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On the Structure of Monodromy Algebras and Drinfeld Doubles

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Abstract

We give a review and some new relations on the structure of the monodromy algebra (also called loop algebra) associated with a quasitriangular Hopf algebra H. It is shown that as an algebra it coincides with the so-called braided group constructed by S. Majid on the dual of H. Gauge transformations act on monodromy algebras via the coadjoint action. Applying a result of Majid, the resulting crossed product is isomorphic to the Drinfeld double $\mathcal{D}(H)$. Hence, under the so-called factorizability condition given by N. Reshetikhin and M. Semenov-Tian-Shansky, both algebras are isomorphic to the algbraic tensor product $H \otimes H$. It is indicated that in this way the results of Alekseev et al. on lattice current algebras are consistent with the theory of more general Hopf spin chains given by K. Szlachányi and the author. In the Appendix the multi-loop algebras $\mathcal{L}_m$ of Alekseev and Schomerus [AS] are identified with braided tensor products of monodromy algebras in the sense of Majid, which leads to an explanation of the "bosonization formula" of [AS] representing $\mathcal{L}_m$ as $H \otimes \ldots \otimes H$.

1 Introduction

Monodromy algebras and Drinfeld doubles have appeared as quantum symmetry algebras in several models of low dimensional quantum field theory during the last few years.

Monodromy or loop algebras associated with quasitriangular Hopf algebras H play an important rôle in lattice approaches to Chern-Simons theory [AGS], topological quantum field theory [AS] and current algebras on a circle [AFFS]. In these models they commonly appear as algebras generated by the matrix elements of quantum holonomy operators $\mathbf{M} = (M^I_{ab})$ around closed loops. Their center is spanned by generators obeying the Verlinde algebra and commuting with all other link operators. Their represenation theory has been used in [AGS,AS] to study quantum Chern-Simons algebras ("graph algebras") on Riemann surfaces with coloured

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punctures. As a particular result the authors show that under the assumption on H to be a modular Hopf algebra the representation category of the associated monodromy algebra coincides with that of H itself [AS].

In [AFFS] these results have been used and further developped to study current algebras on a periodic lattice chain. The authors introduce "gauged loop algebras" $\mathcal{K}$ given as crossed products of a monodromy algebra with a copy of the gauge quantum group sitting on the initial ($\equiv$ end) point of the loop. It turns out [AFFS] that the center of $\mathcal{K}$ already coincides with the center of the whole lattice current algebra $\mathcal{A}$, thereby governing again its representation theory. Under the same assumptions as above it is proven in [AFFS] that the irreducible representations of $\mathcal{K}$ - and therefore of $\mathcal{A}$ - are in one-to-one correspondence with those of $H \otimes H$. [1]

On the other hand, in non-abelian spin systems [SzV] and their generalizations to Hopf spin sytems [NSz] it is the Drinfeld double $\mathcal{D}(H)$ [Dr] which plays a similarly central rôle. In fact, it appears as the appropriate generalization of the familiar order-disorder symmetry group $G \times G$ known from abelian G-spin models. As a rather complete classification result it has been proven in [NSz] that in the infinite lattice limit the category of DHR-superselection sectors of these models is precisely given by Rep $(\mathcal{D}(H))$. This has been done by constructing an equivalence class of localized transportable coactions $\rho : \mathcal{A} \to \mathcal{A} \otimes \mathcal{D}(H)$, where here $\mathcal{A}$ denotes the quasi-local C^*-algebra generated by all local observables on the infinite lattice chain. One may then show that these coactions are universal in the sense that any charged represenation π_r of $\mathcal{A}$ satisfying the DHR-selection criterion relative to some Haag-dual vacuum representation π_0 is equivalent to a representation $(\pi_0 \otimes D_r) \circ \rho$ where D_r is a representation of $\mathcal{D}(H)$ and where the correspondence $\pi_r \leftrightarrow D_r$ is one-to-one on equivalence classes. The associated statistics operators are $\mathbf{C}$-matrix valued and are precisely given by the representation matrices of the canonical quasi-triangular R-matrix of $\mathcal{D}(H)$. The above coaction is in a way dual to the symmetry action of $\mathcal{D}(H)$, which leaves the observables invariant and only acts on the order-disorder fields.

All this confirms with the fact that the Drinfeld double has also been discovered in 2-dimensional continuum models to describe the quantum symmetry associated with nonabelian order and disorder (kink) fields [DPR,Mü].

Not obviously being related, Majid has developed his theory of braided groups during the last 5 years [M3-M7] by also constructing new algebra and coalgebra structures from a given quasitriangular Hopf algebra H. Particular examples of physical interest are among others the Sklynanin Algebra and the quantum Lorentz group [M5]. As a guiding construction principle, braided groups always appear as (co-)algebras in a category of H-modules, i.e. all structural maps are built so as to become H-module morphisms. In the cases of interest for us one takes H (or its dual $\hat{H}$) itself as the underlying space and considers the (co-)adjoint H-action. The given structures of H and $\hat{H}$ must then be deformed in order to provide H-module morphisms under this action. Based on this philosophy one is also naturally lead to depart from ordinary tensor products to braided tensor products [M4]. Majid also provides what he calls "bosonization formulas" [M4,M5,M7,M8] by showing that various crossed product constructions by nontrivial Hopf actions lead to algebras which are nevertheless isomorphic to the ordinary tensor product of their constituents. This might already give a first hint to some hidden relations with the above mentioned results of [AFFS] on the representation theory of their algebra $\mathcal{K}$.

[1] This holds true on finite lattices. The discussion about the infinite lattice limit in [AFFS] is heuristic and inconclusive in various respects.

Digging further into the literature one discovers that already in 1988 Reshetikhin and Semenov-Tian-Shansky [RS1] have provided a somewhat related "bosonization formula" by noting that under a what they called "factorizability condition" on the R-matrix of a quasi-triangular Hopf algebra H the associated Drinfeld double $\mathcal{D}(H)$ is algebraically isomorphic to $H \otimes H$ [2] (to make this an isomorphism of Hopf algebras one has to twist the coproduct on $H \otimes H$, see also below and Section 7). Around the same time Majid had developed his theory of the Drinfeld double as a double crossed product of H with $\widehat{H_{op}}$. He then discovered that without any additional conditions the Drinfeld double of a quasi-triangular Hopf algebra H is always isomorphic as an algebra to an ordinary crossed product $\hat{H}_R \rtimes H$ [M2] by constructing a Hopf module action of H on a newly invented algebra $\hat{H}_R$, which only latter became to be described as the "braided group function algebra" [M4] dual to the braided group introduced in [M3].

Putting all these pieces together one could already arrive at the conjecture that Majid's crossed product $\hat{H}_R \rtimes H$ might be the same as the algebra $\mathcal{K}$ in [AFFS], the latter also being a crossed product (i.e. $\{$monodromy algebra$\} \rtimes H$), and that both algebras are therefore isomorphic to the Drinfeld double $\mathcal{D}(H)$. The identification with the representation category of $H \otimes H$ would then be a consequence of the results of [RS1], since their factorizability condition is in fact nothing but the invertibility property of the linear "monodromy map"

$$\mathsf{mon}_R \,:\, \hat{H} \ni \varphi \mapsto (\varphi \otimes id)(R_{op}R) \in H$$

which was proven (although somewhat hiddenly) to hold for modular Hopf algebras in Section 5.2 of [AS].

This picture may be further supported by the observation, that apart from the periodic boundary condition the lattice current algebra of [AFFS] is in fact a reformulation of the Hopf spin chain of [NSz]. To see this let me shortly review that the latter is defined by placing a copy of a finite dimensional Hopf algebra $\mathcal{A}_{2i} \cong H$ on each site and a copy of the dual algebra $\mathcal{A}_{2i+1} \cong \hat{H}$ on each link of a one dimensional lattice. Non-vanishing commutation relations are then postulated only on neighbouring site-link pairs, where one requires [NSz]

$$A_{2i}(a)A_{2i-1}(\varphi) \;=\; A_{2i-1}(\varphi_{(1)})\,\langle \varphi_{(2)} \mid a_{(1)}\rangle\, A_{2i}(a_{(2)}) \tag{1.1}$$

$$A_{2i+1}(\varphi)A_{2i}(a) \;=\; A_{2i}(a_{(1)})\,\langle a_{(2)} \mid \varphi_{(1)}\rangle\, A_{2i+1}(\varphi_{(2)}) \tag{1.2}$$

Here sites (links) are numbered by even (odd) integers and $H \ni a \mapsto A_{2i}(a) \in \mathcal{A}_{2i} \subset \mathcal{A}$, $\hat{H} \ni \varphi \mapsto A_{2i+1}(a) \in \mathcal{A}_{2i+1} \subset\; A$ denote the embeddings of the single site (link) algebras into the global quantum chain $\mathcal{A}$. As usual, $a_{(1)} \otimes a_{(2)} \equiv \sum_i a^i_{(1)} \otimes a^i_{(2)} = \Delta(a)$ denotes the coproduct on H (and analogously on $\hat{H}$) and $\langle a \mid \varphi\rangle \equiv \langle \varphi \mid a\rangle$ denotes the dual pairing $H \otimes \hat{H} \to \mathbf{C}$.

Let now $R \in H \otimes H$ be quasitriangular and put $\mathbf{R}_{2i} := (id \otimes A_{2i})(R_{op}^{-1}) \in H \otimes \mathcal{A}_{2i}$, where R_{op} denotes the image of R under the permutation of tensor factors. Let us also introduce

$$\mathbf{L}_{2i+1} := (id \otimes A_{2i+1})(\mathbf{E}) \in H \otimes \mathcal{A}_{2i+1} \tag{1.3}$$

where $\mathbf{E} = \sum e_\nu \otimes e^\nu \in H \otimes \hat{H}$ is the canonical element given by any basis $e_\nu \in H$ with dual basis $e^\nu \in \hat{H}$. One may then define "lattice currents"

$$\mathbf{J}_{2i+1} := \mathbf{R}_{2i}^{-1}\mathbf{L}_{2i+1} \in H \otimes \mathcal{A} \tag{1.4}$$

[2] Actually, this is more then just a bosonization, since the copy of the dual algebra $\hat{H} \subset \mathcal{D}(H)$ is traded for a copy of H. We will make the relations more explicit by dividing this transformation into two steps in Section 7, see also Chapter 7 of [M8]

which are immediately verified to satisfy the lattice current algebra of [AFFS]

$$\begin{aligned} [\mathbf{1}_H \otimes A_{2i}(a)]\mathbf{J}_{2i-1} &= \mathbf{J}_{2i-1}[a_{(1)} \otimes A_{2i}(a_{(2)})] && (1.5)\\ \mathbf{J}_{2i+1}[\mathbf{1}_H \otimes A_{2i}(a)] &= [a_{(1)} \otimes A_{2i}(a_{(2)})]\mathbf{J}_{2i+1} && (1.6)\\ \mathbf{J}^{13}_{2i+1}\mathbf{J}^{23}_{2i+1} &= R^{12}(\Delta \otimes id)(\mathbf{J}_{2i+1}) && (1.7)\\ \mathbf{J}^{13}_{2i-1}R^{12}\mathbf{J}^{23}_{2i+1} &= \mathbf{J}^{23}_{2i+1}\mathbf{J}^{13}_{2i-1} && (1.8) \end{aligned}$$

where the last two lines are understood as identities in $H \otimes H \otimes \mathcal{A}$, the upper indices indicating the canonical embeddings of tensor factors.

Hence, under the additional data given by a quasitriangular R the lattice algebras of [NSz] and [AFFS] are isomorphic. Moreover, periodic boundary conditions may be imposed in both versions by identifying $\mathcal{A}_0 \equiv \mathcal{A}_{2N}$.

The discovery of this isomorphism was the starting motivation for the present work. In fact, it seemed hard to believe that the different results on the representation theory of this model (i.e. isomorphic to $\mathrm{Rep}\,\mathcal{K} \equiv \mathrm{Rep}\,H \otimes H$ in [AFFS] and to $\mathrm{Rep}\,\mathcal{D}(H)$ in [NSz]) should be caused by the different approaches of the authors and/or the different boundary conditions.

As it turns out now, the true explanation is indeed given by the chain of algebra isomorphisms mentioned above

$$\mathcal{K} \cong \hat{H}_R \rtimes H \cong \mathcal{D}(H) \cong H \otimes H$$

(the last one holding provided mon_R is invertible). Moreover, all of this could in principle be deduced from the literature as quoted, provided one realizes that the braided group function algebra $\hat{H}_R$ of Majid is in fact the same object as the monodromy algebra introduced in [AGS,AS,AFFS]. So with this point of view the paper could end with just proving this last statement.

However I would say that even the experts agree that the relevant informations are spread in an rather unrelated way over the literature and often have very different appearances and notations making it hard to compare or even identify them. Due to the importance of these structures in the various mathematical and physical contexts mentioned above I therefore consider it worthwhile to give a selfcontained and unifying account of the whole story here.

So let me start in Section 2 by identifying the monodromy algebras of [AGS,AS,AFFS] as (duals of) braided groups $\hat{H}_R$ in the sense of Majid [M4,M5]. In Section 3 I will then review Majid's construction of a coadjoint H-action on $\hat{H}_R$ and identify the resulting crossed product $\mathcal{M}_R(H) := \hat{H}_R \rtimes H$ with the gauged monodromy algebra $\mathcal{K}$ given in Definition 5 of [AFFS]. Section 4 gives a complete proof of an observation of [AFFS] on the existence of commuting left and right monodromies inside $\mathcal{M}_R(H)$. In the course of this proof we find a copy of $\hat{H}$ naturally embedded as a subalgebra in $\mathcal{M}_R(H)$. In this way we recover as a Corollary in Section 5 the algebra isomorphism $\mathcal{M}_R(H) \cong \mathcal{D}(H)$ previously obtained in [M2]. In Section 6 we use methods of [M5] to show that the monodromy map mon_R extends to a homomorphism of crossed products $\mathsf{Mon}_R : \mathcal{M}_R(H) \to H \rtimes_{Ad} H$ which becomes an isomorphism iff mon_R is bijective (here Ad denotes the adjoint action of H on itself). Since by a simple bosonization formula the crossed product $H \rtimes_{Ad} H$ is naturally isomorphic to the algebraic tensor product $H \otimes_{alg} H$, this clarifies the results of [AFFS] on the representation theory of $\mathcal{K} \equiv \mathcal{M}_R(H)$ without relying on any semisimplicity assumptions.

In Section 7 we proceed to the Hopf algebra structure induced on $\mathcal{M}_R(H) \equiv \mathcal{K}$ by the isomorphism with $\mathcal{D}(H)$ and investigate its image on $H \otimes_{alg} H \cong H \rtimes_{Ad} H$ under the map Mon_R. We define on $H \otimes_{alg} H$ a cocycle deformed coproduct $\delta_{H\otimes H}$ and an associated quasitriangular

R-matrix $\mathcal{R}_{H\otimes H}$. In this way the map $\mathcal{D}(H) \cong \mathcal{M}_R(H) \to H \otimes_{alg} H$ induced by Mon_R provides a homomorphism (isomorphism, if mon_R is bijective) of quasitriangular Hopf algebras, which turns out to coincide with the one already given in [RS1]. It is conjectured that after some corrections [3] the Hopf algebra structure obtained on $\mathcal{K}$ by [AFFS] should agree with these results. Furthermore, with these corrections the $\mathcal{K}$-coaction on $\mathcal{A}$ discovered by [AFFS] should also be expected to become an example from the class of universal localized $\mathcal{D}(H)$-coactions on $\mathcal{A}$ established in [NSz].

A more detailed account of this will be given elsewhere, where it will also be shown that, independently of the existence of R, the model (1.1), (1.2) with periodic boundary conditions always contains $\mathcal{D}(H)$ as a global subalgebra governing its representation theory, similarly as in [AFFS].

The Appendix concludes with an independent investigation of the multi-loop algebras $\mathcal{L}_m$ of [AS]. They are shown to be braided tensor products in the sense of Majid of one-loop ($\equiv$ monodromy) algebras. Hence, using bosonization ideas of [M5-M8], they are naturally homomorphic (isomorphic, if mon_R is bijective) to the algebraic tensor product $H^{\otimes m}$. This explains the representation theory of $\mathcal{L}_m$ given in [AS].

In summary, I would once more like to emphasize that in most parts this paper should be considered as a review, unifying different notations and approaches and treating seemingly unrelated results within a common formalism. In particular, Majid's notion of braided groups is brought together with the techniques of generating matrices as advocated by the St. Petersburg school. In view of the fact that the various relations between monodromy algebras, Drinfeld doubles, braided groups and bosonization formulas seemingly have been overlooked in [AGS,AS,AFFS], this article is hoped to add a piece of clarity and also to serve a broader community working in the field.

I should point out that a rather extensive review on braided groups and braided algebras can be found in [M7] and the relation with the Drinfeld double is also reviewed in [M8], however without relating it to monodromy algebras and without using the techniques of generating matrices.

Throughout this paper all algebras are taken to be finite dimensional, but no assumptions on semi-simplicity nor on $*$-structures are made. The reader is supposed to be familiar with standard Hopf algebra theory and notation, see e.g. [Sw]. Elements of H will be denoted by $a, b, c, \ldots$ and elements of $\hat{H}$ by $\varphi, \psi, \chi, \ldots$. The counit is denoted by $\varepsilon : H \to \mathbf{C}$ and the antipode by $S : H \to H$.

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2 Monodromy Algebras and Braided Groups

Defining relations of so-called monodromy or loop algebras have been given in [AGS,AS], where they appear as algebras generated by the matrix elements of quantum holonomy operators around closed loops. For an earlier version based on quadratic relations see also [RS2]. In

[3] There were some inconsistencies in [AFFS], see Section 7

order to establish that they are indeed well defined associative algebras we start by identifying their dual coalgebras with the braided groups introduced by Majid in [M3].

Proposition 2.1 [M3] *Let $R = \sum_i x^i \otimes y^i \in H \otimes H$ be quasitriangular with respect to Δ and let $\Delta_R : H \to H \otimes H$ be a deformed coproduct given by*

$$\Delta_R(a) = \sum_{i,j} (x^i \otimes S(y^j)y^i)\Delta(a)(x^j \otimes \mathbf{1}), \;\; a \in H \tag{2.1}$$

Then Δ_R is coassociative with counit ε.

Proof: Recall that quasitriangularity of R means [Dr2]

$$\begin{aligned} (\Delta \otimes id)(R) &= R^{13}R^{23} && (2.2)\\ (id \otimes \Delta)(R) &= R^{13}R^{12} && (2.3)\\ R\,\Delta(a)\,R^{-1} &= \Delta_{op}(a) && (2.4) \end{aligned}$$

This implies $(\varepsilon \otimes id)(R) = (id \otimes \varepsilon)(R) = \mathbf{1}$ and therefore ε is a counit for Δ_R. Moreover, (2.2)-(2.4) also imply the *cocycle property* [Dr2]

$$R^{12}(\Delta \otimes id)(R) = R^{23}(id \otimes \Delta)(R) \tag{2.5}$$

which is sufficient for $\Delta' : H \to H \otimes H$

$$\Delta'(a) := R\Delta(a) \tag{2.6}$$

to be coassociative. Now Δ' obeys

$$\Delta'(ab) = \Delta'(a)\Delta(b) = \Delta_{op}(a)\Delta'(b) \tag{2.7}$$

for all $a, b \in H$, from which Proposition 2.1 follows by using

Lemma 2.2: *Let $\Delta' : H \to H \otimes H$ be a coassociative coproduct satisfying (2.7). Then $\Delta_R(a) := \sum_j (\mathbf{1} \otimes S(y^j))\Delta'(a)(x^j \otimes \mathbf{1})$ is also coassociative.*

Proof: Using (2.7) and (2.2) we compute for $a \in H$

$$\begin{aligned} (\Delta_R \otimes id)(\Delta_R(a)) &= \sum_{i,j} [\mathbf{1} \otimes S(y^i) \otimes S(y^j)]\Delta'^{(2)}(a)[x^j_{(1)}x^i \otimes x^j_{(2)} \otimes \mathbf{1}]\\ &= \sum_{i,j,k} [\mathbf{1} \otimes S(y^i) \otimes S(y^j y^k)]\Delta'^{(2)}(a)[x^j x^i \otimes x^k \otimes \mathbf{1}] \end{aligned}$$

where $\Delta'^{(2)} = (\Delta' \otimes id) \circ \Delta' = (id \otimes \Delta') \circ \Delta'$. Similarly, using (2.7) and (2.3),

$$\begin{aligned} (id \otimes \Delta_R)(\Delta_R(a)) &= \sum_{i,j} [\mathbf{1} \otimes S(y^i_{(1)}) \otimes S(y^k)S(y^i_{(2)})]\Delta'^{(2)}(a)[x^i \otimes x^k \otimes \mathbf{1}]\\ &= \sum_{i,j,k} [\mathbf{1} \otimes S(y^i) \otimes S(y^k)S(y^j)]\Delta'^{(2)}(a)[x^j x^i \otimes x^k \otimes \mathbf{1}]\\ &= (\Delta_R \otimes id)\Delta_R(a) \qquad \blacksquare \end{aligned}$$

It has been remarked in equ. (32) of [M6] that the passage from Δ' to Δ_R can also be viewed as a cocycle transformation in a generalized sense, i.e. by considering according to (2.7) (H, Δ')

as an $(H \otimes H)$-Hopf right-module coalgebra with right action $a \triangleleft (b \otimes c) := S(b)ac$, and by acting with the $H \otimes H$-right cocycle $R_{23} \in (H \otimes H) \otimes (H \otimes H)$ (see Lemma 7.1) to transform $\Delta' \to \Delta_R$.

Up to a change of conventions (i.e. replacing (H, Δ, S, R) by $(H, \Delta_{op}, S^{-1}, R_{op})$) the deformed coproduct (2.1) has first appeared in [M3], providing H with the structure of what the author calls a "braided group". One should also point out here that the deformed coproducts Δ' and Δ_R are no longer algebra homomorphisms. However, as has been emphasized by Majid [M3,M4], there exists a natural deformed algebra structure on $H \otimes H$, the "braided tensor product" $H \otimes_R H$ (see Appendix A), such that $\Delta_R : H_{cop} \to H_{cop} \otimes_{R_{op}} H_{cop}$ does provide an algebra homomorphism (see equ.(A.11)), where H_{cop} is the Hopf algebra H with opposite coproduct Δ_{op}.

Following Theorem 4.1 of [M4], see also equ. (8) of [M5], we now pass to the algebra dual to (H, Δ_R).

Definition 2.3 [M4] *The "braided group function algebra" $\hat{H}_R$ is defined to be the vector space $\hat{H}$ with multiplication $\times_R : \hat{H} \otimes \hat{H} \to \hat{H}$ induced by Δ_R, i.e.*

$$\langle \varphi \times_R \psi \mid a \rangle := \langle \varphi \otimes \psi \mid \Delta_R(a) \rangle \tag{2.8}$$

for $\varphi, \psi \in \hat{H}$ and $a \in H$.

Note that Δ_R being coassociative, the new product $\times_R$ on $\hat{H}$ is clearly associative with unit given by $\varepsilon \in \hat{H}$. Let me now show that the Definition 2.3 coincides with the description of monodromy algebras given by [AGS,AS,AFFS] in terms of a generating matrix $\mathbf{M}$.

In what follows, $\mathcal{A}$ always denotes an arbitrary target algebra. The basic idea behind generating matrices is given by the observation that if $\hat{\mathcal{B}}$ is the algebra dual to some given finite dimensional coalgebra $(\mathcal{B}, \Delta_{\mathcal{B}}, \varepsilon_{\mathcal{B}})$, then the relation $f(\varphi) = (\varphi \otimes id)(\mathbf{F}_{\mathcal{A}})$, $\varphi \in \hat{\mathbf{B}}$, provides a one-to-one correspondence between algebra homomorphisms $f : \hat{\mathcal{B}} \to \mathcal{A}$ and "generating matrices" $\mathbf{F}_{\mathcal{A}} \in \mathcal{B} \otimes \mathcal{A}$ obeying

$$\mathbf{F}_{\mathcal{A}}^{13} \mathbf{F}_{\mathcal{A}}^{23} = (\Delta_{\mathcal{B}} \otimes id)(\mathbf{F}_{\mathcal{A}}) \in \mathcal{B} \otimes \mathcal{B} \otimes \mathcal{A} \tag{2.9}$$

Clearly, f is unital iff $\mathbf{F}_{\mathcal{A}}$ is unital in the sense $(\varepsilon_{\mathcal{B}} \otimes id)(\mathbf{F}_{\mathcal{A}}) = \mathbf{1}_{\mathcal{A}}$. If $\mathcal{B}$, and therefore $\hat{\mathcal{B}}$, are bialgebras, then $\mathcal{B}$ has an antipode $S_{\mathcal{B}}$ if and only if all unital generating matrices are invertible, the relation being given by

$$\mathbf{F}_{\mathcal{A}}^{-1} = (S_{\mathcal{B}} \otimes id_{\mathcal{A}})(\mathbf{F}_{\mathcal{A}}) \tag{2.10}$$

If also $\mathcal{A}$ is a bialgebra then f defines a bialgebra homomorphism if and only if in addition to (2.9) $\mathbf{F}_{\mathcal{A}}$ satisfies

$$\mathbf{F}_{\mathcal{A}}^{12} \mathbf{F}_{\mathcal{A}}^{13} = (id \otimes \Delta_{\mathcal{A}})(\mathbf{F}_{\mathcal{A}}) \in \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{A} \tag{2.11}$$

Putting $\mathcal{A} = \hat{\mathcal{B}}$ and $f = id_{\hat{\mathcal{B}}}$ one obtains $\mathbf{F} \equiv \mathbf{E} = \sum_\nu e_\nu \otimes e^\nu \in \mathcal{B} \otimes \hat{\mathcal{B}}$, which justifies the statement that the algebra $\hat{\mathcal{B}}$ is generated by the entries $(\varphi \otimes id)(\mathbf{F})$, $\varphi \in \hat{\mathcal{B}}$, with abstract relations (2.9). Applying all this to the algebra $\hat{H}_R$ we get

Proposition 2.4 *The relation $f(\varphi) = (\varphi \otimes id)(\mathbf{M}_{\mathcal{A}})$ provides a one-to-one correspondence between algebra homomorphisms $f : \hat{H}_R \to \mathcal{A}$ and elements $\mathbf{M}_{\mathcal{A}} \in H \otimes \mathcal{A}$ satisfying*

$$\mathbf{M}_{\mathcal{A}}^{13} R^{12} \mathbf{M}_{\mathcal{A}}^{23} = R^{12} (\Delta \otimes id)(\mathbf{M}_{\mathcal{A}}) \in H \otimes H \otimes \mathcal{A} \tag{2.12}$$

Proof: According to (2.8) and (2.9) we have to verify that (2.12) is equivalent to

$$\mathbf{M}_{\mathcal{A}}^{13}\mathbf{M}_{\mathcal{A}}^{23} = (\Delta_R \otimes id)(\mathbf{M}_{\mathcal{A}}) \tag{2.13}$$

To this end we use (2.3) to get the identity

$$\sum_{i,j} x^i x^j \otimes S(y^j) y^i = \sum_i x^i \otimes S(y^i_{(1)}) y^i_{(2)} = \mathbf{1} \otimes \mathbf{1} \tag{2.14}$$

Hence (2.12) implies

$$\mathbf{M}_{\mathcal{A}}^{13}\mathbf{M}_{\mathcal{A}}^{23} = \sum_j [\mathbf{1} \otimes S(y^j) \otimes \mathbf{1}] \mathbf{M}_{\mathcal{A}}^{13} R^{12} \mathbf{M}_{\mathcal{A}}^{23} [x^j \otimes \mathbf{1} \otimes \mathbf{1}] = (\Delta_R \otimes id)(\mathbf{M}_{\mathcal{A}})$$

Conversely, (2.13) implies

$$\mathbf{M}_{\mathcal{A}}^{13} R^{12} \mathbf{M}_{\mathcal{A}}^{23} = \sum_i [\mathbf{1} \otimes y^j \otimes \mathbf{1}] (\Delta_R \otimes id)(\mathbf{M}_{\mathcal{A}}) [x^j \otimes \mathbf{1} \otimes \mathbf{1}] = R^{12} (\Delta \otimes id)(\mathbf{M}_{\mathcal{A}})$$

where, similarly as above, we have used

$$\sum_{i,j} x^i x^j \otimes y^j S(x^i) = \sum_i x^i \otimes y^i_{(1)} S(y^i_{(2)}) = \mathbf{1} \otimes \mathbf{1} \tag{2.15}$$

■

Putting $\mathcal{A} = \hat{H}_R$ and $f = id$, equation (2.12) coincides with the defining "monodromy relations" given in [AFFS]. The convention of [AGS,AS] differs by a multiplication of $\mathbf{M}_{\mathcal{A}}$ with $(\kappa^{-1} \otimes \mathbf{1}_{\mathcal{A}})$, where $\kappa \in H$ is a certain central square root of a ribbon element associated with R.

Hence, from now on we will call $\times_R$ the *monodromy product* on $\hat{H}$ and $\hat{H}_R$ the *monodromy algebra* associated with (H, R).

3 Gauged Monodromy Algebras

Following Majid [M4,M5] we now provide a coadjoint action of H on $\hat{H}_R$ which becomes a Hopf module action with respect to the monodromy product $\times_R$. Again we start with the dual point of view by first recalling that Δ_R is an H-module map with respect to the right adjoint action of H on itself (this is a basic ingredient in Majid's construction of braided groups by what he calls "transmutation").

Lemma 3.1 *For $a, b \in H$ we have*

$$\Delta_R(S(a_{(1)}) b a_{(2)}) = [S(a_{(1)}) \otimes S(a_{(3)})] \Delta_R(b) [a_{(2)} \otimes a_{(4)}] \tag{3.1}$$

Proof: By straight forward verification, using (2.4), (2.7) and the identity $\Delta \circ S = (S \otimes S) \circ \Delta_{op}$.

Corollary 3.2 *Let $\triangleright : H \otimes \hat{H}_R \ni a \otimes \varphi \mapsto a \triangleright \varphi \in \hat{H}_R$ be given by*

$$\langle a \triangleright \varphi \mid b \rangle := \langle \varphi \mid S(a_{(1)}) b a_{(2)} \rangle \tag{3.2}$$

where $a, b \in H$ and $\varphi \in \hat{H}_R$. Then $\triangleright$ defines a Hopf module left action of H on $\hat{H}_R$.

Proof: The identities $a \triangleright (b \triangleright \varphi) = (ab) \triangleright \varphi$, $\mathbf{1} \triangleright \varphi = \varphi$ and $a \triangleright \varepsilon = \varepsilon(a)\varepsilon$ are obvious. The Hopf module property

$$a \triangleright (\varphi \times_R \psi) = (a_{(1)} \triangleright \varphi) \times_R (a_{(2)} \triangleright \psi) \tag{3.3}$$

follows by pairing both sides with $b \in H$ and using (2.8) and (3.1). ■

Definition 3.3 *The "gauged monodromy algebra" $\mathcal{M}_R(H)$ is defined to be the crossed product*

$$\mathcal{M}_R(H) := \hat{H}_R \rtimes H$$

Thus, recalling the definition of a crossed product, $\mathcal{M}_R(H)$ is the linear space $\hat{H} \otimes H$ with associative multiplication

$$(\varphi \otimes a)(\psi \otimes b) := (\varphi \times_R (a_{(1)} \triangleright \psi) \otimes a_{(2)} b), \tag{3.4}$$

where $\varphi, \psi \in \hat{H}$ and $a, b \in H$. To distinguish the new algebraic structure we denote the images of $\hat{H}_R$ and H in $\mathcal{M}_R(H)$ by $i_M(a) \equiv \varepsilon \otimes a$, $a \in H$, and $M(\varphi) \equiv \varphi \otimes \mathbf{1}_H$, $\varphi \in \hat{H}$. We also denote $\mathbf{M} := \sum e_\nu \otimes M(e^\nu) \in H \otimes \mathcal{M}_R(H)$.

The crossed product $\mathcal{M}_R(H)$ has first appeared in [M2,M5]. Using the generating-matrix formalism we now show that $\mathcal{M}_R(H)$ coincides with the algebra $\mathcal{K}$ given in Definition 5 of [AFFS] (see also [AGS,AS]) as the subalgebra generated inside a current algebra by a monodromy operator $\mathbf{M}$ together with the algebra of gauge transformations sitting at its initial ($\equiv$ end) point.

Proposition 3.4 *Let $f : H \to \mathcal{A}$ be an algebra homomorphism. Then the relation*

$$f_M(\varphi \otimes a) := (\varphi \otimes id)(\mathbf{M}_\mathcal{A}) f(a)$$

provides a one-to-one correspondence between algebra homomorphisms $f_M : \mathcal{M}_R(H) \to \mathcal{A}$ extending f and elements $\mathbf{M}_\mathcal{A} \in H \otimes \mathcal{A}$ satisfying (2.12) together with $(\varepsilon \otimes id)(\mathbf{M}_\mathcal{A}) = f(\mathbf{1}_H)$ and

$$[a_{(1)} \otimes f(a_{(2)})] \mathbf{M}_\mathcal{A} = \mathbf{M}_\mathcal{A} [a_{(1)} \otimes f(a_{(2)})], \quad \forall a \in H. \tag{3.5}$$

Proof: Since $\mathbf{1}_{\mathcal{M}_R(H)} = \varepsilon \otimes \mathbf{1}_H$, the condition $(\varepsilon \otimes id)(\mathbf{M}_\mathcal{A}) = f(\mathbf{1}_H)$ is equivalent to $f_M(\mathbf{1}_{\mathcal{M}_R(H)}) = f(\mathbf{1}_H)$. We are left to show that (3.5) holds if and only if f_M respects (3.4), or equivalently, if and only if

$$f(a_{(1)})(\varphi \otimes id_\mathcal{A})(\mathbf{M}_\mathcal{A}) f(S(a_{(2)})) = ((a \triangleright \varphi) \otimes id)(\mathbf{M}_\mathcal{A}) \tag{3.6}$$

for all $\varphi \in \hat{H}$ and all $a \in H$. Now (3.6) is equivalent to

$$(\mathbf{1}_H \otimes f(a_{(1)})) \mathbf{M}_\mathcal{A} (\mathbf{1}_H \otimes f(S(a_{(2)})) = (S(a_{(1)}) \otimes \mathbf{1}_\mathcal{A})(\mathbf{M}_\mathcal{A})(a_{(2)} \otimes \mathbf{1}_\mathcal{A}) \tag{3.7}$$

which further implies

$$\begin{aligned}
[a_{(1)} \otimes f(a_{(2)})] \mathcal{M}_\mathcal{A} &= [a_{(1)} \otimes f(a_{(2)})] \mathbf{M}_\mathcal{A} [\mathbf{1}_H \otimes f(S(a_{(3)}) a_{(4)})] \\
&= [a_{(1)} S(a_{(2)}) \otimes \mathbf{1}_\mathcal{A}] \mathbf{M}_\mathcal{A} [a_{(3)} \otimes f(a_{(4)})] \\
&= \mathbf{M}_\mathcal{A} [a_{(2)} \otimes f(a_{(2)})]
\end{aligned}$$

and therefore (3.5). Conversely, given (3.5) we conclude

$$\begin{aligned}
[\mathbf{1}_H \otimes f(a_{(1)})] \mathbf{M}_\mathcal{A} [\mathbf{1}_H \otimes f(S(a_{(2)}))] &= [S(a_{(1)}) a_{(2)} \otimes f(a_{(3)})] \mathbf{M}_\mathcal{A} [\mathbf{1}_H \otimes f(S(a_{(4)}))] \\
&= [S(a_{(1)}) \otimes \mathbf{1}_\mathcal{A}] \mathbf{M}_\mathcal{A} [a_{(2)} \otimes f(a_{(3)} S(a_{(4)}))] \\
&= [S(a_{(1)}) \otimes \mathbf{1}_\mathcal{A}] \mathbf{M}_\mathcal{A} [a_{(2)} \otimes \mathbf{1}_\mathcal{A}]
\end{aligned}$$

and therefore (3.7). ■

4 Left and Right Monodromies

Here we review the observation of [AFFS] on a right monodromy algebra $\hat{H}_R^r \subset \mathcal{M}_R(H)$ commuting with $\hat{H}_R^\ell \equiv \hat{H}_R \subset \mathcal{M}_R(H)$. First we define $\hat{H}_R^r$ abstractly as in Definition 2.3, where we replace (H, Δ, R, S) by $(H, \Delta_{op}, R_{op}, S^{-1})$, i.e. the quasitriangular Hopf algebra H with opposite coproduct. Similarly, to define a left Hopf module action of $H_{cop} \equiv (H, \Delta_{op})$ on $\hat{H}_R^r$ we replace (3.2) by

$$\langle a \triangleright' \varphi \mid b \rangle := \langle \varphi \mid S^{-1}(a_{(2)}) b a_{(1)} \rangle \tag{4.1}$$

leading to a crossed product

$$\mathcal{M}_R^r(H) := \hat{H}_R^r \rtimes H_{cop} \tag{4.2}$$

for which Propositions 2.4 and 3.4 hold analogously, with (Δ, R) replaced by (Δ_{op}, R_{op}). Putting in particular $\mathcal{A} = \mathcal{M}_R^\ell(H) \equiv \mathcal{M}_R(H)$ we now get an isomorphism $\mathcal{M}_R^r(H) \to \mathcal{M}_R^\ell(H)$ restricting to the identity on H by choosing in Proposition 3.4 $\mathbf{M}_\mathcal{A} = \mathbf{M}^r$,

$$\mathbf{M}^r := \mathbf{R}\,(\mathbf{M}^\ell)^{-1} \mathbf{R}_{op} \in H \otimes \mathcal{M}_R^\ell(H) \tag{4.3}$$

where $\mathbf{M}^\ell \equiv \mathbf{M} := \sum e_\nu \otimes M(e^\nu) \in H \otimes \mathcal{M}_R^\ell(H)$ and where $\mathbf{R}_{(op)} := (id \otimes i_M)(R_{(op)}) \in H \otimes \mathcal{M}_R^\ell(H)$. This is the formula of [AFFS], which is proven by equ. (4.6) below. Note however that for (4.3) to be well defined we first have to assure that $\mathbf{M}^\ell$ is invertible in $H \otimes \mathcal{M}_R^\ell(H)$.

Proposition 4.1 i) *For any target algebra $\mathcal{A}$ let $f : H \to \mathcal{A}$ be a unital homomorphism and let $\mathbf{M}_\mathcal{A}^\ell \equiv \mathbf{M}_\mathcal{A} \in H \otimes \mathcal{A}$ be unital and satisfy (2.12) and (3.5). Then*

$$\mathbf{M}_\mathcal{A}^{-1} = (S \otimes id)\Big((id \otimes f)(R_{op}^{-1})\mathbf{M}_\mathcal{A}\Big)\,(id \otimes f)(R_{op}^{-1}) \in H \otimes \mathcal{A} \tag{4.4}$$

ii) *Putting $\mathbf{M}_\mathcal{A}^r := (id \otimes f)(R)(\mathbf{M}_\mathcal{A}^\ell)^{-1}(id \otimes f)(R_{op})$ we have*

$$\begin{aligned} [a_{(2)} \otimes f(a_{(1)})]\mathbf{M}_\mathcal{A}^r &= \mathbf{M}_\mathcal{A}^r [a_{(2)} \otimes f(a_{(1)})] && (4.5)\\ (\mathbf{M}_\mathcal{A}^r)^{13} R^{21} (\mathbf{M}_\mathcal{A}^r)^{23} &= R^{21} (\Delta_{op} \otimes id)(\mathbf{M}_\mathcal{A}^r) && (4.6)\\ (\mathbf{M}_\mathcal{A}^\ell)^{13} (\mathbf{M}_\mathcal{A}^r)^{23} &= (\mathbf{M}_\mathcal{A}^r)^{23} (\mathbf{M}_\mathcal{A}^\ell)^{13} && (4.7) \end{aligned}$$

Proof: i) Putting

$$\mathbf{D}_\mathcal{A} := (id \otimes f)(R_{op}^{-1})\mathbf{M}_\mathcal{A} \in H \otimes \mathcal{A} \tag{4.8}$$

and using (2.5) we have in $H \otimes H \otimes \mathcal{A}$

$$R^{12}(\Delta \otimes id)(\mathbf{M}_\mathcal{A}) = [(R \otimes \mathbf{1})(\Delta \otimes id)(R)]^{312}(\Delta \otimes id)(\mathbf{D}_\mathcal{A}) \tag{4.9}$$

On the other hand (2.2) and (2.4) imply

$$\begin{aligned} \mathbf{M}_\mathcal{A}^{13} R^{12} \mathbf{M}_\mathcal{A}^{23} &= \mathbf{M}_\mathcal{A}^{13} (\Delta \otimes id)(R)^{132} \mathbf{D}_\mathcal{A}^{23} \\ &= [(\Delta \otimes id)(R)(R_{op} \otimes \mathbf{1})]^{132} \mathbf{D}_\mathcal{A}^{13} \mathbf{D}_\mathcal{A}^{23} \\ &= [(R \otimes \mathbf{1})(\Delta \otimes id)(R)]^{312} \mathbf{D}_\mathcal{A}^{13} \mathbf{D}_\mathcal{A}^{23} && (4.10) \end{aligned}$$

where in the second line we have used (3.5) and where by a convenient abuse of notation we have dropped the symbol f. Hence comparing (4.9) and (4.10) and using (2.12) we conclude

$$\mathbf{D}_\mathcal{A}^{13} \mathbf{D}_\mathcal{A}^{23} = (\Delta \otimes id_\mathcal{A})(\mathbf{D}_\mathcal{A}) \tag{4.11}$$

Now $\mathbf{D}_{\mathcal{A}}$ is unital since f and $\mathbf{M}_{\mathcal{A}}$ are unital and, therefore, (4.11) implies by (2.10)

$$\mathbf{D}_{\mathcal{A}}^{-1} = (S \otimes id_{\mathcal{A}})(\mathbf{D}_{\mathcal{A}}) \tag{4.12}$$

from which (4.4) follows.

ii) Equ.(4.5) immediately follows from (2.4) and (3.5). To prove (4.6) we compute, omitting the symbol f,

$$\begin{aligned}
(\mathbf{M}^r_{\mathcal{A}})^{23} R^{12} (\mathbf{M}^r_{\mathcal{A}})^{13} &= R^{23} (\mathbf{M}_{\mathcal{A}}^{-1})^{23} R^{32} R^{12} R^{13} (\mathbf{M}_{\mathcal{A}}^{-1})^{13} R^{31} \\
&= R^{23} (\mathcal{M}_{\mathcal{A}}^{-1})^{23} (id \otimes \Delta)(R)(R^{-1})^{12} (\Delta \otimes id)(R)^{132} (\mathbf{M}_{\mathcal{A}}^{-1})^{13} R^{31} \\
&= R^{12} (\Delta \otimes id)(R) [(\mathbf{M}_{\mathcal{A}}^{-1})^{23} (R^{-1})^{12} (\mathbf{M}_{\mathcal{A}}^{-1})^{13}] (\Delta \otimes id)(R)^{132} R^{31} \\
&= R^{12} (\Delta \otimes id)(\mathbf{M}^r_{\mathcal{A}} R_{op}^{-1})(R^{12})^{-1} R^{12} R^{32} R^{31} \\
&= R^{12} (\Delta \otimes id)(\mathbf{M}^r_{\mathcal{A}})
\end{aligned}$$

Here we have used (2.2)-(2.4) in the second line, (3.5) and (2.5) in the third line, (2.2), (4.3) and the inverse of (2.12) in the fourth line and again (2.3) in the last line. Applying the permutation $(12) \to (21)$ to both sides this proves (4.6). Finally, to prove (4.7) we put $\mathbf{M} \equiv M^\ell_{\mathcal{A}}$ and $\Omega \equiv (M^r_{\mathcal{A}})^{-1}$ and compute

$$\begin{aligned}
\mathbf{M}^{13} \Omega^{23} &= \mathbf{M}^{13} (R^{12} R^{32})^{-1} R^{12} \mathbf{M}^{23} (R^{23})^{-1} \\
&= (\Delta \otimes id)(R^{-1})^{132} R^{12} (\Delta \otimes id)(\mathbf{M})(R^{23})^{-1} \\
&= (R^{32})^{-1} (\Delta \otimes id)(\mathbf{M})(R^{23})^{-1}
\end{aligned}$$

where in the second line we have used (2.2), (3.5) and (2.12), and in the third line again (2.2). Similarly one obtains

$$\begin{aligned}
\Omega^{23} \mathbf{M}^{13} &= (R^{32})^{-1} \mathbf{M}^{23} R^{21} (R^{23} R^{21})^{-1} \mathbf{M}^{13} \\
&= (R^{32})^{-1} R^{21} (\Delta_{op} \otimes id)(\mathbf{M})(id \otimes \Delta)(R^{-1})^{213} \\
&= (R^{32})^{-1} R^{21} (\Delta_{op} \otimes id)(\mathbf{M})(R^{21})^{-1} (R^{23})^{-1} \\
&= M^{13} \Omega^{23}
\end{aligned} \tag{4.13}$$

which proves (4.7). ■

Note that (4.7) implies that the image of $\hat{H}^r_R$ in $\mathcal{M}^\ell_R(H)$ indeed commutes with $\hat{H}^\ell_R$. Part ii) of Proposition 4.1 has been taken over from [AFFS].

5 The Drinfeld Double

The Drinfeld double $\mathcal{D}(H)$ (also called quantum double) has been introduced in [Dr1] and is meanwhile well understood as a double crossed product of H with $\hat{H}$ [M1,M8], see also [RS1,K].

Definition 5.1 *The Drinfeld double $\mathcal{D}(H)$ over a finite dimensional Hopf algebra H is the linear space $\hat{H} \otimes H$ with multiplication given for $a, b \in H$ and $\varphi, \psi \in \hat{H}$ by*

$$(\varphi \otimes a)(\psi \otimes b) := (\varphi \psi_{(2)} \otimes a_{(2)} b) \ \langle a_{(1)} \mid \psi_{(3)} \rangle \ \langle \psi_{(1)} \mid S^{-1}(a_{(3)}) \rangle \tag{5.1}$$

Putting $i_D(a) := (\mathbf{1}_{\hat{H}} \otimes a)$ and $D(\varphi) := (\varphi \otimes \mathbf{1}_H)$ one can rewrite this equivalently as [NSz]

$$\begin{aligned}
i_D(a) i_D(b) &= i_D(ab) &(5.2)\\
D(\varphi) D(\psi) &= D(\varphi \psi) &(5.3)\\
D(\varphi_{(1)}) \ \langle \varphi_{(2)} \mid a_{(1)} \rangle \ i_D(a_{(2)}) &= i_D(a_{(1)}) \ \langle a_{(2)} \mid \varphi_{(1)} \rangle \ D(\varphi_{(2)}) &(5.4)
\end{aligned}$$

Hence, as an algebra $\mathcal{D}(H) = \mathcal{D}(\hat{H})$. Based on a more general setting given by [Ra] it has first been noticed in [M2] (see also Proposition 4.1 of [M5]) that $\mathcal{D}(H)$ is in fact isomorphic to $\mathcal{M}_R(H)$ for all quasitriangular $R \in H \otimes H$.

Using our formalism of generating matrices let me now demonstrate that this result reduces to a Corollary of the calculation leading to (4.11). First we need an analogue of Proposition 3.4. Introducing $\mathbf{D} := \sum_\nu e_\nu \otimes D(e^\nu) \in H \otimes \mathcal{D}(H)$ we note that (5.3) is equivalent to

$$\mathbf{D}^{13}\mathbf{D}^{23} = (\Delta \otimes id)(\mathbf{D})$$

and (5.4) is equivalent to

$$\mathbf{D}[a_{(1)} \otimes i_D(a_{(2)})] = [a_{(2)} \otimes i_D(a_{(1)})]\mathbf{D}$$

More generally this leads to

Lemma 5.2 *Let $f : H \to \mathcal{A}$ be an algebra homomorphism. Then the relation*

$$f_D(\varphi \otimes a) = (\varphi \otimes id_{\mathcal{A}})(\mathbf{D}_{\mathcal{A}})f(a), \;\; a \in H, \varphi \in \hat{H}$$

provides a one-to-one correspondence between algebra homomorphisms $f_D : \mathcal{D}(H) \to \mathcal{A}$ extending f and elements $\mathbf{D}_{\mathcal{A}} \in H \otimes \mathcal{A}$ obeying $(\varepsilon \otimes id)(\mathbf{D}_{\mathcal{A}}) = f(\mathbf{1}_H)$ and

$$\mathbf{D}_{\mathcal{A}}^{13}\mathbf{D}_{\mathcal{A}}^{23} = (\Delta \otimes id)(\mathbf{D}_{\mathcal{A}}) \tag{5.5}$$

$$\mathbf{D}_{\mathcal{A}}[a_{(1)} \otimes f(a_{(2)})] = [a_{(2)} \otimes f(a_{(1)})]\mathbf{D}_{\mathcal{A}}, \quad a \in H \tag{5.6}$$

Proof: First, by (2.9) the $\mathbf{D}_{\mathcal{A}}$'s satisfying (5.5) are in one-to-one correspondence with homomorphisms $\hat{H} \to \mathcal{A}$. Since $\mathbf{1}_{\mathcal{D}(H)} = i_D(\mathbf{1}_H)$, the condition $(\varepsilon \otimes id)(\mathbf{D}_{\mathcal{A}}) = f(\mathbf{1}_H)$ is equivalent to $f_D(\mathbf{1}_{\mathcal{D}(H)}) = f(\mathbf{1}_H)$. We are left to show that f_D respects (5.4) if and only if $\mathbf{D}_{\mathcal{A}}$ obeys (5.6), which may immediately be realized by applying $(\varphi \otimes id_{\mathcal{A}})$ to (5.6). ■

Inspired by (4.8) and (4.11) we now put $\mathcal{A} = \mathcal{M}_R(H)$, $f = i_M : H \to \mathcal{M}_R(H)$ the canonical embedding and

$$\mathbf{D}_M := \mathbf{R}_{op}^{-1}\mathbf{M} \in H \otimes \mathcal{M}_R(H) \tag{5.7}$$

where $\mathbf{R}_{op} := (id \otimes i_M)(R_{op}) \in H \otimes \mathcal{M}_R(H)$ and $\mathbf{M} = \sum e_\nu \otimes M(e^\nu)$ as before. Then we have

Corollary 5.3 [M2,M5] *The element $\mathbf{D}_M$ (5.7) defines an algebra isomorphism $\lambda_R : \mathcal{D}(H) \to \mathcal{M}_R(H)$ restricting to the identity on H by putting*

$$\lambda_R(\varphi \otimes a) := (\varphi \otimes id)(\mathbf{D}_M)\, i_M(a) \;.$$

Proof: We apply Lemma 5.2. Equ. (5.5) has already been verified in (4.11). Also $\mathbf{D}_M$ is unital since $\mathbf{M}$ is unital, which fits with $f \equiv i_M$ being unital. Equ. (5.6) follows from (3.5) and (2.4). Finally, λ_R is invertible with λ_R^{-1} given according to Proposition 3.4 by

$$\mathbf{M}_D := (id \otimes i_D)(R_{op})\mathbf{D} \in H \otimes \mathcal{D}(H) \tag{5.8}$$

■

Denoting $\mathbf{M}_D^\ell \equiv \mathbf{M}_D$ and looking at (4.3) we also get an immediate formula for a right monodromy $\mathbf{M}_D^r \in H \otimes \mathcal{D}(H)$

$$\mathbf{M}_D^r := (id \otimes i_D)(R)\mathbf{D}^{-1} \tag{5.9}$$

Equ. (4.7) then implies that the subalgebras $\langle \hat{H} \otimes id \mid \mathbf{M}_D^r \rangle \cong \hat{H}_R^r$ and $\langle \hat{H} \otimes id \mid \mathbf{M}_D^\ell \rangle \cong \hat{H}_R^\ell$ commute inside $\mathcal{D}(H)$. In the next section we will review the factorization condition of [RS1] guaranteeing $\mathcal{M}_R(H) \equiv \mathcal{D}(H) \cong \hat{H}_R^\ell \otimes \hat{H}_R^r$.

6 The Monodromy Homomorphism

Following Propositions 2.1 and 2.2 of [M5] we now provide what may be called the monodromy homomorphism $\hat{H}_R \to H$, which as a linear map has already been discussed in [RS1].

Proposition 6.1 [M5] *Let* $\mathsf{mon}_R : \hat{H} \to H$ *be the linear map given by*

$$\mathsf{mon}_R\,(\varphi) := (\varphi \otimes id)(R_{op}R), \quad \varphi \in \hat{H}$$

Then mon_R *provides an algebra homomorphism* $\mathsf{mon}_R : \hat{H}_R \to H$ *satisfying for* $a \in H$ *and* $\varphi \in \hat{H}$

$$\mathsf{mon}_R\,(a \triangleright \varphi) = a_{(1)}\,\mathsf{mon}_R\,(\varphi)\,S(a_{(2)}) \tag{6.1}$$

Proof: Putting $\mathcal{A} = H$ in Proposition 2.4 we have to check

$$(R^{31}R^{13})R^{12}(R^{32}R^{23}) = R^{12}(\Delta \otimes id)(R_{op}R) \tag{6.2}$$

which is straight forward from the quasitriangularity of R. To prove (6.1) we use the definition (3.2) to compute

$$\begin{aligned}
\mathsf{mon}_R\,(a \triangleright \varphi) &= \langle \varphi \otimes id \mid (S(a_{(1)}) \otimes \mathbf{1})R_{op}R(a_{(2)} \otimes \mathbf{1})\rangle \\
&= \langle \varphi \otimes id \mid (S(a_{(1)}) \otimes \mathbf{1})R_{op}R(a_{(2)} \otimes a_{(3)}S(a_{(4)}))\rangle \\
&= \langle \varphi \otimes id \mid (S(a_{(1)})a_{(2)} \otimes a_{(3)})R_{op}R(\mathbf{1} \otimes S(a_{(4)}))\rangle \\
&= \langle \varphi \otimes id \mid (\mathbf{1} \otimes a_{(1)})R_{op}R(\mathbf{1} \otimes S(a_{(2)}))\rangle \\
&= a_{(1)}\,\mathsf{mon}_R\,(\varphi)\,S(a_{(2)})
\end{aligned}$$

where in the third line we have used that $R_{op}R$ commutes with $\Delta(H)$. ■

Proposition 6.1 has implicitely been used in [AS] when studying representations of the monodromy algebra $\hat{H}_R$ in terms of representations of H.

Next, since $(\mathrm{Ad}\,a)\,b := a_{(1)}bS(a_{(2)}),\ a, b \in H$, defines a Hopf module action of H on itself, Proposition 6.1 immediately implies that mon_R extends to a homomorphism Mon_R of the associated crossed products.

Corollary 6.2 [M5] *The map* $\mathsf{Mon}_R : \hat{H} \otimes H \to H \otimes H$

$$\mathsf{Mon}_R\,(\varphi \otimes a) := \mathsf{mon}_R\,(\varphi) \otimes a \tag{6.3}$$

provides a homomorphism of algebras $\mathsf{Mon}_R : \mathcal{M}_R(H) \to H \rtimes_{Ad} H$.

We now note a simple "bosonization formula" [4] showing that as an algebra the crossed product $H \rtimes_{Ad} H$ is in fact isomorphic to $H \otimes_{alg} H$.

Lemma 6.3 *Let* $U : H \otimes H \to H \otimes H$ *be given by* $U(a \otimes b) := ab_{(1)} \otimes b_{(2)}$. *Then* U *defines an algebra isomorphism* $H \rtimes_{Ad} H \to H \otimes_{alg} H$.

Proof: U is invertible with $U^{-1}(a \otimes b) = aS(b_{(1)}) \otimes b_{(2)}$. Now the multiplication in $H \rtimes_{Ad} H$ is given by

$$(a \otimes_{Ad} b)(c \otimes_{Ad} d) = \left(a(\mathrm{Ad}\,b_{(1)})\,c \otimes_{Ad} b_{(2)}d\right). \tag{6.4}$$

[4] This terminology is not meant to indicate physical interpretations here. It just memorizes the fact, that the two non-commuting copies of H generating $H \rtimes_{Ad} H$ are traded for the two commuting copies $U^{-1}(H \otimes_{alg} \mathbf{1}_H)$ and $U^{-1}(\mathbf{1}_H \otimes_{alg} H)$.

Applying U to the r.h.s of (6.4) gives

$$U(ab_{(1)}cS(b_{(2)})\otimes_{Ad} b_{(3)}d) = ab_{(1)}cd_{(1)} \otimes b_{(2)}d_{(2)} = [U(a\otimes_{Ad} b)][U(c\otimes_{Ad} d)]$$

where the last product is taken in $H\otimes_{alg} H$. ■

Note that Lemma 6.3 immediately generalizes to any crossed product by inner actions. Putting $\pi_R := U\circ \mathsf{Mon}_R$ we now arrive at

Theorem 6.4 *Let $\mathbf{M}^\ell \equiv \mathbf{M}$ and $\mathbf{M}^r = \mathbf{R}(\mathbf{M}^\ell)^{-1}\mathbf{R}_{op}$ as in (4.3) and consider the conditions i)-iii) on an algebra homomorphism $\pi_R : \mathcal{M}_R(H)\to H\otimes_{alg} H$*

$$\begin{array}{rrcl} i) & \pi_R\circ i_M & = & \Delta_H \\ ii) & (id_H\otimes\pi_R)(\mathbf{M}^\ell) & = & R^{21}R^{12} \\ iii) & (id_H\otimes\pi_R)(\mathbf{M}^r) & = & R^{13}R^{31} \end{array}$$

Then given i) properties ii) and iii) are equivalent with unique solution $\pi_R = U\circ\mathsf{Mon}_R$. In this case π_R is an isomorphism if and only if mon_R is bijective implying $\pi_R^{-1}(H\otimes_{alg}\mathbf{1}) = \hat{H}^\ell_R$, $\pi_R^{-1}(\mathbf{1}\otimes_{alg} H) = \hat{H}^r_R$, and therefore $\mathcal{M}_R(H) = \hat{H}^\ell_R\otimes_{alg}\hat{H}^r_R$.

Proof: According to (6.3), (6.2) and Proposition 3.4 Mon_R is the unique homomorphism $\mathcal{M}_R(H)\to H\rtimes_{Ad} H$ satisfying $\mathsf{Mon}_R\circ i_M = \mathbf{1}_H\otimes id_H$ and $(id_H\otimes\mathsf{Mon}_R)(\mathbf{M}^\ell) = R^{21}R^{12}\in H\otimes(H\otimes\mathbf{1}_H)\subset H\otimes(H\rtimes_{Ad}H)$. Hence $\pi_R = U\circ\mathsf{Mon}_R$ is the unique homomorphism $\mathcal{M}_R(H)\to H\otimes_{alg}H$ satisfying i) and ii). Moreover, given i) conditions ii) and iii) are equivalent, since by (2.2)-(2.4)

$$(id\otimes\Delta)(R)(R^{21}R^{12})^{-1}(id\otimes\Delta)(R_{op}) = R^{13}R^{31}$$

Finally, by (6.3) Mon_R and therefore π_R are isomorphisms if and only if mon_R is bijective. In this case ii) implies $\pi_R(\hat{H}^\ell_R) = H\otimes_{alg}\mathbf{1}_H$ and $\pi_R(\hat{H}^r_R) = \mathbf{1}_H\otimes_{alg}\mathsf{mon}_{R_{op}}(\hat{H})$, where

$$\mathsf{mon}_{R_{op}}(\varphi) := (\varphi\otimes id)(RR_{op}) \equiv (id\otimes\varphi)(R_{op}R)$$

provides the monodromy homomorphism $\hat{H}^r_R\to H$. Since for $\varphi,\psi\in\hat{H}$

$$\langle\varphi\mid\mathsf{mon}_{R_{op}}(\psi)\rangle = \langle\mathsf{mon}_R(\varphi)\mid\psi\rangle$$

we conclude that $\mathsf{mon}_{R_{op}}$ is bijective iff mon_R is bijective, in which case $\pi_R(\hat{H}^r_R) = \mathbf{1}_H\otimes_{alg} H$. ■

Theorem 6.4 is the true reason underlying the observation of [AFFS] on the equivalence of the categories Rep $(\mathcal{M}_R(H))$ and Rep $(H\otimes_{alg} H)$ provided the map mon_R is invertible. In fact, all representations of $\mathcal{M}_R(H)$ given in [AFFS] are of the form $\tau\circ\pi_R$, $\tau\in$ Rep $(H\otimes_{alg} H)$. Note, however, that in our approach no assumptions on semi-simplicity have been made.

7 The Hopf Algebra Structure

In this section we use the isomorphism $\lambda_R : \mathcal{D}(H)\to\mathcal{M}_R(H)$ of Corollary 5.3 to induce a homomorphism $\Lambda_R := \pi_R\circ\lambda_R : \mathcal{D}(H)\to H\otimes_{alg} H$. We then show that up to cocycle equivalence Λ_R provides a homomorphism of quasitriangular Hopf algebras, which becomes an isomorphism if and only if mon_R is bijectiv. In this way we recover the result of [RS1], where the invertibility property of mon_R has been called *factorizability*. The results of this section are also reviewed in Chapter 7 of [M8], however without using our formalism of generating matrices.

First we recall [Dr1] that $\mathcal{D}(H)$ always is a quasitriangular Hopf algebra with coproduct Δ_D, antipode S_D and R-matrix R_D given for $a \in H$ and $\varphi \in \hat{H}$ by

$$\begin{aligned} \Delta_D(i_D(a)) &= i_D(a_{(1)}) \otimes i_D(a_{(2)}) && (7.1)\\ \Delta_D(D(\varphi)) &= D(\varphi_{(2)}) \otimes D(\varphi_{(1)}) && (7.2)\\ S_D(i_D(a)) &= i_D(S(a)) && (7.3)\\ S_D(D(\varphi)) &= D(\hat{S}^{-1}(\varphi)) && (7.4)\\ R_D &= (i_D \otimes id_{\mathcal{D}(H)})(\mathbf{D}) && (7.5) \end{aligned}$$

This structure may now immediately be transported to $\mathcal{M}_R(H)$ via λ_R to give Δ_M, S_M and R_M, the formulae for which however turn out to look less transparent.

$$\begin{aligned} \Delta_M \circ i_M &= i_M \circ \Delta && (7.6)\\ (id \otimes \Delta_M)(\mathbf{M}) &= \mathbf{R}_{op}^{12}\mathbf{M}^{13}(\mathbf{R}_{op}^{12})^{-1}\mathbf{M}^{12} && (7.7)\\ S_M \circ i_M &= i_M \circ S && (7.8)\\ (id \otimes S_M)(\mathbf{M}) &= (S^{-1} \otimes id)(\mathbf{R}_{op}^{-1}\mathbf{M}\mathbf{R}_{op}) && (7.9)\\ R_M &= (i_M \otimes id_{\mathcal{M}_R(H)})(\mathbf{R}_{op}^{-1}\mathbf{M}) && (7.10) \end{aligned}$$

where $\mathbf{R}_{op}$ and $\mathbf{M}$ have the same meaning as in (5.7). As it is explained at the end of this section these results correct some inconsistencies in [AFFS].

We now skip the intermediate object $\mathcal{M}_R(H)$ and study immediately the composition $\Lambda_R = \pi_R \circ \lambda_R$. To show that up to cocycle equivalence Λ_R provides a Hopf algebra map we first recall the natural coproduct $\Delta_{H\otimes H}$ given on $H \otimes_{alg} H$ by

$$\Delta_{H\otimes H}(a \otimes b) := (a_{(1)} \otimes b_{(1)}) \otimes (a_{(2)} \otimes b_{(2)})$$

Lemma 7.1: *Let $T := R_{23}^{-1} \in (H \otimes H) \otimes (H \otimes H)$* [5]*. Then T is a left $\Delta_{H\otimes H}$-cocycle and therefore*

$$\delta_{H\otimes H} := Ad\, T \circ \Delta_{H\otimes H} \qquad (7.11)$$

is also coassociative.

Proof: Putting $R^{-1} = \sum_i u^i \otimes v^i$ we have in $(H \otimes H)^{\otimes 3}$

$$\begin{aligned} T^{12}(\Delta_{H\otimes H} \otimes id)(T) &= R_{23}^{-1} \sum [(\mathbf{1} \otimes u^i_{(1)}) \otimes (\mathbf{1} \otimes u^i_{(2)}) \otimes (v^i \otimes \mathbf{1})]\\ &= R_{23}^{-1} R_{45}^{-1} R_{25}^{-1}\\ &= R_{45}^{-1} R_{23}^{-1} R_{25}^{-1}\\ &= R_{45}^{-1} \sum [(\mathbf{1} \otimes u^i) \otimes (v^i_{(1)} \otimes \mathbf{1}) \otimes (v^i_{(2)} \otimes \mathbf{1})]\\ &= T^{23}(id \otimes \Delta_{H\otimes H})(T) \end{aligned}$$

where we have used (2.2) and (2.3). Hence T is a left $\Delta_{H\otimes H}$-cocycle. ∎

[5] From now on lower indices refer to the embedding $H \otimes H \to H^{\otimes n}, n \geq 3$, and upper indices – as before – refer to the embeddings $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}^{\otimes 3}$ or $H \otimes \mathcal{A} \to \mathcal{H} \otimes \mathcal{A} \otimes \mathcal{A}$, respectively, where $\mathcal{A} = H \otimes H$. Also, from now on $\otimes$ always means $\otimes_{alg}$.

Replacing (Δ, R) by (Δ_{op}, R_{op}), the coproduct $\delta_{H\otimes H}$ has already appeared in Theorem 2.7 of [RS1] see also 7.3 of [M8]. Next, we have

Lemma 7.2 *Let $\mathcal{R}_{H\otimes H} \in (H\otimes H)\otimes(H\otimes H)$ be given by*

$$\mathcal{R}_{H\otimes H} := R_{41}^{-1}R_{42}^{-1}R_{13}R_{23} \tag{7.12}$$

Then $\mathcal{R}_{H\otimes H}$ is quasi-triangular with respect to $\delta_{H\otimes H}$.

Proof: Since R and R_{op}^{-1} are quasi-triangular w.r.t. Δ, $\mathcal{R}'_{H\otimes H} := R_{42}^{-1}R_{13} \in (H\otimes H)\otimes(H\otimes H)$ is quasitriangular w.r.t. $\Delta_{H\otimes H}$. The claim now follows from the twist equivalence

$$\mathcal{R}_{H\otimes H} = T_{op}\mathcal{R}'_{H\otimes H}T^{-1} \tag{7.13}$$

■

We are now in the position to prove that Λ_R provides a homomorphism of quasitriangular Hopf algebras $(\mathcal{D}(H), R_D, \Delta_D) \to (H\otimes H, \delta_{H\otimes H}, \mathcal{R}_{H\otimes H})$, which in fact coincides with the homomorphism given in Theorem 2.9 of [RS1], see also Chapter 7 of [M8].

Theorem 7.3 *For any quasitriangular $R\in H\otimes H$ denote $\Lambda_R := \pi_R\circ\lambda_R : \mathcal{D}(H)\to H\otimes H$. Then*

$$\begin{array}{rrcl} i) & \Lambda_R\circ i_D & = & \Delta \\ ii) & (id\otimes\Lambda_R)(\mathbf{D}) & = & R_{31}^{-1}R_{12} \\ iii) & \delta_{H\otimes H}\circ\Lambda_R & = & (\Lambda_R\otimes\Lambda_R)\circ\Delta_D \\ iv) & (\Lambda_R\otimes\Lambda_R)(R_D) & = & \mathcal{R}_{H\otimes H} \end{array}$$

Moreover, Λ_R is uniquely determined by i) and ii) and Λ_R is an isomorphism if and only if mon_R is bijective.

Proof: Part i) follows from Theorem 6.4i). To prove ii) we use Theorem 6.4ii), (5.7) and (2.2) to compute

$$(id\otimes\Lambda_R)(\mathbf{D}) = (id\otimes\Delta)(R_{op}^{-1})R_{21}R_{12} = R_{31}^{-1}R_{12} \tag{7.14}$$

The uniqueness of Λ_R under conditions i) and ii) follows from Lemma 5.2. Next, using i), the claim iii) holds on $i_D(H)\subset\mathcal{D}(H)$ since

$$\delta_{H\otimes H}(\Delta(a)) = R_{23}^{-1}(a_{(1)}\otimes a_{(3)}\otimes a_{(2)}\otimes a_{(4)})R_{23} = \Delta(a_{(1)})\otimes\Delta(a_{(2)})$$

It remains to check that iii) holds on all $D(\varphi)$, $\varphi\in\hat{H}$, where Δ_D is given by $\hat{\Delta}_{op}$, according to (7.2). Hence, by (2.12) we have to show

$$\mathbf{F}_{H\otimes H}^{13}\mathbf{F}_{H\otimes H}^{12} = (id\otimes\delta_{H\otimes H})(\mathbf{F}_{H\otimes H}) \in H\otimes(H\otimes H)^{\otimes 2} \tag{7.15}$$

where $\mathbf{F}_{H\otimes H} = (id\otimes\Lambda_R)(\mathbf{D}) \in H\otimes(H\otimes H)$. Using (7.14) we have

$$\mathbf{F}_{H\otimes H}^{13}\mathbf{F}_{H\otimes H}^{12} = R_{51}^{-1}R_{14}R_{31}^{-1}R_{12}$$

and

$$\begin{aligned} (id\otimes\delta_{H\otimes H})(\mathbf{F}_{H\otimes H}) &= R_{34}^{-1}\sum(v^ix^j\otimes y^j_{(1)}\otimes u^i_{(1)}\otimes y^j_{(2)}\otimes u^i_{(2)})R_{34} \\ &= R_{34}^{-1}R_{51}^{-1}R_{31}^{-1}R_{14}R_{12}R_{34} \\ &= R_{51}^{-1}(R_{34}^{-1}R_{31}^{-1}R_{14}R_{34})R_{12} \\ &= R_{51}^{-1}R_{14}R_{31}^{-1}R_{12} \\ &= \mathbf{F}_{H\otimes H}^{13}\mathbf{F}_{H\otimes H}^{12} \end{aligned}$$

where in the second line we have used (2.2), (2.3) and in the fourth line the Yang-Baxter equations in the form

$$R_{14}R_{34}R_{31} = R_{31}R_{34}R_{14}$$

which follow from (2.2) - (2.5). This proves (7.15) and therefore iii). To prove iv) we use (7.5), i) and ii) to get

$$\begin{aligned} (\Lambda_R \otimes \Lambda_R)(R_D) &= \Delta(e_\nu) \otimes (e^\nu \otimes id \otimes id)(R_{31}^{-1}R_{12}) \\ &= (\Delta \otimes id \otimes id)(R_{31}^{-1}R_{12}) \\ &= R_{41}^{-1}R_{42}^{-1}R_{13}R_{23} \\ &= \mathcal{R}_{H\otimes H} \end{aligned}$$

■

Let me close by pointing out that [AFFS] seem to provide a coproduct $\Delta_{\mathcal{K}}$ on $\mathcal{K} \equiv \mathcal{M}_R(H)$ such that

$$\Delta'_{H\otimes H} := (\pi_R \otimes \pi_R) \circ \Delta_{\mathcal{K}} \circ \pi_R^{-1}$$

satisfies

$$\Delta'_{H\otimes H}(a \otimes \mathbf{1}) = (a_{(1)} \otimes \mathbf{1}) \otimes (a_{(2)} \otimes \mathbf{1}) \ . \tag{7.16}$$

Indeed, assuming π_R invertible and defining as in [AFFS]

$$\mathbf{M}_+ := (id_H \otimes \pi_R^{-1})(R_{op} \otimes \mathbf{1}_H) \in H \otimes \mathcal{M}_R(H) \tag{7.17}$$

$$\mathbf{M}_- := (id_H \otimes \pi_R^{-1})(R^{-1} \otimes \mathbf{1}_H) \in H \otimes \mathcal{M}_R(H) \tag{7.18}$$

equ. (7.16) would be equivalent to

$$(id_H \otimes \Delta_{\mathcal{K}})(\mathbf{M}_\pm) = \mathbf{M}_\pm^{12}\mathbf{M}_\pm^{13} \tag{7.19}$$

or

$$(id_H \otimes \Delta_{\mathcal{K}})(\mathbf{M}) = \mathbf{M}_+^{12}\mathbf{M}^{13}(\mathbf{M}_-^{12})^{-1} \tag{7.20}$$

which are the formulas given in Section 4 of [AFFS]. On the other hand, equ. (4.15) of [AFFS] implies

$$\Delta'_{H\otimes H}(a_{(1)} \otimes a_{(2)}) = (a_{(1)} \otimes a_{(2)}) \otimes (a_{(3)} \otimes a_{(4)}) \tag{7.21}$$

which would be consistent with our $\delta_{H\otimes H}$. However, equ. (7.16) is manifestly inconsistent with (7.21). In fact, writing

$$(\mathbf{1} \otimes a) = (S(a_{(1)}) \otimes \mathbf{1})(a_{(2)} \otimes a_{(3)})$$

it is easy to see that a map $\Delta'_{H\otimes H} : H \otimes H \to (H \otimes H) \otimes (H \otimes H)$ obeying (7.16) and (7.21) cannot consistently be extended to an algebra homomorphism.

After presenting these results I have been informed [S] that there will be a revised version of [AFFS] reproducing $\delta_{H\otimes H}$ – or equivalently Δ_M given in (7.6),(7.7) – at least up to cocycle equivalence.

A Multi-Loop Algebras

Having identified the monodromy algebra $\hat{H}_R$ as a braided group we show in this Appendix that the multi-loop algebras of [AS] arise as braided tensor products of braided groups in the sense of Majid [M4,M7]. We recall that these are defined in the natural way so as to obtain all structural maps as H-module morphisms with respect to the coadjoint action $\triangleright$. More generally one has

Definition A.1 [M4, M7] Let $(H, \Delta, R = \sum x^i \otimes y^i)$ be a quasitriangular Hopf algebra and let $\mathcal{A}$ and $\mathcal{B}$ be left H-module algebras with both actions denoted by $\triangleright$. The braided tensor product algebra $\mathcal{A} \otimes_R \mathcal{B}$ is defined to be the vector space $\mathcal{A} \otimes \mathcal{B}$ with multiplication given for $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$ by

$$(a \otimes_R b)(a' \otimes_R b') := \sum a(y^i \triangleright a') \otimes_R (x^i \triangleright b)b' \tag{A.1}$$

One immediately checks that $\mathcal{A} \otimes_R \mathcal{B}$ is a again an H-module algebra with action

$$h \triangleright (a \otimes_R b) := (h_{(1)} \triangleright a) \otimes_R (h_{(2)} \triangleright b) \tag{A.2}$$

Moreover, the braided tensor product is associative in the sense that $(\mathcal{A}_1 \otimes_R \mathcal{A}_2) \otimes_R \mathcal{A}_3$ and $\mathcal{A}_1 \otimes_R (\mathcal{A}_2 \otimes_R \mathcal{A}_3)$ define the same algebra structure on $\mathcal{A}_1 \otimes \mathcal{A}_2 \otimes \mathcal{A}_3$.

We now apply this to the multi-loop algebras introduced in Section 6 of [AS] and show that with respect to the coadjoint action they are indeed the m-fold braided tensor product of the associated one-loop ($\equiv$ monodromy) algebras. To this end let us denote

$$\mathcal{L}_m := \hat{H}_R \otimes_R \cdots \otimes_R \hat{H}_R \tag{A.3}$$

the m-fold braided tensor product and let $\mathbf{M}_\nu$, $1 \le \nu \le m$, denote the m copies of monodromy matrices in $H \otimes \mathcal{L}_m$

$$\mathbf{M}_\nu = (id_H \otimes \iota_\nu)(\mathbf{M})$$

where $\iota_\nu : \hat{H}_R \to \mathcal{L}_m$ is the obvious embedding into the ν-th tensor factor. We then have

Proposition A.2 *For $\mu < \nu$ the following relations hold in $H \otimes H \otimes \mathcal{L}_m$*

$$\mathbf{M}_\nu^{13} R^{12} \mathbf{M}_\mu^{23} = R^{12} \mathbf{M}_\mu^{23} (R^{-1})^{12} \mathbf{M}_\nu^{13} R^{12} \tag{A.4}$$

Proof: We only proof the case $m = 2$, from which the general case follows straight forwardly. Putting $M_\nu(\varphi) := (\varphi \otimes id)(\mathbf{M}_\nu)$, $\varphi \in \hat{H}$, we conclude from (A.1)

$$M_2(\psi) M_1(\xi) = \sum_i M_1(y^i \triangleright \xi) M_2(x^i \triangleright \psi)$$

which by (3.2) implies

$$\begin{aligned}
\mathbf{M}_2^{13}\mathbf{M}_1^{23} &= \sum_i [S(x^i_{(1)}) \otimes S(y^i_{(1)}) \otimes \mathbf{1}_{\mathcal{A}}] \mathbf{M}_1^{23} \mathbf{M}_2^{13} [x^i_{(2)} \otimes y^i_{(2)} \otimes \mathbf{1}_{\mathcal{A}}] \\
&= \sum_{i,j,k,\ell} [S(x^i x^j) \otimes S(y^j y^\ell) \otimes \mathbf{1}_{\mathcal{A}}] \mathbf{M}_1^{23} \mathbf{M}_2^{13} [x^k x^\ell \otimes y^i y^k \otimes \mathbf{1}_A] \\
&= \sum_\ell [\mathbf{1}_H \otimes S(y^\ell) \otimes \mathbf{1}_{\mathcal{A}}] R^{12} \mathbf{M}_1^{23} (R^{-1})^{12} \mathbf{M}_2^{13} R^{12} [x^\ell \otimes \mathbf{1}_H \otimes \mathbf{1}_{\mathcal{A}}]
\end{aligned} \tag{A.5}$$

from which the claim follows as in the proof of Proposition 2.4. Here we have used (2.2) and (2.3) in the second line and the identities $(S \otimes id)(R) = R^{-1}$ and $(S \otimes S)(R) = R$ in the last line. ■

A comparison with equ.(6.1) of [AS] shows that up to an ordering convention our definition of $\mathcal{L}_m$ coincides with their multiloop algebras.

Next we note that the monodromy homomorphism mon_R naturally extends to multi-loop algebras. As a matter of fact, being an H-module map intertwining the coadjoint action on $\hat{H}_R$ with the adjoint action on H according to Proposition 6.1, we can use its ordinary m-fold tensor product ${\mathsf{mon}_R}^{\otimes m} = \mathsf{mon}_R \otimes \cdots \otimes \mathsf{mon}_R$ to obtain an H-module algebra morphism

$$ {\mathsf{mon}_R}^{\otimes m} : \mathcal{L}_m \to H \otimes_R \cdots \otimes_R H \tag{A.6} $$

where on the r.h.s. the braided tensor product with respect to the left adjoint action of H on itself is understood.

Next, similarly as crossed products with respect to inner actions are isomorphic to algebraic tensor products (see e.g. Lemma 6.3), we now have the following "bosonization formula".

Proposition A.3 *Consider H as a left H-module algebra under the adjoint action and let $\iota_{\mathcal{A}} : H \to \mathcal{A}$ be a unital algebra map inducing an inner left H-action $\triangleright$ on $\mathcal{A}$ by*

$$ h \triangleright a := \iota_{\mathcal{A}}(h_{(1)})\, a\, \iota_{\mathcal{A}}(S(h_{(2)})), \ h \in H, \ a \in \mathcal{A} \ . \tag{A.7} $$

Then the linear map $V_{\mathcal{A}} : \mathcal{A} \otimes_R H \to \mathcal{A} \otimes_{alg} H$

$$ V_{\mathcal{A}}(a \otimes_R h) := \sum_{i,j} a\, \iota_{\mathcal{A}}(y^i y^j) \otimes_{alg} x^i h S(x^j) \tag{A.8} $$

defines an algebra isomorphism satisfying for all $a \in \mathcal{A}$ and $h, h' \in H$

$$ V_{\mathcal{A}}(h \triangleright (a \otimes_R h')) = \Delta_{\mathcal{A}}(h_{(1)}) V_{\mathcal{A}}(a \otimes_R h') \Delta_{\mathcal{A}}(S(h_{(2)})) \tag{A.9} $$

where $\Delta_{\mathcal{A}} := (\iota_{\mathcal{A}} \otimes id_H) \circ \Delta$.

Proof: Using $(S \otimes id)(R) = R^{-1}$ and (2.14) the inverse of $V_{\mathcal{A}}$ is given by

$$ V_{\mathcal{A}}^{-1}(a \otimes_{alg} h) = \sum_{i,j} a\, \iota_{\mathcal{A}}(y^j S(y^i)) \otimes_R x^i h x^j \tag{A.10} $$

Now $V_{\mathcal{A}}(a \otimes_R \mathbf{1}_H) = (a \otimes_{alg} \mathbf{1}_H)$ and

$$ V_{\mathcal{A}}(\mathbf{1}_{\mathcal{A}} \otimes_R h) = (\iota_{\mathcal{A}} \otimes id)(R_{op}(\mathbf{1}_H \otimes_{alg} h) R_{op}^{-1}). $$

proving that the restrictions of $V_{\mathcal{A}}$ to the subalgebras $\mathcal{A} \otimes_R \mathbf{1}_H$ and $\mathbf{1}_{\mathcal{A}} \otimes_R H$ are algebra maps. We are left to check that $V_{\mathcal{A}}$ respects the commutation relations between $\mathcal{A} \otimes_R \mathbf{1}_H$ and $\mathbf{1}_{\mathcal{A}} \otimes_R H$. To this end we compute (implying a summation convention over doubled indices)

$$ \begin{aligned} V_{\mathcal{A}}((\mathbf{1}_{\mathcal{A}} \otimes_R h)(a \otimes_R \mathbf{1}_H)) &= V_{\mathcal{A}}(y^i \triangleright a \otimes_R x^i \triangleright h) \\ &= y^i_{(1)} a S(y^i_{(2)}) y^m y^n \otimes_{alg} x^m x^i_{(1)} h S(x^i_{(2)}) S(x^n) \\ &= y^j y^\ell a S(y^i y^k) y^m y^n \otimes_{alg} x^m x^i x^j h S(x^n x^k x^\ell) \\ &= y^j y^\ell a \otimes_{alg} x^j h S(x^\ell) \\ &= V_{\mathcal{A}}(\mathbf{1}_{\mathcal{A}} \otimes_R h)\, V_{\mathcal{A}}(a \otimes_R \mathbf{1}_H) \end{aligned} $$

where we have used (2.2) and (2.3) in the third line and (2.14) in the fourth line, and where by a convient abuse of notation we have dropped the symbol $\iota_{\mathcal{A}}$. This proves that $V_{\mathcal{A}} : \mathcal{A} \otimes_R H \to \mathcal{A} \otimes_{alg} H$ provides an algebra map. To prove (A.9) we use that the general definition (A.2) provides a Hopf module action. Hence, it is enough to check (A.9) separately on the generating factors $\mathcal{A} \otimes_R \mathbf{1}_H \cong \mathcal{A}$ and $\mathbf{1}_{\mathcal{A}} \otimes_R H \cong H$. Now, dropping again the symbol $\iota_{\mathcal{A}}$ we have in $\mathcal{A} \otimes_{alg} H$

$$\begin{aligned}\Delta_{\mathcal{A}}(h_{(1)})(a \otimes_{alg} \mathbf{1}_H)\Delta_{\mathcal{A}}(S(h_{(2)})) &= h_{(1)} a S(h_{(4)}) \otimes_{alg} h_{(2)} S(h_{(3)}) \\ &= (h \triangleright a) \otimes_{alg} \mathbf{1}_H\end{aligned}$$

proving (A.9) on $(\mathcal{A} \otimes_R \mathbf{1}_H)$. On $(\mathbf{1}_{\mathcal{A}} \otimes_R H)$ we get

$$\begin{aligned}V_{\mathcal{A}}(\mathbf{1}_{\mathcal{A}} \otimes_R h \triangleright h') &= y^i y^j \otimes_{alg} x^i h_{(1)} h' S(h_{(2)}) S(x^j) \\ &= y^i y^j h_{(3)} S(h_{(4)}) \otimes_{alg} x^i h_{(1)} h' S(x^j h_{(2)}) \\ &= h_{(1)} y^i y^j S(h_{(4)}) \otimes_{alg} h_{(2)} x^i h' S(x^j) S(h_{(3)}) \\ &= \Delta_{\mathcal{A}}(h_{(1)}) V_{\mathcal{A}}(\mathbf{1}_{\mathcal{A}} \otimes_R h') \Delta_{\mathcal{A}}(S(h_{(2)}))\end{aligned}$$

This proves (A.9) and therefore Proposition A.3. ■

Corollary A.4 *Under the conditions of Proposition A.3 put* $\delta_{\mathcal{A}} := V_{\mathcal{A}}^{-1} \circ \Delta_{\mathcal{A}} : H \to \mathcal{A} \otimes_R H$. *Then* $\delta_{\mathcal{A}}$ *implements the* H*-action on* $\mathcal{A} \otimes_R H$, *i.e.*

$$h \triangleright (a \otimes_R h') = \delta_{\mathcal{A}}(h_{(1)})(a \otimes_R h')\delta_{\mathcal{A}}(S(h_{(2)}))$$

Note that we may in particular put $\mathcal{A} = H$ and $\iota_{\mathcal{A}} = id$ to obtain, using $(S \otimes S)(R) = R$

$$\begin{aligned}\delta_{\mathcal{A}}(a) &= \sum_{i,j} y^j a_{(2)} y^i \otimes S^{-1}(x^i) x^j a_{(1)} \\ &= \sum_{i,j} a_{(1)} y^j S(y^i) \otimes x^i a_{(2)} x^j \qquad \text{(A.11)}\end{aligned}$$

which coincides with Δ_R given in (2.1) up to a change of conventions (i.e. replacing (H, Δ, S, R) by $(H, \Delta_{op}, S^{-1}, R_{op})$. This shows that Δ_R provides an algebra map $H \to H_{cop} \otimes_{R_{op}} H_{cop}$ as remarked after Lemma 2.2.

We now put $\iota_{\mathcal{A} \otimes_R H} = \delta_{\mathcal{A}}$ and proceed inductively to get isomorphisms

$$V_{\mathcal{A},m} : \mathcal{A} \otimes_R H \otimes_R \cdots \otimes_R H \to \mathcal{A} \otimes_{alg} H \otimes_{alg} \cdots \otimes_{alg} H$$

where the tensor factors H appear m-times. Choosing in particular $\mathcal{A} = \mathbf{C}$ and $\iota_{\mathcal{A}} = \varepsilon$ and denoting $V_{\mathbf{C},m} \equiv V_m$ we have proven

Theorem A.5 *The map*

$$\mathsf{mon}_{R\,,m} := V_m \circ \mathsf{mon}_R{}^{\otimes m} : \mathcal{L}_m \to H^{\otimes m}$$

provides a homomorphism of algebras such that

$$\mathsf{mon}_{R\,,m}(h \triangleright \underline{a}) = \Delta^{(m)}(h_{(1)}) \mathsf{mon}_{R\,,m}(\underline{a}) \Delta^{(m)}(S(h_{(2)}))$$

where $\underline{a} \in \mathcal{L}_m$, $h \in H$ *and* $\Delta^{(m)} : H \to H^{\otimes m}$ *denotes the* m*-fold coproduct* [6]. *Moreover,* $\mathsf{mon}_{R,m}$ *is an isomorphism if and only if* mon_R *is bijectiv.*

Generalizing Lemma 6.3 in the obvious way we further have $H^{\otimes m} \rtimes_{Ad} H \cong H^{\otimes(m+1)}$ and therefore also a homomorphism

$$\mathsf{Mon}_{R,m} : \mathcal{L}_m \rtimes H \to H^{\otimes(m+1)},$$

which is bijective iff mon_R is bijective.

Theorem A.5 explains the representation theory of $\mathcal{L}_m$ given by [AS] without having to rely on any semi-simplicity assumptions. To see this explicitly we show

Proposition A.6 *Let* $\mathbf{M}_\nu = (id_H \otimes \iota_\nu)(\mathbf{M}) \in H \otimes \mathcal{L}_m$, $1 \le \nu \le m$, *be the* ν*-th monodromy matrix. Then*

$$(id_H \otimes \mathsf{mon}_{R,m})(\mathbf{M}_\nu) = \mathbf{N}_\nu \otimes \mathbf{1}_H^{\otimes(m-\nu)} \in H \otimes H^{\otimes m} \tag{A.12}$$

where

$$\mathbf{N}_\nu := (id_H \otimes \Delta^{(\nu-1)} \otimes id_H)(R^{32}R^{31}R^{13}(R^{-1})^{32}) \in H \otimes H^{\otimes \nu} \tag{A.13}$$

Proof: We proceed by induction over $m \ge 1$. The case $m = 1$ holds by definition of mon_R. Now suppose the claim holds for m_0 and all $1 \le \nu \le m_0$. Putting $\mathcal{A} = H \otimes_R \cdots \otimes_R H$ (m_0 factors) we have

$$\mathsf{mon}_{R,m_0+1} = (V_{m_0} \otimes id_H) \circ V_{\mathcal{A}} \circ \mathsf{mon}_R{}^{\otimes(m_0+1)} \tag{A.14}$$

Since the restriction of $V_{\mathcal{A}}$ for $\mathcal{A} \otimes_R \mathbf{1}_H$ is the identity we get with respect to the identification $\mathcal{L}_m \cong \mathcal{L}_m \otimes_R \hat{\mathbf{1}} \subset \mathcal{L}_{m+1}$

$$\mathsf{mon}_{R,m_0+1}|\mathcal{L}_{m_0} = \mathsf{mon}_{R,m_0} \tag{A.15}$$

This proves (A.12) for $m = m_0 + 1$ and $1 \le \nu \le m_0$. For $\nu = m_0 + 1$ we use

$$(id_H \otimes \mathsf{mon}_R{}^{\otimes(m_0+1)})(\mathbf{M}_\nu) = R^{31}R^{13} \in H \otimes_{alg} (\mathcal{A} \otimes_{alg} H) \tag{A.16}$$

Applying $(id_H \otimes V_{\mathcal{A}})$ to (A.16) we get in $H \otimes_{alg} (\mathcal{A} \otimes_{alg} H)$

$$(id_H \otimes V_{\mathcal{A}})(R^{31}R^{13}) = (id_H \otimes \iota_{\mathcal{A}} \otimes id_H)(R^{32}R^{31}R^{13}(R^{32})^{-1}) \tag{A.17}$$

Finally, according to Corollary A.4 and the inductive definition of V_m we have $V_m \circ \iota_{\mathcal{A}} = \Delta^{(m)}$, which together with (A.14), (A.16) and (A.17) proves (A.12) for $m = m_0$ and $\nu = m_0 + 1$. ■

Comparing (A.12) with the representations of $\mathcal{L}_m$ given in Section 6 of [AS] we realize that they are in fact all of the form $\tau \circ \mathsf{mon}_{R,m}$, where $\tau \in \mathrm{Rep}\,(H^{\otimes m})$.

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[6] i.e. $\Delta^{(0)} = \varepsilon, \Delta^{(1)} = id_H$ and $\Delta^{(m+1)} = (id^{\otimes(m-1)} \otimes \Delta) \circ \Delta^{(m)}$

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