Holomorphic deformation of Hopf algebras and applications to quantum groups

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Abstract

In this article we propose a new and so-called holomorphic deformation scheme for locally convex algebras and Hopf algebras. Essentially we regard converging power series expansion of a deformed product on a locally convex algebra, thus giving the means to actually insert complex values for the deformation parameter. Moreover we establish a topological duality theory for locally convex Hopf algebras. Examples coming from the theory of quantum groups are reconsidered within our holomorphic deformation scheme and topological duality theory. It is shown that all the standard quantum groups comprise holomorphic deformations. Furthermore we show that quantizing the function algebra of a (Poisson) Lie group and quantizing its universal enveloping algebra are topologically dual procedures indeed. Thus holomorphic deformation theory seems to be the appropriate language in which to describe quantum groups as deformed Lie groups or Lie algebras.

Contents

Introduction

In this paper we propose a new deformation scheme which we call **holomorphic deformation** and which seems to recapture what is actually done in the context of describing quantum groups as deformed Lie groups or Lie algebras. Although it appears to be new as an explicitely formulated concept, we are convinced that our holomorphic deformation theory is in fact very close to many aspects of existing deformation procedures in mathematical physics.

The reason to consider holomorphic deformations instead of the by now classical formal deformations of GERSTENHABER (cf. [9, 10, 11]) is twofold. First, one likes to obtain concrete deformations, i.e. deformations of the structure on a given vector space which are defined on this vector space and not only on a suitable extension. Secondly, it is a well-known fact within the theory of infinite dimensional Hopf algebras that one is often forced to change the usual tensor product and/or the concept of the dual space. This change is well understood by introducing a (locally convex) topology on the Hopf algebra in question as has been shown in the work [4] of P. BONNEAU, M. FLATO, M. GERSTENHABER and G. PINCZON. They consider nuclear Hopf algebras H and work with the locally convex ring $\mathbb{C}[[T]]$ of formal power series as a ring extension of $\mathbb{C}$ in order to formulate the concept of a deformation: A deformation of H in their setup is a certain Hopf algebra structure on $H_T := \mathbb{C}[[T]]\hat{\otimes} H$ over $\mathbb{C}[[T]]$, where $\hat{\otimes}$ denotes the completed π-tensor product of locally convex spaces. The change of definition we propose (in sections 3 and 4) is simply to replace $\mathbb{C}[[T]]$ by the locally convex algebra $\mathcal{O}(\Omega)$ of holomorphic functions on a domain $\Omega \subset \mathbb{C}$ (or a complex manifold Ω): A holomorphic deformation of H thus is a certain Hopf algebra structure on $H_\Omega := \mathcal{O}(\Omega)\hat{\otimes} H \cong \mathcal{O}(\Omega, H)$ over $\mathcal{O}(\Omega)$, where $\mathcal{O}(\Omega, H)$ denotes the locally convex space of holomorphic functions on Ω with values in H. The advantage of this approach lies, among other things, in the fact that it is possible to actually insert values $z \in \Omega$ into the holomorphic deformation in order to get a deformed Hopf algebra structure on H (and not merely on H_T resp. H_Ω). Furthermore, the structure maps of our concrete deformations of H are evaluations of mappings depending holomorphically on z.

Of course, in order to show that this variation of a deformation concept is reasonable and useful, one has to give interesting examples. In the present paper we will show that a large part of the actually studied deformations of Hopf algebras, in particular those arising in the context of quantum groups, can in fact be interpreted as being holomorphic deformations. This will be done in the final section of the present paper where we show that the Drinfeld and the FRT models can be regarded as being holomorphic deformations of the respective Lie algebras or Lie groups.

In the course of preparing this paper we also found it useful to discuss the problem of what kind of locally convex topologies on the Hopf algebra in question one should consider and whether or not there exist interesting and natural such topologies. These questions are dealt with in the second section after we have introduced in the first section some basic definitions concerning the compatibility of algebraic structures on a vector space with a given locally convex topology. The rather elementary considerations concerning locally convex topologies compatible with a given Hopf algebra, leads to more general topologies than the ones studied in [4]. Our discussion in section 2 furthermore shows that – contrary to what might be expected – introducing a topology on a Hopf algebra H does not necessarily require an additional structure: There exist useful and natural topologies on H, even nuclear ones, which are completely determined by the algebraic data of H.

One of the most useful topologies on a given (Hopf) algebra H is, for instance, the locally convex projective limit topology with respect to the family of all finite dimensional representations of H. Working with this projective limit topology indicates the close connection of a holomorphic deformation with the concept of a formal deformation. Indeed, given a formal deformation of e.g. a universal enveloping algebra $\mathcal{U}\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$, it is in general not allowed to insert numerical values replacing the formal parameter. But after fixing a finite dimensional representation ρ of $\mathcal{U}\mathfrak{g}$ this can often be done with respect to the finite dimensional algebra $\rho(\mathcal{U}\mathfrak{g})$. In particular this procedure works for deformations considered in inverse scattering theory or conformal field theory. Now regarding these important examples it is natural to introduce the projective limit topology of finite dimensional representations. Investigating the reason why it is possible to evaluate the deformation parameter in the representation $\rho(\mathcal{U}\mathfrak{g})$ at any element of a complex domain one discovers holomorphic expressions. We thus arrive at a holomorphic deformation of $\mathcal{U}\mathfrak{g}$, or more precisely, of the completion of $\mathcal{U}\mathfrak{g}$ with respect to the projective limit topology of all finite dimensional representations.

Our work is motivated by the desire to understand physicists work on deformation quantization and inverse scattering, in particular the papers [8] by Fock, Rosly and [1, 2] by Alekseev, Grosse, Schomerus, where algebras are "concretely deformed" by a real parameter $\hbar$ or q and not only by a formal one. We hope that the concept of holomorphic deformations will give the means to better understand the deformation quantization of the moduli space of flat connections on a given connected, oriented and compact surface with marked points as described in [8, 1, 2], and to compare it with the quantization in Scheinost, Schottenloher [22].

Let us also mention that our notion of a deformation creates the means to consider formal and holomorphic deformations in one common language. For more details about this approach we refer the interested reader to Pflaum [18, 19].

The main ingredient to our theory of holomorphic deformations is not so much an algebraic viewpoint, but a functional analytic one. In particular we will often use the notions of locally convex spaces, nuclearity, and topological tensor products. For the convenience of the reader we therefore explain in the appendix the most important functional analytic concepts used in this article.

The problem of classifying holomorphic deformations is not considered here. This will be the subject of a forthcoming paper together with the study of a suitable cohomology theory.

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1 Topological and algebraic structures

In order to formulate our deformation concept in the next section we need a topological structure on a given vector space. Therefore, in this section we study locally convex structures on a vector space E which are compatible with additional algebraic structures on E such as the structure of an algebra, of a coalgebra or of a Hopf algebra. A collection of definitions and results on locally convex spaces is provided in the Appendix.

In the following $\mathbb{K}$ will denote one of the fields $\mathbb{R}$ and $\mathbb{C}$ together with the Euclidean topology. Furthermore, the locally convex topologies on a given $\mathbb{K}$-vector space E are always assumed to be Hausdorff and complete. The collection of the (complete and Hausdorff) locally convex spaces together with the $\mathbb{K}$-linear continuous maps form a category $\underline{\mathsf{Lcs}}$. For two locally convex spaces E and F the completion of the tensor product $E \otimes F$ endowed with the π-topology is denoted by $E\hat{\otimes}F$. The significance of $E\hat{\otimes}F$ is apparent by the following universal property: There is a natural bijection between the space of continuous bilinear maps $E \times F \to G$ into a third locally convex space G and the space of continuous linear maps $E\hat{\otimes}F \to G$. With $\hat{\otimes}$ as tensor product functor $\underline{\mathsf{Lcs}}$ becomes a symmetric monoidal category.

Furthermore, a nuclear space is always supposed to be a locally convex $\mathbb{K}$-vector space E which is Hausdorff, complete and nuclear. The completion $E\hat{\otimes}F$ of $E \otimes F$ for two nuclear spaces E and F is again a nuclear space (see the Appendix for precise definitions and examples as well as PIETSCH [20] and TRÈVES [29] for proofs and details about nuclear spaces). Thus nuclear spaces form a monoidal subcategory $\underline{\mathsf{Nuc}}$ of $\underline{\mathsf{Lcs}}$.

We call a nuclear space E **strictly nuclear** if its (strong) dual E' is nuclear as well, if E is reflexive (i.e. the strong dual of E' is isomorphic to E as a locally convex space) and if it fulfills the **duality condition** (cf. (87) in the Appendix), i.e. the canonical linear mapping $E' \otimes E' \to (E\hat{\otimes}E)'$ extends to an algebraic and topological isomorphism

$$E'\hat{\otimes}E' \to (E\hat{\otimes}E)'.$$

Nuclear Fréchet spaces or duals of nuclear Fréchet spaces are strictly nuclear (cf. TRÈVES [29]) as well as nuclear LF-spaces. Restricting the object class of $\underline{\mathsf{Nuc}}$ to strictly nuclear spaces we receive a proper subcategory $\underline{\mathsf{sNuc}}$ of $\underline{\mathsf{Nuc}}$.

Algebraic structures can now be formulated within $\underline{\mathsf{Lcs}}$, (resp. $\underline{\mathsf{Nuc}}$ or $\underline{\mathsf{sNuc}}$) in order to obtain locally convex (resp. nuclear or strictly nuclear) algebras, Hopf algebras, etc. In this section we will give detailed definitions for these objects and will examine them. In the next section we present natural examples.

Definition 1.1 *A* **locally convex algebra** *is a locally convex space A together with continuous linear mappings $\mu : A\hat{\otimes}A \to A$ and $\eta : \mathbb{K} \to A$ such that μ fulfills the associativity constraint $\mu\circ(\mu\hat{\otimes}\mathrm{id}_A) = \mu\circ(\mathrm{id}_A\hat{\otimes}\mu)$ and η gives rise to a unit: $\mu\circ(\mathrm{id}_A\hat{\otimes}\eta) = \mu \circ (\eta\hat{\otimes}\mathrm{id}_A) \cong \mathrm{id}_A$. A homomorphism between locally convex algebras A and $\tilde{A}$ is just a continuous linear map $f : A \to \tilde{A}$ such that $\tilde{\mu} \circ (f\hat{\otimes}f) = f \circ \mu$ and $\tilde{\eta} \circ f = \eta$.*

A **locally m-convex algebra** *is a locally convex algebra A for which there exists a defining family of multiplicative seminorms (see Appendix).*

A **nuclear algebra** *(resp. a* **strictly nuclear algebra** *) is a locally convex algebra for which the underlying locally convex space A is a nuclear space (resp. a strictly nuclear space).*

Similarly one defines the concepts of **locally convex coalgebra**, **locally convex bialgebra** *and* **locally convex Hopf algebra***: These are locally convex spaces together with appropriate continuous structure maps such that, respectively, the axioms of coassociativity, counit, compatibility of multiplication and comultipication and the axiom of the antipode are fulfilled with $\hat{\otimes}$ as tensor product functor. Likewise there exists an appropriate notion of morphism of locally convex coalgebras, locally convex bialgebras and locally convex Hopf algebras, i.e. continuous linear maps which leave the structure maps invariant.*

*Finally, a (***strictly***)* **nuclear Hopf algebra** *H is a locally convex Hopf algebra such that the underlying locally convex space H is (strictly) nuclear. Similarly define the notions of (***strictly***)* **nuclear coalgebras** *and* **bialgebras**.

Remark 1.2 A nuclear space A comprises a nuclear algebra if and only if it has an underlying structure of a $\mathbb{K}$-algebra such that the multiplication $\mu : A \times A \to A$ is continuous. A similar result does not hold for coalgebras. Namely there exist nuclear coalgebras C which do not have an underlying structure of a coalgebra. In other words this means that the map $\Delta : C \to C\hat{\otimes}C$ need not have its image in $C \otimes C$. An example is given by the quantized $\mathfrak{sl}(N+1,\mathbb{C})$ of section 5.

Within the topological setting we also have to define appropriate topological versions of R-matrix, triangularity, coquasitriangularity, etc.

Definition 1.3 *Let H be a locally convex Hopf algebra or bialgebra. It is called* **topologically quasitriangular** *if there exists an invertible element $\mathcal{R} \in H\hat{\otimes}H$ such that the following conditions hold:*

$$\begin{aligned} \tau \circ \Delta(a) &= \mathcal{R}\,\Delta(a)\,\mathcal{R}^{-1} && (1)\\ (\Delta \otimes \mathrm{id})(\mathcal{R}) &= \mathcal{R}_{13}\,\mathcal{R}_{23} && (2)\\ (\mathrm{id} \otimes \Delta)(\mathcal{R}) &= \mathcal{R}_{13}\,\mathcal{R}_{12}, && (3) \end{aligned}$$

where $\tau : H\hat{\otimes}H \to H\hat{\otimes}H$ is the flip-morphism and $\mathcal{R}_{12}$, $\mathcal{R}_{13}$, $\mathcal{R}_{23}$ are the obvious extensions of $\mathcal{R}$ to $H\hat{\otimes}H\hat{\otimes}H$ which are trivial on the third, resp. second, resp. first factor. In this case $\mathcal{R}$ is called the **topological universal R-matrix** *of H. If additionally*

$$\mathcal{R}^{-1} = \tau \circ \mathcal{R} \qquad (4)$$

then H is called **topologically triangular**.

Dually H is called **topologically coquasitriangular**, *if there exists a continuous bilinear map* $<\ |\ >: H\hat{\otimes}H \to \mathbb{C}$, *the* **braiding form**, *such that for all $a, b, c \in H$*

$$\sum_{(a),(b)} < a_1|b_1 > \ a_2\, b_2 \quad = \quad \sum_{(a),(b)} a_1\, b_1 \ < a_2|b_2 >, \tag{5}$$

$$< a|bc > \quad = \quad \sum_{(a)} < a_1|b >< a_2|c >, \tag{6}$$

$$< ab|c > \quad = \quad \sum_{(c)} < a|c_1 >< b|c_2 > . \tag{7}$$

One of the main reasons to consider strictly nuclear Hopf algebras instead of just Hopf algebras lies in the fact that the category of strictly nuclear Hopf algebras has duals.

Proposition 1.4 *Let H be a strictly nuclear Hopf algebra. Then the dual space H' carries in a natural way the structure of a nuclear Hopf algebra. Moreover, H can be recovered as H''.*

If H is topologically quasitriangular (resp. coquasitriangular) then H' is topologically coquasitriangular (resp. quasitriangular).

The same properties hold for reflexive locally convex Hopf algebras fulfilling the duality condition.

Proof: For the first part of the proposition just apply the isomorphism $(H\hat{\otimes}H)' \cong H'\hat{\otimes}H'$ to obtain the coproduct Δ' on H' as the pull-back

$$\mu^* : \ H' \longrightarrow (H\hat{\otimes}H)', \quad f \longmapsto f \circ \mu. \tag{8}$$

The other structure maps of H' are directly defined by transposition.

For the second part note that an element $\mathcal{R} \in H\hat{\otimes}H$ induces a continuous bilinear form $<\ |\ >_{\mathcal{R}}: H'\hat{\otimes}H' \to \mathbb{C}$ by

$$f \otimes g \mapsto f \otimes g(\mathcal{R}). \tag{9}$$

Furthermore, by the isomorphism $(H\hat{\otimes}H)' \cong H'\hat{\otimes}H'$ every continuous bilinear form $<\ |\ >: H \otimes H \to \mathbb{C}$ can be interpreted as an element $\mathcal{R}_{<\,|\,>} \in H'\hat{\otimes}H'$. The proof of the required algebraic properties for the thus defined braiding form $<\ |\ >_{\mathcal{R}}$ resp. R-matrix $\mathcal{R}_{<\,|\,>}$ follows exactly like in the well-known finite dimensional case. □

Let us finally mention that all the nuclear spaces given in the Appendix, in particular the function spaces $\mathcal{E}(\Omega)$ and $\mathcal{O}(\Omega)$ as well as all finite dimensional algebras (resp. coalgebras, bialgebras and Hopf algebras), comprise examples of nuclear algebras (resp. coalgebras, bialgebras and Hopf algebras).

2 Natural locally convex and nuclear Hopf algebras

In this section we will consider some general constructions and examples of locally convex Hopf algebras. Note that analoguous topological constructions can be carried through for locally convex algebras and bialgebras as well.

2.1 Inductive limit topologies on Hopf algebras. On a given Hopf algebra H over $\mathbb{K}$ we can always consider the finest locally convex topology. Then H is a complete locally convex Hausdorff space and, since all structure maps are automatically continuous for the finest locally convex topology, H is a locally convex Hopf algebra. But this locally convex Hopf algebra H is locally m-convex only if H is finite dimensional and nuclear only if H is of countable dimension. The dual H' with the strong topology carries the coarsest locally convex topology. Since $H \cong \mathbb{K}^{(\Lambda)}$ satisfies the duality condition (see Appendix (88)) H' is, according to proposition 1.4, a locally convex Hopf algebra as well. For the special case of a group algebra $H = \mathbb{K}G$ of a group G this dual is the Hopf algebra $H' \cong \mathbb{K}^G$ of all functions on G.

2.2 Projective limit topologies on Hopf algebras. Alternatively one could provide a given Hopf algebra H over $\mathbb{K}$ with the locally convex projective limit **topology of finite dimensional representations**, i.e. with the coarsest locally convex topology leaving continuous all finite dimensional representations $\varphi : H \to \operatorname{End} V$.

Let us assume that these representations separate the points of H. According to the later proved Proposition 2.4 this is e.g. the case for finitely generated Hopf algebras H. It also holds for the universal enveloping algebra $H = \mathcal{U}\mathfrak{g}$ of a finite dimensional Lie algebra $\mathfrak{g}$. Then H is Hausdorff locally convex space. H is complete only if H is finite dimensional. The completion $\hat{H}$ of H however lies in $\underline{\mathsf{Nuc}}$ since it is a locally convex projective limit of finite dimensional spaces. All structure maps of the Hopf algebra H are continuous since the topology is adapted to the finite dimensional representations. Therefore, they can be uniquely extended to $\hat{H}$ and thus turn $\hat{H}$ into a nuclear Hopf algebra. In addition $\hat{H}$ is locally m-convex.

The dual $\hat{H}'$ of $\hat{H}$ (or of H) is the space of matrix coefficients on H:

$$\hat{H}' \quad = \quad \{\xi \circ \hat{\varphi} \mid \varphi : H \to \operatorname{End} V \text{ finite dim. representation, } \xi \in (\operatorname{End} V)'\}\,. \tag{10}$$

Hence, $\hat{H}'$ coincides with the restricted dual H° of H (see Chari, Pressley [5] Chapter 4.1.D). The strong topology on $\hat{H}'$ is given by the locally convex inductive limit topology of the maps

$$\varphi' : (\operatorname{End} V)' \to \hat{H}', \quad \xi \mapsto \xi \circ \hat{\varphi}, \tag{11}$$

where φ runs through all finite dimensional representations of H. Thus the strong topology on $\hat{H}'$ is the finest locally convex topology. Although the locally convex space $\hat{H}' = H^\circ$ is in general not nuclear, it satisfies the duality condition (cf. Appendix, Eq. (88)). Therefore, as in the Proposition 1.4 the transpositions of the structure maps of the locally convex

Hopf algebra $\hat{H}$ define the structure of a locally convex Hopf algebra on $\hat{H}'$. Dualizing again one gets the Hausdorff completion $H^{\circ\prime}$ of H (which is $\hat{H}$ if H is assumed to be Hausdorff).

The locally convex space $\hat{H}'$ will be nuclear, even strictly nuclear, whenever countably many of the finite dimensional representations generate the topology of H, i.e. if $\hat{H}$ is Fréchet. This is the case e.g. for the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ of a semi-simple Lie algebra $\mathfrak{g}$ (see below).

In the same spirit one can attach to H the projective limit topology with respect to all homomorphisms $H \to A$, where A is a nuclear locally m-convex Fréchet algebra. We call the resulting projective limit topology the **topology of nuclear Fréchet representations**. The completion $\check{H}$ of H with respect to this topology again is a nuclear Hopf algebra. Furthermore, one has a natural continuous inclusion $\check{H} \to \hat{H}$.

More generally, one can consider other projective systems of representations of H in order to define appropriate locally convex topologies on H which have their origin in purely algebraic properties of H.

Remark 2.3 In the case $A = \mathbb{C}[x_1, ..., x_n]$ the topology of finite dimensional representations is the topology of pointwise convergence of all derivatives. Hence, $\hat{A}$ is not a Fréchet space. Similarly, the tensor algebra $\mathrm{T}V$ endowed with the topology of finite dimensional representations is a nuclear m-convex algebra which is not metrizable.

However, the completion of $A = \mathbb{C}[x_1, ..., x_n]$ with respect to the topology of nuclear Fréchet representations is isomorphic to the nuclear m-convex Fréchet algebra $\mathcal{O}(\mathbb{C}^n)$ of entire holomorphic functions on $\mathbb{C}^n$.

We will determine in the following the topologies of finite dimensional representations and of Fréchet representations for some important algebras.

Proposition 2.4 *Let A be a finitely generated algebra and $\mathrm{T}V \xrightarrow{\pi} A$ a presentation of A, where $\mathrm{T}V$ is the tensor algebra of a finite dimensional $\mathbb{K}$-vector space V. Denote by $\hat{\mathrm{T}}V$ and $\hat{A}$ (resp. $\check{\mathrm{T}}V$ and $\check{A}$) the completions of $\mathrm{T}V$ and A with repsect to the topology of finite dimensional representations (resp. of nuclear Fréchet representations). Then the algebras $\hat{\mathrm{T}}V$, $\hat{A}$, $\check{\mathrm{T}}V$ and $\check{A}$ are locally m-convex nuclear Hausdorff spaces. The spaces $\check{\mathrm{T}}V$ and $\check{A}$ are even Fréchet. Furthermore, the presentation $\mathrm{T}V \xrightarrow{\pi} A$ extends uniquely to surjective and open maps $\hat{\mathrm{T}}V \xrightarrow{\hat{\pi}} \hat{A}$ and $\check{\mathrm{T}}V \xrightarrow{\check{\pi}} \check{A}$, i.e. to topological presentations of $\hat{A}$ and $\check{A}$.*

PROOF: By the universal property of the complete hull we have unique homomorphisms $\hat{\mathrm{T}}V \xrightarrow{\hat{\pi}} \hat{A}$ and $\check{\mathrm{T}}V \xrightarrow{\check{\pi}} \check{A}$ both extending $\mathrm{T}V \xrightarrow{\pi} A$. We will show that they have the claimed properties.

First consider the topology of finite dimensional representations. Let us show that this topology is Hausdorff or in other words that the finite dimensional representations of A separate the points of A. Consider the ideals $I_n = \oplus_{k \geq n} V^{\otimes k}$ in $\mathrm{T}V$. Their images in

A define ideals J_n in A. Now for two elements $a, b \in A$, $a \neq b$ there exists an $n \in \mathbb{N}$ large enough such that $a - b$ dos not vanish in A/J_n. But A/J_n is finite dimensional and an A-module. Thus the points a and b are separated by the representation of A on A/J_n.

The continuous homomorphism $\hat{\mathrm{T}}V \rightarrow \hat{A}$ is an open map. To see this choose an open set $U \subset \hat{\mathrm{T}}V$. We can assume that there exists a finite dimensional representation $\varphi : \hat{\mathrm{T}}V \rightarrow \operatorname{End} W$ and an open $O \subset \operatorname{End} W$ such that $U = \varphi^{-1}(O)$. Let $\hat{I}$ be the kernel of $\hat{\pi}$, and $\tilde{W}$ be the algebra $\operatorname{im} \varphi / \varphi(\hat{I})$. Then φ induces a representation $\tilde{\varphi} : A \rightarrow \tilde{W} \subset \operatorname{End} \tilde{W}$. As projections between finite dimensional spaces are open there exists an open $\tilde{O} \subset \operatorname{End} \tilde{W}$ such that $\tilde{O} \cap \operatorname{im} \tilde{\varphi} = O + \varphi(\hat{I})$. We then have $\hat{\pi}(U) = \tilde{\varphi}^{-1}(\tilde{O})$, i.e. $\hat{\pi}(U)$ is open in $\hat{A}$.

As $\hat{\mathrm{T}}V \rightarrow \hat{A}$ is continuous, open and has dense image, it is surjective, hence, provides a continuous presentation of $\hat{A}$.

Now let us consider the case of nuclear Fréchet representations. Choose a basis $(x_1, ..., x_n)$ of V and recall that $\mathrm{T}V$ is canonically isomorphic to the linear space $\mathbb{C}^{(<x_1,...,x_n>)}$, where $< x_1, ..., x_n >$ is the free half group generated by $x_1, ..., x_n$. For any n-tupel $\alpha \in (\mathbb{Q}^+)^n$ define the seminorm p_α on V by

$$p_\alpha \left(\sum_{1 \leq k \leq n} \lambda_k \, x_k \right) = \sum_{1 \leq k \leq n} |\lambda_k| \, \alpha_k. \tag{12}$$

This seminorm naturally extends to a seminorm on $V^{\otimes k}$, $k \in \mathbb{N}$ and then to a seminorm on $\mathrm{T}V$. By definition $p_\alpha(v \otimes w) = p_\alpha(v) \, p_\alpha(w)$ holds for every $v, w \in \mathrm{T}V$. Hence, the countable system of seminorms p_α gives rise to a metrizable locally m-convex topology on $\mathrm{T}V$. Consequently the completion E of $\mathrm{T}V$ with respect to the topology generated by the seminorms p_α is locally m-convex and Fréchet. Now assume that $\rho : \mathrm{T}V \rightarrow F$ is a representation with F a nuclear locally m-convex Fréchet algebra. Let p be a multiplicative seminorm on F. Choose $\alpha_k \in \mathbb{Q}^+$, $k = 1, ..., n$ so large that $p(\rho(x_k)) \leq p_\alpha(x_k)$. By the multiplicativity of p and the definition of p_α this implies $p(\rho(v)) \leq p_\alpha(v)$ for every $v \in \mathrm{T}V$. Hence, ρ extends to a continuous homomorphism $E \rightarrow F$.

Let us suppose for a moment that E is nuclear. Since the inclusion $\mathrm{T}V \rightarrow E$ is a continuous Fréchet representation, the nuclearity of E entails the relation $E \cong \check{T}V$. Denoting by $\check{I}$ the completion of the kernel I of π, the quotient algebra $E/\check{I}$ then is nuclear as well, locally m-convex and Fréchet. As A lies densely in $E/\check{I}$ the same argument like for E shows that $E/\check{I}$ is the completion $\check{A}$. Hence, the claim follows.

So, it remains to show that E is nuclear. We will achieve this by an argument using the sequence spaces of Pietsch [20]. Denote for every $\beta \in < x_1, ..., x_n >$ by $\ell(\beta)$ the multilength of β; that means $\ell(\beta)$ is the n-tupel $(\ell_1, ..., \ell_n)$ where ℓ_k counts how many times x_k appears in β. The absolute length $\ell_1 + ... + \ell_n$ of β is denoted by $|\ell(\beta)|$. Now choose a bijection $\xi : \mathbb{N} \rightarrow < x_1, ..., x_n >$ such that for $\beta, \tilde{\beta} \in < x_1, ..., x_n >$ fulfilling $|\ell(\beta)| < |\ell(\tilde{\beta})|$ the relation $\xi^{-1}(\beta) < \xi^{-1}(\tilde{\beta})$ holds. By definition E is then isomorphic to the sequence space (cf. Pietsch [20], Chapter 6) generated by the system of sequences

$(\lambda_k^\alpha)_{k\in\mathbb{N}}$ with $\lambda_k^\alpha = \alpha^{\ell(\xi(k))}$ and $\alpha \in (\mathbb{Q}^+)^n$. For $0 < q < \frac{1}{n}$ we now have

$$\sum_{\beta\in<x_1,\ldots,x_n>} q^{|\ell(\beta)|} = \sum_{k\in\mathbb{N}} (n\,q)^k < \infty. \tag{13}$$

By Pietsch [20], Theorem 6.1.2. this implies that E is nuclear. This proves the claim. $\square$

2.5 Matrix coefficients of group representations. For a group G let us consider the Hopf algebra of complex-valued matrix coefficients $\mathcal{R}_0(G) := \{\xi \circ \varphi |\, \varphi : G \to \mathrm{GL}\, V$ finite dimensional representation, $\xi \in (\mathrm{End}\, V)'\}$ in the light of 2.2 and 2.1. (Here we assume V always to be finite dimensional **complex** vector spaces.) $\mathcal{R}_0(G)$ with the finest locally convex topology is a locally convex Hopf algebra fulfilling the duality condition. This topology can also be described as the locally convex inductive limit of the maps $\varphi' : (\mathrm{End}\, V)' \to \mathcal{R}_0(G)$ where φ runs through all finite dimensional representations of G.

The dual $\mathcal{R}_0(G)'$ of $\mathcal{R}_0(G)$ endowed with the strong topology is a locally convex projective limit of finite dimensional algebras and thus is a nuclear Hopf algebra which is locally m-convex. By the duality condition for $\mathcal{R}_0(G)$ the dual $\mathcal{R}_0(G)'$ obtains as in Proposition 1.4 the structure of a nuclear Hopf algebra.

In case of a topological group G we replace $\mathcal{R}_0(G)$ by the continuous matrix coefficients $\mathcal{R}(G) \subset \mathcal{R}_0(G)$. $\mathcal{R}(G)$ with the finest locally convex topology is a locally convex Hopf algebra as well and the dual $\mathcal{R}(G)'$ is a nuclear m-convex algebra. In general, $\mathcal{R}(G)$ is not nuclear. For compact groups, however, $\mathcal{R}(G)$ is strictly nuclear since by the theorem of Peter and Weyl it is of countable dimension. Moreover, the dual $\mathcal{R}(G)'$ is a Fréchet nuclear locally m-convex algebra.

The dual $\mathcal{R}_0(G)'$ contains all the evaluations $\delta_x : \mathcal{R}_0(G) \to \mathbb{C}$, $f \mapsto f(x)$, where $x \in G$. The map $\delta : G \to \mathcal{R}_0(G)'$ defines a Hopf algebra morphism $\delta : \mathbb{C}G \to \mathcal{R}_0(G)'$. In the case of a topological group one analogously has a natural continuous Hopf algebra map $\delta : \mathbb{C}G \to \mathcal{R}(G)'$. For compact groups δ is injective, open onto its image, and $\mathrm{span}(\delta(G))$ is dense in $\mathcal{R}(G)'$ (cf. [4]).

Other models of locally convex Hopf algebras consisting of representative functions can be considered by fixing a suitable class of representations of G in locally convex vector spaces being closed under dualizing, (finite) direct sums and (finite) completed tensor products.

2.6 Universal enveloping algebras. Starting with a Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ is a Hopf algebra over $\mathbb{K}$. One can study $\mathcal{U}\mathfrak{g}$ as a topological Hopf algebra with respect to the finest locally convex topology as in 2.1. However, the projective limit topologies of 2.2 are more interesting in some aspects.

So consider the coarsest locally convex topology on $\mathcal{U}\mathfrak{g}$ such that all finite dimensional representations $\rho : \mathcal{U}\mathfrak{g} \to \mathrm{End}_{\mathbb{C}} V$ are continuous. Let us assume that these finite dimensional representations separate the points of $\mathcal{U}\mathfrak{g}$. This is always the case for a finite dimensional Lie algebra by Ado's theorem. Then the topology of finite dimensional representations on $\mathcal{U}\mathfrak{g}$ is Hausdorff, and the completion $\hat{\mathcal{U}}\mathfrak{g}$ of $\mathcal{U}\mathfrak{g}$ is a nuclear Hopf algebra as in 2.2. Moreover, $\hat{\mathcal{U}}\mathfrak{g}$ is locally m-convex.

For a Lie group G with Lie algebra $\mathfrak{g}$ there is a natural map i relating $\hat{\mathcal{U}}\mathfrak{g}$ and the nuclear (cf. 2.5) Hopf algebra $\mathcal{R}(G)'$. $i : \mathcal{U}\mathfrak{g} \to \mathcal{R}(G)'$ is defined by $i(X)(f) = L_X f(e)$, $f \in \mathcal{R}(G)$, where L_X is the left invariant differential operator on G given by $X \in \mathcal{U}\mathfrak{g}$ and where e is the unit of G. Comparing the topologies one sees that i is continuous since every finite dimensional representation φ of G induces a representation $\dot{\varphi}$ on $\mathfrak{g}$ by differentiation and since $i(X)(\xi \circ \varphi) = \xi(\dot{\varphi}(X))$. Hence, i can be extended to a continuous $\mathbb{R}$-linear map $i : \hat{\mathcal{U}}\mathfrak{g} \to \mathcal{R}(G)'$ which is a morphism of nuclear Hopf algebras. i is not injective in general. For connected and simply connected Lie groups, however, i is injective. Moreover, in that case i is an open map onto its image $i(\hat{\mathcal{U}}\mathfrak{g})$, since the finite dimensional complex representations of $\mathfrak{g}$ and G are in one-to-one correspondence. Therefore, $\hat{\mathcal{U}}\mathfrak{g}$ can be considered as a closed nuclear sub-Hopf algebra of $\mathcal{R}(G)'$. But in general i does not need to be open for every Lie group G. Take for example $G = \mathrm{U}(1)$ and its Lie algebra $\mathfrak{g} \cong \mathbb{R}$. Then $\mathfrak{g}$ is also the Lie algebra of $\mathbb{R}$ and $\mathcal{U}\mathfrak{g} \cong \mathbb{R}[T]$. Though $\hat{\mathcal{U}}\mathfrak{g}$ is a closed sub-Hopf algebra of $\mathcal{R}(\mathbb{R})' \cong \mathbb{C}^{\mathbb{R}}$ the map $i : \mathcal{U}\mathfrak{g} \to \mathcal{R}(\mathrm{U}(1))' \cong \mathbb{C}^{\mathbb{N}}$ cannot be open since the topology of finite dimensional representations on $\mathcal{U}\mathfrak{g}$ is not metrizable (cf. 2.3).

2.7 Simple Lie algebras. If $\mathfrak{g}$ is a simple Lie algebra over $\mathbb{C}$ then finitely many of the finite dimensional representations already generate all finite dimensional representations of $\mathfrak{g}$ (via finite sums and tensor products; e.g. for $\mathfrak{g} = \mathfrak{s}l(N, \mathbb{C})$ the finite dimensional representations are generated by the fundamental representation $\mathfrak{s}l(N, \mathbb{C}) \subset \mathfrak{g}l(N, \mathbb{C})$). As a consequence, the topology of finite dimensional representations on $\hat{\mathcal{U}}\mathfrak{g}$ is metrizable and hence Fréchet. Therefore, in the simple case $\hat{\mathcal{U}}\mathfrak{g}$ is in particular strictly nuclear.

For a simple complex Lie algebra $\mathfrak{g}$ with corresponding connected and simply connected Lie group G the image $i(\mathcal{U}\mathfrak{g})$ is dense in $\mathcal{R}(G)'$. Hence, by the above, the map $i : \hat{\mathcal{U}}\mathfrak{g} \to \mathcal{R}(G)'$ is an isomorphism of Fréchet algebras. As a consequence, in this situation we have a natural complete duality between $\hat{\mathcal{U}}\mathfrak{g}$ and $\mathcal{R}(G)'$. In particular, $\hat{\mathcal{U}}\mathfrak{s}l(N, \mathbb{C}) \cong \mathcal{R}(\mathrm{SL}(N, \mathbb{C})'$.

In the case of a compact Lie group G the map $i : \mathcal{U}\mathfrak{g} \to \mathcal{R}(G)'$ is injective as well. It can be continued to a $\mathbb{C}$-linear injective map $i : \mathcal{U}\mathfrak{g}^{\mathbb{C}} \to \mathcal{R}(G)'$ by complexification of the Lie algebra $\mathfrak{g}$. The image $i\left(\mathcal{U}\mathfrak{g}^{\mathbb{C}}\right)$ turns out to be dense in $\mathcal{R}(G)'$. However, the induced topology on $i\left(\mathcal{U}\mathfrak{g}^{\mathbb{C}}\right) \subset \mathcal{R}(G)'$ does in general not coincide with the topology coming from the projective topology of finite dimensional representations; see, e.g. the example of $U(1)$. Instead of this, the inclusion i induces a new locally convex topology on $\mathcal{U}\mathfrak{g}^{\mathbb{C}}$ which can be described as the locally convex projective limit of all representations $\dot{\varphi}$ which are derivatives of finite dimensional continuous representations φ of the group G.

This topology depends on the group in question and not only on the Lie algebra $\mathfrak{g}$. It is always metrizable and nuclear. Hence, the completion – which we denote by $\tilde{\mathcal{U}}\mathfrak{g}^{\mathbb{C}}$ – is a Fréchet nuclear Hopf algebra naturally isomorphic to $\mathcal{R}(G)'$.

As the completion $\check{\mathcal{U}}\mathfrak{g}$ of $\mathcal{U}\mathfrak{g}$ with respect to the topology of nuclear Fréchet representations naturally lies in $\hat{\mathcal{U}}\mathfrak{g}$ we have a continuous inclusion $\check{\mathcal{U}}\mathfrak{g} \to \mathcal{R}(G)'$ as well. Note that $\hat{\mathcal{U}}\mathfrak{g}$, $\check{\mathcal{U}}\mathfrak{g}$, $\tilde{\mathcal{U}}\mathfrak{g}^{\mathbb{C}}$ and $\mathcal{R}(G)'$ are locally m-convex algebras as projective limits of locally m-convex algebras.

2.8 Function and distribution spaces on Lie groups All the above examples of locally convex and nuclear Hopf algebras are based essentially on the algebraic structures of the Hopf algebra in question. In many cases they carry the finest or the coarsest locally convex topology. In the case of a Lie group G other topological structures can be imposed on certain Hopf algebras of smooth functions on G. These topologies arise from the analytical structure on G. Similarly new locally convex Hopf algebra structures on the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$ are of interest.

For example, on the space of complex valued smooth functions $\mathcal{E}(G) := \mathcal{C}^\infty(G)$ the topology of uniform convergence of all derivatives on the compact sets of G induces on $\mathcal{E}(G)$ the structure of a strictly nuclear Hopf algebra. $\mathcal{E}(G)$ is a locally m-convex algebra. Its dual $\mathcal{E}'(G)$ contains G by the evaluations δ_x, $x \in G$, and hence it contains the group algebra $\mathbb{C}G$. $\mathcal{E}'(G)$ also contains $\mathcal{U}\mathfrak{g}$ via the injective Hopf algebra map $i : \mathcal{U}\mathfrak{g} \to \mathcal{E}'(G)$, where $i(X) = \delta_e \circ L_X$ now acts on the bigger space $\mathcal{E}(G)$. Note that this map is not continuous in general. It gives rise to yet another locally convex topology on $\mathcal{U}\mathfrak{g}$ whose completion is a strictly nuclear Hopf algebra.

Another dual pair of strictly nuclear Hopf algebras is given by the test functions $\mathcal{D}(G) = \mathcal{C}_c^\infty(G) \subset \mathcal{E}(G)$ and its dual $\mathcal{D}'(G)$, the space of distributions on G. We now have the following continuous inclusions:

$$\begin{array}{ccccc} \mathcal{R}(G) & \subset & \mathcal{E}(G)\ ,\ \mathcal{E}'(G) & \subset & \mathcal{R}(G)' \\ \mathbb{K}G & \subset & \mathcal{E}'(G) & \subset & \mathcal{D}'(G) \\ \mathcal{D}(G) & \subset & \mathcal{E}(G) & \subset & \mathbb{K}^G \cong \mathbb{K}G'. \end{array} \tag{14}$$

2.9 FRT-bialgebras. In the following we will show how the construction of quantum matrix algebras by FADDEEV, RESHETIKHIN, TAKHTAJAN [7] can be implemented in our concept of nuclear Hopf algebras and comprises a further nontrivial example.

First we introduce some notation. Let A be an arbitrary algebra. The space of $n \times m$ matrices with entries from A is denoted by $\mathrm{M}(n \times m, A)$. The Kronecker product is then the mapping

$$\odot : \mathrm{M}(n \times m, A) \otimes \mathrm{M}(k \times l, A) \to \mathrm{M}(nk \times ml, A) \tag{15}$$

defined by

$$(M \odot N)_{\alpha\iota,\beta\kappa} = M_{\alpha,\beta}\, N_{\iota,\kappa}. \tag{16}$$

Now let V be a n-dimensional complex vector space and let $R : \Omega \to \mathrm{End}\,(V) \otimes \mathrm{End}\,(V)$ be a holomorphic map defined on an open domain $\Omega \subset \mathbb{C}$ such that $1 \in \Omega$ and $R(1) = 1$.

Choosing a basis $(x_1, ..., x_n)$ of V we can regard R as a $n \times n$ matrix with entries in $\mathcal{O}(\Omega)$. The basis $(x_1, ..., x_n)$ naturally induces a basis (t_ι^κ) of the coalgebra $C = (\mathrm{End}\ V)'$. Furthermore, it gives rise to the matrix $T \in \mathrm{M}(n \times n, \mathrm{T}C)$ the entries of which are the t_ι^κ. The tensor algebra $\mathrm{T}C$ shall carry the inductive limit topology of all finite dimensional subspaces. Thus it is a nuclear bialgebra with coproduct

$$\Delta(t_\iota^\kappa) = \sum_{1 \leq \alpha \leq n} t_\iota^\alpha \otimes t_\alpha^\kappa. \tag{17}$$

and counit

$$\epsilon(t_\iota^\kappa) = \delta_\iota^\kappa. \tag{18}$$

We now construct the **quantum matrix algebra** or **FRT-bialgebra** $A(R)$ as the quotient of $\mathrm{T}C$ modulo the closed ideal I generated by the coefficients of the matrix

$$(T \odot 1)\,(1 \odot T)\,R \;-\; R\,(1 \odot T)\,(T \odot 1). \tag{19}$$

By Larson and Tauber [15] or Manin [16] I is a biideal, so $A(R)$ becomes a nuclear bialgebra with a continuous coaction on $\mathcal{O}(\Omega) \otimes V$ given by

$$\Psi_{V,R} : \mathcal{O}(\Omega) \otimes V \to A(R) \otimes V, \quad x_\iota \mapsto t_\iota^\kappa \otimes x_\kappa. \tag{20}$$

Using Theorem and Definition 3.1 of Larson and Tauber [15] it follows that $A(R)$ is the initial object with respect to the following properties:

(FRT1) $\mathcal{O}(\Omega) \otimes V$ is a topological left comodule over $A(R)$ via the structure map $\Psi_{V,R}$.

(FRT2) R is an $A(R)$-comodule map with respect to the natural structure of a topological left $A(R)$-comodule on $\mathcal{O}(\Omega) \otimes V \otimes V$.

If R is a Yang-Baxter operator, i.e. R is non-degenerate and fulfills the quantum Yang-Baxter equation (QYBE)

$$R_{12} \circ R_{13} \circ R_{23} \;=\; R_{23} \circ R_{13} \circ R_{12}, \tag{21}$$

then Theorem and Definition 5.1 of Larson and Tauber [15] imply that $A(R)$ is topologically coquasitriangular. The braiding form $< \mid >$: $A(R) \otimes A(R) \to \mathbb{C}$ can furthermore be calculated by the relations

$$< t_\iota^i | t_\kappa^j > = \; R_{\iota\kappa}^{ji}. \tag{22}$$

3 Holomorphic deformation of locally convex algebras

In our approach to deformation theory we replace the ring $\mathbb{C}[[h]]$ used in the formal deformation theory of Gerstenhaber [9, 11, 4] by the ring $\mathcal{O}(\Omega)$ of holomorphic functions on an open domain $\Omega \subset \mathbb{C}^n$. Note that in the case of $\Omega = \mathbb{C}$ we have the following relations and inclusions:

$$\begin{array}{ccccc} \mathbb{C}[h] & \longmapsto & \mathbb{C}[[h]] & \longmapsto & \mathbb{C}[h] \\ \cap & & \cup & & \cap \\ \mathcal{O}(\mathbb{C}) & \longmapsto & \mathcal{O}\{z\} & \longmapsto & \mathcal{O}(\mathbb{C}) \\ \cap & & \cup & & \cap \\ \mathbb{C}[[h]] & \longmapsto & \mathbb{C}[h] & \longmapsto & \mathbb{C}[[h]], \end{array} \tag{23}$$

where the arrows denote passing to the topological dual, and where $\mathcal{O}\{z\}$ denotes the algebra of convergent power series in z with the natural locally convex inductive limit topology.

In the following let Ω be an open domain in $\mathbb{C}^n$ and let $\mathcal{O} = \mathcal{O}(\Omega)$ be the nuclear Fréchet algebra of holomorphic functions on Ω. For a complete locally convex Hausdorff space E let

$$E_\Omega \;=\; \mathcal{O}(\Omega, E) \tag{24}$$

be the space of holomorphic E-valued functions $f : \Omega \to E$ equipped with the compact open topology. Every E can locally be represented by a convergent power series $f(z) = \sum_{\alpha\in\mathbb{N}^n} f_\alpha z^\alpha$, $z \in \Omega$ where $f_\alpha \in E$ (cf. Appendix). E_Ω is a complete locally convex Hausdorff space which is isomorphic to the completion of the tensor product $\mathcal{O}(\Omega) \otimes E$ with respect to the π-topology (cf. Appendix):

$$E_\Omega \;=\; \mathcal{O}(\Omega, E) \;\cong\; \mathcal{O}(\Omega)\hat{\otimes}E. \tag{25}$$

Hence, E_Ω is nuclear (resp. Fréchet), if E is nuclear (resp. Fréchet). The pointwise multiplication $\mathcal{O}(\Omega) \times E_\Omega \to E_\Omega$, $(\lambda, f) \mapsto \lambda f$ is continuous and thus defines on E_Ω the structure of a **topological $\mathcal{O}(\Omega)$-module**. Because of Eq. (25) we sometimes call E_Ω a **topologically free $\mathcal{O}(\Omega)$-module**.

For a locally convex algebra A the space

$$A_\Omega \;=\; \mathcal{O}(\Omega, A) \;\cong \mathcal{O}(\Omega)\hat{\otimes}A \tag{26}$$

is a locally convex algebra and a topological module over $\mathcal{O}(\Omega)$. The algebra structure on A_Ω which we call the **constant algebra structure** is given by pointwise multiplication and trivial extension of the unit: If $\mu : A\hat{\otimes}A \to A$ and $\eta : \mathbb{C} \to A$ are multiplication and unit of A respecticely, the corresponding structure maps of the constant algebra structure on A_Ω are given by

$$\mu = \mu_\Omega : \quad A_\Omega \times A_\Omega \to A_\Omega, \quad (f, g) \mapsto (\Omega \ni z \mapsto \mu_\Omega(f, g)(z) = \mu(f(z), g(z)) \in A) \tag{27}$$

and

$$\eta = \eta_\Omega : \quad \mathbb{C} \to A_\Omega, \quad \lambda \mapsto \lambda \otimes \eta(1). \tag{28}$$

Evidently the mappings μ_Ω and η_Ω are continuous and fulfill the axioms of associativity and unit. They are in fact holomorphic in an obvious sense. Under the isomorphism $\mathcal{O}(\Omega, A) \cong \mathcal{O}(\Omega)\hat{\otimes}A$ the product μ_Ω is given by

$$A_\Omega \otimes A_\Omega \ni (f \otimes a) \otimes (g \otimes b) \mapsto (fg \otimes ab) \in A_\Omega. \tag{29}$$

In case A is a complete locally m-convex algebra we have an obvious functional calculus on A_Ω, i.e. for any holomorphic function $f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha z^\alpha$ in $\mathcal{O}(\mathbb{C})$ and any element $a \in A$ there is a unique $f(a) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha a^\alpha \in A$ (cf. Appendix).

In deformation theory we are now interested in algebra structures on A_Ω different from the constant one. Let us describe this in more detail. Define a **topological $\mathcal{O}(\Omega)$-algebra structure** on E_Ω to be an algebra structure on $E_\Omega = \mathcal{O}(\Omega, E)$ given by a continuous $\mathcal{O}(\Omega)$-bilinear map

$$\tilde{\mu} : E_\Omega \times E_\Omega \longrightarrow E_\Omega \tag{30}$$

fulfilling the associativity constraint and a continuous $\mathcal{O}(\Omega)$-linear unit

$$\tilde{\eta} : \mathcal{O}(\Omega) \longrightarrow E_\Omega. \tag{31}$$

We often denote this algebra $(E_\Omega, \tilde{\mu}, \tilde{\eta})$ by $\tilde{E}$. $\tilde{\eta}$ is determined by $\tilde{\eta}(1) \in E_\Omega$.

Two such algebra structures $(\tilde{\mu}, \tilde{\eta})$ and $(\check{\mu}, \check{\eta})$ on E_Ω are called **equivalent** if there exists a $\mathcal{O}(\Omega)$-linear isomorphism $\varphi : E_\Omega \longrightarrow E_\Omega$ (of locally convex spaces) such that the relations

$$\varphi \circ \check{\mu} \;=\; \tilde{\mu} \circ (\varphi \times \varphi), \tag{32}$$
$$\varphi \circ \check{\eta} \;=\; \tilde{\eta} \tag{33}$$

hold.

The multiplication $\tilde{\mu}$ of an $\mathcal{O}(\Omega)$-algebra $(E_\Omega, \tilde{\mu}, \tilde{\eta})$ can be regarded to be a continuous $\mathcal{O}(\Omega)$-linear map $\tilde{\mu} : E_\Omega \hat{\otimes}_{\mathcal{O}(\Omega)} E_\Omega \to E_\Omega$, where $E_\Omega \hat{\otimes}_{\mathcal{O}(\Omega)} E_\Omega$ is isomorphic (as a topological $\mathcal{O}(\Omega)$-module) to $\mathcal{O}(\Omega, E\hat{\otimes}E)$ (cf. Appendix). Consequently $\tilde{\mu}$ can be described as a holomorphic map $\tilde{\mu} : \Omega \to \mathcal{L}(E\hat{\otimes}E, E)$, where $\mathcal{L}(E, F)$ is the space of continuous linear maps from E to F equipped with the topology of uniform convergence on bounded sets of E. Thus around any $z_0 \in \Omega$ the product map $\tilde{\mu}$ can locally be expanded in the form

$$\tilde{\mu}(z) \;=\; \sum_{\alpha \in \mathbb{N}^n} \mu_\alpha (z - z_0)^\alpha, \tag{34}$$

where $z \in \Omega$ is close enough to z_0 and the μ_α are continuous bilinear mappings on E with values in E and which depend on the base point z_0. Similarly $\tilde{\eta} : \mathbb{C} \to E_\Omega$ with

$\tilde{\eta}(\lambda) = \lambda\,\tilde{\eta}(1)$ can be viewed as the map $\tilde{\eta}(1) : \Omega \to E$. In case $E = A$ is a locally convex algebra the constant algebraic structure on A_Ω is then given by $\tilde{\mu}(z) = \mu$ and $\tilde{\eta}(\lambda)(z) = \eta(\lambda)$, $\lambda \in \mathbb{C}$, $z \in \Omega$.

Before giving the definition of a deformation let us mention that in the following $\mathfrak{m}_z$ denotes the maximal ideal of $\mathcal{O}(\Omega)$ at a point $z \in \Omega$, i.e. $\mathfrak{m}_z$ is the ideal of all holomorphic functions on Ω vanishing at z.

Definition 3.1 *A* **topologically free holomorphic deformation** *of a locally convex algebra (A, μ, η) over a complex domain Ω is an $\mathcal{O}(\Omega)$ -algebra structure $(\tilde{\mu}, \tilde{\eta})$ on A_Ω such that for a distinguished point $* \in \Omega$ the quotient $\mathcal{O}(\Omega)$-module $A_\Omega/\mathfrak{m}_* A_\Omega$ is isomorphic to A as a locally convex algebra. Equivalently $\mu_0 = \mu$ in the expansion (34) and $\tilde{\eta}(1)(*) = \eta(1)$.*

The deformation is called **trivial** *if $(A_\Omega, \tilde{\mu}, \tilde{\eta})$ is equivalent to the constant algebra structure on A_Ω. The distinguished point $*$ is called the* **base point** *of the deformation.*

In the sequel we shall call a topologically free holomorphic deformation simply a holomorphic deformation, although there exist more general schemes of holomorphic deformations (see PFLAUM [18, 19]). However, these are not needed for our purposes of deformation of Hopf algebra structures.

An important advantage of holomorphic deformations in comparison to formal deformations lies in the fact that for every parameter $z \in \Omega$ one receives a **concrete** deformed algebra structure on the underlying linear space of the original algebra A: Simply take – for any value $z \in \Omega$ – $\tilde{\mu}(z) \in \mathcal{L}(A \hat{\otimes} A, A)$ as the deformed multiplication and $\tilde{\eta}(z) \in \mathcal{L}(\mathbb{C}, A)$ as the deformed unit. Then for $a, b \in A$

$$a *_z b \;=\; \tilde{\mu}(z)\,(a, b) \in A \tag{35}$$

is the new product of a and b and

$$e_z \;=\; \tilde{\eta}(z)\,(1) \in A \tag{36}$$

the new unit. Both, the new product and the new unit are contained in the original space A and not only in a nontrivial extension of A.

Of course, the question now is, whether there exist interesting and nontrivial holomorphic deformations of locally convex algebras. The main purpose of this paper is to answer this question in the positive.

Example 3.2 (i) **Quantum vector spaces** (cf. FADDEEV, RESHETIKHIN, TAKHTAJAN, [7] MANIN [16]). Let $\Omega = \mathbb{C}^*$ be the set of all nonzero complex numbers and consider the n-dimensional complex vector space $V = \mathbb{C}^n$ with the canonical basis $(x_1, ..., x_n)$. Then construct the tensor algebra $\mathrm{T}V$ of V or in other words the free $\mathbb{C}$-algebra in n generators. By completion with respect to the projective limit topology of all Fréchet representations $\mathrm{T}V$ becomes a nuclear locally m-convex Fréchet algebra $\check{\mathrm{T}}V$. The functions

$$f_{\iota\kappa} : \Omega \to \mathrm{T}V, \quad z \mapsto x_\iota\, x_\kappa - z\, x_\kappa\, x_\iota \quad 1 \le \iota < \kappa \le n \tag{37}$$

then generate a unique closed ideal I in the nuclear algebra $\mathcal{O}(\Omega)\hat{\otimes}\check{\mathrm{T}}V$. The quotient $\mathcal{O}(\mathbb{C}_q^n) = \mathcal{O}(\Omega)\hat{\otimes}\check{\mathrm{T}}V/I$ is called the **algebra of entire functions on the quantum n-vector space**. It comprises a holomorphic deformation of the algebra $\mathcal{O}(\mathbb{C}^n)$ of entire functions on $\mathbb{C}^n$: The homomorphism $\mathcal{O}(\mathbb{C}_q^n) \to \mathcal{O}(\mathbb{C}^n)$ defined by $[f \otimes x_\iota] \mapsto f(1)x_\iota$ is well-defined, surjective and has kernel $\mathfrak{m}_1\mathcal{O}(\mathbb{C}_q^n)$. Therefore $\mathcal{O}(\mathbb{C}_q^n)/\mathfrak{m}_1\mathcal{O}(\mathbb{C}_q^n) \cong \mathcal{O}(\mathbb{C}^n)$ holds. Because of the relations (37) the $\mathcal{O}(\mathbb{C}^*)$-linear combinations of the family $(x_1^{m_1}, ..., x_n^{m_n})_{m_1,...,m_n\in\mathbb{N}}$ are dense in $\mathcal{O}(\mathbb{C}_q^n)$. Since $(x_1^{m_1}, ..., x_n^{m_n})_{m_1,...,m_n\in\mathbb{N}}$ furthermore is free over $\mathcal{O}(\mathbb{C}^*)$, $\mathcal{O}(\mathbb{C}_q^n)$ is isomorphic to $\mathcal{O}(\mathbb{C}^*)\hat{\otimes}\mathcal{O}(\mathbb{C}^n) \cong \mathcal{O}(\mathbb{C}^* \times \mathbb{C}_q^n)$ as a nuclear space. This proves the claim.

Alternatively one could give $\mathrm{T}V$ the inductive topology of all finite dimensional subspaces. Then $\mathrm{T}V$ is already a strictly nuclear algebra. By the same procedure as above but now applied to $\mathrm{T}V$ one is lead to the algebra $\mathcal{P}(\mathbb{C}_q^n)$ of **polynomial functions on the quantum n-vector space**. $\mathcal{P}(\mathbb{C}_q^n)$ comprises a deformation of the algebra $\mathbb{C}[x_1, ..., x_n]$ of polynomials in n complex variables.

(ii) **Quantum exterior algebra** (cf. Wess, Zumino [30], Manin [16]). In the spirit of the preceeding example it is also possible to deform the exterior algebra on $\mathbb{C}^n$. Let V' be the dual of V, $(\xi_1, ..., \xi_n)$ the dual basis of $(x_1, ..., x_n)$ and let $\mathrm{T}V'$ be given the finest locally convex topology. Then consider the closed ideal $J \subset \mathcal{O}(\mathbb{C}^*)\hat{\otimes}\mathrm{T}V'$ generated by the relations

$$\xi_\iota^2 = 0, \quad \xi_\iota\,\xi_\kappa \;=\; -z^{-1}\,\xi_\kappa\,\xi_\iota \quad 1 \le \iota < \kappa \le n. \tag{38}$$

The corresponding quotient $\Lambda(\mathbb{C}_q^n) = \mathrm{T}V'/J$ is the **exterior algebra of the quantum n-vector space**. Exactly like above it is shown that $\Lambda(\mathbb{C}_q^n)$ is a holomorphic deformation over $\mathbb{C}^*$ of the exterior algebra $\Lambda(\mathbb{C}^n)$. Note that unlike $\mathcal{O}(\mathbb{C}_q^n)$ and $\mathcal{P}(\mathbb{C}_q^n)$ the algebra $\Lambda(\mathbb{C}^n)$ is finite dimensional. The tensor product algebra $\mathcal{O}(\mathbb{C}_q^n) \otimes \Lambda(\mathbb{C}^n)$ can be interpreted as the algebra of **entire holomorphic quantum differential forms on $\mathbb{C}_q^n$**, the tensor product $\mathcal{P}(\mathbb{C}_q^n) \otimes \Lambda(\mathbb{C}^n)$ as the algebra of **algebraic quantum differential forms on $\mathbb{C}_q^n$**.

3.3 Holomorphic Quantization. In mathematical physics one is not only interested in deformations of algebras but also in their quantization. More precisely we want to quantize so-called **locally convex Poisson algebras**, i.e. locally convex $\mathbb{K}$-algebras A on which a continuous bilinear bracket $\{\ ,\ \} : A \times A \to A$ is defined which is antisymmetric, fulfills the Jacobi identity and the Leibniz rule. A **topologically free holomorphic quantization** of A over the domain $\Omega \subset \mathbb{C}$ with base point $0 \in \Omega$ then is a topologically free holomorphic deformation $(A_\Omega, \tilde{\mu}, \tilde{\eta})$ of A such that the relation

$$\tilde{\mu}(f, g) - \tilde{\mu}(g, f) \;=\; -i\,z\,\{f, g\} \;+\; o(z^2) \tag{39}$$

holds for all $f, g \in A$ and $z \in \Omega$.

Remark 3.4 Suppose A to be a locally convex Poisson algebra having a holomorphic deformation $(\tilde{A}, \tilde{\mu})$. If now $\tilde{A}\hat{\otimes}\mathbb{C}[[\hbar]]$ is a formal quantization of A, then $\tilde{A}$ obviously is a holomorphic quantization of A.

Let us give an important example of a holomorphic quantization. Consider the space $\mathrm{S}^{\infty}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ of classical symbols on $\mathbb{R}^n$ together with the topology of asymptotic convergence (cf. HÖRMANDER [13] or PFLAUM [19] for details on symbol spaces). The Poisson bracket on $\mathbb{R}^{2n}$ naturally induces a Poisson bracket on the symbol space $\mathrm{S}^{\infty}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$. We want to quantize this Poisson space. By the asymptotic expansion

$$(a *_\hbar b)\,(x,\xi) \sim \sum_{\alpha,\beta\in\mathbb{N}^n} (-1)^{|\beta|} \left(\frac{-i\hbar}{2}\right)^{|\alpha+\beta|} \left(\frac{\partial^{|\alpha|}}{\partial\xi^\alpha}\frac{\partial^{|\beta|}}{\partial x^\beta} a(x,\xi)\right) \left(\frac{\partial^{|\beta|}}{\partial\xi^\beta}\frac{\partial^{|\alpha|}}{\partial x^\alpha} b(x,\xi)\right) \quad (40)$$

we define the **Moyal-product** $a *_\hbar b$ of two symbols $a, b \in \mathrm{S}^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$. Unfortunately – as asymptotic expansions are only unique up to smoothing symbols $\in \mathrm{S}^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ – Eq. (40) does not define an associative product on $\mathrm{S}^{\infty}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$. But the Moyal product gives a holomorphic deformation of the quotient algebra $\mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ carrying the topology of asymptotic convergence. As is well-known the Moyal product induces a formal quantization of the Poisson space $\mathcal{C}^{\infty}(\mathbb{R}^n \times \mathbb{R}^n)$, hence, of $\mathrm{S}^{\infty}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$. As the Poisson bracket on $\mathrm{S}^{\infty}_{\rho,\delta}(\mathbb{R}^n \times \mathbb{R}^n)$ can be pushed down to the quotient $\mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$, the above proposition entails that the Moyal-product induces a holomorphic quantization of $\mathrm{S}^{\infty}_{\rho,\delta}/\mathrm{S}^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$.

Note that in the same way the Wick quantization procedure leads to a holomophic quantization. See also PFLAUM [19] for more general quantization schemes which fit into the above holomorphic picture.

4 Holomorphic deformation of nuclear Hopf algebras

The concept of a (topologically free) holomorphic deformation can easily be transferred to the case of deformations of nuclear coalgebra structures, bialgebra and Hopf algebra structures, as well as Lie bialgebra structures. In particular a **(topologically free) holomorphic deformation** of a nuclear Hopf algebra H with multiplication μ, comultiplication Δ, unit η, counit ε, and antipode S is given by the following data:

$$\begin{array}{llll} \tilde{\mu} \in \mathcal{O}(\Omega, \mathcal{L}(H\hat{\otimes}H, H) & \cong \mathcal{L}_{\mathcal{O}(\Omega)}(H_\Omega\hat{\otimes}_{\mathcal{O}(\Omega)}H_\Omega, H_\Omega) & \text{with} & \tilde{\mu}(*) = \mu, \\ \tilde{\Delta} \in \mathcal{O}(\Omega, \mathcal{L}(H, H\hat{\otimes}H) & \cong \mathcal{L}_{\mathcal{O}(\Omega)}(H_\Omega, H_\Omega\hat{\otimes}_{\mathcal{O}(\Omega)}H_\Omega) & \text{with} & \tilde{\Delta}(*) = \Delta, \\ \tilde{\eta} \in \mathcal{O}(\Omega, \mathcal{L}(\mathbb{C}, H)) & \cong \mathcal{L}_{\mathcal{O}(\Omega)}(\mathcal{O}(\Omega), H_\Omega) & \text{with} & \tilde{\eta}(*) = \eta, \\ \tilde{\epsilon} \in \mathcal{O}(\Omega, \mathcal{L}(H, \mathbb{C})) & \cong \mathcal{L}_{\mathcal{O}(\Omega)}(H_\Omega, \mathcal{O}(\Omega)) & \text{with} & \tilde{\epsilon}(*) = \epsilon, \\ \tilde{S} \in \mathcal{O}(\Omega, \mathcal{L}(H, H)) & \cong \mathcal{L}_{\mathcal{O}(\Omega)}(H_\Omega, H_\Omega) & \text{with} & \tilde{S}(*) = S, \end{array} \tag{41}$$

such that $(H_\Omega, \tilde{\mu}, \tilde{\Delta}, \tilde{\eta}, \tilde{\epsilon}, \tilde{S})$ is a nuclear Hopf algebra. (The isomorphisms in the above formulas are explained in the Appendix.) Two such Hopf algebra deformations are **equivalent** if the corresponding Hopf algebra structures on H_Ω are isomorphic. Any holomorphic deformation of a nuclear Hopf algebra turns out to be equivalent to a holomorphic deformation with constant unit and counit, i.e. with $\tilde{\eta}(z) = \eta$ and $\tilde{\epsilon}(z) = \epsilon$ for all $z \in \Omega$.

Our first result concerns the dual of the holomorphic deformation $\tilde{H}$ of a strictly nuclear Hopf algebra H: Transposing all the structure maps one gets a holomorphic deformation of the dual nuclear Hopf algebra H'. The underlying topological $\mathcal{O}(\Omega)$-module is

$${H_\Omega}' = \mathcal{L}_{\mathcal{O}(\Omega)}(H_\Omega, \mathcal{O}(\Omega)) = \{\phi : H_\Omega \to \mathcal{O}(\Omega) \mid \phi \text{ is continuous and } \mathcal{O}(\Omega)\text{-linear}\} \tag{42}$$

with the topology of uniform convergence on bounded (or equivalently compact) sets of $H_\Omega \cong \mathcal{O}(\Omega)\hat{\otimes}H$. This space is isomorphic as an $\mathcal{O}(\Omega)$-module to

$$\mathcal{O}(\Omega)\hat{\otimes}H' \cong \mathcal{O}(\Omega, H') = H'_\Omega. \tag{43}$$

The structure maps on H'_Ω are

$$\tilde{\Delta}' = (\tilde{\mu})^* : H'_\Omega \longrightarrow (H_\Omega\hat{\otimes}H_\Omega)' \cong H'_\Omega\hat{\otimes}H'_\Omega \cong \mathcal{O}(\Omega, H'\hat{\otimes}H'), \tag{44}$$

hence $\tilde{\Delta}' \in \mathcal{O}(\Omega, \mathcal{L}(H', H'\hat{\otimes}H'))$,

$$\tilde{\mu}' = (\tilde{\Delta})^* : (H_\Omega\hat{\otimes}H_\Omega)' \cong H'_\Omega \otimes H'_\Omega \to H'_\Omega, \tag{45}$$

hence $\tilde{\mu}' \in \mathcal{O}(\Omega, \mathcal{L}(H'\hat{\otimes}H', H'))$, and similarly

$$\tilde{S}' = \tilde{S}^*, \quad \tilde{\eta}' = \tilde{\varepsilon}^*, \quad \tilde{\varepsilon}' = \tilde{\eta}^*. \tag{46}$$

The following duality theorem is now obvious.

Theorem 4.1 *Let $\tilde{H} = (H_\Omega, \tilde{\mu}, \tilde{\Delta}, \tilde{\varepsilon}, \tilde{\eta}, \tilde{S})$ be a holomorphic deformation of the strictly nuclear Hopf algebra $(H, \mu, \Delta, \varepsilon, \eta, S)$. Then the $\mathcal{O}(\Omega)$-dual H'_Ω of H_Ω with the structure maps $\tilde{\mu}'$, $\tilde{\Delta}'$, $\tilde{\eta}'$, $\tilde{\varepsilon}'$ and $\tilde{S}'$ is a holomorphic deformation of the nuclear Hopf algebra H'. Moreover, $\tilde{H}$ and $\check{H}$ are equivalent if and only if $\tilde{H}'$ and $\check{H}'$ are equivalent. In addition, the bidual $(\tilde{H}')'$ is canonically isomorphic to $\tilde{H}$.*

The above theorem is a direct generalization of the corresponding result for the formal case (cf. [4]). Similarly, the next fact about the construction of a twisting matrix carries over directly to the holomorphic case (cf. [4], Proposition 4.2.4).

Theorem 4.2 *Let G be a compact group and let H denote the strictly nuclear Hopf algebra $\mathcal{R}(G)' \cong \tilde{\mathcal{U}}\mathfrak{g}^{\mathbb{C}}$ respectively $\mathcal{D}(G) = \mathcal{E}(G)'$ (cf. 2.6 resp. 2.8). Let $(H_\Omega, \tilde{\Delta})$ be a holomorphic deformation of the nuclear bialgebra H leaving invariant the algebra product on H. Then there exists $\mathcal{F} \in \big(H \hat{\otimes} H\big)_\Omega$ such that $\tilde{\Delta}\,\mathcal{F} = \mathcal{F}\,\Delta$.*

Proof: Because of $\tilde{\Delta} \in \mathcal{L}_{\mathcal{O}(\Omega)}(H_\Omega, H_\Omega \otimes_{\mathcal{O}(\Omega)} H_\Omega) \cong \mathcal{O}(\Omega, \mathcal{L}(H, H \hat{\otimes} H))$ we get a "twisting matrix" $\mathcal{F} \in \mathcal{O}(\Omega, H \hat{\otimes} H) \cong (H \hat{\otimes} H)_{\mathcal{O}(\Omega)}$ by the following integral:

$$\mathcal{F} \;=\; \int_G \tilde{\Delta}(g)\; (\Delta(g))^{-1}\; d\mu(g), \tag{47}$$

where μ is the Haar measure on G and $g \in G$ stands for δ_g, the Dirac distribution with support in g. By left- and right-invariance of the Haar measure

$$\begin{aligned} \mathcal{F}\,\Delta(h) \;&=\; \int_G \tilde{\Delta}(hg)\,(\Delta(hg))^{-1}\,\Delta(h)\,d\mu(hg) \;= \\ &=\; \int_G \tilde{\Delta}(h)\,\tilde{\Delta}(g)\,(\Delta(g))^{-1}\,d\mu(g) \;=\; \tilde{\Delta}(h)\,\mathcal{F} \end{aligned} \tag{48}$$

holds. Hence the claim follows from the fact that the Dirac distributions $g = \delta_g$ lie densely in $\mathcal{R}(G)'$(*cf.* 2.5). □

Remark 4.3 We expect in all important examples $\mathcal{F}(z)$ to be invertible at least in a neighborhood of the base point. In that case we get a dual version of Theorem 4.2 which leads to a holomorphic coquasitriangular Hopf algebra deformation of $\mathcal{R}(G)$.

Theorem 4.4 *Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a cocommutative nuclear Hopf algebra and let $\mathcal{F} : \Omega \to H \hat{\otimes} H$ be a* **twisting matrix** *for H, i.e. a continuous map on an open domain $\Omega \subset \mathbb{C}$ such that the following conditions hold:*

(*i*) *$\mathcal{F}(z)$ is an invertible element of $H \hat{\otimes} H$ for every $z \in \Omega$.*

(*ii*) *$\mathcal{F}$ fulfills the equations*

$$\mathcal{F}_{12}\,(\Delta\otimes 1)\,\mathcal{F} \;=\; \mathcal{F}_{23}\,(1\otimes\Delta)\,\mathcal{F} \tag{49}$$

$$(\varepsilon\otimes 1)\mathcal{F} \;=\; 1 \;=\; (1\otimes\varepsilon)\mathcal{F}, \tag{50}$$

where $\mathcal{F}_{12}$ and $\mathcal{F}_{23}$ are defined as usual.

Then $\mathcal{F}$ induces a new nuclear Hopf algebra structure $(H_\Omega,\mu^{\mathcal{F}},\eta^{\mathcal{F}},\Delta^{\mathcal{F}},\varepsilon^{\mathcal{F}},S^{\mathcal{F}})$ on $H_\Omega = \mathcal{O}(\Omega)\hat{\otimes}H$ by defining $\mu^{\mathcal{F}}=\mu$, $\eta^{\mathcal{F}}=\eta$, $\Delta^{\mathcal{F}}(z\otimes h)=\mathcal{F}(z)\,\Delta(h)\mathcal{F}^{-1}(z)$, $\varepsilon^{\mathcal{F}}=\varepsilon$ and $S^{\mathcal{F}}(z\otimes h) = v(z)\,S(h)\,v^{-1}(z)$. Hereby is $z\in\mathcal{O}(\Omega)$, $h\in H$ and $v=\mu(1\otimes S)\mathcal{F}$. $(H_\Omega,\mu,\eta,\Delta^{\mathcal{F}},\varepsilon,S^{\mathcal{F}})$ is topologically triangular with universal R-matrix $\mathcal{R}=\mathcal{F}_{21}\,\mathcal{F}^{-1}$.

Now assume H to be topologically quasitriangular with universal R-matrix $\mathcal{R}$ and, additionally to the above, the following:

(*iii*) *The quantum Yang-Baxter equation (QYBE) is fulfilled:*

$$\mathcal{F}_{12}\circ\mathcal{F}_{13}\circ\mathcal{F}_{23} \;=\; \mathcal{F}_{23}\circ\mathcal{F}_{13}\circ\mathcal{F}_{12}. \tag{51}$$

(*iv*) $\mathcal{F}_{21}=\mathcal{F}^{-1}$.

Then $(H_\Omega,\mu,\eta,\Delta^{\mathcal{F}},\varepsilon,S^{\mathcal{F}})$ is a topologically quasitriangular nuclear Hopf algebra with universal R-matrix $\mathcal{R}^{\mathcal{F}}=\mathcal{F}^{-1}\mathcal{R}\mathcal{F}^{-1}$.

If the twisting matrix $\mathcal{F}$ fulfills $\mathcal{F}(*)=1\otimes 1$ for a point $*\in\Omega$, then in both of the above cases $(H_\Omega,\mu,\eta,\Delta^{\mathcal{F}},\varepsilon,S^{\mathcal{F}})$ comprises a topologically free holomorphic deformation over Ω of the Hopf algebra H with base point $*$.

Proof: The proof of the theorem can be taken almost literally from the corresponding one in the formal case. Confer for example Chari, Presley [5]. □

The purpose of the following considerations is to show how one can construct under certain conditions a holomorphic deformation of an algebra (resp. bialgebra or Hopf algebra) out of a formal one. We will carry out the details only for the algebra case; the ones for bialgebras and Hopf algebras are analogous.

So let A be a finitely generated $\mathbb{C}$-algebra and assume that there exists a formal deformation $A_\hbar \cong (A[[\hbar]],\mu,\eta)$ of A. In other words μ is a $\mathbb{C}[[\hbar]]$-bilinear multiplication map on $A[[\hbar]]$ and η a unit such that $A[[\hbar]]/\hbar A[[\hbar]]\cong A$. We now choose a finite dimensional vector space V and a surjective homomorphism $\mathrm{T}V=\bigoplus_{n\in\mathbb{N}}V^{\otimes n}\to A$. According to 2.2 this gives rise to a topological presentation $\hat{\mathrm{T}}V\to\hat{A}$, where $\hat{A}$ and $\hat{\mathrm{T}}V$ are the completions of A resp. $\mathrm{T}V$ with respect to the topology of all finite dimensional representations (resp. all nuclear Fréchet representations). Using the universal property of the tensor algebra $\mathrm{T}V$ there exists a unique morphism of $\mathbb{C}[[\hbar]]$-algebras $\pi:\mathrm{T}V[[\hbar]]\to A_\hbar$ such that $\mathrm{T}V\to\mathrm{T}V[[\hbar]]\to A_\hbar\to A$ is the presentation $\mathrm{T}V\to A$. Let I be the kernel of π and $\hat{I}$ its completion in $\hat{\mathrm{T}}V[[\hbar]]=\mathbb{C}[[\hbar]]\,\hat{\otimes}\,\hat{\mathrm{T}}V$. Denoting the nuclear algebra $\hat{\mathrm{T}}V[[\hbar]]/\hat{I}$ by $\hat{A}_\hbar$ we then have a commutative diagram

$$\begin{tikzcd} \mathrm{T}V[[\hbar]] \arrow[r, "\iota"] \arrow[d, "\pi"'] & \hat{\mathrm{T}}V[[\hbar]] \arrow[d, "\hat{\pi}"] \\ A_\hbar \arrow[r, "\hat{\iota}"'] & \hat{A}_\hbar \end{tikzcd}$$

with injective horizontal and surjective vertical arrows. Only the injectivity of $\hat{\iota}$ is not immediately clear. It follows from the fact that I is closed in $\mathrm{T}V[[\hbar]]$: every image of I under a projection of $\mathrm{T}V[[\hbar]]$ to a finite dimensional vector space is closed. Note that the morphisms in the above diagram are also filtered with respect to the filtrations induced by the maximal ideals generated by $\hbar$. By these considerations we now have $\hat{A}_\hbar/\hbar\hat{A}_\hbar \cong \overline{A}$. Further the vector space $A[[\hbar]] \cong A_\hbar$ lies densely in $\hat{A}_\hbar$, hence $\hat{A}_\hbar \cong A[[\hbar]]$ and the algebra $\hat{A}_\hbar$ comprises a formal deformation of $\hat{A}$.

In the next step assume Ω to be a connected open domain in $\mathbb{C}$ containing the origin. This gives us a continuous and injective map $\rho : \hat{\mathrm{T}}V_\Omega = \mathcal{O}(\Omega)\hat{\otimes}\hat{\mathrm{T}}V \to \hat{\mathrm{T}}V[[\hbar]]$ by power series expansion around the origin. We now say that the formal deformation $A_\hbar$ has **holomorphic initial data** if there exists a system of generators Y of I such that $Y \in \mathrm{im}(\rho)$. Assuming this is the case indeed let $\tilde{Y}$ be the preimage of Y under ρ. Let I_Ω be the closed ideal in $\hat{\mathrm{T}}V_\Omega$ generated by $\tilde{Y}$. We then have $I_\Omega = \rho^{-1}(\hat{I})$, which induces an injective and filtered homomorphism $\tilde{A} := \mathcal{O}(\Omega)\hat{\otimes}\hat{\mathrm{T}}V/I_\Omega \to \hat{\mathrm{T}}V[[\hbar]]/\hat{I} \cong \hat{A}[[\hbar]]$. This map is surjective from $\mathfrak{m}^n\tilde{A}/\mathfrak{m}^{n+1}\tilde{A}$ to $\hbar^n\hat{A}[[\hbar]]/\hbar^{n+1}\hat{A}[[\hbar]]$ where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}(\Omega)$ of functions vanishing at the origin. Thus $\tilde{A}/\mathfrak{m}\tilde{A} \cong \hat{A}$. As furthermore $\tilde{A}$ is dense in $\hat{A}[[\hbar]]$ we finally have $\tilde{A} \cong \hat{A}_\Omega$ as a nuclear space. Thus $\tilde{A}$ is a topologically free holomorphic deformation of $\hat{A}_\Omega$.

We subsume these results in the following proposition.

Proposition 4.5 *Let A be a finitely generated complex algebra (resp. bialgebra or Hopf algebra) and let $\hat{A}$ be the completion of A with respect to the topology of finite dimensional representations (resp. nuclear Fréchet representations). Then every formal deformation $A_\hbar$ of A induces a formal deformation $\hat{A}_\hbar$ of $\hat{A}$ together with a canonical filtered embedding $A_\hbar \to \hat{A}_\hbar$. If the deformation $A_\hbar$ has holomorphic initial data over some open connected domain $\Omega \subset \mathbb{C}$ containing the origin, then there exist a topologically free holomorphic deformation $\tilde{A}$ of A over Ω and a canonical filtered embedding $\tilde{A}_\Omega \to \hat{A}_\hbar$. These constructions are unique up to isomorphy.*

Remark 4.6 Suppose $H_\hbar$ to be a formal deformation of a finitely generated Hopf algebra H with holomorphic initial data. Give $H_\hbar$ the projective limit topology with respect to all projections $H_\hbar \to H_\hbar/\hbar^m H_\hbar \cong H^m \subset \hat{H}^m$. Assume further that $H_\hbar$ is topologically (quasi-) triangular with universal R-matrix $\mathcal{R}_\hbar \in H_\hbar\hat{\otimes}H_\hbar$. The question now is whether the universal R-matrix on $H_\hbar$ can be pushed down to one on $\tilde{H}_\Omega$ or in other words whether it induces on $\tilde{H}_\Omega$ the structure of a topologically (quasi-) triangular Hopf algebra.

Obviously this is the case if and only if the formal power series expansion of $\mathcal{R}_\hbar$ with respect to $\hbar$ in $\hat{H}\hat{\otimes}\hat{H}$ is converging over Ω.

Remark 4.7 Let Ω be an open domain in $\mathbb{C}^n$ or even more generally a complex manifold. Then it is possible to define holomorphic deformations over Ω exactly like in the case of one complex variable. Such deformations are called **holomorphic multiparameter deformations**. We do not give examples here but mention that a formal multiparameter deformation (cf. RESHETIKHIN [21]) with holomorphic initial data obviously induces a holomorphic multiparameter deformation. The theory of holomorphic deformations give sense also for infinite dimensional domains Ω, at least if $\mathcal{O}(\Omega)$ is a nuclear Fréchet algebra. This is the case, for example, if Ω is a domain in a suitable Köthe sequence space E which is the dual of a Fréchet nuclear space. (cf. SCHOTTENLOHER [24]).

5 Quantum groups as holomorphically deformed Hopf algebras

By discussing some important examples we want to convince the reader in this section that the theory of quantum groups can be understood as the theory of holomorphic deformations of universal enveloping algebras respectively of Hopf algebras of functions on a Lie group. In particular, we want to show that these two approaches are dual to each other in the context of holomorphic deformation. But this is the precise mathematical meaning of the physicists claim that quantizing the algebra of matrix coefficients on a (Poisson) Lie group is dual to quantizing the corresponding universal enveloping algebra.

5.1 Quantized $\mathcal{U}\mathfrak{sl}(N+1,\mathbb{C})$. Consider the Lie algebra $\mathfrak{sl}(N+1,\mathbb{C})$. Its Lie algebra structure is given by the basis $X_\iota^+, X_\iota^-, H_\iota$ for $1 \leq \iota \leq N$ together with the commutation relations

$$[H_\iota, H_\kappa] = 0 \quad [H_\iota, X_\kappa^\pm] = \pm a_{\iota\kappa} X_\kappa^\pm \tag{52}$$

$$[X_\iota^+, X_\kappa^-] = \delta_{\iota\kappa} H_\iota, \quad [X_\iota^\pm, X_\kappa^\pm] = 0 \text{ for } |\iota - \kappa| > 1 \tag{53}$$

$$X_\iota^{\pm 2} X_\kappa^\pm - 2 X_\iota^\pm X_\kappa^\pm X_\iota^\pm + X_\kappa^\pm X_\iota^{\pm 2} = 0 \text{ for } |\iota - \kappa| = 1. \tag{54}$$

where $(\alpha_{\iota\kappa})$ is the Cartan matrix of $\mathfrak{sl}(N+1,\mathbb{C})$, i.e. $a_{\iota\iota} = 2$, $a_{\iota\kappa} = 0$ for $|\iota - \kappa| > 1$ and $a_{\iota\kappa} = 1$ for $|\iota - \kappa| = 1$. From the universal enveloping algebra $\mathcal{U}\mathfrak{sl}(N+1,\mathbb{C})$ we can construct by completion with respect to the projective limit topology of all finite dimensional representations (resp. all nuclear Fréchet representations) the nuclear algebra resp. nuclear Hopf algebra $\hat{\mathcal{U}}\mathfrak{sl}(N+1,\mathbb{C})$ (cf. 2.6).

We now want to construct a holomorphic deformation of $\hat{\mathcal{U}}\mathfrak{sl}(N+1,\mathbb{C})$ (resp. one of $\check{\mathcal{U}}\mathfrak{sl}(N+1,\mathbb{C})$) over the domain $\Omega = \{z \in \mathbb{C} : \; z \neq k\pi i, \; k \in \mathbb{Z}^*\}$ by applying Proposition 4.5. It is a well-known fact from the theory of quantum groups (cf. Chari, Pressley [5]) that the following relations define a formal deformation of the nuclear Hopf algebra $\mathcal{U}\mathfrak{sl}(N+1,\mathbb{C})$:

$$[H_\iota, H_\kappa] = 0, \quad [H_\iota, X_\kappa^\pm] = \pm a_{\iota\kappa} X_\kappa^\pm, \tag{55}$$

$$[X_\iota^+, X_\kappa^-] = \delta_{\iota\kappa} \frac{\sinh \frac{z}{2} H}{\sinh \frac{z}{2}}, \quad [X_\iota^\pm, X_\kappa^\pm] = 0 \text{ for } |\iota - \kappa| > 1, \tag{56}$$

$$X_\iota^{\pm 2} X_\kappa^\pm - \left(\mathrm{e}^{z/2} + \mathrm{e}^{-z/2}\right) X_\iota^\pm X_\kappa^\pm X_\iota^\pm + X_\kappa^\pm X_\iota^{\pm 2} = 0 \text{ for } |\iota - \kappa| = 1, \tag{57}$$

$$\Delta(H_\iota) = H_\iota \otimes 1 + 1 \otimes H_\iota, \tag{58}$$

$$\Delta(X_\iota^+) = X_\iota^+ \otimes \mathrm{e}^{z\,H_\iota} + 1 \otimes X_\iota^+, \quad \Delta(X_\iota^-) = X_\iota^- \otimes 1 + \mathrm{e}^{-z\,H_\iota} \otimes X_\iota^-, \tag{59}$$

$$S(H_\iota) = -H_\iota, \quad S(X_\iota^+) = -X_\iota^+ \,\mathrm{e}^{-z\,H_\iota}, \quad S(X_\iota^-) = -\mathrm{e}^{z\,H_\iota} X_\iota^-, \tag{60}$$

$$\varepsilon(H_\iota) = \varepsilon(X_\iota^\pm) = 0. \tag{61}$$

By power series expansion around 0 it is clear that $\frac{\sinh \frac{z}{2} H}{\sinh \frac{z}{2}}$ comprises a holomorphic function on Ω; the other relations are obviously holomorphic in z as well. Hence, the formal deformation of $\mathcal{U}\mathfrak{sl}(N+1,\mathbb{C})$ has holomorphic initial data. By Proposition 4.5 the relations (55) to (61) thus generate a holomorphic deformation $\mathcal{U}_q\mathfrak{sl}(N+1,\mathbb{C}) = \hat{\mathcal{U}}_q\mathfrak{sl}(N+1,\mathbb{C})$ of the nuclear Hopf algebra $\hat{\mathcal{U}}\mathfrak{sl}(N+1,\mathbb{C})$ and a holomorphic deformation $\check{\mathcal{U}}_q\mathfrak{sl}(N+1,\mathbb{C})$ of $\check{\mathcal{U}}\mathfrak{sl}(N+1,\mathbb{C})$.

Using the Drinfeld double (cf. DRINFEL'D [6]) one can construct a (topological) R-matrix $\mathcal{R}_\hbar$ on formally quantized $\mathcal{U}\mathfrak{sl}(N+1,\mathbb{C})$ such that $\mathcal{R}_\hbar$ has an expansion of the form

$$\mathcal{R}_\hbar = \sum_{\beta\in\mathbb{N}^N} \left(\exp\left(\hbar\left[\frac{1}{2}t_0 + \frac{1}{4}(H_\beta\otimes 1 + 1\otimes H_\beta)\right]\right)\right) P_\beta, \tag{62}$$

where $t_0 \in \mathfrak{sl}(N+1,\mathbb{C})\otimes\mathfrak{sl}(N+1,\mathbb{C})$ is chosen appropriate, $H_\beta = \sum_\iota \beta_\iota H_\iota$ and the P_β are polynomials homogeneous of degree β_ι in $X_\iota^+\otimes 1$ and $1\otimes X_\iota^-$. Hence $\mathcal{R}_\hbar$ has a converging power series expansion, so Remark 4.6 entails that $\mathcal{U}_q\mathfrak{sl}(N+1,\mathbb{C})$ is topologically quasitriangular as well.

Remark 5.2 The quantum group models defined by JIMBO [14] do not comprise a holomorphic deformation of $\hat{\mathcal{U}}\mathfrak{sl}(N+1,\mathbb{C})$, though a holomorphic deformation of a certain extended Hopf algebra, for example for $N=1$, of $\hat{\mathcal{U}}\mathfrak{sl}(2,\mathbb{C})\otimes\mathbb{C}[T]/(T^2-1)$ ([4]).

5.3 Quantizing $\mathrm{SL}(N,\mathbb{C})$ **by dualizing.** Now we want to quantize the dual of the preceding example, or in other words the algebra $\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))$ of representation functions on the Lie group $\mathrm{SL}(N,\mathbb{C})$. Recall from 2.7 that $\hat{\mathcal{U}}\mathfrak{sl}(N,\mathbb{C})$ and $\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))$ are strictly nuclear and that $\hat{\mathcal{U}}\mathfrak{sl}(N,\mathbb{C})$ is topologically isomorphic to $\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))'$. Hence, according to Proposition 4.5 the deformation $\mathcal{U}_q\mathfrak{sl}(N,\mathbb{C})$ gives rise to a holomorphic deformation $\mathcal{U}_q\mathfrak{sl}(N,\mathbb{C})'$ of $\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))$. Unfortunately we do not yet have a concrete representation of $\mathcal{U}_q\mathfrak{sl}(N,\mathbb{C})'$ by generators and relations. As will be shown in the next paragraph, the well-known FRT-deformation of $\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))$ does exactly provide that.

5.4 Quantizing $\mathrm{SL}(N,\mathbb{C})$ **according to FRT.** First let us recall the FRT-construction of quantized $\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))$. Any finite dimensional representation of $\mathrm{SL}(N,\mathbb{C})$ can be realized as a subrepresentation of a sum of tensor products of the fundamental representation of $\mathrm{SL}(N,\mathbb{C})$. This means that $\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))$ is generated by the matrix coefficients $T = \left(t_i^j\right)_{1\le i,j\le N}$ of the fundamental representation. The t_i^j hereby fulfill the relation

$$\det T \ = \ 1. \tag{63}$$

It can even be shown that the algebra generated by t_i^j modulo the determinant relation is in fact isomorphic to $\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))$. From 2.5 and 2.7 we infer the finest locally convex topology on $\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))$ being the natural one, in particular, because of the natural duality between $\mathcal{U}\mathfrak{g}$ and $\mathcal{R}(G)$. To deform the nuclear Hopf algebra $\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))$ we will now use the FRT-construction as described in 2.9. Let V be the complex vector space spanned by x_i, $i=1,...,N$ and consider the R-matrix

$$
\begin{aligned}
R:\ \mathbb{C}^* &\to \mathrm{End}\,(V)' \otimes \mathrm{End}\,(V)', \\
z &\mapsto z \sum_{i=1}^{n} t_i^i \otimes t_i^i + \sum_{\substack{i,j=1 \\ i\neq j}}^{n} t_i^i \otimes t_j^j + (z - z^{-1}) \sum_{\substack{i,j=1 \\ i>j}}^{n} t_i^j \otimes t_j^i,
\end{aligned} \tag{64}
$$

where (t_i^j) is the basis of $\mathrm{End}\,(V)'$ dual to the basis (e_i^j) of $\mathrm{End}\,(V)$, and e_i^j is the endomorphism of V mapping x_i to x_j and vanishing on all the other elements of $(x_1, ..., x_N)$. This R-matrix fulfills the quantum Yang-Baxter equation and is nondegenerate with inverse $R(z)^{-1} = R(z^{-1})$. (see for example TAKHTAJAN [28]). Hence, by the FRT-construction 2.9 we receive the bialgebra $A(R)$. It is well-known that the **quantum determinant**

$$
\det\nolimits_{\mathrm{q}} T = \sum_{\sigma \in \mathrm{S}_N} (-z)^{\ell(\sigma)}\, t_1^{\sigma(1)} \cdot ... \cdot t_N^{\sigma(N)} \tag{65}
$$

belongs to the center of $A(R)$. Denoting by I the ideal $I = A(R)(\det_{\mathrm{q}} T - 1)$ the quotient bialgebra $A(R)/I$ then is even a Hopf algebra with antipode given by

$$
S(t_i^j) = (-z)^{i-j}\, \tilde{t}_i^j, \tag{66}
$$

where the $\tilde{t}_i^j$ are the so-called quantum-cofactors

$$
\tilde{t}_i^j = \sum_{\sigma \in \mathrm{S}_{N-1}} (-z)^{\ell(\sigma)}\, t_1^{\sigma_1} \cdot ... \cdot t_{i-1}^{\sigma_{i-i}} \cdot t_{i+1}^{\sigma_{i+i}} \cdot ... \cdot t_N^{\sigma_N} \tag{67}
$$

with

$$
(\sigma_1, ..., \sigma_{i-1}, \sigma_{i+1}, ..., \sigma_N) = (\sigma(1), ..., \sigma(j-1), \sigma(j+1), ..., \sigma(N)). \tag{68}
$$

We denote the quotient $A(R)/I$ by $\mathcal{R}(\mathrm{SL}_{\mathrm{q}}(N, \mathbb{C}))$ and call it the **algebra of matrix coefficients on quantized** $\mathrm{SL}(N, \mathbb{C})$. As we will show $\mathcal{R}(\mathrm{SL}_{\mathrm{q}}(N, \mathbb{C}))$ (endowed with the locally convex inductive limit topology of all finite dimensional subspaces) comprises the holomorphic deformation of $\mathcal{R}(\mathrm{SL}(N, \mathbb{C}))$ we are looking for. In particular $\mathcal{R}(\mathrm{SL}_{\mathrm{q}}(N, \mathbb{C}))$ will be isomorphic to $\mathcal{U}_q'\mathfrak{sl}(N, \mathbb{C})$.

In the second step we will show that the FRT-bialgebra $A(R)$ is a holomorphic deformation of the bialgebra of polynomial functions on the semigroup $\mathrm{M}(N \times N, \mathbb{C})$. Though the proof seems to be quite technical we will give it in some detail, as it provides an example for constructing a holomorphic deformation without first having a formal one.

Let us determine the relations between the t_i^j. These relations are given by the entries of the matrix $T_1 T_2 R - R T_2 T_1$, where $T_1 = T \odot 1$ and $T_2 = 1 \odot T$. Equivalently we can calculate the entries of the matrix

$$
T_2 T_1 - R(z^{-1})\, T_1 T_2\, R(z). \tag{69}
$$

They are written down in the following table:

variables i,j,k,l	$t_i^j \otimes t_k^l \;-\; (R(z^{-1})\, T_1\, T_2\, R(z))_{ik,jl}$
$i=k,\; j<l$	$t_i^j \otimes t_i^l \;-\; z\, t_i^l \otimes t_i^j$
$i=k,\; j>l$	$t_i^j \otimes t_i^l \;-\; z^{-1}\, t_i^l \otimes t_i^j$
$i<k,\; j=l$	$t_i^j \otimes t_k^j \;-\; z\, t_k^j \otimes t_i^j$
$i>k,\; j=l$	$t_i^j \otimes t_k^j \;-\; z^{-1}\, t_k^j \otimes t_i^j$
$i<k,\; j>l$	$t_i^j \otimes t_k^l \;-\; t_k^l \otimes t_i^j$
$i<k,\; j<l$	$t_i^j \otimes t_k^l \;-\; \left(t_k^l \otimes t_i^j + (z-z^{-1})\, t_k^j \otimes t_i^l\right)$
$i>k,\; j>l$	$t_i^j \otimes t_k^l \;-\; \left(t_k^l \otimes t_i^j - (z-z^{-1})\, t_i^l \otimes t_k^j\right)$
$i>k,\; j<l$	$t_i^j \otimes t_k^l \;-$ $(t_k^l \otimes t_i^j + (z-z^{-1})\left(t_k^j \otimes t_i^l - t_i^l \otimes t_k^j - (z-z^{-1})\, t_i^j \otimes t_k^l\right))$

(70)

Check that the first, the third, the sixth and last entry in this table are linear combination of the other ones. Hence, $A(R)$ is generated by the t_i^j modulo the relations

$$\left\{ \begin{array}{l} \left(t_i^j \otimes t_k^l \;-\; z^{-1}\, t_k^l \otimes t_i^j\right)_{i=k,\, j>l},\; \left(t_i^j \otimes t_k^l \;-\; t_k^l \otimes t_i^j\right)_{i>k,\, j<l}, \\ \left(t_i^j \otimes t_k^l \;-\; \left(t_k^l \otimes t_i^j - (z-z^{-1})\, t_k^j \otimes t_i^l\right)\right)_{i>k,\, j>l},\; \left(t_i^j \otimes t_k^l \;-\; z^{-1}\, t_k^l \otimes t_i^j\right)_{i>k,\, j=l} \end{array} \right\}. \tag{71}$$

Now it is straightforward to show that $A(R)$ is free over $\mathcal{O}(\mathbb{C}^*)$ with basis

$$(T^m)_{m\in\mathbb{N}^{N^2}} \;:=\; \left((t_1^1)^{m_1^1} \cdot \ldots \cdot \left(t_1^N\right)^{m_1^N} \cdot (t_2^1)^{m_2^1} \cdot \ldots \cdot \left(t_N^N\right)^{m_N^N}\right)_{m=(m_1^1, m_1^2, \ldots, m_N^N)\in\mathbb{N}^{N^2}}. \tag{72}$$

Consequently $A(R)$ is isomorphic as $\mathcal{O}(\mathbb{C}^*)$-module to $\mathcal{O}(\mathbb{C}^*) \otimes A(R)_1 = \mathcal{O}(\mathbb{C}^*)\hat{\otimes} A(R)_1$, where the undeformed algebra $A(R)_1 = A(R)/\mathfrak{m}_1 A(R)$ is isomorphic to the algebra $\mathbb{C}[t_1^1, \ldots, t_1^N, \ldots, t_N^N]$ of polynomial functions on $\mathrm{M}(N \times N, \mathbb{C})$. The t_i^j can hereby be interpreted as the functions giving the (i,j) entry of a matrix. Altogether these considerations prove that $A(R)$ is a holomorphic deformation of $\mathbb{C}[t_1^1, \ldots, t_1^N, \ldots, t_N^N]$ indeed. Note that $A(R)$ does not have zero divisors.

Remark 5.5 The basis of $A(R)$ given in (72) could as well be derived using the diamond lemma of Bergman [3].

Next let us prove that the monomials $(T^m + I)_{m\in M}$ with

$$M = \left\{(m_1^1, \ldots, m_N^N) \in \mathbb{N}^{N^2} \;:\; \text{one element of } \left\{m_1^N, \ldots, m_N^1\right\} \text{ vanishes}\right\} \tag{73}$$

form a basis of $\mathcal{R}\left(\mathrm{SL_q}(N,\mathbb{C})\right) = A(R)/I$. Denote for $m \in \mathbb{N}^{N^2}$ by $\operatorname{val} m$ the **value**

$$\operatorname{val} m \;=\; m_1^N + \ldots + m_N^1. \tag{74}$$

To prove that $\operatorname{span}\left\{(T^m + I)_{m\in M}\right\} = A(R)/I$ it suffices to show that for $m \in \mathbb{N}^{N^2} \setminus M$ we can write the monomial T^m modulo I as a $\mathcal{O}(\mathbb{C}^*)$-linear combination of monomials T^r, $r \in \mathbb{N}^{N^2}$ with $\operatorname{val} r < \operatorname{val} m$. Indeed, repeating this process for the r and proceeding inductively we can after finitely many steps expand T^m modulo I as a combination of monomials T^s, $s \in \mathbb{N}^{N^2}$ with $\operatorname{val} s < \operatorname{val} m$ and $s \in M$. The idea now is to move for any pair (i,j) with $i+j = N+1$ one t_i^j appearing in T^m to the right while controlling the value of the monomials created by this process. Denote by $a_k^l \in \mathbb{N}^{N^2}$ the multiindex the entry of which at the $\binom{l}{k}$ position is 1 and 0 otherwise. Suppose we already have for an index $i \in \{1,\ldots,N\}$ the expansion

$$T^m \;=\; \sum_{n\in\mathbb{N}^{N^2}} c(n,i)\,T^n \;+\; d(i)\,T^{m(i)}\cdot t_i^{N+1-i}\cdot\ldots\cdot t_N^1, \tag{75}$$

where $c(n,i), d(i) \in \mathcal{O}(\mathbb{C}^*)$, $\operatorname{val} n < \operatorname{val} m$ for every $n \in \mathbb{N}^{N^2}$ with $c(n,i) \neq 0$ and $m(i) = m - \sum_{k\geq i} a_k^{N+1-k}$. Now consider one t_{i-1}^{N+2-i} in T^m. By the commutation relations (70) we have

$$T^{m(i)} \;=\; \tilde{d}(i)\,T^{m(i)-a_{i-1}^{N+2-i}}\,t_{i-1}^{N+2-i} \;+\; \sum_{n\in M(i)} \tilde{c}(n,i)\,T^n, \tag{76}$$

where $\tilde{d}(i), \tilde{c}(i,n) \in \mathcal{O}(\mathbb{C}^*)$ and

$$M(i) \;=\; \left\{ n\in\mathbb{N}^{N^2} : \begin{array}{l} n_1^1 + n_1^2 + \ldots + n_N^N = m(i)_1^1 + m(i)_1^2 + \ldots + m(i)_N^N, \\ n_i^{N+1-i} = m(i)_i^{N+1-i} - 1, \quad n_k^{N+1-k} = m(i)_k^{N+1-k} \text{ for } k \neq i \end{array} \right\}. \tag{77}$$

By definition $\operatorname{val} n < \operatorname{val} m(i)$ holds for every $n \in M(i)$. Hence, by (70) the expansion of $T^n \cdot t_{i-1}^{N-i}\cdot\ldots\cdot t_1^N$, $n \in M(i)$ with respect to the basis $(T^{\tilde{m}})_{\tilde{m}\in\mathbb{N}^{N^2}}$ contains only monomials $T^{\tilde{m}}$ with $\operatorname{val}\tilde{m} < \operatorname{val} m$. Therefore the relation

$$T^m \;=\; \sum_{n\in\mathbb{N}^{N^2}} c(n,i-1)\,T^n \;+\; d(i-1)\,T^{m(i-1)}\,t_{i-1}^{N+2-i}\cdot\ldots\cdot t_N^1 \tag{78}$$

is true, and $\operatorname{val} n < \operatorname{val} m$ for every $n \in \mathbb{N}^{N^2}$ with $c(n,i-1) \neq 0$. Hence, by induction we can set $i = 1$ in (75). Modulo I this implies that

$$T^m \;=\; \sum_{n\in\mathbb{N}^{N^2}} c(n,1)\,T^n \;+\; d\,T^{m(1)} \;+\; \sum_{\sigma\in S_n\setminus\{\sigma_{\mathrm{inv}}\}} d(\sigma)\,T^{m(1)}\,t_1^{\sigma(1)}\cdot\ldots\cdot t_N^{\sigma(N)}, \tag{79}$$

where σ_{inv} is the permutation fulfilling $\sigma_{\mathrm{inv}}(i) = N+1-i$ for every $i = 1,\ldots,N$ and $c(n), d, d(\sigma)$ are elements of $\mathcal{O}(\mathbb{C}^*)$ such that $c(n) = 0$ for $n \in \mathbb{N}^{n^2}$ with $\operatorname{val} n \geq \operatorname{val} m$.

Now we want to determine the expansion of $T^{m(1)}\, t_1^{\sigma(1)} \cdot \ldots \cdot t_N^{\sigma(N)}$ with respect to the basis $\left(T^{\tilde m}\right)_{\tilde m \in \mathbb{N}^{N^2}}$ by "moving the $t_i^{\sigma(i)}$ to the left". Check that for every $\sigma \in S_n \setminus \{\sigma_{inv}\}$ there exists an index i such that $i + \sigma(i) > N + 1$. Moving the corresponding $t_i^{\sigma(i)}$ in the monomial $T^{m(1)}\, t_1^{\sigma(1)} \cdot \ldots \cdot t_N^{\sigma(N)}$ to the left by using the commutation relations (70) one does not generate a further t_k^{N+1-k}. Moving another $t_j^{\sigma(j)}$ to the left one generates at most one t_k^{N+1-k}. This entails that $T^{m(1)}\, t_1^{\sigma(1)} \cdot \ldots \cdot t_N^{\sigma(N)}$ can be expanded by monomials $T^{\tilde m}$ with $\operatorname{val} \tilde m \leq m(1) + N - 1 = \operatorname{val} m - 1$. But then T^m itself can modulo I be expanded by such monomials, hence $\operatorname{span}\left\{ (T^m + I)_{m \in M} \right\} = A(R)/I$.

We still have to prove that $(T^m + I)_{m \in M}$ is a linear independant family. So let

$$\sum_{m \in M} d(m)\, T^m \;=\; a \,\left(\det_{\mathrm{q}} T - 1\right) \tag{80}$$

with $d(m) \in \mathcal{O}(\mathbb{C}^*)$ and $a \in A(R)$, and assume that both sides do not vanish. But then by (70) and the definition of $\det_{\mathrm{q}} T$ the expansion of the right hand side contains at least one monomial T^m, $m \notin M$ with nonvanishing coefficient in $\mathcal{O}(\mathbb{C}^*)$. Thus the left and right side of (80) cannot be equal unlike they are both 0. This shows the linear independance.

As the monomials T^m, $m \in M$ form a $\mathbb{C}$-basis of the undeformed algebra $\mathcal{R}(\mathrm{SL}(N, \mathbb{C}))$, it follows that as (topological) vector spaces $\mathcal{O}(\mathbb{C}^*) \otimes \mathcal{R}(\mathrm{SL}(N, \mathbb{C}))$ and $\mathcal{R}(\mathrm{SL}_{\mathrm{q}}(N, \mathbb{C}))$ are isomorphic. Furthermore we have the following chain of Hopf algebra isomorphisms:

$$\begin{aligned} &\mathcal{R}(\mathrm{SL}_{\mathrm{q}}(N, \mathbb{C}))/\mathfrak{m}_1 \mathcal{R}(\mathrm{SL}_{\mathrm{q}}(N, \mathbb{C})) \cong \\ &\quad \cong \;\; A(R)/\mathfrak{m}_1 A(R) \,/\, I/\mathfrak{m}_1 I \cong A(R)_1 \,/\, A(R)_1 (\det T - 1) \cong \mathcal{R}(\mathrm{SL}(N, \mathbb{C})). \end{aligned} \tag{81}$$

This finally proves the claim, i.e. $\mathcal{R}(\mathrm{SL}_{\mathrm{q}}(N, \mathbb{C}))$ is a holomorphic deformation of $\mathcal{R}(\mathrm{SL}(N, \mathbb{C}))$.

Next we want to sketch the proof for the isomorphy of $\mathcal{R}(\mathrm{SL}_{\mathrm{q}}(N, \mathbb{C}))$ and $\mathcal{U}_q\mathfrak{s}l(N, \mathbb{C})'$ or in other words for the duality between $\mathcal{R}(\mathrm{SL}_{\mathrm{q}}(N, \mathbb{C}))$ and $\mathcal{U}_q\mathfrak{s}l(N, \mathbb{C})$. Denote for $z \in \mathbb{C}$ by $\mathcal{R}(\mathrm{SL}_z(N, \mathbb{C}))$ the Hopf algebra $\mathcal{R}(\mathrm{SL}_{\mathrm{q}}(N, \mathbb{C}))/\mathfrak{m}_z\, \mathcal{R}(\mathrm{SL}_{\mathrm{q}}(N, \mathbb{C}))$. Similarly let $\mathcal{U}_z\mathfrak{s}l(N, \mathbb{C}) = \mathcal{U}_q\mathfrak{s}l(N, \mathbb{C})/\mathfrak{m}_z\, \mathcal{U}_q\mathfrak{s}l(N, \mathbb{C})$ for $z \in \Omega$. We claim that $\mathcal{R}(\mathrm{SL}_{\mathrm{e}^z}(N, \mathbb{C}))$, $z \in \Omega$ is topologically isomorphic to the restricted Hopf dual $\mathcal{U}_z\mathfrak{s}l(N, \mathbb{C})^\circ$ with the finest locally convex topology. Check that there is a canonical N-dimensional indecomposable $\mathcal{U}_z\mathfrak{s}l(N, \mathbb{C})$-module $\rho : \mathcal{U}_z\mathfrak{s}l(N, \mathbb{C}) \to \operatorname{End}(W)$. Define the matrix coefficients A_{ij} by

$$\rho(h) \;=\; \begin{pmatrix} A_{11}(h) & \cdot & \cdot & \cdot & A_{1n}(h) \\ \cdot & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ A_{n1}(h) & \cdot & \cdot & \cdot & A_{nn}(h) \end{pmatrix} \tag{82}$$

for any $h \in \mathcal{U}_z$. Therefore, we can define a homomorphism $\Phi : \mathrm{T}C \to \mathcal{U}_z\mathfrak{s}l(N, \mathbb{C})^\circ$, where $C = \operatorname{End}(W)'$. Obviously, Φ is continuous with respect to the finest locally convex topology on $\mathrm{T}C$. By a lengthy calculation one can show that Φ is surjective and $\ker \Phi = I_z := \ker(\mathrm{T}C \to \mathcal{R}(\mathrm{SL}_{\mathrm{e}^z}(N,\mathbb{C}))$. (More details for this are given in CHARI, PRESSLEY [5], Theorem 7.1.4. for the case of $\mathcal{U}\mathfrak{s}l(2,\mathbb{C})$ and in TAKEUCHI [27, 26] for the general case. See also DRINFEL'D [6].) Hence, Φ induces a linear isomorphism $\overline{\Phi} : \mathcal{R}(\mathrm{SL}_{\mathrm{e}^z}(N,\mathbb{C})) \to \mathcal{U}_z\mathfrak{s}l(N,\mathbb{C})^\circ$ which is topological since both spaces are endowed with the finest locally convex topology.

We now have proven the main part of the following theorem.

Theorem 5.6 *The FRT-algebra* $\mathcal{R}(\mathrm{SL}_\mathrm{q}(N,\mathbb{C})) = A(R)/(\det_\mathrm{q} - 1)A(R)$ *corresponding to the R-matrix (64) comprises a holomorphic quantization of the Poisson algebra* $\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))$ *of matrix coefficients on the Lie group* $\mathrm{SL}(N,\mathbb{C})$. *Moreover, it coincides with the deformation* $\mathcal{U}_\mathrm{q}\mathfrak{s}l(N,\mathbb{C})'$ *dual to the quantization of* $\mathcal{U}\mathfrak{s}l(N,\mathbb{C})$. *For every* $z \in \Omega$ *the Hopf algebra* $\mathcal{R}(\mathrm{SL}_{\mathrm{e}^z}(N,\mathbb{C}))$ *is topologically isomorphic to the restricted Hopf dual* $\mathcal{U}_z\mathfrak{s}l(N,\mathbb{C})^\circ$, *and* $\mathcal{U}_z\mathfrak{s}l(N,\mathbb{C})$ *to* $\mathcal{R}(\mathrm{SL}_{\mathrm{e}^z}(N,\mathbb{C}))'$.

PROOF: It has been shown above that $A(R)/(\det_\mathrm{q} - 1)A(R)$ is a holomorphic deformation of $\mathcal{R}(\mathrm{SL}(N,\mathbb{C}))$. That it is even a holomorphic quantization follows from Theorem 3.4 and [7].

As $\overline{\Phi} : \mathcal{R}(\mathrm{SL}_{\mathrm{e}^z}(N,\mathbb{C})) \to \mathcal{U}_z\mathfrak{s}l(N,\mathbb{C})^\circ$ is a topological isomorphism, paragraph 2.2 entails that $\mathcal{R}(\mathrm{SL}_{\mathrm{e}^z}(N,\mathbb{C}))$ is isomorphic to $\mathcal{U}_z\mathfrak{s}l(N,\mathbb{C})'$. By strict nuclearity $\mathcal{U}_z\mathfrak{s}l(N,\mathbb{C}) \cong \mathcal{R}(\mathrm{SL}_{\mathrm{e}^z}(N,\mathbb{C}))'$ follows as well. $\square$

A Nuclear spaces and holomorphic vector-valued functions

Locally convex spaces

A seminorm on a (real or complex) vector space E is a function $p : E \to \mathbb{R}$ satisfying $p \geq 0$, $p(0) = 0$, $p(x+y) \leq p(x)+p(y)$ and $p(\lambda x) = |\lambda| p(x)$ for all $x, y \in E$ and $\lambda \in \mathbb{K}$. Any family P of seminorms on E defines a locally convex topology on E: A subbasis of this topology is, for example, the system of all "open balls" $\mathrm{B}_p(a, r) = \{x \in E : p(a - x) < r\}$, where $a \in E$, $r > 0$ and $p \in P$. E together with such a topology is called a locally convex space, and the set of all continuous seminorms on E will be denoted by $\mathrm{cs}(E)$. A locally convex space can also be described as a vector space E together with a translation invariant topology for which the structure maps $E \times E \to E$ and $\mathbb{K} \times E \to E$ are continuous such that $0 \in E$ has a neighborhood basis $\mathcal{B}$ consisting of absolutely convex open subsets $U \subset E$. Each such U corresponds to a seminorm

$$p_U(x) := \inf \{\lambda \in \mathbb{R} : x \in \lambda U\}, \tag{83}$$

and the topology is defined by the family $\{p_U : U \in \mathcal{B}\}$ of seminorms. In general, P as above is called a defining family of seminorms of the locally convex space E.

A locally convex space E is metrizable (resp. normalizable), if E is Hausdorff and possesses a countable (resp. finite) defining family P of seminorms. E is Hausdorff, if to every $x \in E \setminus \{0\}$ there corresponds a $p \in P$ with $p(x) \neq 0$. We are mainly interested in locally convex Hausdorff spaces which are complete with respect to the uniformity given by $\mathrm{cs}(E)$. A complete metrizable locally convex space is called a Fréchet space.

Locally m-convex spaces

A locally convex algebra over $\mathbb{K}$ is an associative $\mathbb{K}$-algebra A with a locally convex topology such that the multiplication $A \times A \to A$ is continuous. In order to have a reasonable functional calculus on A one needs an additional property, namely m-convexity. Recall that a seminorm p on an algebra A is called multiplicative, if

$$p(xy) \leq p(x)\, p(y) \tag{84}$$

for all $x, y \in A$. A locally convex algebra A is called locally m-convex (cf. Michael [17]) if there exists a defining family P of multiplicative seminorms. On a complete locally m-convex algebra over $\mathbb{C}$ there exists a functional calculus as an action of the space $\mathcal{O}(\mathbb{C})$ of entire functions on the algebra A: To each $f \in \mathcal{O}(\mathbb{C})$ with power series expansion $f(z) = \sum c_n z^n$ and each $a \in A$ the series $\sum c_n a^n$ yields a well-defined element $\hat{f}(a) = \sum c_n a^n \in A$. The convergence of this series follows from $p(a^n) \leq p(a)^n$ for multiplicative seminorms p on A:

$$p\left(\sum_{m\le n\le m+k}\right) \le \sum_{m\le n\le m+k} |c_n|\, p(a)^n \longrightarrow 0 \tag{85}$$

for $m, k \to \infty$ and $p \in P$. Hence, $\sum c_n a^n$ is a Cauchy series. An example for a locally convex but not locally m-convex algebra is given by $\mathbb{C}[T]$ (or $\mathcal{U}\mathfrak{g}$) with the finest locally convex topology. Otherwise, $\sum \frac{1}{n!} a^n$ would converge for all $a \in A$. But for $a = T$ the sequence $\sum \frac{1}{n!} T^n$ does not have a limit in $\mathbb{C}[T]$. The usual topologies on function algebras are locally m-convex since they are defined by seminorms given by the supremum of the function (resp. their derivatives) on a family of subsets of the common domain of the functions in question. Clearly, those seminorms are multiplicative. In particular the algebra $\mathcal{O}(\Omega)$ of holomorphic functions on an open domain $\Omega \subset \mathbb{C}$ with the compact open topology is locally m-convex. This holds true for an open subset $\Omega \subset \mathbb{C}^n$ as well, for a complex manifold Ω or even for an infinite dimensional domain in a locally convex space E.

Every normed algebra over $\mathbb{K}$ is locally m-convex, and so is every subalgebra of a product of normed algebras. Hence, the locally convex inverse limit topology on A of a family $\varphi_i : A \to B_i$ of homomorphisms into normed algebras is locally m-convex, as is the locally convex projective limit of a projective system of m-convex algebras. Hence, the projective limit topologies considered in section 2 define locally m-convex algebras.

Nuclear Spaces

On a given tensor product $E \otimes F$ of two locally convex spaces E and F one can consider many different locally convex topologies arising from the topologies on E and F. The most natural one is the π-topology, i.e. the finest locally convex topology on $E \otimes F$ for which the natural mapping

$$\otimes :\; E \times F \to E \otimes F \tag{86}$$

is continuous. $E \otimes F$ with this topology is denoted by $E \otimes_\pi F$, its completion by $E \hat{\otimes} F$. Another useful topology can be described by the embedding $E \otimes F \to \mathcal{B}(E'_{\mathrm{s}}, F'_{\mathrm{s}})$, where E'_{s} is the dual E' equipped with the topology of simple convergence. The ε-topology is the topology induced from $\mathcal{B}_\varepsilon(E'_{\mathrm{s}}, F'_{\mathrm{s}})$, where the subscript ε denotes the topology of uniform convergence on all products $A \times B \subset E' \times F'$ of equicontinuous subsets $A \subset E', B \subset F'$. Both, the π-topology and the ε-topology are compatible with $\otimes$ in the following sense:

(*i*) $\otimes :\; E \times F \to E \otimes F$ is continuous,

(*ii*) for all $(e, f) \in E' \times F'$ the linear form $e \otimes f \;:\; E \otimes F \to \mathbb{K}$, $x \otimes y \mapsto e(x)\, f(y)$ is continuous.

In fact, the π-topology is the strongest and the ε-topology is the weakest topology on $E \otimes F$ compatible with $\otimes$.

A locally convex space E is called **nuclear**, if all the compatible topologies on $E \otimes F$ agree for all locally convex spaces F (cf. GROTHENDIECK [12]). All finite dimensional vector spaces are nuclear. Subspaces and Hausdorff quotients of nuclear spaces are nuclear, as well as the completion of a nuclear space. In general, locally convex projective limits of nuclear spaces and countable locally convex inductive limits of nuclear spaces are also nuclear. The direct sum $\mathbb{C}^{(\Lambda)}$ for an uncountable Λ provides an example of an inductive limit of nuclear spaces which is not nuclear. It also provides an example of a nuclear space E, namely $E = \mathbb{C}^{\Lambda}$, such that the dual $E' \cong \mathbb{C}^{(\Lambda)}$ is not nuclear with respect to the strong topology on E', i.e. the topology of uniform convergence on bounded subsets of E. In the following we will denote by $E\hat{\otimes}F$ the completion of $E \otimes_\pi F$. Note, that $E\hat{\otimes}F$ is nuclear for complete nuclear spaces E and F.

Definition A.1 *We call a reflexive nuclear space E* **strictly nuclear** *if the dual E' with the strong topology is nuclear as well and if the inclusion $E' \otimes F' \to (E\hat{\otimes}F)'$ induces an isomorphism*

$$E'\hat{\otimes}E' \;\cong\; (E\hat{\otimes}E)', \tag{87}$$

algebraically and topologically. Hereby E', F' and $(E\hat{\otimes}F)'$ are endowed with the strong topology which coincides with the compact open topology for complete nuclear spaces E.

Formula (87) holds true for nuclear Fréchet spaces (cf.TRÈVES [29]) and therefore, by dualizing, also for duals of nuclear Fréchet spaces. It also can be proven for nuclear LF-spaces E (and their duals), i.e. nuclear spaces E which can be described as an inductive limit $E_n \subset E_{n+1} \subset ... \subset E$ of Fréchet spaces such that E_n has the topology induced from E_{n+1}. As a consequence many of the important function spaces in Analysis are strictly nuclear, e.g. the space $\mathcal{O}(\Omega)$ of holomorphic functions on a complex manifold Ω, the space $\mathcal{E}(M) = \mathcal{C}^\infty(M)$ of infinitely differentiable functions on a manifold M, the space $\mathcal{D}(M)$ of test functions on a manifold M and the dual, the space of distributions. The latter two examples are in general neither Fréchet nor dual Fréchet. Note that for a strictly nuclear space E the dual E' is strictly nuclear as well. This can be seen by simply dualizing (87).

The space $\mathbb{K}^{(\Lambda)}$ of functions $\Lambda \to \mathbb{K}$ with finite support for an arbitrary set Λ is the prototype of a locally convex space E with the finest locally convex topology (simply choose a Hamel basis of E indexed by Λ). The space $\mathbb{K}^{(\Lambda)}$ satisfies the duality condition (87) as well, although this space is only nuclear (and strictly nuclear) for countable Λ. The reason for this is essentially the fact that for $E := \mathbb{K}^{(\Lambda)}$ the tensor product $E \otimes E$ with the π-topology is already complete and isomorphic to $\mathbb{K}^{(\Lambda\times\Lambda)}$, which follows from the observation that the product topology on $E \times E$ is the finest locally convex topology. Moreover, the strong dual of $\mathbb{K}^{(\Lambda)}$ is isomorphic to $\mathbb{K}^{\Lambda}$ with the product topology. Hence,

$$(E\hat{\otimes}E)' \cong (\mathbb{K}^{(\Lambda\times\Lambda)})' \cong \mathbb{K}^{\Lambda\times\Lambda} \cong \mathbb{K}^{\Lambda}\hat{\otimes}\mathbb{K}^{\Lambda} \cong (\mathbb{K}^{(\Lambda)})'\hat{\otimes}(\mathbb{K}^{(\Lambda)})',$$

i.e.

$$(\mathbb{K}^{(\Lambda)})'\hat{\otimes}(\mathbb{K}^{(\Lambda)})' \cong (\mathbb{K}^{(\Lambda)}\hat{\otimes}\mathbb{K}^{(\Lambda)})' \tag{88}$$

and similarly or by dualizing

$$(\mathbb{K}^{\Lambda})'\hat{\otimes}(\mathbb{K}^{\Lambda})' \cong (\mathbb{K}^{\Lambda}\hat{\otimes}\mathbb{K}^{\Lambda})'. \tag{89}$$

Holomorphic vector-valued functions

The theory of holomorphic functions on an open subset Ω of $\mathbb{C}$ with values in a complete locally convex space E over $\mathbb{C}$ parallels the theory of holomorphic functions $g : \Omega \to \mathbb{C}$. The following is easy to show.

Proposition A.2 *For a continuous $f : \Omega \to E$ the following properties are equivalent:*

(i) f is holomorphic, i.e. there is a continuous $f' : \Omega \to E$ with

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = f'(z), \quad z \in \Omega. \tag{90}$$

(ii) f is analytic, i.e. for each $z \in \Omega$ there are $c_n \in E$ such that

$$f(z+h) = \sum_{n=0}^{\infty} c_n\, h^n \tag{91}$$

for small $h \in \mathbb{C}$.

(iii) For all continuous linear forms $\alpha \in E'$ the function $\alpha \circ f : \Omega \to \mathbb{C}$ is holomorphic.

Recall that for a given sequence (c_n) of points in E the power series $\sum c_n T^n$ converges in E for $|h| < r$, if and only if for all $p \in \mathrm{cs}(E)$ and all $s < r$ the series $\sum p(c_n)\, s^n$ converges (or equivalently: is bounded). Of course, the proposition has a straightforward generalization to open subsets Ω of $\mathbb{C}^n$.

Let $\mathcal{O}(\Omega, E)$ denote the space of holomorphic functions $f : \Omega \to E$ endowed with the compact open topology. Since E is complete, $\mathcal{O}(\Omega, E)$ is complete as well. It can be shown that $\mathcal{O}(\Omega, E)$ induces on $\mathcal{O}(\Omega) \otimes E$ the ε-topology via the canonical map $\mathcal{O} \otimes E \ni f \otimes x \mapsto (z \mapsto f(z)\, x \in E)$. In the case of $\Omega \subset \mathbb{C}$ (or $\Omega \subset \mathbb{C}^n$) the nuclearity of $\mathcal{O}(\Omega)$ implies that $\mathcal{O}(\Omega)\hat{\otimes}E \cong \mathcal{O}(\Omega, E)$. Moreover, the space $E_\Omega := \mathcal{O}(\Omega, E)$ is a $\mathcal{O}(\Omega)$-module with continuous action $\mathcal{O}(\Omega) \times \mathcal{O}(\Omega, E) \to \mathcal{O}(\Omega, E)$.

Proposition A.3 *For complete locally convex spaces E and F the $\mathcal{O}(\Omega)$-module $(E\hat{\otimes}F)_\Omega = \mathcal{O}(\Omega, E\hat{\otimes}F)$ has the following universal property: Every continuous $\mathcal{O}(\Omega)$-bilinear map $\beta : E_\Omega \times F_\Omega \to G_\Omega$ can factored by a continuous $\mathcal{O}(\Omega)$-linear mapping $\tilde{\beta} :\ (E\hat{\otimes}F)_\Omega \to G_\Omega$, i.e. $\beta = \tilde{\beta} \circ \otimes$.*

Of course, any $\mathcal{O}(\Omega)$-linear $\lambda : (E\hat{\otimes}F)_\Omega \to G_\Omega$ determines an $\mathcal{O}(\Omega)$-bilinear mapping $\lambda \circ \otimes$ on $E_\Omega \times F_\Omega$. Because of the property stated in the proposition, we denote $(E\hat{\otimes}F)_\Omega$ also by $E_\Omega \hat{\otimes}_{\mathcal{O}(\Omega)} F_\Omega$. Indeed, $(E\hat{\otimes}F)_\Omega$ is the completion of $E_\Omega \otimes_{\mathcal{O}(\Omega)} F_\Omega$.

PROOF: β is determined by $\beta|_{E\times F} : E \times F \to G_\Omega$. By the universal property of $E\hat{\otimes}F$ there is a continuous $\mathbb{C}$-linear $\lambda : E\hat{\otimes}F \to G_\Omega$ with $\beta|_{E\times F} = \lambda \circ \otimes$. λ determines a $\mathcal{O}(\Omega)$-bilinear map $\tilde{\beta}_0$ by $\tilde{\beta}_0(f \otimes (x \otimes y)) = f \cdot \lambda\,(x \otimes y)$ for all $f \in \mathcal{O}(\Omega)$, $(x, y) \in E \times F$. Hereby "$\cdot$" means pointwise multiplication of functions. $\tilde{\beta}_0$ is continuous on $\mathcal{O}(\Omega) \otimes (E \otimes F)$ and therefore has a continuous $\mathcal{O}(\Omega)$-linear extension $\tilde{\beta}$ to $\mathcal{O}(\Omega)\hat{\otimes}(E\hat{\otimes}F) = (E\hat{\otimes}F)_\Omega$ with $\beta = \tilde{\beta} \circ \otimes$. □

Another useful result for our presentation in section 3 is the following

Proposition A.4 $\mathcal{L}_{\mathcal{O}(\Omega)}(E_\Omega, F_\Omega)$ *is algebraically and topologically isomorphic to* $\mathcal{O}(\Omega, \mathcal{L}(E, F))$. *Hereby* $\mathcal{L}_{\mathcal{O}(\Omega)}(E_\Omega, F_\Omega)$ *is the* $\mathcal{O}(\Omega)$*-module of* $\mathcal{O}(\Omega)$*-linear and continuous maps* $E_\Omega \to F_\Omega$ *with the topology of uniform convergence on the bounded sets of* $E_\Omega = \mathcal{O}(\Omega, E)$.

PROOF: We first show that $T \mapsto T|_E$ defines an isomorphism $\mathcal{L}_{\mathcal{O}(\Omega)}(E_\Omega, F_\Omega) \to \mathcal{L}(E, F_\Omega)$ of locally convex spaces. Clearly, $t := T|_E : E \to F_\Omega$ is $\mathbb{C}$-linear and continuous. Hence, $t \in \mathcal{L}(E, F_\Omega)$, and the map $T \mapsto t$ is $\mathcal{O}(\Omega)$-linear, injective and continuous. For a given $s \in \mathcal{L}(E, F_\Omega)$

$$S(f \otimes x) \;:=\; f \cdot s(x) \tag{92}$$

holds for all $f \in \mathcal{O}(\Omega)$, and $x \in E$ defines an element of $\mathcal{L}_{\mathcal{O}(\Omega)}(E_\Omega, F_\Omega)$ with $S|_E = s$. It remains to show that $t \mapsto T$ is continuous. The topology on $\mathcal{L}_{\mathcal{O}(\Omega)}(E_\Omega, F_\Omega)$ is generated by the seminorms

$$q_{K,B}(T) := \sup\{q(Tf(z)) : f \in B, z \in K\}, \quad T \in \mathcal{L}_{\mathcal{O}(\Omega)}(E_\Omega, F_\Omega), \tag{93}$$

where K is a compact subset of Ω and B is a bounded subset of E_Ω. For such K and B the set $B(K) := \{f(z) : f \in B \text{ and } z \in K\}$ is a bounded subset of E. Since T is $\mathcal{O}(\Omega)$-linear one obtains $T(\lambda \otimes x)(z) = \lambda(z)(Tx)(z) = t(\lambda(z)x)(z)$ whenever $z \in \Omega$, $\lambda \in \mathcal{O}(\Omega)$ and $x \in E$. Therefore, $(Tf)(z) = t(f(z))(z)$ for all $z \in \Omega$. As a consequence,

$$\begin{aligned} q_{K,B}(T) = \sup\{q(t(f(z))(z)) : z \in K, f \in B\} \leq \\ \leq \sup\{q(tx(z)) : z \in K, f \in B(K)\} = q_{K,B(K)}(t), \end{aligned} \tag{94}$$

i.e. $t \mapsto T$ is continuous.

In order to prove $\mathcal{L}(E, F_\Omega) \cong \mathcal{O}(\Omega, \mathcal{L}(E, F))$, one can proceed in a similar manner. However, this isomorphism can also be deduced from general results on the ε-product (cf. [25]) which is useful for product formulas for spaces of holomorphic functions (cf. [23]): Let E'_c be the dual of the complete locally convex space E endowed with the topology of compact convergence. Then $E\varepsilon F := \mathcal{L}(E'_c, F)$, endowed with the topology of uniform convergence on the equicontinuous subsets of E', is (by transposition) canonically isomorphic to $F\varepsilon E$, and this new product is associative. By the theorem of Mackey, $\mathcal{L}(E,F) \cong E'_c\varepsilon F$, and for a nuclear space E one can show $E\varepsilon F \cong E\hat{\otimes}F$. Hence, $\mathcal{O}(\Omega,F) \cong \mathcal{O}(\Omega)\varepsilon F$ and we conclude

$$\mathcal{L}(E,F_\Omega) \;\cong\; E'_c\varepsilon\,(\mathcal{O}(\Omega)\varepsilon F) \;\cong\; \mathcal{O}(\Omega)\varepsilon(E'_c\varepsilon F) \;\cong\; \mathcal{O}(\Omega)\varepsilon\mathcal{L}(E,F) \;\cong\; \mathcal{O}(\Omega,\mathcal{L}(E,F)).$$

□

Corollary A.5 *The following identifications hold for a complete locally convex space.*

(i) $\mathcal{L}_{\mathcal{O}(\Omega)}(E_\Omega,\mathcal{O}(\Omega)) \cong \mathcal{O}(\Omega,E')$, *or in a shorter way:* $(E_\Omega)' \cong (E')_\Omega = E'_\Omega$.

(ii) $E_\Omega\hat{\otimes}_{\mathcal{O}(\Omega)}E_\Omega \cong (E\hat{\otimes}E)_\Omega$ *and therefore*

$$\mathcal{L}_{\mathcal{O}(\Omega)}(E_\Omega\hat{\otimes}_{\mathcal{O}(\Omega)}E_\Omega, E_\Omega) \;\cong\; \mathcal{O}(\Omega,\mathcal{L}(E\hat{\otimes}E,E) \tag{95}$$

$$\mathcal{L}_{\mathcal{O}(\Omega)}(E_\Omega, E_\Omega\hat{\otimes}_{\mathcal{O}(\Omega)}E_\Omega) \;\cong\; \mathcal{O}(\Omega,\mathcal{L}(E,E\hat{\otimes}E). \tag{96}$$

(iii) If E is strictly nuclear $E'_\Omega\hat{\otimes}_{\mathcal{O}(\Omega)}E'_\Omega \cong (E_\Omega\hat{\otimes}_{\mathcal{O}(\Omega)}E_\Omega)'$.

Proof: The first two relations follow directly from the above proposition. The last isomorphism is a consequence of the following chain of isomorphisms:

$$E'_\Omega\hat{\otimes}_{\mathcal{O}(\Omega)}E'_\Omega \;\cong\; \mathcal{O}(\Omega,E'\hat{\otimes}E') \;\cong\; \mathcal{O}(\Omega,(E\hat{\otimes}E)') \tag{97}$$

since E is strictly nuclear. Furthermore,

$$\mathcal{O}(\Omega,(E\hat{\otimes}E)') \;\cong\; (E\hat{\otimes}E)'_\Omega \;\cong\; \big((E\hat{\otimes}E)_\Omega\big)' \;\cong\; \big(E_\Omega\hat{\otimes}_{\mathcal{O}(\Omega)}E_\Omega\big)'. \tag{98}$$

□

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