$SU(N)$ Matrix Difference Equations and a Nested Bethe Ansatz

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Abstract

A system of $SU(N)$-matrix difference equations is solved by means of a nested version of a generalized Bethe Ansatz, also called "off shell" Bethe Ansatz [1]. The highest weight property of the solutions is proved.

1 Introduction

Difference equations play a role in various contexts of mathematical physics (see e.g. [2] and references therein). We are interested in the application to the form factor program in the exact integrable 1+1 dimensional field theory, which was formulated in 1978 by one of the authors (M.K.) and Weisz [3]. Form factors are matrix elements of local operators $\mathcal{O}(x)$

$$F(i\pi - \theta) = \langle\, p' \,|\, \mathcal{O}(0) \,|\, p \,\rangle$$

where $p'p = m^2 \cosh\theta$. Difference equations for these functions are obtained by Watson's equations [4]

$$F(\theta) = S(\theta)\, F(-\theta)\ , \quad F(i\pi - \theta) = F(i\pi + \theta) \tag{1.1}$$

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where S is the S-matrix. For several models these equations have been solved in [3] and many later publications (see e.g. [5, 6] and references therein). Generalized form factors are matrix elements for many particle states. For generalized form factors Watson's equations lead typically to matrix difference equations, which can be solved by a generalized Bethe Ansatz, also called "off-shell Bethe Ansatz". The conventional Bethe Ansatz introduced by Bethe [7] is used to solve eigenvalue problems. The algebraic formulation, which is also used in this article, was worked out by Faddeev and coworkers (see e.g. [8]). The "off-shell Bethe Ansatz" has been introduced by one of the authors (H.B.) to solve the Knizhnik-Zamolodchikov equations which are differential equations. In [9] a variant of this technique has been formulated to solve matrix difference equations of the form

$$f(x_1, \ldots, x_i + 2, \ldots, x_n) = Q(x_1, \ldots, x_n, ; i)\, f(x_1 \ldots, x_i, \ldots, x_n)\ , \quad (i = 1, \ldots, n)$$

where $f(\underline{x})$ is a vector valued function and the $Q(\underline{x}, i)$ are matrix valued functions to be specified later. For higher rank internal symmetry groups the nested version of this Bethe Ansatz has to be applied. The nested Bethe Ansatz to solve eigenvalue problems was introduced by Yang [10] and further developed by Sutherland [11] (see also [12] for the algebraic formulation). The very interesting generalization of this technique, which is applicable to difference equations, is developed in this article for the $SU(N)$ - symmetry group. This generalization demonstrates the power of the Bethe Ansatz even more beautifully than the conventional applications. Other methods to solve such matrix difference equations have been discussed in [5, 13, 14, 15]. In part II we will solve the $U(N)$ case, which will be used in [16] to solve the form factor problem for the $SU(N)$ chiral Gross-Neveu model.

The article is organized as follows. In Section 2 we recall some well known results concerning the $SU(N)$ R-matrix, the monodromy matrix and some commutation rules. In Section 3 we introduce the nested generalized Bethe Ansatz to solve a system of $SU(N)$ difference equations and present the solutions in terms of "Jackson-type Integrals". The proof of the main theorem avoids the decomposition of the monodromy matrix, as used e.g. in [9]. Instead we introduce a new type of monodromy matrix fulfilling a new type of Yang-Baxter relation and which is adapted to the difference problem. In particular this yields an essential simplification of the proof of the main theorem. In Section 4 we prove the highest weight property of the solutions and calculate the weights. Section 5 contains some examples of solutions of the $SU(N)$ difference equations.

2 The $SU(N)$ - R-matrix

Let $V^{1\dots n}$ be the tensor product space

$$V^{1\dots n} = V_1 \otimes \ldots \otimes V_n \tag{2.1}$$

where the vector spaces $V_i \cong \mathbf{C}^N$, $(i = 1, \ldots, n)$ are considered as fundamental (vector) representation spaces of $SU(N)$. It is straightforward to generalize the results of this paper to the case where the V_i are vector spaces for other representations. We denote the canonical basis vectors by

$$|\,\alpha_1 \ldots \alpha_n\,\rangle \in V^{1\dots n}, \quad (\alpha_i = 1, \ldots, N). \tag{2.2}$$

A vector $v^{1\dots n} \in V^{1\dots n}$ is given in term of its components by

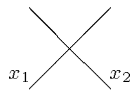
$$v^{1\dots n} = \sum_{\underline{\alpha}} |\,\alpha_1 \ldots \alpha_n\,\rangle\, v^{\alpha_1,\ldots,\alpha_n}. \tag{2.3}$$

A matrix acting in $V^{1\dots n}$ by is denoted by

$$A_{1\dots n} \;:\; V^{1\dots n} \to V^{1\dots n}. \tag{2.4}$$

The $SU(N)$ spectral parameter dependent R-matrix [17] acts on the tensor product of two (fundamental) representation spaces of $SU(N)$. It may be written and depicted as

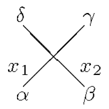
$$R_{12}(x_1 - x_2) = b(x_1 - x_2)\,\mathbf{1}_{12} + c(x_1 - x_2)\,P_{12} \quad =$$



$$: V^{12} \to V^{21}, \tag{2.5}$$

where P_{12} is the permutation operator. Here and in the following we associate a variable (spectral parameter) $x_i \in \mathbf{C}$ to each space V_i which is graphically represented by a line labeled by x_i (or simply by i). The components of the R-matrix are

$$R^{\delta\gamma}_{\alpha\beta}(x_1 - x_2) = \delta_{\alpha\gamma}\delta_{\beta\delta}\, b(x_1 - x_2) + \delta_{\alpha\delta}\delta_{\beta\gamma}\, c(x_1 - x_2) \quad =$$



$$, \tag{2.6}$$

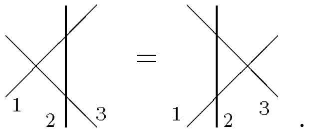
and the functions

$$b(x) = \frac{x}{x - 2/N}, \quad c(x) = \frac{-2/N}{x - 2/N} \tag{2.7}$$

are obtained as the rational solution of the Yang-Baxter equation

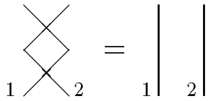
$$R_{12}(x_1 - x_2)\, R_{13}(x_1 - x_3)\, R_{23}(x_2 - x_3) = R_{23}(x_2 - x_3)\, R_{13}(x_1 - x_3)\, R_{12}(x_1 - x_2),$$

where we have employed the usual notation [10]. This relation is depicted as



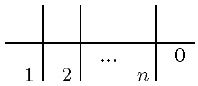
The "unitarity" of the R-matrix reads and may depicted as

$$R_{21}(x_2 - x_1)\, R_{12}(x_1 - x_2) = 1 \quad :$$



As usual we define the monodromy matrix (with $\underline{x} = x_1, \ldots, x_n$)

$$T_{1\ldots n,0}(\underline{x}, x_0) = R_{10}(x_1 - x_0)\, R_{20}(x_2 - x_0) \ldots R_{n0}(x_n - x_0) = \tag{2.8}$$



as a matrix acting in the tensor product of the "quantum space" $V^{1\ldots n}$ and the "auxiliary space" V_0 (all $V_i \cong \mathbf{C}^N$). The Yang-Baxter algebra relations

$$T_{1\ldots n,a}(\underline{x}, x_a)\, T_{1\ldots n,b}(\underline{x}, x_b)\, R_{ab}(x_a - x_b) = R_{ab}(x_a - x_b)\, T_{1\ldots n,b}(\underline{x}, x_b)\, T_{1\ldots n,a}(\underline{x}, x_a) \tag{2.9}$$

imply the basic algebraic properties of the sub-matrices w.r.t the auxiliary space defined by

$$T_{1\ldots n}{}^{\alpha}_{\beta}(\underline{x}, x) \equiv \begin{pmatrix} A_{1\ldots n}(\underline{x}, x) & B_{1\ldots n\beta}(\underline{x}, x) \\ C_{1\ldots n}{}^{\alpha}(\underline{x}, x) & D_{1\ldots n}{}^{\alpha}_{\beta}(\underline{x}, x) \end{pmatrix} . \tag{2.10}$$

The indices α, β on the left hand side run from 1 to N and on the right hand side from 2 to N. The commutation rules which we will need later are

$$B_{1\ldots n\alpha}(\underline{x}, x')\, B_{1\ldots n\beta}(\underline{x}, x) = B_{1\ldots n\beta'}(\underline{x}, x)\, B_{1\ldots n\alpha'}(\underline{x}, x')\, R^{\alpha'\beta'}_{\beta\alpha}(x - x'), \tag{2.11}$$

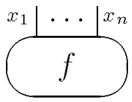
$$\begin{aligned} A_{1\ldots n}(\underline{x}, x')\, B_{1\ldots n\beta}(\underline{x}, x) &= \frac{1}{b(x' - x)}\, B_{1\ldots n\beta}(\underline{x}, x)\, A_{1\ldots n}(\underline{x}, x') \\ &\quad - \frac{c(x' - x)}{b(x' - x)}\, B_{1\ldots n\beta}(\underline{x}, x')\, A_{1\ldots n}(\underline{x}, x) \end{aligned} \tag{2.12}$$

and

$$\begin{aligned} D_{1\ldots n}{}^{\gamma'}_{\gamma}(\underline{x}, x')\, B_{1\ldots n\beta}(\underline{x}, x) &= \frac{1}{b(x - x')}\, B_{1\ldots n\beta'}(\underline{x}, x)\, D_{1\ldots n}{}^{\gamma'}_{\gamma''}(\underline{x}, x')\, R^{\gamma''\beta'}_{\beta\gamma}(x - x') \\ &\quad - \frac{c(x - x')}{b(x - x')}\, B_{1\ldots n\gamma}(\underline{x}, x')\, D_{1\ldots n}{}^{\gamma'}_{\beta}(\underline{x}, x). \end{aligned} \tag{2.13}$$

3 The $SU(N)$ - difference equation

Let

$$f^{1\dots n}(\underline{x}) =$$  $$\in V^{1\dots n}$$

be a vector valued function of $\underline{x} = x_1, \dots, x_n$ with values in $V^{1\dots n}$. The components of this vector are denoted by

$$f^{\alpha_1 \dots \alpha_n}(\underline{x}) \,, \quad (1 \leq \alpha_i \leq N).$$

Conditions 3.1 *The following symmetry and periodicity conditions of the vector valued function $f^{1\dots n}(\underline{x})$ are supposed to be valid:*

(i) *The symmetry property under the exchange of two neighboring spaces V_i and V_j and the variables x_i and x_j, at the same time, is given by*

$$f^{\dots ji \dots}(\dots, x_j, x_i, \dots) = R_{ij}(x_i - x_j)\, f^{\dots ij \dots}(\dots, x_i, x_j, \dots). \tag{3.1}$$

(ii) *The* **system of matrix difference equations** *holds*

$$\boxed{f^{1\dots n}(\dots, x_i + 2, \dots) = Q_{1\dots n}(\underline{x}; i)\, f^{1\dots n}(\dots, x_i, \dots) \,, \quad (i = 1, \dots, n)} \tag{3.2}$$

where the matrices $Q_{1\dots n}(\underline{x}; i) \in End(V^{1\dots n})$ are defined by

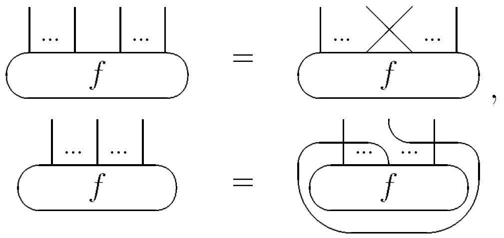
$$Q_{1\dots n}(\underline{x}; i) = R_{i+1i}(x_{i+1} - x_i') \dots R_{ni}(x_n - x_i')\, R_{1i}(x_1 - x_i) \dots R_{i-1i}(x_{i-1} - x_i) \tag{3.3}$$

with $x_i' = x_i + 2$.

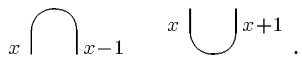
The Yang-Baxter equations for the R-matrix guarantee that these conditions are compatible. The shift of 2 in eq. (3.2) could be replaced by an arbitrary κ. For the application to the form factor problem, however, it is fixed to 2 because of crossing symmetry. Conditions 3.1 (i) and (ii) may be depicted as

(i)

(ii)



with the graphical rule that a line changing the "time direction" changes the spectral parameters $x \to x \pm 1$ as follows



The $Q_{1\dots n}(\underline{x}; i)$ fulfill the commutation rules

$$\begin{aligned} &Q_{1\dots n}(\dots x_i \dots x_j + 2 \dots; i)\, Q_{1\dots n}(\dots x_i \dots x_j \dots; j) \\ &\qquad = Q_{1\dots n}(\dots x_i + 2 \dots x_j \dots; j)\, Q_{1\dots n}(\dots x_i \dots x_j \dots; i). \end{aligned} \tag{3.4}$$

The following Proposition is obvious

Proposition 3.2 *Let the vector valued function $f^{1\dots n}(\underline{x}) \in V^{1\dots n}$ fulfill Condition 3.1 (i), then Conditions 3.1 (ii) for all $i = 1, \dots, n$ are equivalent to each other and also equivalent to the following periodicity property under cyclic permutation of the spaces and the variables*

$$f^{12\dots n}(x_1, x_2, \dots, x_n + 2) = f^{n1\dots n-1}(x_n, x_1, \dots, x_{n-1}). \tag{3.5}$$

Remark 3.3 *The equations (3.1,3.5) imply Watson's (1.1) equations for the form factors [16].*

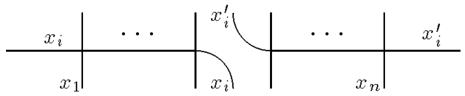
For later convenience we write the matrices

$$Q_{1\dots n}(\underline{x}; i) = \mathrm{tr}_0\, T^Q_{1\dots n,0}(\underline{x}; i) \tag{3.6}$$

as the trace of of a new type of monodromy matrices where to the horizontal line two different spectral parameters are associated, namely one to the right hand side and the other one to the left hand side. However, both are related to a spectral parameter of one of the vertical lines. This new monodromy matrix is given by the following

Definition 3.4 *For $i = 1, \dots, n$*

$$T^Q_{1\dots n,0}(\underline{x}; i) = R_{10}(x_1 - x_i) \dots R_{i-10}(x_{i-1} - x_i)\, P_{i0}\, R_{i+10}(x_{i+1} - x_i') \dots R_{n0}(x_n - x_i') \tag{3.7}$$

$=$ 

with $x_i' = x_i + 2$.

Note that for $i = n$ one has simply $T^Q_{1\dots n,0}(\underline{x}; n) = T_{1\dots n,0}(\underline{x}, x_n)$ since $R(0)$ is the permutation operator P.

The new type of monodromy matrix fulfills a new type of Yang-Baxter relation. Instead of eq. (2.9) we have for $i = 1, \ldots, n$

$$T^Q_{1\ldots n,a}(\underline{x}; i)\, T_{1\ldots n,b}(\underline{x}, u)\, R_{ab}(x'_i - u) = R_{ab}(x_i - u)\, T_{1\ldots n,b}(\underline{x}', u)\, T^Q_{1\ldots n,a}(\underline{x}; i) \tag{3.8}$$

with $\underline{x}' = x_1, \ldots, x'_i, \ldots, x_n$ and $x'_i = x_i + 2$. This relation follows from the Yang-Baxter equation for the R-matrix and the obvious relation for the permutations operator P

$$P_{ia}\, R_{ib}(x_i - u)\, R_{ab}(x'_i - u) = R_{ab}(x_i - u)\, R_{ib}(x'_i - u)\, P_{ia}.$$

Correspondingly to eq. (2.10) we introduce (suppressing the indices $1 \ldots n$)

$$T^{Q\,\alpha}{}_{\beta}(\underline{x}; i) \equiv \begin{pmatrix} A^Q(\underline{x}; i) & B^Q{}_{\beta}(\underline{x}; i) \\ C^{Q\,\alpha}(\underline{x}; i) & D^{Q\,\alpha}{}_{\beta}(\underline{x}; i) \end{pmatrix} . \tag{3.9}$$

with the commutation rules with respect to the usual A, B, C, D

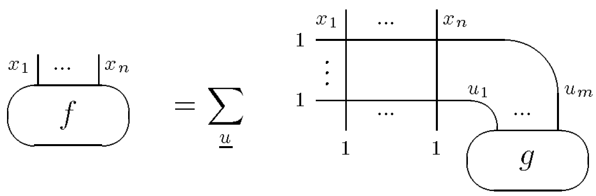
$$A^Q(\underline{x}; i)\, B_b(\underline{x}, u) = \frac{1}{b(x'_i - u)}\, B_b(\underline{x}', u)\, A^Q(\underline{x}; i) - \frac{c(x'_i - u)}{b(x'_i - u)}\, B^Q{}_b(\underline{x}; i)\, A(\underline{x}, u), \tag{3.10}$$

$$\begin{aligned} D^Q{}_a(\underline{x}; i)\, B_b(\underline{x}, u) \quad = \quad & \frac{1}{b(u - x'_i)}\, B_b(\underline{x}', u)\, D^Q{}_a(\underline{x}; i)\, R_{ba}(u - x'_i) \\ & - \frac{c(u - x_i)}{b(u - x_i)}\, B^Q{}_b(\underline{x}; i)\, D_a(\underline{x}, u)\, P_{ab}. \end{aligned} \tag{3.11}$$

The system of difference equations (3.2) can be solved by means of a generalized ("off-shell") nested Bethe Ansatz. The first level is given by the

BETHE ANSATZ 3.5

$$f^{1\ldots n}(\underline{x}) = \sum_{\underline{u}}\; B_{1\ldots n\beta_m}(\underline{x}, u_m) \ldots B_{1\ldots n\beta_1}(\underline{x}, u_1)\, \Omega^{1\ldots n}\; g^{\beta_1 \ldots \beta_m}(\underline{x}, \underline{u}) \tag{3.12}$$



where summation over $\beta_1, \ldots, \beta_m$ is assumed and $\Omega^{1\ldots n} \in V^{1\ldots n}$ is the reference state defined by $C_{1\ldots n}{}^{\beta}\, \Omega^{1\ldots n} = 0$ for $1 < \beta \le N$. The summation over $\underline{u}$ is specified by

$$\underline{u} = (u_1, \ldots, u_m) = (\tilde{u}_1 - 2l_1, \ldots, \tilde{u}_m - 2l_m)\ , \quad l_i \in \mathbf{Z}, \tag{3.13}$$

where the $\tilde{u}_i$ are arbitrary constants.

The reference state is

$$\Omega^{1\dots n} = |1\dots 1\rangle, \tag{3.14}$$

a basis vector with components $\Omega^{\alpha_1\dots\alpha_n} = \prod_{i=1}^n \delta_{\alpha_i 1}$. It is an eigenstate of $A_{1\dots n}$ and $D_{1\dots n}$

$$A_{1\dots n}(\underline{x},u)\Omega^{1\dots n} = \Omega^{1\dots n}\ , \quad D_{1\dots n}{}^{\alpha}_{\beta}(\underline{x},u)\Omega^{1\dots n} = \Omega^{1\dots n}\,\delta_{\alpha\beta}\prod_{i=1}^n b(x_i-u). \tag{3.15}$$

The sums (3.12) are also called "Jackson-type Integrals" (see e.g. [9] and references therein). Note that the summations over β_i run only over $1 < \beta_i \le N$. Therefore the $g^{\beta_1\dots\beta_m}$ are the components of a vector $g^{1\dots m}$ in the tensor product of smaller spaces $V^{(1)1\dots m} = V_1^{(1)}\otimes\dots\otimes V_m^{(1)}$ with $V_i^{(1)} \cong \mathbf{C}^{N-1}$.

Definition 3.6 *Let the vector valued function $f^{(1)1\dots m}(\underline{u}) \in V^{(1)1\dots m}$ be given by*

$$g^{1\dots m}(\underline{x},\underline{u}) = \prod_{i=1}^n\prod_{j=1}^m \psi(x_i-u_j)\prod_{1\le i<j\le m}\tau(u_i-u_j)\,f^{(1)1\dots m}(\underline{u}) \tag{3.16}$$

where the functions $\psi(x)$ and $\tau(x)$ fulfill the functional equations

$$b(x)\,\psi(x) = \psi(x-2)\ , \quad \frac{\tau(x)}{b(x)} = \frac{\tau(x-2)}{b(2-x)}. \tag{3.17}$$

Using the definition of $b(x)$ (2.7) we get the solutions of eqs. (3.17)

$$\psi(x) = \frac{\Gamma(1-\frac{1}{N}+\frac{x}{2})}{\Gamma(1+\frac{x}{2})}, \quad \tau(x) = \frac{x\,\Gamma(\frac{1}{N}+\frac{x}{2})}{\Gamma(1-\frac{1}{N}+\frac{x}{2})} \tag{3.18}$$

where the general solutions are obtained by multiplication with arbitrary periodic functions with period 2. Just as $g^{1\dots m}(\underline{x},\underline{u})$ also the vector valued function $f^{(1)1\dots m}(\underline{u})$ is an element of the tensor product of the smaller spaces $V_i \cong \mathbf{C}^N$

$$f^{(1)1\dots m}(\underline{u}) \in V^{(1)1\dots m}.$$

We say $f^{(1)1\dots m}(\underline{u})$ fulfills Conditions 3.1 (i)$^{(1)}$ and (ii)$^{(1)}$ if eqs. (3.1) and (3.2) hold in this space. We are now in a position to formulate the main theorem of this paper.

Theorem 3.7 *Let the vector valued function $f^{1\dots n}(\underline{x})$ be given by the Bethe Ansatz 3.5 and let $g^{1\dots m}(\underline{x},\underline{u})$ be of the form of Definition 3.6. If in addition the vector valued function $f^{(1)1\dots m}(\underline{u}) \in V^{(1)1\dots m}$ fulfills the Conditions 3.1* (i)$^{(1)}$ and (ii)$^{(1)}$, *then also $f^{1\dots n}(\underline{x}) \in V^{1\dots n}$ fulfills the Conditions 3.1* (i) and (ii), *i.e. $f^{1\dots n}(\underline{x})$ is a solution of the set of difference equations (3.2).*

Remark 3.8 *For $SU(2)$ (see e.g.[9]) the problem is already solved by 3.16 since then $f^{(1)}$ is a constant.*

Proof: Condition 3.1 (i) follows directly from the definition and the normalization of the R-matrix (2.5)

$$R_{ij}(x_i - x_j)\,\Omega^{\dots ij\dots} = \Omega^{\dots ij\dots},$$

the symmetry of $g^{1\dots m}(\underline{x},\underline{u})$ given by eq. (3.16) under the exchange of $x_1,\dots,x_n$ and

$$B_{\dots ji\dots,\beta}(\dots x_j, x_i \dots, u)\,R_{ij}(x_i - x_j) = R_{ij}(x_i - x_j)\,B_{\dots ij\dots,\beta}(\dots x_i, x_j \dots, u)$$

which is a consequence of the Yang-Baxter relations for the R-matrix.

Because of Proposition 3.2 it is sufficient to prove Condition 3.1 (ii) only for $i = n$

$$Q(\underline{x};n)\,f(\underline{x}) = \mathrm{tr}_a\,T_a^Q(\underline{x};n)\,f(\underline{x}) = f(\underline{x}')\ ,\quad (\underline{x}' = x_1,\dots,x_n' = x_n + 2).$$

where the indices $1\dots n$ have been suppressed. For the first step we apply a technique quite analogous to that used for the conventional algebraic Bethe Ansatz which solves eigenvalue problems. We apply the trace of $T_a^Q(\underline{x};n)$ to the vector $f(\underline{x})$ as given by eq. (3.12) and push $A^Q(\underline{x};n)$ and $D_a^Q(\underline{x};n)$ through all the B's using the commutation rules (3.10) and (3.11). Again with $\underline{x}' = x_1,\dots,x_n' = x_n + 2$ we obtain

$$A^Q(\underline{x};n)\,B_{b_m}(\underline{x},u_m)\dots B_{b_1}(\underline{x},u_1) = B_{b_m}(\underline{x}',u_m)\dots B_{b_1}(\underline{x}',u_1)\prod_{j=1}^m \frac{1}{b(x_n' - u_j)} \times A^Q(\underline{x};n) + \mathrm{uw}_A$$

$$D_a^Q(\underline{x};n)\,B_{b_m}(\underline{x},u_m)\dots B_{b_1}(\underline{x},u_1) = B_{b_m}(\underline{x}',u_m)\dots B_{b_1}(\underline{x}',u_1)\prod_{j=1}^m \frac{1}{b(u_j - x_n)} \times D_a^Q(\underline{x};n)\,R_{b_1 a}(u_1 - x_n')\dots R_{b_m a}(u_m - x_n') + \mathrm{uw}_{D_a}$$

The "wanted terms" written explicitly originate from the first term in the commutations rules (3.10) and (3.11); all other contributions yield the "unwanted terms". If we insert these equations into the representation (3.12) of $f(\underline{x})$ we find that the wanted contribution from A^Q already gives the desired result. The wanted contribution from D^Q applied to Ω gives zero. The unwanted contributions can be written as as difference which vanishes after summation over the u's. These three facts can be seen as follows. We have

$$A^Q(\underline{x};n)\,\Omega = \Omega\ ,\quad D_a^Q(\underline{x};n)\,\Omega = 0$$

which follow from eq. (3.15) since $T^Q(\underline{x};n) = T(\underline{x},x_n)$ and $b(0) = 0$. The defining relation of $\psi(x)$ (3.17) implies that the wanted term from A yields $f(\underline{x}')$. The commutation relations (3.10), (3.11), (2.12) and (2.13) imply that the unwanted terms are proportional to a product of B-operators, where exactly one $B_{b_j}(\underline{x}, u_j)$ is replaced by $B^Q_{b_j}(\underline{x}; n)$. Because of the commutation relations of the B's (2.11) and the symmetry property given by Condition 3.1 (i)$^{(1)}$ of $g^{1\dots m}(\underline{x}, \underline{u})$ it is sufficient to consider only the unwanted terms for $j = m$ denoted by uw^m_A and uw^m_D. They come from the second term in (3.10) if $A^Q(\underline{x}; n)$ is commuted with $B_{b_m}(\underline{x}, u_m)$ and then the resulting $A(\underline{x}, u_m)$ pushed through the other B's taking only the first terms in (2.12) into account and correspondingly for $D^Q_a(\underline{x}; u_m)$.

$$\mathrm{uw}^m_A = -\frac{c(x'_n - u_m)}{b(x'_n - u_m)} B^Q_{b_m}(\underline{x}; m) \dots B_{b_1}(\underline{x}, u_1) \prod_{j<m} \frac{1}{b(u_m - u_j)} A(\underline{x}, u_m)$$

$$\mathrm{uw}^m_{D_a} = -\frac{c(u_m - x_n)}{b(u_m - x_n)} B^Q_{b_m}(\underline{x}; m) \dots B_{b_1}(\underline{x}, u_1) \prod_{j<m} \frac{1}{b(u_j - u_m)} D_a(\underline{x}, u_m)\, T^{Q(1)}_{b_1\dots b_m,a}(\underline{u}; m)$$

where $T^{Q(1)}$ is the new type of monodromy matrix

$$T^{Q(1)}_{b_1\dots b_m,a}(\underline{u}; m) = R_{b_1 a}(u_1 - u_m) \dots R_{b_{m-1} a}(u_{m-1} - u_m)\, P_{b_m a} \tag{3.19}$$

analogous to (3.6) whose trace over the auxiliary space $V^{(1)}_a$ yields the shift operator $Q^{(1)}(\underline{u}; m)$. With $D_a(\underline{x}, u_m)\Omega = \mathbf{1}_a \prod_{i=1}^n b(x_i - u_m)\,\Omega$ (see (3.15)), by the assumption

$$Q^{(1)}(\underline{u}; m)\, f^{(1)}(\underline{u}) = f^{(1)}(\underline{u}') \,, \quad (\underline{u}' = u_1, \dots, u'_m = u_m + 2)$$

and the defining relations (3.17) of $\psi(x)$ and $\tau(x)$, we obtain

$$\mathrm{tr}_a\, \mathrm{uw}^m_{D_a}(\underline{u})\, \Omega\, g(\underline{x}, \underline{u}) = -\mathrm{uw}^m_A(\underline{u}')\, \Omega\, g(\underline{x}, \underline{u}')$$

where $c(-x)/b(-x) = -c(x)/b(x)$ has been used. Therefore the sum of all unwanted terms yield a difference analog of a total differential which vanishes after summation over the u's.

Iterating Theorem 3.7 we obtain the nested generalized Bethe Ansatz with levels $k = 1, \dots, N-1$. The Ansatz of level k reads

$$\begin{aligned} f^{(k-1)^{1\dots n_{k-1}}}\left(\underline{x}^{(k-1)}\right) = \sum_{\underline{x}^{(k)}} B^{(k-1)}_{1\dots n_{k-1}\beta_{n_k}}\left(\underline{x}^{(k-1)}, x^{(k)}_{n_k}\right) \dots \\ \dots B^{(k-1)}_{1\dots n_{k-1}\beta_1}\left(\underline{x}^{(k-1)}, x^{(k)}_1\right) \Omega^{(k-1)^{1\dots n_{k-1}}}\, g^{(k-1)^{\beta_1\dots\beta_{n_k}}}\left(\underline{x}^{(k-1)}, \underline{x}^{(k)}\right) \end{aligned} \tag{3.20}$$

The functions $f^{(k)}$ and $g^{(k)}$ are vectors with

$$f^{(k)^{1\dots n_k}}, g^{(k-1)^{1\dots n_k}} \in V^{(k)^{1\dots n_k}} = V^{(k)}_1 \otimes \dots \otimes V^{(k)}_{n_k} \,, \quad (V^{(k)}_i \cong \mathbf{C}^{N-k}).$$

The basis vectors of these spaces are $|\,\alpha_1 \dots \alpha_{n_k}\rangle^{(k)} \in V^{(k)^{1\dots n_k}}$ and $k < \alpha_i \leq N$.

Analogously to Definition 3.6 we write

$$g^{(k-1)^{1\ldots n_k}}(\underline{x}^{(k-1)},\underline{x}^{(k)}) = \prod_{i=1}^{n_{k-1}}\prod_{j=1}^{n_k}\psi(x_i^{(k-1)}-x_j^{(k)})\prod_{1\le i<j\le n_k}\tau(x_i^{(k)}-x_j^{(k)})\, f^{(k)^{1\ldots n_k}}(\underline{x}^{(k)}) \tag{3.21}$$

where the functions $\psi(x)$ and $\tau(x)$ fulfill the functional equations (3.17) with the solutions (3.18). Then the start of the iteration is given by a $k_{max} \le N$ with

$$f^{(k_{max}-1)^{1\ldots n_{k_{max}-1}}} = |\, k_{max}\ldots k_{max}\,\rangle\,, \quad \text{and} \quad n_k = 0 \ \text{ for } k \ge k_{max} \tag{3.22}$$

which is the reference state of level $k_{max}-1$ and trivially fulfills the Conditions 3.1.

Corollary 3.9 *The system of $SU(N)$ matrix difference equations (3.2) is solved by the nested Bethe Ansatz (3.20) with (3.21), (3.22) and $f^{1\ldots n}(\underline{x}) = f^{(0)^{1\ldots n}}(\underline{x})$.*

4 Weights of generalized $SU(N)$ Bethe vectors

In this section we analyze some group theoretical properties of generalized Bethe states. We calculate the weights of the states and show that they are highest weight states. The first result does not depend on any restriction to the states. On the other hand the second result is not only true for the conventional Bethe Ansatz, which solves an eigenvalue problem and which is well known, but also, as we will show, for the generalized one which solves a difference equation (or a differential equation).

By asymptotic expansion of the R-matrix and the monodromy matrix T (cf. eqs.(2.5) and(2.8)) we get for $u \to \infty$

$$\begin{aligned} R_{ab}(u) &= \mathbf{1}_{ab} - \frac{2}{Nu}\,P_{ab} + O(u^{-2}) & (4.1)\\ T_{1\ldots n,a}(\underline{x},u) &= \mathbf{1}_{1\ldots n,a} + \frac{2}{Nu}\,M_{1\ldots n,a} + O(u^{-2}). & (4.2)\end{aligned}$$

Explicitly we get from eq. (2.8)

$$M_{1\ldots n,a} = P_{1a} + \ldots + P_{na} \tag{4.3}$$

where the P's are the permutation operators. The matrix elements of $M_{1\ldots n,a}$ as a matrix in the auxiliary space are the $su(N)$ Lie algebra generators. In the following we will consider only operators acting in the fixed tensor product space $V = V^{1\ldots n}$ of (2.1); therefore we will omit the indices $1\ldots n$. In terms of matrix elements in the auxiliary space V_a the generators act on the basis states as

$$M_\alpha^{\alpha'}\,|\,\alpha_1,\ldots,\alpha_i,\ldots,\alpha_n\,\rangle = \sum_{i=1}^{n}\delta_{\alpha'\alpha_i}\,|\,\alpha_1,\ldots,\alpha,\ldots,\alpha_n\,\rangle. \tag{4.4}$$

The Yang-Baxter relations (2.9) yield for $x_a \to \infty$

$$[M_a + P_{ab}, T_b(x_b)] = 0 \tag{4.5}$$

and if additionally $x_b \to \infty$

$$[M_a + P_{ab}, M_b] = 0 \tag{4.6}$$

or for the matrix elements

$$\begin{aligned} [M_\alpha^{\alpha'}, T_\beta^{\beta'}(u)] &= \delta_{\alpha'\beta}\, T_\alpha^{\beta'}(u) - \delta_{\alpha\beta'}\, T_\beta^{\alpha'}(u) &(4.7)\\ [M_\alpha^{\alpha'}, M_\beta^{\beta'}] &= \delta_{\alpha'\beta}\, M_\alpha^{\beta'} - \delta_{\alpha\beta'}\, M_\beta^{\alpha'}. &(4.8)\end{aligned}$$

Equation (4.8) represents the structure relations of the $su(N)$ Lie algebra and (4.7) the $SU(N)$-covariance of T. In particular the transfer matrix is invariant

$$[M_\alpha^{\alpha'}, \operatorname{tr} T(u)] = 0. \tag{4.9}$$

We now investigate the action of the lifting operators $M_\alpha^{\alpha'}$ $(\alpha' > \alpha)$ to generalized Bethe vectors.

Lemma 4.1 *Let $F[g](\underline{x}) \in V$ be a Bethe Ansatz vector given in terms of a vector $g(\underline{x}, \underline{u}) \in V^{(1)} \cong \mathbf{C}^{(N-1)\otimes m}$ by*

$$F[g](\underline{x}, \underline{u}) = B_{\beta_m}(\underline{x}, u_m) \dots B_{\beta_1}(\underline{x}, u_1)\, \Omega\; g^{\underline{\beta}}(\underline{x}, \underline{u}) \tag{4.10}$$

with $\underline{\beta} = \beta_1, \dots, \beta_m$. Then $M_\alpha^{\alpha'} F[g]$ is of the form

$$M_\alpha^{\alpha'} F[g] = \begin{cases} \sum_{j+1}^m B_{\beta_m} \dots \delta_{\alpha'\beta_m} \dots B_{\beta_1}\, \Omega\; G_j^{\underline{\beta}}(\underline{x}, \underline{u}) & \text{for } \alpha' > \alpha = 1 \\ F[M^{(1)}{}_\alpha^{\alpha'}\, g] & \text{for } \alpha' > \alpha > 1, \end{cases} \tag{4.11}$$

where the $M^{(1)}{}_\alpha^{\alpha'}$ are the $su(N-1)$ generators represented in $V^{(1)}$ (analogously to (4.3)) and

$$G_m(\underline{x}, \underline{u}) = \left(\frac{1}{\prod_{j=1}^m b(u_m - u_j)} - \frac{\prod_{i=1}^n b(x_i - u_m)}{\prod_{j=1}^m b(u_j - u_m)}\, Q^{(1)}(\underline{u}; m) \right) g(\underline{x}, \underline{u}). \tag{4.12}$$

The operator $Q^{(1)}(\underline{u}; m) \in End(V^{(1)})$ is a next level Q-matrix given by the trace

$$Q^{(1)}(\underline{u}; m) = \operatorname{tr}_a T_a^{Q(1)}(\underline{u}; m) \tag{4.13}$$

(see eq. (3.19)). The other G_j are obtained by Yang-Baxter relations.

Proof: First we consider the case $\alpha = 1$. The commutation rule (4.7) reads for $\beta' = 1$ and $\alpha' \to \alpha$

$$[M_1^\alpha, B_\beta(u)] = \delta_{\alpha\beta}\, A(u) - D_\beta^\alpha(u).$$

We commute M_1^α through all the B's of (4.10) and use $M_1^\alpha\,\Omega = 0$ for $\alpha > 1$ (cf. (4.4)). The A's and D's appearing are also commuted through all the B's using the commutation rules (2.12) and (2.13). In each summand exactly one B-operator disappears. Therefore the result is of the form of eq. (4.11). Contributions to G_m arise when we commute M_1^α through $B_{\beta_m}(u_m)$ and then push the $A(u_m)$ and $D(u_m)$ through the other B's $(j < m)$, only taking the first terms of (2.12) and (2.13) into account. All other terms would contain a $B(u_m)$ and would therefore contribute to one of the other G_j $(j < m)$. Finally we apply $A(u_m)$ and $D(u_m)$ to Ω

$$A(u_m)\,\Omega = \Omega\ , \quad D_\beta^\alpha(u_m)\,\Omega = \delta_{\alpha\beta}\prod_{i=1}^n b(x_i - u_m)\,\Omega$$

and get eq. (4.12).

For $\alpha' > \alpha > 1$ we again use the commutation rule (4.7)

$$[M_\alpha^{\alpha'}, B_\beta(u)] = \delta_{\alpha'\beta}\, B_\alpha(u)$$

and get

$$M_\alpha^{\alpha'}\, B_\beta(u_m)\dots B_\beta(u_1) = B_{\beta_m}(u_m)\dots B_{\beta_1}(u_1)\, M_\alpha^{\alpha'} + B_{\beta'_m}(u_m)\dots B_{\beta'_1}(u_1)\, M^{(1)}{}_{\underline{\beta},\alpha}^{\underline{\beta}',\alpha'}$$

with $M^{(1)}_{1\dots m,a} = P^{(1)}_{1a} + \dots + P^{(1)}_{ma}$ analogously to (4.3). Because of $M_\alpha^{\alpha'}\,\Omega = 0$ for $\alpha' > 1$ (cf. (4.4)) we get eq. (4.11)

The diagonal elements of M are the weight operators $W_\alpha = M_\alpha^\alpha$, they act on the basis vectors in V as

$$W_\alpha\,|\,\alpha_1,\dots,\alpha_n\,\rangle = \sum_{i=1}^n \delta_{\alpha_i\alpha}\ |\,\alpha_1,\dots,\alpha_n\,\rangle \tag{4.14}$$

which follows from $P_{i\alpha}^{\alpha'}\,|\,\alpha_i\,\rangle = \delta_{\alpha\alpha_i}\,|\,\alpha'\,\rangle$. In particular we get for the Bethe Ansatz reference state (3.14)

$$W_\alpha\,\Omega = \delta_{\alpha 1}\, n\,\Omega. \tag{4.15}$$

Lemma 4.2 *Let $F[g] \in V^{1\dots n}$ be as in Lemma 4.1. Then*

$$W_\alpha\, F[g] = \begin{cases} (n-m)\,F[g] & \text{for } \alpha = 1 \\ F[W_\alpha^{(1)}\, g] & \text{for } \alpha > 1, \end{cases} \tag{4.16}$$

where the $W_\alpha^{(1)}$'s are the $su(N-1)$ weight operators acting in $V^{(1)}$, i.e. the diagonal elements of generator matrix $M^{(1)}{}_\alpha^{\alpha'}$ (analogously to (4.3)).

Proof: By means of the commutation relation (4.7) for $\alpha' = \alpha = \beta' = 1,\ \beta > 1$

$$[W_1, B_\beta] = -B_\beta$$

we commute W_1 through all m B's of eq. (4.10) and with (4.15) we get the first equation. For the second equation we again use (4.7) now for $\alpha' = \alpha > 1,\ \beta' = 1,\ \beta > 1$

$$[W_\alpha, B_\beta] = \delta_{\alpha\beta}\, B_\beta.$$

Again commuting W_α through all the B's of eq. (4.10) we get with eq. (4.15)

$$\begin{aligned} W_\alpha\, B_{\beta_m} \dots B_{\beta_1}\, \Omega\; g^{\beta_1 \dots \beta_m} &= B_{\beta_m} \dots B_{\beta_1} \left(W_\alpha + \sum_{i=1}^{m} \delta_{\beta_i \alpha} \right) \Omega\; g^{\beta_1 \dots \beta_m} \\ &= B_{\beta_m} \dots B_{\beta_1}\, \Omega \left(W_\alpha^{(1)}\, g \right)^{\beta_1 \dots \beta_m} \end{aligned}$$

which concludes the proof.

Theorem 4.3 *Let the vector valued function $f(\underline{x}) \in V$ be given by the BETHE ANSATZ 3.5 fulfilling the assumptions of Theorem 3.7. If in addition $f^{(1)}$ is a highest weight vector and an eigenvector of the weight operators with*

$$W_\alpha^{(1)}\, f^{(1)} = w_\alpha^{(1)}\, f^{(1)}, \tag{4.17}$$

then also f is a highest weight vector

$$M_\alpha^{\alpha'}\, f = 0\ , \quad (\alpha' > \alpha) \tag{4.18}$$

and an eigenvector of the weight operators

$$W_\alpha\, f = w_\alpha\, f\ , \qquad w_\alpha = \begin{cases} n - m & \text{for } \alpha = 1 \\ w_\alpha^{(1)} & \text{for } \alpha > 1 \end{cases} \tag{4.19}$$

with

$$w_\alpha \geq w_\beta\ , \quad (1 \leq \alpha < \beta \leq N). \tag{4.20}$$

Proof: To prove the highest weight property we apply Lemma 4.1. By assumption $f^{(1)}$ fulfills the difference equation

$$f^{(1)}(u_1, \dots, u_m + 2) = Q^{(1)}(\underline{u}; m)\, f^{(1)}(u_1, \dots, u_m).$$

Together with eq. (3.16), (3.17) and (4.12) we obtain after summation $\sum_{u_m} G_m(\underline{x}, \underline{u}) = 0$, if $u_m = \tilde{u}_m - 2l_m$ ($l_m \in \mathbf{Z}$). The same is true for the other G_i in eq. (4.11), since g fulfills the symmetry property of Condition 3.1 (i) and thereby $F[g](\underline{x}, \underline{u})$ of eq. (4.10) is symmetric with respect to the u_i. Therefore in eq. (4.12) we have $M_\alpha^{\alpha'} f = 0$ for $\alpha' > \alpha > 1$ and for $\alpha' > \alpha = 1$ by assumption on $f^{(1)}$. The weights of f follow from Lemma 4.2 and also by assumption on $f^{(1)}$. From the commutation rule (4.8) and $M_\alpha^{\beta\dagger} = M_\beta^\alpha$ follows

$$0 \leq M_\alpha^\beta \, M_\beta^\alpha = M_\beta^\alpha \, M_\alpha^\beta + W_\alpha - W_\beta$$

which implies (4.20).

Since the states $f^{(k_{max}-1)}$ of eq. (3.22) are highest weight states in $V^{(k_{max}-1)}$ with weight $w_{k_{max}}^{(k_{max}-1)} = n_{k_{max}-1}$ we have the

Corollary 4.4 *If $f(\underline{x})$ is a solution of the system of $SU(N)$ matrix difference equations (3.2)*

$$f(\ldots, x_i + 2, \ldots) = Q(\underline{x}; i) \, f(\ldots, x_i, \ldots) \;, \quad (i = 1, \ldots, n)$$

given by the generalized nested Bethe Ansatz of Corollary 3.9, then f is a highest weight vector with weights

$$w = (w_1, \ldots, w_N) = (n - n_1, n_1 - n_2, \ldots, n_{N-2} - n_{N-1}, n_{N-1}), \tag{4.21}$$

where n_k is the number of $B^{(k-1)}$ operators in the Bethe Ansatz of level k, $(k = 1, \ldots, N-1)$. Further non-highest weight solutions of (3.2) are given by

$$f_\alpha^{\alpha'} = M_\alpha^{\alpha'} f \;, \quad (\alpha' < \alpha). \tag{4.22}$$

The interpretation of eq. (4.21) is that each $B^{(k)}$-operator reduces w_k and lifts a w_l $(l > k)$ by one.

5 Examples

From a solution of the matrix difference equations (3.2) one gets a new solution by multiplication of a scalar function which is symmetric with respect to all variables x_i and periodic with period 2. Therefore the solutions of the following examples may be multiplied by such functions.

Example 5.1 *The simplest example is obtained for $k_{max} = 1$ which means the trivial solution of the difference equations*

$$f^{1\ldots n} = \Omega^{1\ldots n}$$

The weights of $f^{1\ldots n}$ are $w = (n, 0, \ldots, 0)$.

In the language of spin chains this case corresponds to the ferro-magnetic ground state.

Example 5.2 *For the case $k_{max} = 1$ and $n^{(1)} = 1$ the solution reads*

$$f^{1\dots n}(\underline{x}) = \sum_u B_{1\dots n,\beta}(\underline{x}, u)\, \Omega^{1\dots n}\; g^{\beta}(\underline{x}, u).$$

with $u = \tilde{u} - 2l$ ($l \in \mathbf{Z}$, $\tilde{u}$ an arbitrary constant) and

$$g^{\beta}(\underline{x}, u) = \delta_{\beta 2} \prod_{i=1}^{n} \psi(x_i - u).$$

The weights of this vector $f^{1\dots n}$ are $w = (n-1, 1, 0, \dots, 0)$. The action of the creation operator $B_{1\dots n\beta}(x, y; u)$ on the reference state is easily calculated with help of eqs. (2.6) (2.8) and (2.10).

As a particular case of this example we determine explicitly the solution for the following

Example 5.3 *The action of the B-operator on the reference state for the case of $n = 2$ of Example 5.2 yields*

$$B_{12,\beta}(x, y; u)\,|\,11\,\rangle = c(x-u)\,b(y-u)\;|\,\beta 1\,\rangle + c(y-u)\;|\,1\beta\,\rangle.$$

Therefore we obtain

$$f^{12}(x, y) = \sum_u \psi(x-u)\,\psi(y-u)\{c(x-u)\,b(y-u)\;|\,\beta 1\,\rangle + c(y-u)\;|\,1\beta\,\rangle\}$$

with $u = \tilde{u} - 2l$, ($l \in \mathbf{Z}$). Using the expressions for the functions b, c, ψ given by eqs. (2.7) and (3.18) we get up to a constant

$$f^{12}(x, y) = \Big(\sin\pi\left(\tfrac{x-\tilde{u}}{2} - \tfrac{1}{N}\right)\,\sin\pi\left(\tfrac{y-\tilde{u}}{2} - \tfrac{1}{N}\right)\,\Gamma\left(\tfrac{y-x}{2} - \tfrac{1}{N}\right)\,\Gamma\left(1 + \tfrac{x-y}{2} - \tfrac{1}{N}\right)\Big)^{-1}(\,|\,21\,\rangle - |\,12\,\rangle).$$

This solution could also be obtained by means of the method used in [3], namely by diagonalization of the R-matrix. One obtains the scalar difference equations

$$f_-(x, y) = R_-(x-y)\, f_-(y, x)\;, \qquad f_-(x, y) = f_-(y, x+2)$$

with the eigenvalue $R_-(x) = (x + 2/N)(x - 2/N)$ of the antisymmetric tensor representation.

Example 5.4 *Next we consider for $N > 2$ the case of the quantum space $V_{123} = V_1 \otimes V_2 \otimes V_3$ and the case that the nested Bethe Ansatz has only two levels with two creation operators in the first level and one in the second level. This means $k_{max} = 3$, $n = 3$, $n^{(1)} = 2$, $n^{(2)} = 1$ and the weights $w = (1, 1, 1, 0, \dots, 0)$. The first level Bethe Ansatz is given by*

$$f^{123}(x, y, z) = \sum_{u,v} B_{123,\beta}(x, y, z; v)\, B_{123,\alpha}(x, y, z; u)\, \Omega^{123}\; g^{\alpha\beta}(x, y, z; u, v).$$

where the summation is specified by $u = \tilde{u} - 2k,\ v = \tilde{u} - 2l,\ (k, l \in \mathbf{Z})$*. By eq. (3.16)* g^{12} *is related to the next level function* ${f^{(1)}}^{12}$ *by*

$$g^{12}(x, y, z; u, v) = \prod_{x_i = x,y,z} \prod_{u_j = u,v} \psi(x_i - u_j)\, \tau(u - v)\, {f^{(1)}}^{12}(u, v).$$

The second level Bethe Ansatz reads

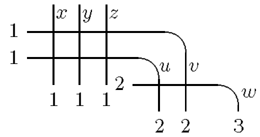
$${f^{(1)}}^{12}(u, v) = \sum_{w} B^{(1)}_{12\gamma}(u, v; w)\, {\Omega^{(1)}}^{12}\ {g^{(1)}}^{\gamma}(u, v; w)$$

where $w = \tilde{w} - 2m,\ (m \in \mathbf{Z})$*. The second level reference state is* ${\Omega^{(1)}}^{12} = |\,22\,\rangle^{(1)} \in {V^{(1)}}^{12}$*. Again according to eq. (3.16)*

$${g^{(1)}}^{\gamma}(u, v; w) = \psi(u - w)\, \psi(v - w)\, {f^{(2)}}^{\gamma},$$

with ${f^{(2)}}^{\gamma} = \delta_{\gamma 3}$*. Similar as in Example 5.3 the action of the operators* B *and* $B^{(1)}$ *on their reference states may be calculated.*

For this example the two level nested Bethe Ansatz may be depicted as



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