T) \subset \mathcal{A}(G_T)$ reads:

$$
\begin{aligned}
X_{t+1,n} \quad &:= \quad G_{t,n} G_{t+1,n-1} = G_{t,n} R_k(P_{t,n-1}) G_{t-1,n-1} R_k(P_{t,n-1})^{-1} e^{i\xi_k(P^B_{t,n-1})} \qquad (22) \\
&= \quad R_k(P_{t,n-1}) X_{t,n} R_k(P_{t,n-1})^{-1} e^{i\xi_k(P^B_{t,n-1})} = \frac{R_k(P_{t,n-1})}{R_k(q^m P_{t,n-1})} X_{t,n} e^{i\xi_k(P^B_{t,n-1})} \\
X_{t+1,n-1} \quad &:= \quad G_{t+1,n-1} G_{t,n-2} = R_k(P_{t,n-1}) G_{t-1,n-1} R_k(P_{t,n-1})^{-1} G_{t,n-2} e^{i\xi_k(P^B_{t,n-1})} \\
&= \quad R_k(P_{t,n-1}) X_{t,n-1} R_k(P_{t,n-1})^{-1} e^{i\xi_k(P^B_{t,n-1})} = \frac{R_k(P_{t,n-1})}{R_k(q^m P_{t,n-1})} X_{t,n-1} e^{i\xi_k(P^B_{t,n-1})} \\
P_{t+1,n} \quad &:= \quad q^{-\frac{m}{2}} G^{-1}_{t+1,n+1} G_{t+1,n-1} \qquad (23) \\
&= \quad q^{-\frac{m}{2}} R_k(P_{t,n+1}) G^{-1}_{t-1,n+1} R_k(P_{t,n+1})^{-1} e^{-i\xi_k(P^B_{t,n+1})} \qquad (24) \\
&\quad\quad R_k(P_{t,n-1}) G_{t-1,n-1} R_k(P_{t,n-1})^{-1} e^{i\xi_k(P^B_{t,n-1})} \\
&= \quad \frac{R_k(P_{t,n-1})}{R_k(q^m P_{t,n-1})} \frac{R_k(P_{t,n+1})}{R_k(q^{-m} P_{t,n+1})} P_{t-1,n} e^{i\xi_k(P^B_{t,n-1})} e^{-i\xi_k(P^B_{t,n+1})} \\
&=: \quad \mathbf{E}_{t,n+1} \mathbf{E}_{t,n-1}(P_{t-1,n}) \qquad (25)
\end{aligned}
$$

($n-t$ even).

Now $I^P_{2T} = \{(P_{2T-1,2n})_{n\in\{0..p-1\}}, (P_{2T,2n+1})_{n\in\{0..p-1\}}\}$ or I^P_{2T-1}, respectively, shortly denoted by I^P_T, is an initial configuration of arbitrary unitary periodic (face) operators, i.e. $P_{t,2p+n} = P_{t,n} \in \mathcal{A}(P_T)$, $n \in \mathbb{Z}$ which obey the commutation rules:

$$
\begin{aligned}
[P_{t,n}, P_{t,j}] &= 0 \quad n \neq j \\
[P_{2T\pm 1,n}, P_{2T,n+j}] &= 0 \qquad \text{for} \quad j \in \{2...2p-2\} \quad mod\, 2p \\
P_{t,n}P_{\tilde{t},n+1} &= q^{-m} P_{\tilde{t},n+1}P_{t,n}
\end{aligned}
\tag{26}
$$

$t, \tilde{t} \in \{2T, 2T \pm 1\}, |t-\tilde{t}| = 1$

Consider the automorphism $\mathbf{E}_{t,n+1}\mathbf{E}_{t,n-1} : \mathcal{A}(P_T) \to \mathcal{A}(P_T)$ which was recursively defined by:

$$
\begin{aligned}
& \mathbf{E}_{t,n+1}\mathbf{E}_{t,n-1}(P_{t-1,k}) \\
:= \; & R_k(P_{t,n+1})R_k(P_{t,n-1})P_{t-1,n}R_k(P_{t,n-1})^{-1}R_k(P_{t,n+1})^{-1}e^{i\xi_k(P^B_{t,n-1})}e^{-i\xi_k(P^B_{t,n+1})}
\end{aligned}
\tag{27}
$$

and $\mathbf{E}_{t,n+1}\mathbf{E}_{t,n-1}$ acting trivially on all other faces along the corresponding Cauchy zig zag $\mathcal{C}_t$. Since

$$P_{t+1,n} := \mathbf{E}_{t,n+1}\mathbf{E}_{t,n-1}(P_{t-1,n}).$$

it follows by the definition (28) that :

$$
\begin{aligned}
& R_k(P_{t+1,n}) \\
= \; & R_k(R(P_{t,n+1})R_k(P_{t,n-1})P_{t-1,n}R_k(P_{t,n+1})^{-1}R_k(P_{t,n+1})^{-1}e^{i\xi_k(P^B_{t,n-1})}e^{-i\xi_k(P^B_{t,n+1})}) \\
= \; & R_k(P_{t,n+1})R_k(P_{t,n-1})R_k(e^{i\xi_k(P^B_{t,n-1})}e^{-i\xi_k(P^B_{t,n+1})}P_{t-1,n})R_k(P_{t,n-1})^{-1}R_k(P_{t,n+1})^{-1}.
\end{aligned}
\tag{28}
$$

Define the operators ($t \in \mathbb{Z}$)

$$\mathbf{R}_{2t-1} := \prod_{n=0}^{p-1} R_k(P_{2t-1,2n}) \qquad \mathbf{R}_{2t} := \prod_{n=0}^{p-1} R_k(P_{2t,2n+1}).$$

Proposition 3.1 *The operators $\mathbf{R}_t \in \mathcal{A}(P)$ defined as above evolve as:*

$$
\begin{aligned}
\mathbf{R}_{2t+1} &= \mathbf{R}_{2t} \prod_{n=0}^{p-1} R_k(e^{i\xi_k(P^B_{2t,2n-1})}e^{-i\xi_k(P^B_{2t,2n+1})}P_{2t-1,2n})\mathbf{R}^{-1}_{2t} \\
\mathbf{R}_{2t} &= \mathbf{R}_{2t-1} \prod_{n=0}^{p-1} R_k(e^{i\xi_k(P^B_{2t-1,2n})}e^{-i\xi_k(P^B_{2t-1,2n+2})}P_{2t-2,2n+1})\mathbf{R}^{-1}_{2t-1}
\end{aligned}
$$

Proof: Using the commutation relations in (26) and (28) and the periodicity of the face operators the proof is straightforward. □

Corollary 3.2 *If $e^{i\xi_k(P^B_{t,n-1})}e^{-i\xi_k(P^B_{t,n+1})} = \mathbb{1}$ f.a. $t, x \in \mathbb{Z}$ ($k-t$ even) then $\mathbf{R}_{t+1}\mathbf{R}_t = \mathbf{R}_t\mathbf{R}_{t-1}$ is constant.*

Since in this case the evolution for the face variables is given by conjugation

$$P_{t+1,n} = \mathbf{R}_t\mathbf{R}_{t-1}P_{t-1,n}\mathbf{R}^{-1}_{t-1}\mathbf{R}^{-1}_t$$

the operator $\mathbf{R}_t\mathbf{R}_{t-1}$ can be viewed as the discrete analog of a continous hamiltonian time evolution, i.e.

$$\mathbf{R}_t\mathbf{R}_{t-1} \leftrightarrow e^{iH\Delta t_0} \qquad \Delta t_0 \text{ fixed.}$$

This should justify, why we called the automorphism constructed in (20),(21) *almost* hamiltonian.

4 The quantum discrete sine-Gordon model

4.1 Constructing the evolution automorphism

Theorem 4.1 *Let $\hat{x}$ be an element of a $*$-algebra over $\mathbb{C}$, such that $x^{-1} = x^*$ and $\hat{x}^B := \underbrace{\hat{x} \cdot \ldots \cdot \hat{x}}_{B\ times}$ is a multiple of the identity element $\mathbb{1}$ in the algebra, i.e. $\hat{x}^B = x^B \mathbb{1}$, with $x^B \in S^1 \subset \mathbb{C}$. Let $k \in [0,1), B = \frac{N}{gcd(m,N)}, q = e^{\frac{2\pi i}{N}}$. Choose any root*

$$e^{i\xi_k} = e^{i\xi_k(\hat{x}^B)} := \left(\frac{1 + k^B x^B (-1)^{B-1}}{k^B + x^B(-1)^{B-1}}\right)^{\frac{1}{B}} \mathbb{1} \quad \in S^1 \mathbb{1}\,. \tag{29}$$

Define $R_k(\hat{x}) := \sum_{j=0}^{B-1} l_j \hat{x}^j$ with

$$l_j := \prod_{n=1}^{j} \frac{e^{i\xi_k} q^{m(n-1)} - k}{1 - e^{i\xi_k} k q^{mn}}$$

then $R_k(\hat{x})$ satisfies the functional equation:

$$R_k(\hat{x}) = \frac{k + \hat{x}}{1 + k\hat{x}}\, R_k(q^m \hat{x}) e^{i\xi_k}. \tag{30}$$

Proof: By straightforward verification.
□

The above theorem is a generalization of the results found in [V] and [FZ], where it was assumed that $\hat{x}^B = \mathbb{1}$. As it turns out a generalization to general Casimirs $\hat{x}^B$ is important for obtaining an evolution with nontrivial classical background as first described in [BBR], where a solution for the above functional equation (30)was indicated for the case $m = 2, N =$ odd. Solutions to the above equation for q^m not being a root of unity can also be found in [BBR].

Unfortunately, despite the suggestive notation the numbers $e^{i\xi_k(\hat{x}^B)}$ depend not only on the Casimirs $\hat{x}^B$ and the real constant $k \in [0,1)$, but also on the choice of a B'th root. Clearly once a choice is made (for all times) one can use the operators $R_k(\hat{x})$ together with the chosen roots (now viewed as functions in the Casimirs) for defining an almost hamiltonian quantum evolution for the sine-Gordon model as in (20).

However the fixing of the roots $e^{i\xi_k(\hat{x}^B)}$ for all times contradicts the idea of defining an evolution, by a local process. In the following we want to show that given an initial choice of roots, it is possible to define an unique evolution for the above roots. Nevertheless in order to accomplish this task we need to extend the algebra $\mathcal{A}(X_T)$ by the central elements $e^{i\xi_k(P_T^B)}$ and will denote this new algebra by $\mathcal{A}(X_T)^e$.

4.2 Light cone shifts

The doubly discrete sine-Gordon equation, as well as the above described equations of sine-Gordon type (see e.g. (14)) are, as in the continous case, invariant under light cone shifts, i.e. if $g_{t,k} : \mathbb{Z}^2 \to \mathbb{R}$ is a solution to (3) then $g_{t\pm n,k\pm n}$ is also a solution. In this sense space time shifts can be lifted to automorphisms on covariant phase space and can be interpreted as symplectomorphisms [EK].

In the previous section we found a quantization of another (yet trivial) symplectomorphism ([AM, GS]) on phase space, namely time evolution. It would be now only consequent to find quantized analogs of the above mentioned light cone shifts. This will be done by defining quantized space translations of half the lattice spacing distance and then applying the time automorphism. Since translations of half the lattice spacing distance are hard to define on the vertex operators, as one would have to go over to the dual lattice, one has to restrict oneself to the edge algebra $\mathcal{A}(X_T)$.

For constructing the above mentioned space shifts, we will follow an idea developped in [FV3], where such space shifts were suggested for the case of a special choice of vertex monodromies. As it will turn out the treatment of the more general case will result in a possibility to fix the above roots in a very natural way.

Lemma 4.2 *The quotient of the two vertex monodromies*

$$\begin{aligned} M^{(1)}(M^{(2)})^{-1} &= P_{2t\pm 1,0}P_{2t\pm 1,2}\dots P_{2t\pm 1,2p-2}P^{-1}_{2t,2p-1}P^{-1}_{2t,2p-3}\dots P^{-1}_{2t,1} \\ &= q^{-2mp+m}X^2_{2t,0}X^{-2}_{2t,1}X^2_{2t,2}\dots X^2_{2t,2p-1}M_X \\ &= q^{-2mp+m}X^2_{2t+1,0}X^{-2}_{2t+1,1}X^2_{2t+1,2}\dots X^2_{2t+1,2p-1}M_X \end{aligned}$$

is a Casimir in $\mathcal{A}(X_T)$.

Demand 4.3 *For establishing quantum space time shifts demand*

a.) *The Casimirs*

$$P_{2t-1,0}P_{2t-1,2}\dots P_{2t-1,2p-2}P^{-1}_{2t,2p-1}P^{-1}_{2t,2p-3}\dots P^{-1}_{2t,1}$$

and $P^B_{t,k-1}$ *(B as before) shall be multiples of the identity within* $\mathcal{A}(X_T)$.

b.) *The roots* $e^{i\xi_0(P^B_{t,k-1})} = (P_{t,k-1}(-1)^{B-1})^{-\frac{1}{B}}$ *(see 29) shall be fixed in such a way that*

$$\prod_{k=0}^{p-1}(P^B_{2t-1,2k}(-1)^{B-1})^{\frac{1}{B}}\prod_{k=0}^{p-1}(P^B_{2t,2k-1}(-1)^{B-1})^{-\frac{1}{B}} = \prod_{k=0}^{p-1}P_{2t-1,2k}\prod_{k=0}^{p-1}P^{-1}_{2t,2k-1}$$

By the definition of an almost hamiltonian quantum evolution demand a.) holds automatically for all times t, if it is true at an initial time T. The same is valid for demand b.) which will become evident soon, therefore the index t in the above demands refers to all times.

Define

$$S_t^{-1} := \prod_{k=2}^{2p \leftarrow} R_0(X^{-1}_{t,k}X_{t,k-1}) = R_0(X^{-1}_{t,2p}X_{t,2p-1})R_0(X^{-1}_{t,2p-1}X_{t,2p-2})\dots R_0(X^{-1}_{t,2}X_{t,1}) \tag{31}$$

Proposition 4.4 *For all* $k \in \mathbb{Z}$

$$S_t^{-1}X_{t,k}S_t = q^{-m}e^{i\xi_0((X^{-1}_{t,k}X_{t,k-1})^B)}X_{t,k-1}$$

where $e^{i\xi_0(x^B)} = (x^B(-1)^{B-1})^{-\frac{1}{B}}$ *as in (29).*

Proof:
If $n \in \{2\dots 2p\}$ then by the commutation rules of the edge variables

$$\begin{aligned} &S_t^{-1}X_{t,n}S_t = \\ &= \prod_{k=n}^{2p \leftarrow} R_0(X^{-1}_{t,k}X_{t,k-1})X_{t,n}\prod_{k=n}^{2p \rightarrow} R_0(X^{-1}_{t,k}X_{t,k-1})^{-1} \end{aligned}$$

$$\begin{aligned}
&= \prod_{k=n+1}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})\frac{R_0(X_{t,n}^{-1}X_{t,n-1})}{R_0(q^m X_{t,n}^{-1}X_{t,n-1})}X_{t,n}\prod_{k=n+1}^{2p\rightarrow} R_0(X_{t,k}^{-1}X_{t,k-1})^{-1} \\
&= \prod_{k=n+1}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})X_{t,n}^{-1}X_{t,n-1}X_{t,n}\prod_{k=n+1}^{2p\rightarrow} R_0(X_{t,k}^{-1}X_{t,k-1})^{-1}e^{i\xi_0((X_{t,n}^{-1}X_{t,n-1})^B)} \\
&= q^{-m}\prod_{k=n+1}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})X_{t,n-1}\prod_{k=n+1}^{2p\rightarrow} R_0(X_{t,k}^{-1}X_{t,k-1})^{-1}e^{i\xi_0((X_{t,n}^{-1}X_{t,n-1})^B)} \\
&= q^{-m}e^{i\xi_0((X_{t,n}^{-1}X_{t,n-1})^B)}X_{t,n-1}
\end{aligned}$$

Analogously one obtains

$$\begin{aligned}
& S_t^{-1}X_{t,1}S_t = \\
&= \prod_{k=3}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})X_{t,2}^{-1}X_{t,1}^2\prod_{k=3}^{2p\rightarrow} R_0(X_{t,k}^{-1}X_{t,k-1})^{-1}e^{i\xi_0((X_{t,2}^{-1}X_{t,1})^B)} \\
&= \prod_{k=4}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})\frac{R_0(X_{t,3}^{-1}X_{t,2})}{R_0(q^{-m}X_{t,3}^{-1}X_{t,2})}X_{t,2}^{-1}X_{t,1}^2\prod_{k=4}^{2p\rightarrow} R_0(X_{t,k}^{-1}X_{t,k-1})e^{i\xi_0((X_{t,2}^{-1}X_{t,1})^B)} \\
&= \prod_{k=4}^{2p\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})q^{2m}X_{t,3}X_{t,2}^{-2}X_{t,1}^2\prod_{k=4}^{2p\rightarrow} R_0(X_{t,k}^{-1}X_{t,k-1})e^{-i\xi_0((X_{t,3}^{-1}X_{t,2})^B)+i\xi_0((X_{t,2}^{-1}X_{t,1})^B)} \\
&= q^{2m(p-1)}X_{t,2p}^{-1}\prod_{k=1}^{2p-1}X_{t,k}^{2(-1)^{k+1}}\prod_{k=1}^{2p}e^{(-1)^k i\xi_0((X_{t,k}^{-1}X_{t,k-1})^B)}\,e^{i\xi_0((X_{t,1}^{-1}X_{t,0})^B)} \\
&\overset{b.)}{=} q^{-m}e^{i\xi_0((X_{t,1}^{-1}X_{t,0})^B)}X_{t,0} \qquad (32)
\end{aligned}$$

if we suppose that demand b.) holds. With $S_t^{-1}M_X S_t = M_X$ the assertion follows.
□
It is easy to show that:

$$(S_t^{(l)})^{-1}X_{t,n}S_t^{(l)} = q^{-m}e^{i\xi_0((X_{t,n}^{-1}X_{t,n-1})^B)}X_{t,n-1}$$

where

$$(S_t^{(l)})^{-1} := \prod_{k=2+l}^{2p+l\leftarrow} R_0(X_{t,k}^{-1}X_{t,k-1})$$

Therefore the index l is irrelevant and will be skipped, also if $(S_t^{(l)})^{-1}$ instead of $(S_t^{(0)})^{-1}$ is used. Clearly one can also define an automorphism of the above kind by redefining

$$\hat{S}_t^{-1} := \prod_{k=2+l}^{2p+l\leftarrow} R_0(\alpha X_{t,k}^{-1}X_{t,k-1}) \qquad \alpha\in S^1,$$

which will act as:

$$\hat{S}_t^{-1}X_{t,n}\hat{S}_t = \alpha q^{-m}e^{i\xi_0((X_{t,n}^{-1}X_{t,n-1})^B)}X_{t,n-1},$$

a fact to be used later.
The automorphism $\mathbf{S}_t^{-1}:\mathcal{A}(X_T)\to\mathcal{A}(X_T)$ defined on the operators $X_{t,n}$ as

$$\mathbf{S}_t^{-1}(X_{t,n}) := q^m e^{-i\xi_0((X_{t,n}^{-1}X_{t,n-1})^B)}S_t^{-1}X_{t,n}S_t = X_{t,n-1}$$

can be interpreted as a shift of these edge operators in space direction. As an automorphism on $\mathcal{A}(P_T)$, the picture of the action of $\mathbf{S}_t^{-1}$ is a little different, we find

$$\begin{aligned} S_t^{-1} P_{t,k-1} S_t &= P_{t-1,k-2}\, e^{-i\xi_0(P^B_{t,k-1})} + e^{i\xi_0(P^B_{t-1,k-2})} \\ S_t^{-1} P_{t-1,k-2} S_t &= P_{t,k-3}\, e^{-i\xi_0(P^B_{t,k-2})} + e^{i\xi_0(P^B_{t-1,k-3})} \end{aligned}$$

Hence $\mathbf{S}_t$ applied to face operators is rather an up and down shift in lightcone direction than a shift in space:

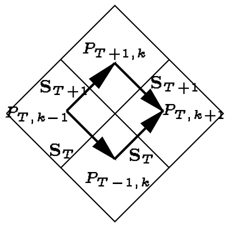


Figure 2.1

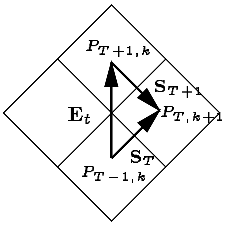


Figure 2.2

Fix an initial time e.g. $T =$ odd. Then by definition the operators

$$\begin{aligned} S_T^{-1} &= R_0(P_{T-1,2p-1}) R_0(P_{T,2p-2}) \dots R_0(P_{T-1,1}) \\ S_{T+1}^{-1} &= R_0(P_{T+1,2p-1}) R_0(P_{T,2p-2}) \dots R_0(P_{T+1,1}) \end{aligned}$$

are given by the choice of initial operators $\left\{(P_{T-1,2k+1})_{k\in\{0\dots p-1\}}, (P_{T,2k})_{k\in\{0\dots p-1\}}\right\}$ and the time evolved face operators $(P_{T+1,2k+1})_{k\in\{0\dots p-1\}}$. The operator $P_{T+1,2k+1}$ can now be obtained by shifting the operator $P_{T,2k}$ or by applying the time evolution to $P_{T-1,2k+1}$, i.e. the in figure 2.2 depictured shifts have to commute.
By the commutation relations of the face operators it is straightforward to find:

Lemma 4.5

$$\begin{aligned} & R_0(P_{T+1,2p-1}) R_0(P_{T,2p-2}) \dots R_0(P_{T+1,1}) \\ & = \mathbf{R}_T\, R_0(e^{-i\xi_0(P^B_{T,2p})+i\xi_0(P^B_{T,2p-2})} P_{T-1,2p-1}) R_0(P_{T,2p-2}) \\ & \dots R_0(e^{-i\xi_0(P^B_{T,2})+i\xi_0(P^B_{T,0})} P_{T-1,1}) \mathbf{R}_T^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} & S_{T+1}^{-1} P_{T,2n} S_{T+1} \stackrel{!}{=} P_{T+1,2n-1} e^{-i\xi_0(P^B_{T,2n})+i\xi_0(P^B_{T+1,2n-1})} \\ \stackrel{4.5}{=} \quad & \mathbf{R}_T\, R_0(e^{-i\xi_0(P^B_{T,2p})+i\xi_0(P^B_{T,2p-2})} P_{T-1,2p-1}) R_0(P_{T,2p-2}) \\ & \dots R_0(e^{-i\xi_0(P^B_{T,2})+i\xi_0(P^B_{T,0})} P_{T-1,1}) P_{T,2k} \\ & (\mathbf{R}_T\, R_0(e^{-i\xi_0(P^B_{T,2p})+i\xi_0(P^B_{T,2p-2})} P_{T-1,2p-1}) R_0(P_{T,2p-2}) \\ & \dots R_0(e^{-i\xi_0(P^B_{T,2})+i\xi_0(P^B_{T,0})} P_{T-1,1}))^{-1} \\ = \quad & \mathbf{R}_T P_{T-1,2n-1} \mathbf{R}_{T-1}^{-1} e^{-i\xi_0(P^B_{T,2n})+i\xi_0((e^{-i\xi_k(P^B_{T,2n})+i\xi_k(P^B_{T,2n-2})} P_{T-1,2n-1})^B)} \\ & e^{-i\xi_k(P^B_{T,2n})+i\xi_k(P^B_{T,2n-2})} \\ = \quad & P_{T+1,2n-1} e^{-i\xi_0(P^B_{T,2n})+i\xi_0((e^{-i\xi_k(P^B_{T,2n})+i\xi_k(P^B_{T,2n-2})} P_{T-1,2n-1})^B)} \end{aligned}$$

Comparing both sides of the equation, one obtains finally the compatibility condition:

$$e^{i\xi_0(P^B_{T+1,2n-1})} \stackrel{!}{=} e^{-i\xi_k(P^B_{T,2n})+i\xi_k(P^B_{T,2n-2})} e^{i\xi_0(P^B_{T-1,2n-1})} \qquad (33)$$

For defining the evolution we had already fixed the roots on the right hand side of equation (33), so that equation (33) determines the roots at one time step further.

Equation (33) fixes the roots at $k = 0$. It is easy to see that one can stay on one leave, when extending to arbitrary $k \in [0, 1)$. Hence also the evolution of the roots $e^{-i\xi_k(P)}$ has been now defined. As a direct consequence of equation (33), it follows that demand b.) holds for all times since the classical monodromies are as well as their quantum counterparts integrals of motion [K].

Note that if one takes the $B'th$ power of equation (33) then one obtains

$$P^B_{T+1,2n-1} = \frac{k^B + P^B_{T,2n}(-1)^{B-1}}{1 + k^B P^B_{T,2n}(-1)^{B-1}} \frac{1 + k^B P^B_{T,2n-2}(-1)^{B-1}}{k^B + P^B_{T,2n-2}(-1)^{B-1}} P^B_{T-1,2n-1}.$$

Hence the "classical" variables $P^B_{t,k}$ satisfy also an equation of sine-Gordon type. A fact which was first noticed in [BBR] by using the commutation relations of the face operators and computing $P^B_{T+1,2k-1}$.

It is now straightforward to show that as an immediate consequence

$$\mathbf{S}_{T+1}(X_{T+1,k}) = \mathbf{E}_T \circ \mathbf{S}_T \circ \mathbf{E}_T^{-1}(X_{T+1,k})$$

i.e. shifts in time and space direction commute if (33) is satisfied.

The picture of the above developped quantum evolution looks considerably complicate. It simplifies by a great amount, if one restricts oneself to the case of corollary (3.2):

Proposition 4.6 *Let $\mathbf{S}_T$ be an automorphism on $\mathcal{A}(X_T)$ such that at an initial time T*

$$\mathbf{R}_T = \mathbf{S}_T(\mathbf{R}_{T-1}) = \mathbf{S}_T^{-1}(\mathbf{R}_{T-1}) \tag{34}$$

where recursively

$$\mathbf{R}_{t+1} = \mathbf{R}_t \mathbf{R}_{t-1} \mathbf{R}_t^{-1} \tag{35}$$

then $\mathbf{R}_t = \mathbf{R}_T \mathbf{S}_T(\mathbf{R}_{t-1})\mathbf{R}_T^{-1}$ for all $t \in \mathbb{Z} \geq T$.

Proof:
By (34) and (35)

$$\mathbf{R}_T = \mathbf{R}_T \mathbf{S}_T(\mathbf{R}_{T-1})\mathbf{R}_T^{-1} \tag{36}$$

$$\mathbf{R}_{T+1} = \mathbf{R}_T \mathbf{S}_T(\mathbf{R}_T)\mathbf{R}_T^{-1} \tag{37}$$

For completing the inductional argument assume that the following is true

$$\mathbf{R}_{T+n} = \mathbf{R}_T \mathbf{S}_T(\mathbf{R}_{T-1+n})\mathbf{R}_T^{-1} \tag{38}$$

$$\mathbf{R}_{T+1+n} = \mathbf{R}_T \mathbf{S}_T(\mathbf{R}_{T+n})\mathbf{R}_T^{-1} \tag{39}$$

Hence

$$\begin{aligned} \mathbf{R}_{T+n+2} &\overset{(35)}{=} \mathbf{R}_{T+n+1}\mathbf{R}_{T+n}\mathbf{R}_{T+n+1}^{-1} \\ &\overset{(38,39)}{=} \mathbf{R}_T\mathbf{S}_T(\mathbf{R}_{T+n})\mathbf{R}_T^{-1}\mathbf{R}_T\mathbf{S}_T(\mathbf{R}_{T-1+n})\mathbf{R}_T^{-1}(\mathbf{R}_T\mathbf{S}_T(\mathbf{R}_{T+n})\mathbf{R}_T^{-1})^{-1} \\ &\overset{(35)}{=} \mathbf{R}_T\mathbf{S}_T(\mathbf{R}_{T+n+1})\mathbf{R}_T^{-1} \end{aligned} \tag{40}$$

analogously $\mathbf{R}_{T+n+3} \overset{(39,40)}{=} \mathbf{R}_T\mathbf{S}_T(\mathbf{R}_{T+n+2})\mathbf{R}_T^{-1}$.
□

Proposition 4.7 *If* $\mathbf{S}_T$ *is an automorphism such at initial time* T

$$X_{T,k} = \mathbf{S}_T(X_{T,k-1}) = \mathbf{S}_T^{-1}(X_{T,k+1}) \qquad \text{and} \quad X_{t+1,k} = \mathbf{R}_t X_{t,k} \mathbf{R}_t^{-1}$$

for all $t \in \mathbb{N}$ *and with* $\mathbf{R}_t$ *as in proposition (4.6) then*

$$X_{t+1,k} = \mathbf{R}_T \mathbf{S}_T(X_{t,k-1})\mathbf{R}_T^{-1}$$

Proof:
By induction as above and by the use of proposition (4.6).
□
The connection to models of statistical mechanics is now evident. We find

$$\begin{aligned} X_{t+2,k+1} &= \mathbf{R}_T \mathbf{S}_T^{-1}(\mathbf{R}_T \mathbf{S}_T(X_{t,k+1})\mathbf{R}_T^{-1})\mathbf{R}_T^{-1} \\ &= \mathbf{R}_T \mathbf{S}_T^{-1}(\mathbf{R}_T) X_{t,k+1} \mathbf{S}_T^{-1}(\mathbf{R}_T^{-1})\mathbf{R}_T^{-1}. \end{aligned} \tag{41}$$

Moreover $\mathbf{R}_T$ is a product of "local amplitudes" $R(P_{T,k-1})$ associated to the faces at time T within the light cone lattice, hence $\mathbf{S}_T^{-1}(\mathbf{R}_T)$ is a product of "local amplitudes" $R(P_{t,k-1})$ associated to faces which are shifted in lightcone direction of the original faces. Because of (41) this picture is the same all over the lattice, hence we can interpret R_T as a kind of transfermatrix (though with complex weights).

Another fortunate consequence of the above is that for investigating the evolution it suffices to control the first time step, everything else is obtained by applying the light cone shifts $\mathbf{E}_T \circ \mathbf{S}_T$. This is espiacially important for the construction of integrals of motion, since if one finds an operator H_T, which commutes with the above light cone shifts, then this will be automatically an integral of motion.

In the next section an example of such a 'static' quantum field theory will be discussed.

5 Relations to the massive Thirring model

Choose an initial time $T =$ odd.

Let

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \tag{42}$$

The notation is adopted from viewing B as a "**B**oost" and S as a "Shift" operator. Note that

$$CSC^{-1} = -iBS \qquad CBSC^{-1} = -iS.$$

B, S, C are operators acting on the Hilbertspace $\mathcal{H} = \mathbb{C}^2$. Define for any operator A on $\mathcal{H}$ an operator A_l on a "big" Hilbertspace $\mathcal{H}^{2p} = \bigotimes_{k=0}^{2p-1} \mathbb{C}^2$ by

$$A_l = \underbrace{\mathbb{1}}_{2p-1'th\, site} \otimes \mathbb{1} \ldots \underbrace{A}_{l'th\, site} \ldots \mathbb{1} \otimes \underbrace{\mathbb{1}}_{0'th\, site}$$

Let

$$\mathbf{C} := \prod_{l=0}^{2p-1} C_l.$$

Define

$$X_{T,2k} := S_{2k} \prod_{l=0}^{2k-1 \leftarrow} B_l = S_{2k} B_{2k-1} \ldots B_0, \tag{43}$$

$$X_{T,2k+1} := -B_{2k+1} S_{2k+1} \prod_{l=0}^{2k \leftarrow} B_l, \tag{44}$$

$$M_X := \mathbb{1} \otimes \mathbb{1} \ldots \otimes \mathbb{1} \tag{45}$$

where $X_{T,0} = S_0$. Denote shortly $BS_{2k+1} := B_{2k+1}S_{2k+1}$.
The above defined operators satisfy the commutation relations of the edge operators

$$X_{T,k}X_{T,k+1} = q^{-m}X_{T,k+1}X_{T,k} \qquad X_{T,k}M_X = q^{-2m}M_X X_{T,k}$$

for $q^{-m} = -1$. They are periodic and

$$X^2_{T,2k} = \mathbb{1} \quad X^2_{T,2k+1} = -\mathbb{1} \tag{46}$$

The corresponding face operators are almost all bilocal in terms of the tensorproduct:

$$\begin{array}{rcl} P_{T,2k} & := & X^{-1}_{T,2k+1}X_{T,2k} \\ & = & BS_{2k+1}BS_{2k} \\ & & \text{for } k \in \{0\ldots p-1\} \end{array} \qquad \begin{array}{rcl} P_{T-1,2k+1} & := & X^{-1}_{T,2k+2}X_{T,2k+1} \\ & = & -S_{2k+2}S_{2k+1} \\ & & \text{for } k \in \{0\ldots p-2\}, \end{array}$$

$$P_{T-1,2p-1} = BS_{2p-1}\prod_{l=1}^{2p-2} B_l \; BS_0$$

and

$$P^2_{T,2k} = P^2_{T-1,2k+1} = \mathbb{1}\,.$$

The difference of the vertex monodromies is

$$P_{2T-1,2p-1}\ldots P_{T-1,1}P^{-1}_{T,0}\ldots P^{-1}_{T,2p-2} = (-1)^{p-1}\mathbb{1}\,.$$

For defining the shift automorphism $\mathcal{S}_T$ we fix the roots $e^{i\xi_0(iP^B_{t,n}))}$ $(t \in \{T, T-1\}, n \in \{0\ldots 2p-1\})$ as:

$$e^{i\xi_0(iP^B_{T,2n}))} = e^{i\xi_k(iP^B_{T,2n}))} = \left(\frac{1+k^2(iP_{T,2n})^2(-1)}{k^2+(iP_{t,n})^2(-1)}\right)^{\frac{1}{2}} := \mathbb{1}$$

$n \in \{0\ldots p-1\}$ and analogously

$$e^{i\xi_k(iP^B_{T-1,2n+1}))} = \mathbb{1} \qquad n \in \{0\ldots p-1\}$$

f. a. $\quad k \in [0,1)$.
Note that with this definition

$$\begin{array}{rcl} P_{T-1,2p-1}\ldots P_{T-1,1}P^{-1}_{T,0}\ldots P^{-1}_{T,2p-2} & = & (-1)^{p-1}e^{-i\xi_0(iP^B_{T-1,2p-1}))}\ldots e^{-i\xi_0(iP^B_{T-1,0}))} \\ & & e^{i\xi_0(iP^B_{T,0}))}\ldots e^{i\xi_0(iP^B_{T,2p-2}))} \end{array}$$

hence we have to be careful when defing the shift automorphism (compare with (32)). Let

$$c := \frac{2k}{1+k^2} \qquad b = -i\frac{1-k^2}{1+k^2},$$

then using (4.1) and normalizing, the local amplitudes are straightforward obtained as:

$$R_k(iP_{T,2n}) \;\; = \;\; \frac{i}{\sqrt{2(1-c)}}(b+(1-c)P_{T,2n})$$

They obey the functional equations:

$$\frac{R_k(iP_{T,2n})}{R_k(-iP_{T,2n})} = \frac{k+iP_{T,2n}}{1+kiP_{T,2n}};$$

The evolution for the edge operators to the next time step is given by

$$X_{T+1,n} = \mathbf{R}_T X_{T,n} \mathbf{R}_T^{-1} \quad \text{where} \quad \mathbf{R}_T = \prod_{l=0}^{p-1} R_k(iP_{T,2l}).$$

The shift matrix shall be given by:

$$S_T^{-1} := R_0(iP_{T,2p-2}) R_0(iP_{T-1,2p-3}) \dots R_0(iP_{T-1,1}) R_0(iP_{2T,0})$$

which defines the following shiftautomorphism on the edge algebra:

$$\begin{aligned} \mathbf{S}_T^{-1}(X_{T,n}) &:= iS_T^{-1} X_{T,n} S_T = X_{T,n-1} \qquad n \in \{1 \dots 2p-1\} \\ \mathbf{S}_T^{-1}(X_{T,0}) &:= i(-1)^{p-1} S_T^{-1} X_{T,0} S_T = X_{T,2p-1} \end{aligned}$$

Lemma 5.1 *The matrices* $\mathbf{C}P_{T,2n}\mathbf{C}^{-1}$ *and* $\mathbf{C}P_{T,2n+1}\mathbf{C}^{-1}$ $(n \in \{0 \dots p-1\})$ *commute with all generators* $I_T^X = (X_{T,n})_{n \in \{0..2p\}}$ *of* $\mathcal{A}(X_T)$.

Hence

$$\begin{aligned} & \mathbf{R}_T \mathbf{C} \mathbf{R}_T \mathbf{C}^{-1} S_T^{-1} \mathbf{C} S_T^{-1} \mathbf{C}^{-1} (X_{T,n} - i\mathbf{C} X_{T,n} \mathbf{C}^{-1}) \mathbf{C} S_T \mathbf{C}^{-1} S_T \mathbf{C} \mathbf{R}_T^{-1} \mathbf{C}^{-1} \mathbf{R}_T^{-1} \\ = \; & \mathbf{R}_T S_T^{-1} X_{T,n} S_T \mathbf{R}_T^{-1} - i\mathbf{C} \mathbf{R}_T S_T^{-1} X_{T,n} S_T \mathbf{R}_T^{-1} \mathbf{C}^{-1} \\ = \; & -i(\mathbf{R}_T \mathbf{S}_T^{-1}(X_{T,n}) \mathbf{R}_T^{-1} - i\mathbf{C} \mathbf{R}_T \mathbf{S}_T^{-1}(X_{T,n}) \mathbf{R}_T^{-1} \mathbf{C}^{-1}) \end{aligned}$$

$n \in \{1 \dots p-1\}$, analogous for $X_{T,0}$.
Clearly this defines lightcone shifts on the operators

$$\psi_{T,n+1} := \frac{1}{2}(X_{T,n} - i\mathbf{C} X_{T,n} \mathbf{C}^{-1}) = \sigma_n^- \prod_{l=0}^{n-1} \sigma_l^z,$$

i.e.

$$\psi_{T+1,n+1} := \tilde{\mathbf{R}}_T \tilde{\mathbf{S}}_T(X_{T,n}) \tilde{\mathbf{R}}_T^{-1}$$

with $n \in \{0 \dots p-1\}$

$$\tilde{\mathbf{R}}_T := \mathbf{R}_T \mathbf{C} \mathbf{R}_T \mathbf{C}^{-1} \qquad \tilde{\mathbf{S}}_T^{-1}(\psi_{T,n}) := iS_T^{-1} \mathbf{C} S_T^{-1} \mathbf{C}^{-1} \psi_{T,n} S_T \mathbf{C} S_T \mathbf{C}^{-1}$$

and

$$\sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \sigma^z = B.$$

The shift matrices S_T and S_T^{-1}, as well as the evolution matrix $\mathbf{R}_T$ are products of bilocal operators

$$\begin{aligned} R_k(iP_{T,2n}) &= \mathbb{1} \otimes \dots R_k(iBS \otimes BS) \dots \otimes \mathbb{1} \quad n \in \{0 \dots p-1\} \\ R_k(iP_{T-1,2n+1}) &= \mathbb{1} \otimes \dots R_k(-iS \otimes S) \dots \otimes \mathbb{1} \qquad n \in \{0 \dots p-2\}. \end{aligned}$$

Hence the same holds fot $\tilde{\mathbf{R}}_T$ and $\tilde{S}_T$. A straightforward computation gives

$$\begin{aligned} R_k(iBS \otimes BS) \mathbf{C} R_k(iBS \otimes BS) \mathbf{C}^{-1} &= R_k(iBS \otimes BS) R_k(-iS \otimes S) \\ &= \mathbb{1} \otimes \dots \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c & b & 0 \\ 0 & b & c & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \dots \otimes \mathbb{1} = \end{aligned}$$

$$= R_k(-iS \otimes S) \mathbf{C} R_k(-iS \otimes S) \mathbf{C}^{-1}$$

The shift matrices

$$\tilde{S}_T^{-1} = \tilde{R}_0(iP_{T,2p-2})\tilde{R}_0(iP_{T-1,2p-3})\tilde{R}_0(iP_{T,2p-4})\dots\tilde{R}_0(iP_{T,0})$$

act on the fermionic operators $\psi_{T,k}$ as

$$\begin{aligned}
& \tilde{S}_T^{-1}\psi_{T,2n}\tilde{S}_T = \\
= \;& \tilde{R}_0(iP_{T,2n-2})\psi_{T,2n}\tilde{R}_0(iP_{T,2n-2})^{-1} \\
= \;& \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\sigma^-_{2n-1}\otimes\sigma^z_{2n-2}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \prod_{l=0}^{2n-3}\sigma^z_l \\
= \;& -i\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} (\sigma^-_{2n-1}\otimes\sigma^z_{2n-2}) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \prod_{l=0}^{2n-3}\sigma^z_l \\
= \;& -i\mathbf{V}^{-1}\psi_{T,2n}\mathbf{V}
\end{aligned} \tag{47}$$

$n \in \{0\dots p-1\}$ with

$$V_n = V_n^{-1} = \mathbb{1}\otimes\dots\underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{(n+1,n)'th\ site}\dots\otimes\mathbb{1} \qquad \mathbf{V}^{-1} = V_{2p-2}^{-1}V_{2p-3}^{-1}\dots V_0^{-1}$$

In a similar way

$$\begin{aligned}
\tilde{S}_T^{-1}\psi_{T,2n+1}\tilde{S}_T &= -i\mathbf{V}^{-1}\psi_{T,2n+1}\mathbf{V} \quad n\in\{1\dots p-1\} \\
\tilde{S}_T^{-1}\psi_{T,1}\tilde{S}_T &= i(-1)^p\mathbf{V}^{-1}\psi_{T,1}\mathbf{V},
\end{aligned}$$

so that finally

$$\begin{aligned}
\tilde{\mathbf{S}}_T^{-1}(\psi_{T,n}) &= i\tilde{S}_T^{-1}\psi_{T,n}\tilde{S}_T &= \mathbf{V}^{-1}\psi_{T,n}\mathbf{V} \qquad n\in\{2\dots 2p\} \\
\tilde{\mathbf{S}}_T^{-1}(\psi_{T,1}) &= i(-1)^{p-1}\tilde{S}_T^{-1}\psi_{T,n}\tilde{S}_T &= \mathbf{V}^{-1}\psi_{T,1}\mathbf{V}
\end{aligned}$$

which is identical to the shift automorphism constructed in [DV]. Following (4.6,4.7) we know that the construction of the shift automorphism $\tilde{S}$ and the evolution automorphism given by the conjugation with $\tilde{R}$ is sufficient for constructing a hamiltonian quantum evolution in the sense of the previous sections.

Finally comparing with the construction in [DV] one finds that the fermionic operators obey an evolution of free massive fermions. The evolution equations can be derived easily by considering the evolution for the edge variables. Remembering (46) one finds:

$$\begin{aligned}
X_{t+1,2k} := \mathbf{R}_t X_{t,2k}\mathbf{R}_t^{-1} &= \frac{k+iX^{-1}_{t,2k+1}X_{t,2k}}{1-kiX^{-1}_{t,2k+1}X_{t,2k}}\,\frac{1-kiX^{-1}_{t,2k+1}X_{t,2k}}{1-kiX^{-1}_{t,2k+1}X_{t,2k}}X_{t,2k} \\
&= \frac{1}{1+k^2}(2k+i(1-k^2)X^{-1}_{t,2k+1}X_{t,2k})X_{t,2k} \\
&= cX_{t,2k}+bX_{t,2k+1}
\end{aligned} \tag{48}$$

analogously for $X_{t+1,2k+1}$. Since these equations are linear, the evolution equations for the fermionic operators follow immediately:

$$\psi_{t+1,2n-1} = c\psi_{t,2n-1}+b\psi_{t,2n} \tag{49}$$

$$\psi_{t+1,2n} = c\psi_{t,2n}+b\psi_{t,2n-1}. \tag{50}$$

5.1 Conclusion

In the present paper a generalized model of a lattice field theory of sine Gordon type at root of unity was suggested. Among others the aim was to stress the purely local character of the evolution automorphism and finally to derive global features like constant transfermatrices and hence a connection to models in statistical mechanics by restricting to a special case (please refer 4.6,4.7) or by considering symmetries of the models in consideration.

Since the classical phase space belonging to the evolution of e.g. the face variables (current variables) is a Torus S^{2p} the corresponding quantum model can be viewed as a kind of quantization of this torus. Hence a next future project should be the investigation of the above within the framework of noncommutative geometry [Co].

A nice side effect of the study of the above quantum lattice model was the detection of a relation to another quantum lattice model, namely the massive Thirring model in it's reduced version as describing free massive fermions, as given by [DV]. Since relations between these two models are known for the continous case, see e.g. [KM, C] it seems to speak for the self coherence of the above lattice models, that they also exist in the discrete case.

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