$U(N)$ Matrix Difference Equations and a Nested Bethe Ansatz

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Abstract

A system of $U(N)$-matrix difference equations is solved by means of a nested version of a generalized Bethe Ansatz. The highest weight property of the solutions is proved and some examples of solutions are calculated explicitly.

1 Introduction

This is the second part of a series of articles on the generalized nested Bethe Ansatz to solve matrix difference equations for different symmetry groups. The first part [1] (later denoted by [I]) was on the $SU(N)$ problem. In this part we investigate $U(N)$ symmetric matrix difference equations which are considerable more complicated than in the $SU(N)$ case, because the Bethe Ansatz reference states are more complicated. The reader is assumed to be familiar with the techniques and results of part [I]. We concentrate here on the problems which have to be solved in addition to those of part [I]. The results of this paper will be used in [2] to calculate exact form factors for the chiral Gross-Neveu model.

The article is organized similar as [I]. In Section 2 we recall some results concerning the $U(N)$ R-matrix. In Section 3 we introduce the nested generalized Bethe Ansatz to

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solve a system of $U(N)$ difference equations and present the solutions again in terms of "Jackson-type Integrals". In addition to the one used in [I] we have to introduce an additional monodromy matrix of the new type also fulfilling a new type of Yang-Baxter relation. In Section 4 we prove the highest weight property of the solutions and calculate the weights. Section 5 contains some examples of solutions.

2 The $U(N)$ R - matrix

Let $V^{1\dots n}$ be the tensor product space

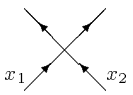
$$V^{1\dots n} = V_1 \otimes \dots \otimes V_n \tag{2.1}$$

where the vector spaces are $V_i = (V \oplus \bar{V})_i$. Here $V \cong \mathbf{C^N}$ is considered as a fundamental representation space of $U(N)$ and $\bar{V} \cong \mathbf{C^N}$ as the conjugate representation space. The vectors of V are sometimes called "particles" with positive $U(1)$ charge and those of $\bar{V}$ "anti-particles" with negative $U(1)$ charge. We will also use instead of $1, 2, \dots$ the indices a, b, etc. to refer to the spaces V_a, V_b, etc. It is straightforward to generalize the results of this paper to the case where the V_i are vector spaces for other representations. We denote the canonical basis vectors by

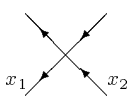
$$|\, \alpha_1 \dots \alpha_n \,\rangle \in V^{1\dots n}, \quad \Big(\alpha_i = (1,+), \dots, (N,+), (1,-), \dots, (N,-)\Big). \tag{2.2}$$

We will later also use the simpler notation $(\alpha, +) \equiv \alpha$ for particles and $(\alpha, -) \equiv \bar{\alpha}$ for anti-particles. A vector in $V^{1\dots n}$ is denoted by $v^{1\dots n}$ and a matrix acting in $V^{1\dots n}$ by $A_{1\dots n} \in End(V^{1\dots n})$.

The $U(N)$ spectral parameter depending R-matrix [3] acting on the tensor product of two particle spaces $V^{12} = V \otimes V$ coincides with that of $SU(N)$ used in [I]. It may be written and depicted as

$$R_{12}(x_1 - x_2) = b(x_1 - x_2)\mathbf{1}_{12} + c(x_1 - x_2)\, P_{12} \quad = \quad  \quad : V^{12} \to V^{21}, \tag{2.3}$$

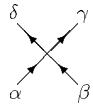
where P_{12} is the permutation operator. For $U(N)$ there is in addition the R-matrix acting on the tensor product of the anti-particle particle spaces $V^{\bar{1}2} = \bar{V} \otimes V$

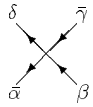
$$R_{\bar{1}2}(x_1 - x_2) = \mathbf{1}_{\bar{1}2} + d(x_1 - x_2)\, K_{\bar{1}2} \quad = \quad  \quad : V^{\bar{1}2} \to V^{2\bar{1}}, \tag{2.4}$$

where $K_{\bar{1}2}$ is the annihilation-creation matrix. There is no particle-anti-particle "backward scattering" (see e.g. [3]). Here and in the following we associate a variable (spectral

parameter) $x_i \in \mathbf{C}$ to each space V_i which is graphically represented by a line labeled by x_i (or simply by i). In addition an arrow on the lines denotes the $U(1)$-charge flow. (If we do not want to specify the direction of charge flow, we draw no arrow.) The components of the R-matrices are

$$R^{\delta\gamma}_{\alpha\beta} = \delta_{\alpha\gamma}\delta_{\beta\delta}\, b + \delta_{\alpha\delta}\delta_{\beta\gamma}\, c \quad = \quad , \quad R^{\delta\bar{\gamma}}_{\bar{\alpha}\beta} = \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\beta}\delta_{\delta\gamma}\, d \quad = \quad \tag{2.5}$$



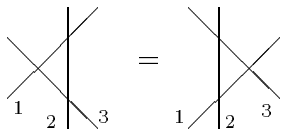


with $\alpha = (\alpha, +)$ etc. and $\bar{\alpha} = (\alpha, -)$ etc. The functions

$$b(x) = \frac{x}{x - 2/N}, \quad c(x) = \frac{-2/N}{x - 2/N}, \quad d(x) = \frac{2/N}{x - 1}. \tag{2.6}$$

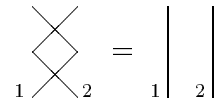
belong to the rational solution of the Yang-Baxter equations [3]

$$R_{12}(x_1 - x_2)\, R_{13}(x_1 - x_3)\, R_{23}(x_2 - x_3) = R_{23}(x_2 - x_3)\, R_{13}(x_1 - x_3)\, R_{12}(x_1 - x_2).$$



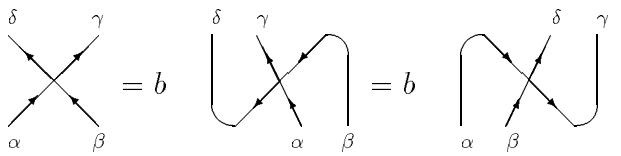
which holds for all possible charge flows. The matrices $R_{1\bar{2}}$ and $R_{\bar{1}\bar{2}}$ have the same matrix elements as $R_{\bar{1}2}$ and R_{12}, respectively. The inversion ("unitarity") relation of the R-matrix reads and may be depicted as

$$R_{21}(x_2 - x_1)\, R_{12}(x_1 - x_2) = 1 \;\; : \qquad = \tag{2.7}$$

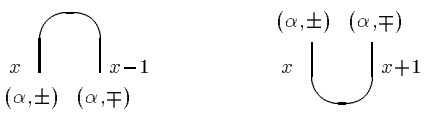


again for all possibilities charge flows. A further property of the $U(N)$ R-matrices is crossing and may be written and depicted as

$$R^{\delta\gamma}_{\alpha\beta}(x_1 - x_2) = b(x_1 - x_2)\, R^{\gamma\bar{\beta}}_{\delta\alpha}((x_2 + 1) - x_1) = b(x_1 - x_2)\, R^{\bar{\alpha}\delta}_{\beta\bar{\gamma}}(x_2 - (x_1 - 1))$$



where again we have used the notation $\alpha = (\alpha, +)$ etc. and $\bar{\alpha} = (\alpha, -)$ etc. We have introduced the graphical rule that a line changing the "time direction" also interchanges particles and anti-particles and changes $x \to x \pm 1$ as follows

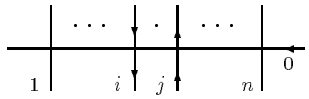


. (2.8)

We introduce a monodromy matrix (with $\underline{x} = x_1, \ldots, x_n$)

$$T_{1\ldots n,0}(\underline{x}; x_0) = R_{10}(x_1 - x_0) \ldots R_{n0}(x_n - x_0) \tag{2.9}$$

$$=$$



as a matrix acting in the tensor product of the "quantum space" $V^{1\ldots n}$ and the "auxiliary space" $V_0 \cong V \cong \mathbf{C}^N$ as a particle space. Since there is no "charge reflection" the positions of the particles and the anti-particles will not change under the application of $T_{1\ldots n,0}$ to a state in $V^{1\ldots n}$. (The construction of the Bethe states will be not symmetric with respect to particles and anti-particles, because we only use the monodromy matrix with particles for the auxiliary space.)

The Yang-Baxter algebra relations

$$T_{1\ldots n,a}(\underline{x}; x_a)\, T_{1\ldots n,b}(\underline{x}; x_b)\, R_{ab}(x_a - x_b) = R_{ab}(x_a - x_b)\, T_{1\ldots n,b}(\underline{x}; x_b)\, T_{1\ldots n,a}(\underline{x}; x_a) \tag{2.10}$$

imply the basic algebraic properties of the sub-matrices w.r.t the auxiliary space defined by

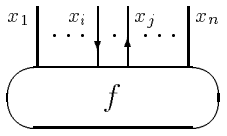
$$T_{1\ldots n}{}^{\alpha}_{\beta}(\underline{x}; x) \equiv \begin{pmatrix} A_{1\ldots n}(\underline{x}; x) & B_{1\ldots n,\beta}(\underline{x}; x) \\ C_{1\ldots n}{}^{\alpha}(\underline{x}; x) & D_{1\ldots n}{}^{\alpha}_{\beta}(\underline{x}; x) \end{pmatrix}. \tag{2.11}$$

The indices α, β on the left hand side run from 1 to N and on the right hand side from 2 to N. The commutation rules of the A, B, C, D are the same as for the $SU(N)$ case of [I].

3 The $U(N)$ - difference equation

Let

$$f^{1\ldots n}(\underline{x}) =$$



$$\in V^{1\ldots n}$$

be a vector valued function of $\underline{x} = x_1, \ldots, x_n$ with values in $V^{1\ldots n}$.

Similar as in [I] we constrain the function $f^{1\ldots n}$ by the

Conditions 3.1 *The following symmetry and periodicity conditions on the vector valued function $f^{1\ldots n}(\underline{x})$ are supposed to be valid:*

(i) *The symmetry property under the exchange of two neighboring spaces V_i and V_j and the variables x_i and x_j, at the same time, is given by*

$$f^{\ldots ji\ldots}(\ldots, x_j, x_i, \ldots) = R_{ij}(x_i - x_j)\, f^{\ldots ij\ldots}(\ldots, x_i, x_j, \ldots). \tag{3.1}$$

(ii) *The* **system of matrix difference equations** *holds*

$$\boxed{f^{1\ldots n}(\ldots,x_i+2,\ldots)=Q_{1\ldots n}(\underline{x};i)\,f^{1\ldots n}(\ldots,x_i,\ldots)\ ,\quad (i=1,\ldots,n)} \tag{3.2}$$

where the matrices $Q_{1\ldots n}(\underline{x};i)\in End(V^{1\ldots n})$ *are defined by*

$$Q_{1\ldots n}(\underline{x};i)=R_{i+1i}(x_{i+1}-x_i')\ldots R_{ni}(x_n-x_i')\,R_{1i}(x_1-x_i)\ldots R_{i-1i}(x_{i-1}-x_i) \tag{3.3}$$

with $x_i'=x_i+2$.

The Yang-Baxter equations for the R-matrix guarantee that these conditions are compatible. The shift of 2 in eq. (3.2) could be replaced by an arbitrary κ for the application to the form factor problem, however, it is fixed to 2 because of crossing symmetry. The $Q_{1\ldots n}(\underline{x};i)$ fulfill the same commutation rules as in [I].

Proposition 3.2 and Remark 3.3 of [I] also hold for the $U(N)$ case. Also we need the new type of monodromy matrix given by Definition 3.4 of [I] with

$$Q_{1\ldots n}(\underline{x};i)=\mathrm{tr}_0\,T^Q_{1\ldots n,0}(\underline{x};i). \tag{3.4}$$

We use this formula if the index i is associated to a particle. If the index i is associated to an anti-particle, we will use a further monodromy matrix of a new type given by the following

Definition 3.2 *For* $i=1,\ldots,n$

$$\tilde{T}^Q_{\bar{0},1\ldots n}(\underline{x};i)=R_{\bar{0}1}(x_i-x_1)\ldots R_{\bar{0}i-1}(x_i-x_{i-1})\,P_{\bar{0}\bar{i}}\,R_{\bar{0}i+1}(x_i'-x_{i+1})\ldots R_{\bar{0}n}(x_i'-x_n) \tag{3.5}$$

with the auxiliary space $V_{\bar{0}}=\bar{V}$ *and* $x_i'=x_i+2$.

By the "unitarity" of the R-matrix (2.7) the trace of this monodromy matrix is the inverse of the Q-matrix

$$Q^{-1}_{1\ldots n}(\underline{x};i)=\mathrm{tr}_{\bar{0}}\,\tilde{T}^Q_{\bar{0},1\ldots n}(\underline{x};i). \tag{3.6}$$

The new type of monodromy matrix $\tilde{T}^Q_{\bar{0},1\ldots n}$ fulfills a new type of Yang-Baxter relation in the form

$$\tilde{T}^Q_{\bar{a},1\ldots n}(\underline{x};i)\,R_{\bar{a}b}(x_i-u)\,T_{1\ldots n,b}(\underline{x}',u)=T_{1\ldots n,b}(\underline{x},u)\,R_{\bar{a}b}(x_i'-u)\,\tilde{T}^Q_{\bar{a},1\ldots n}(\underline{x};i) \tag{3.7}$$

with $\underline{x}'=x_1,\ldots,x_i',\ldots,x_n$ and $x_i'=x_i+2$. In addition to the commutations rules (3.10) and (3.11) of [I] for the case that the indices i and a belong to particles, we have the commutation rules for the case that the indices i and a correspond to anti-particles

$$\tilde{A}^Q(\underline{x};i)\,B_b(\underline{x}',u)=\frac{1-d^2(x_i'-u)}{1+d(x_i-u)}\,B_b(\underline{x}',u)\,\tilde{A}^Q(\underline{x};i)+d(x_i'-u)\,\tilde{B}^Q_{\bar{b}}(\underline{x};i)\,A(\underline{x}',u) \tag{3.8}$$

$$\tilde{D}^Q_{\bar{a}}(\underline{x};i)\, B_b(\underline{x}',u) = B_b(\underline{x},u)\, R_{\bar{a}b}(x'_i-u)\, \tilde{D}^Q_{\bar{a}}(\underline{x};i) - d(x_i-u)\, \tilde{B}^Q_{\bar{a}}(\underline{x};i)\, K_{\bar{a}b}\, D_b(\underline{x}',u). \quad (3.9)$$

The system of difference equations (3.2) can be solved by means of a generalized ("off-shell") nested Bethe Ansatz. The first level is given by the

BETHE ANSATZ 3.3

$$f^{1\dots n}(\underline{x}) = \sum_{\underline{u}}\; B_{1\dots n,\beta_m}(\underline{x},u_m)\dots B_{1\dots n,\beta_1}(\underline{x},u_1)\, g^{1\dots n\beta_1\dots\beta_m}(\underline{x},\underline{u}) \quad (3.10)$$

where summation over $1<\beta_1,\dots,\beta_m\le N$ *is assumed. The summation over* $\underline{u}$ *is specified by*

$$\underline{u} = (u_1,\dots,u_m) = (\tilde{u}_1-2l_1,\dots,\tilde{u}_m-2l_m)\ , \quad l_i\in\mathbf{Z}, \quad (3.11)$$

where the $\tilde{u}_i$ *are arbitrary constants. For a fixed set of indices* $\underline{\beta}=\beta_1,\dots,\beta_m$ $(1<\beta_i\le N)$ *the reference state vector* $g^{1\dots n\underline{\beta}}\in V^{1\dots n}$ *fulfills*

$$C_{1\dots n}{}^{\gamma}\, g^{1\dots n\underline{\beta}} = 0 \quad (1<\gamma\le N) \quad (3.12)$$

Note that here we have denoted by g, what was called $\Omega\, g$ in [I]. This notation is more convenient, since contrasting the $SU(N)$ case, the space of reference states is here higher dimensional for $U(N)$. This space V_{ref} is spanned by all basis vectors of the form

$$|\,\alpha_1,\dots,\alpha_n\,\rangle\ , \quad \alpha_i = \begin{cases} (1,+) & \text{for particles} \\ (\alpha_i,-)\ , \quad (1<\alpha_i\le N) & \text{for antiparticles.}\end{cases}$$

The $U(1)$-charge is left invariant by operators like the monodromy matrices $T_{1\dots n,0}$, $T^Q_{1\dots n,0}$, $\tilde{T}^Q_{\bar{0},1\dots n}$ and also by the operations of Conditions 3.1. Therefore the space $V^{1\dots n}$ decomposes into invariant subspaces of fixed charge, i.e. fixed numbers n_-, n_+ $(n_-+n_+=n)$ of anti-particles and particles, respectively. Moreover because of the symmetry property (i) it is sufficient to consider the even smaller subspaces where the anti-particles and the particles are sitting at fixed places

$$V^{1\dots n}_{n_-} = \bigoplus_{I_-\subset I} V^{1\dots n}_{I_-}\ , \quad (I_-\cup I_+ = I = \{1,\dots,n\},\ |I_-|=n_-,\ |I_+|=n_+). \quad (3.13)$$

Here I_- and I_+ denote the positions of the anti-particles and particles, respectively. The space of reference states in the subspace $V^{1\dots n}_{I_-}$ may be written as

$$V_{ref}\cap V^{1\dots n}_{I_-} \cong \bar{V}^{(1)}_1\otimes\dots\otimes\bar{V}^{(1)}_{n_-}\otimes\Omega^{1\dots n_+} \quad (3.14)$$

where the $\bar{V}^{(1)}_i\cong\mathbf{C}^{N-1}$ contain only anti-particle states with $\alpha>1$ and with the fixed vector

$$\Omega^{1\dots n_+} = |\,1\dots1\,\rangle\in\bigotimes_{i\in I_+} V_i$$

Therefore g may also be considered as a vector

$$g^{1\dots n b_1\dots b_m} \in {V^{(1)}}^{\bar{1}\dots\bar{n}_- b_1\dots b_m} \otimes \Omega^{1\dots n_+}$$

with

$${V^{(1)}}^{\bar{1}\dots\bar{n}_- b_1\dots b_m} = \bigotimes_{i\in I_-} \bar{V}_i^{(1)} \otimes V_{b_1}^{(1)} \otimes \dots \otimes V_{b_m}^{(1)} \tag{3.15}$$

where the $V_{b_i}^{(1)} \cong \mathbf{C}^{N-1}$ are particle spaces. Again as in [I] we define a new vector valued function ${f^{(1)}}^{\bar{1}\dots\bar{n}_- b_1\dots b_m} \in {V^{(1)}}^{\bar{1}\dots\bar{n}_- b_1\dots b_m}$ by

Definition 3.4 *Let $g^{1\dots n\beta}(\underline{x},\underline{u})$ be given in the subspace $V_{I_-}^{1\dots n} \subset V^{1\dots n}$ with fixed positions of particles and anti-particles as*

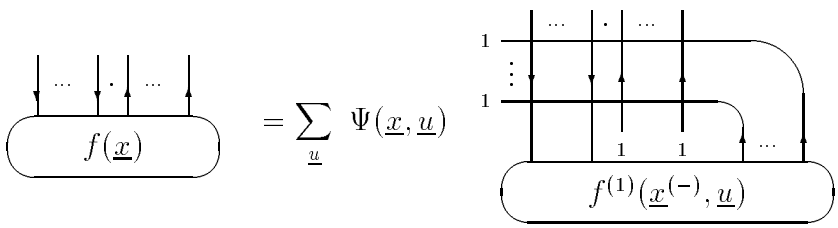
$$g^{1\dots n b_1\dots b_m}(\underline{x},\underline{u}) = \prod_{i\in I_+}\prod_{j=1}^{m} \psi(x_i-u_j) \prod_{1\le i<j\le m} \tau(u_i-u_j)\, {f^{(1)}}^{\bar{1}\dots\bar{n}_- b_1\dots b_m}(\underline{x}^{(-)},\underline{u}) \otimes \Omega^{1\dots n_+} \tag{3.16}$$

where $\underline{x}^{(-)} = \{x_i; i \in I_-\}$ are the spectral parameters of the anti-particles only.

The functions $\psi(x)$ and $\tau(x)$ are the same as in [I]

$$\psi(x) = \frac{\Gamma(1-\frac{1}{N}+\frac{x}{2})}{\Gamma(1+\frac{x}{2})}, \quad \tau(x) = \frac{x\,\Gamma(\frac{1}{N}+\frac{x}{2})}{\Gamma(1-\frac{1}{N}+\frac{x}{2})}. \tag{3.17}$$

For the case that the anti-particles are sitting at the first n_- places $I_- = \{1,\dots,n_-\}$ the BETHE ANSATZ 3.3 may be depicted as



where $\Psi(\underline{x},\underline{u})$ is given by the products of ψ's and τ's of eq. (3.16). The main theorem of this article analogous to the one of [I].

Theorem 3.5 *Let the vector valued function $f^{1\dots n}(\underline{x})$ be given by the BETHE ANSATZ 3.3 where $g^{1\dots m}(\underline{x},\underline{u})$ is of the form of Definition 3.4. If in addition the vector valued function ${f^{(1)}}^{\bar{1}\dots\bar{n}_- b_1\dots b_m}(\underline{x}^{(-)},\underline{u}) \in {V^{(1)}}^{\bar{1}\dots\bar{n}_- b_1\dots b_m}$ fulfills the Conditions 3.1* (i)$^{(1)}$ and (ii)$^{(1)}$, *then also $f^{1\dots n}(\underline{x}) \in V^{1\dots n}$ fulfills the Conditions 3.1* (i) and (ii), *i.e. $f^{1\dots n}(\underline{x})$ is a solution of the set of difference equations (3.2).*

Proof: Condition 3.1 (i) follows as in [I]. Again it is sufficient to prove Condition 3.1 (ii) for $i = n$

$$Q(\underline{x}; n)\, f(\underline{x}) = f(\underline{x}') \ , \quad (\underline{x}' = x_1, \ldots, x'_n = x_n + 2).$$

where the indices $1 \ldots n$ have been suppressed. For the case that the index n belongs to a particle the proof is the same as in [I], with the only difference that here the next level Q-matrix is given by the trace of

$$T_a^{Q(1)}(\underline{x}^{(-)}, \underline{u}; b_m) = \prod_{i \in I_-} R_{ia}(x_i - u_m)\, R_{b_1 a}(u_1 - u_m) \ldots R_{b_{m-1} a}(u_{m-1} - u_m)\, P_{b_m a}. \quad (3.18)$$

For the case that the index n belongs to an anti-particle we will prove the inverse of (ii) using $\tilde{T}^Q_{\bar{a},1\ldots n}$. We apply $Q^{-1}(\underline{x}; n)$ given by the trace of $\tilde{T}^Q_{\bar{a},1\ldots n}(\underline{x}; n)$ to the vector $f^{1\ldots n}(\underline{x}')$ as given by eq. (3.10) and push $\tilde{A}^Q(\underline{x}; n)$ and $\tilde{D}^Q_{\bar{a}}(\underline{x}; n)$ through all the B's using the commutation rules (3.8) and (3.9). Again with $\underline{x}' = x_1, \ldots, x'_n = x_n + 2$ we obtain

$$\begin{aligned} \tilde{A}^Q(\underline{x}; n)\, B_{b_m}(\underline{x}', u_m) \ldots B_{b_1}(\underline{x}', u_1) &= B_{b_m}(\underline{x}, u_m) \ldots B_{b_1}(\underline{x}, u_1) \\ &\quad \times \prod_{j=1}^{m} \frac{1 - d^2(x'_n - u)}{1 + d(x_n - u)}\, \tilde{A}^Q(\underline{x}; n) + \mathrm{uw}_A \end{aligned}$$

$$\begin{aligned} \tilde{D}^Q_{\bar{a}}(\underline{x}; \bar{n})\, B_{b_m}(\underline{x}', u_m) \ldots B_{b_1}(\underline{x}', u_1) &= B_{b_m}(\underline{x}, u_m) \ldots B_{b_1}(\underline{x}, u_1) \\ &\quad \times R_{\bar{a} b_1}(x'_n - u_1) \ldots R_{\bar{a} b_m}(x'_n - u_m)\, \tilde{D}^Q_{\bar{a}}(\underline{x}; \bar{n}) + \mathrm{uw}_{D_a} \end{aligned}$$

The "wanted" and "unwanted" terms have origins analogous to the ones considered in [I]. If we insert these equations into the representation (3.10) of $f(\underline{x}')$ we find, however, that the wanted contribution from A^Q vanish and the wanted contribution from D^Q already gives the desired result. By definition of $\tilde{T}^Q$ and the reference state vector g we have

$$\tilde{A}^Q(\underline{x}; n)\, g = 0 \ , \quad \tilde{D}^Q_{\bar{a}}(\underline{x}; n)\, g = \prod_{i \in I_-} R_{\bar{a} i}(x_n - x_i)\, P_{\bar{a}\bar{n}}\, g$$

if the index $\bar{a}$ correspond to an anti-particle. With

$$Q^{(1)^{-1}}(\underline{x}^{(-)}, \underline{u}; n) = \mathrm{tr}_a \Big(\prod_{i \in I_-} R_{\bar{a} i}(x_n - x_i)\, P_{\bar{a}\bar{n}}\, R_{\bar{a} b_1}(x'_n - u_1) \ldots R_{\bar{a} b_m}(x'_n - u_m) \Big)$$

and the assumption

$$Q^{(1)^{-1}}(\underline{x}^{(-)}, \underline{u}; n)\, f^{(1)}(\underline{x}^{(-)'}, \underline{u}) = f^{(1)}(\underline{x}^{(-)}, \underline{u})$$

it follows that the wanted term from $\tilde{D}^Q$ yields $f(\underline{x})$. Again as in [I] the unwanted contributions can be written as differences which vanish after summation over the u's.

Because of the commutation relations of the B's (see (2.11) of [I]) and the symmetry property (i)$^{(1)}$ of $g^{1\dots nb_1\dots b_m}(\underline{x}^{(-)},\underline{u})$ it is sufficient to consider only the unwanted terms for $j=m$ denoted by uw_A^m and uw_D^m.

$$\begin{aligned}
\mathrm{uw}_A^m(\underline{x},\underline{u}) &= d(x_n'-u_m)\,\tilde{B}^Q_{\bar{b}_m}(\underline{x};m)\dots B_{b_1}(\underline{x},u_1)\prod_{j<m}\frac{1}{b(u_m-u_j)}\,A(\underline{x},u_m)\\
\mathrm{uw}_{D\bar{a}}^m(\underline{x},\underline{u}) &= -d(x_n-u_m)\,\tilde{B}^Q_{\bar{a}}(\underline{x};m)\dots B_{b_1}(\underline{x},u_1)\prod_{j<m}\frac{1}{b(u_j-u_m)}\\
&\quad\times D_{b_m}(\underline{x}',u_m)\,K_{\bar{a}b_m}\,R_{b_1b_m}(u_1-u_m)\dots R_{b_{m-1}b_m}(u_{m-1}-u_m)
\end{aligned}$$

With

$$D_a(\underline{x},u)\,g(\underline{x},\underline{u})=\prod_{i\in I_+}b(x_i-u)\prod_{i\in I_-}R_{ia}(x_i-u)\,g(\underline{x},\underline{u}),$$

$$Q^{(1)}(\underline{x}^{(-)},\underline{u};b_m)=\mathrm{tr}_a\,T_a^{Q(1)}(\underline{x}^{(-)},\underline{u};b_m)$$

(see 3.18)), the assumption $(ii)^{(1)}$, in particular the relation

$$Q^{(1)}(\underline{x}^{(-)},\underline{u};b_m)\,f^{(1)}(\underline{x}^{(-)},\underline{u})=f^{(1)}(\underline{x}^{(-)},\underline{u}')\ ,\quad(\underline{u}'=u_1,\dots,u_m'=u_m+2)$$

and the defining relations of the functions $\psi(x)$ and $\tau(x)$ (3.17) of [I] follows

$$\mathrm{uw}_A^m(\underline{x},\underline{u})\,g(\underline{x},\underline{u})=-\mathrm{tr}_{\bar{a}}\ \mathrm{uw}_{D\bar{a}}^m(\underline{x},\underline{u}')\,g(\underline{x},\underline{u}')$$

which concludes the proof.

As in [I] we can iterate Theorem 3.5 to get the nested generalized Bethe Ansatz with levels $k=1,\dots,N-1$. To simplify the notation we introduce as an extension of $I_\pm$ the index sets I_k with $n_k=|I_k|$ elements for $k=0,\dots,N-1$ and as an extension of eqs. (2.1) and (3.15) the spaces

$$V^{(k)^{I_-I_k}}=\bigotimes_{i\in I_-}\bar{V}_i^{(k)}\bigotimes_{i\in I_k}V_i^{(k)} \tag{3.19}$$

with basis vectors $|\,\bar{\alpha}_1\dots\bar{\alpha}_{n_-}\alpha_1\dots\alpha_{n_k}\,\rangle$ $(k<\bar{\alpha}_i,\alpha_i\le N)$. In terms of the previous notation we identify $I_+=I_0$, $n_+=n_0$, $V^{1\dots n}=V^{(0)^{I_-I_0}}$ and $V^{(1)^{\bar{1}\dots\bar{n}_-b_1\dots b_m}}=V^{(1)^{I_-I_1}}$. The Ansatz of level k reads

$$\begin{aligned}
f^{(k-1)^{I_-I_{k-1}}}\left(\underline{x}^{(-)},\underline{x}^{(k-1)}\right) &= \sum_{\underline{x}^{(k)}}B^{(k-1)}_{I_-I_{k-1}\beta_{n_k}}\left(\underline{x}^{(-)},\underline{x}^{(k-1)},x^{(k)}_{n_k}\right)\dots \\
&\quad\dots B^{(k-1)}_{I_-I_{k-1}\beta_1}\left(\underline{x}^{(-)},\underline{x}^{(k-1)},x^{(k)}_1\right)\,g^{(k-1)^{I_-I_{k-1}\beta_1\dots\beta_{n_k}}}\left(\underline{x}^{(-)},\underline{x}^{(k-1)},\underline{x}^{(k)}\right)
\end{aligned} \tag{3.20}$$

where for $k<N-1$ analogously to Definition 3.4

$$\begin{aligned}
g^{(k-1)^{I_-I_{k-1}I_k}}\left(\underline{x}^{(-)},\underline{x}^{(k-1)},\underline{x}^{(k)}\right) &= \prod_{i=1}^{n_{k-1}}\prod_{j=1}^{n_k}\psi(x_i^{(k-1)}-x_j^{(k)})\\
&\quad\times\prod_{1\le i<j\le n_k}\tau(x_i^{(k)}-x_j^{(k)})\,f^{(k)^{I_-I_k}}(\underline{x}^{(-)},\underline{x}^{(k)})\otimes\Omega^{(k-1)^{I_{k-1}}}.
\end{aligned} \tag{3.21}$$

For $k = 0$ with $f^{(0)} = f$ and $\underline{x}^{(0)} = \underline{x}^{(+)}$ we have to replace $\prod_{i=1}^{n_0}$ by $\prod_{i \in I_+}$. The start of the iteration is given by a $1 \leq k_{max} \leq N$ such that all $n_k = 0$ for $k \geq k_{max}$. We have to construct a vector valued function proportional to a fixed vector which fulfills the assumptions of Theorem 3.5. This is given by

Lemma 3.6 *The vector valued functions for $k_{max} < N$*

$$f^{(k_{max}-1)^{I_- I_{k_{max}-1}}} = |\, \bar{N} \ldots \bar{N} k_{max} \ldots k_{max} \,\rangle \tag{3.22}$$

and for $k_{max} = N$

$$f^{(N-1)^{I_- I_{N-1}}}(\underline{x}^{(-)}, \underline{x}^{(N-1)}) = \prod_{i \in I_-} \prod_{j \in I_{N-1}} \bar{\psi}_{N-1}(x_i^{(-)} - x_j^{(N-1)}) \,|\, \bar{N} \ldots \bar{N} N \ldots N \,\rangle \tag{3.23}$$

fulfill the Conditions 3.1. The function $\bar{\psi}_{N-1}(x)$ has to obey the functional equation

$$(1 + d(x))\, \bar{\psi}_{N-1}(x) = \bar{\psi}_{N-1}(x - 2) \tag{3.24}$$

which is solved by

$$\bar{\psi}_{N-1}(x) = \frac{\Gamma\left(\frac{1}{2} + \frac{x}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{N} + \frac{x}{2}\right)}. \tag{3.25}$$

Again the general solution is obtained by multiplication with an arbitrary periodic function with period 2.

Proof: Conditions 3.1 (i) is fulfilled because of the symmetry with respect to the particles and anti-particles among themselves. Conditions 3.1 (ii) follows from the definition (3.3) of the Q-matrix and (2.4) of the R-matrix. For $k_{max} < N$ Conditions 3.1 (ii) follows since the Q-operators act as the unit operator on the state $f^{(k_{max}-1)^{I_- I_{k_{max}-1}}}$. For $k_{max} = N$ we have to take into account annihilation-creation contributions of the anti-particle-particle R-matrix (2.4). For the last index in I_- we have

$$Q^{(N-1)^{I_- I_{N-1}}}(\underline{x}^{(-)}, \underline{x}^{(N-1)}; n_-) = \prod_{j \in I_{N-1}} \left(1 + d(x_j^{(N-1)} - x_{n_-}^{(-)})\right) \mathbf{1}$$

and for the last index in I_{N-1}

$$Q^{(N-1)^{I_- I_{N-1}}}(\underline{x}^{(-)}, \underline{x}^{(N-1)}; n_{N-1}) = \prod_{j \in I_-} \left(1 + d(x_j^{(-)} - x_{n_{N-1}}^{(N-1)} - 2)\right) \mathbf{1}$$

which, together with eq. (3.24) implies the difference equations (ii) for (3.23).

Corollary 3.7 *The system of $U(N)$ matrix difference equations (3.2) is solved by the generalized nested Bethe Ansatz (3.20) with (3.21-3.24).*

4 Weights of generalized $U(N)$ Bethe vectors

The results of this section and also the techniques used are very similar to the corresponding ones of [I]. Therefore we only point out the main differences. By an asymptotic expansion of the R-matrix and the monodromy matrix T (cf. eqs.(2.3) and(2.9)) we get for $u \to \infty$

$$R_{ab}(u) = \mathbf{1}_{ab} - \frac{2}{Nu} P_{ab} + O(u^{-2}) \tag{4.1}$$

$$T_{1\ldots n,a}(\underline{x}, u) = \mathbf{1}_{1\ldots n,a} + \frac{2}{Nu} M_{1\ldots n,a} + O(u^{-2}). \tag{4.2}$$

Explicitly we get from eq. (2.9)

$$M_{1\ldots n,a} = \sum_{i \in I_+} P_{ia} - \sum_{i \in I_-} K_{ia}\,, \tag{4.3}$$

where $I_\pm$ denote the particles and anti-particles, respectively.

In the following we will suppress the indices like $1\ldots n$. In terms of matrix elements in the auxiliary space V_a the generators act on the basis states as

$$M_\alpha^{\alpha'} \,|\, \alpha_1, \ldots, \alpha_i, \ldots, \alpha_n \,\rangle = \left(\sum_{i \in I_+} \delta_{\alpha' \alpha_i} - \sum_{i \in I_-} \delta_{\alpha' \alpha_i} \right) \,|\, \alpha_1, \ldots, \alpha, \ldots, \alpha_n \,\rangle. \tag{4.4}$$

The commutation relations of the M and T which follow from the Yang-Baxter relations are the same as in [I]. Also Lemma 4.1 of [I] holds here for the $U(N)$ case. However, one has to replace $\Omega\, g$ in eq. (4.10) of [I] by g as in BETHE ANSATZ 3.3, and the Q-matrix here is given by the trace of (3.18).

The slight difference to the $SU(N)$ case arises from the action of the Cartan subalgebra; i.e. from the diagonal elements of M which are the weight operators $W_\alpha = M_\alpha^\alpha$. They act on the basis vectors in V as

$$W_\alpha \,|\, \alpha_1, \ldots, \alpha_n \,\rangle = \left(\sum_{i \in I_+} \delta_{\alpha \alpha_i} - \sum_{i \in I_-} \delta_{\alpha \alpha_i} \right) \,|\, \alpha_1, \ldots, \alpha_n \,\rangle \tag{4.5}$$

which follows from eq. (4.4). In particular we have for the Bethe Ansatz reference state g

$$W_1\, g = n_+\, g. \tag{4.6}$$

This means that also Lemma 4.2 of [I] holds here for the $U(N)$ case where, however, in eq. (4.16) of [I] the number n is replaced by n_+.

Theorem 4.1 *Let the vector valued function $f(\underline{x}) \in V$ be given by the BETHE ANSATZ 3.3 fulfilling the conditions of Theorem 3.5, i.e. eqs. (3.16) and (3.17). If in addition $f^{(1)}$ is a highest weight vector and an eigenvector of the weight operators with*

$$W_\alpha^{(1)} f^{(1)} = w_\alpha^{(1)} f^{(1)}, \tag{4.7}$$

then also f is a highest weight vector

$$M_\alpha^{\alpha'} f = 0 \ , \quad (\alpha' > \alpha) \tag{4.8}$$

and an eigenvector of the weight operators

$$W_\alpha f = w_\alpha f \ , \quad w_\alpha = \begin{cases} n_+ - m & \text{for } \alpha = 1 \\ w_\alpha^{(1)} & \text{for } \alpha > 1 \end{cases} \tag{4.9}$$

with

$$w_\alpha \geq w_\beta \ , \quad (1 \leq \alpha < \beta \leq N), \tag{4.10}$$

The proof of this theorem is again parallel to the corresponding one in [I].

The states $f^{(k_{max}-1)}$ of Lemma 3.6 which define the start of the iteration of the nested Bethe Ansatz are obviously highest weight states in $V^{(k_{max}-1)}$ with weight $w_{k_{max}}^{(k_{max}-1)} = n_{k_{max}-1} - n_-$ by eq. (4.5).

Corollary 4.2 *If $f(\underline{x})$ is a solution of the system of $U(N)$ matrix difference equations (3.3)*

$$f(\ldots, x_i + 2, \ldots) = Q(\underline{x}; i)\, f(\ldots, x_i, \ldots) \ , \quad (i = 1, \ldots, n)$$

given by the generalized nested Bethe Ansatz of Corollary 3.7, then f is a highest weight vector with weights

$$w = (w_1, \ldots, w_N) = (n_+ - n_1, n_1 - n_2, \ldots, n_{N-2} - n_{N-1}, n_{N-1} - n_-), \tag{4.11}$$

where n_k is the number of $B^{(k)}$ operators in the Bethe Ansatz of level k. Further non-highest weight solutions of (3.2) are given by

$$f_\alpha^{\alpha'} = M_\alpha^{\alpha'} f \ , \quad (\alpha' < \alpha). \tag{4.12}$$

Note that in contrast to the $SU(N)$ case of [I], here the weights may also be negative.

5 Examples

Similar as in [I] the solutions of the following examples may be multiplied by a scalar function which is periodic in all x's with period 2 and symmetric with respect to the anti-particle and particle variables.

Example 5.1 Let us consider the trivial case that there is no B-operator in each level of the nested Bethe Ansatz, which means that k_{max} of Section 3 is equal to 1. In the language of the conventional Bethe Ansatz for quantum chains this correspond to the "ferro-magnetic vacuum". By Section 4 this means that $f^{1\ldots n}(\underline{x})$ has the weights

$$w = (n_+, 0, \ldots, 0, -n_-).$$

For fixed positions of the particles I_+ and anti-particles I_- by Lemma 3.6 the vector $f^{1\ldots n} \in V_{I_-}^{1\ldots n}$ is given by

$$f^{\alpha_1\ldots\alpha_n}(\underline{x}) = \prod_{i\in I_-} \delta_{\alpha_i \bar{N}} \prod_{i\in I_+} \delta_{\alpha_i 1}.$$

or if the anti-particles are sitting at the first places

$$f^{1\ldots n} = |\, \bar{N} \ldots \bar{N} 1 \ldots 1 \,\rangle.$$

This $f^{1\ldots n}$ is a highest weight vector in V_{ref}.

Example 5.2 For $N > 2$ and $n_+ > 2$ let us take the case where there is one B-operator in the first level of the nested Bethe Ansatz and no B's in higher levels, which means that the k_{max} of Section 3 is equal to 2. By Section 4 this means that the weights are

$$w = (n_+ - 1, 1, 0, \ldots, 0, -n_-).$$

The first level Ansatz reads

$$f^{1\ldots n}(\underline{x}) = \sum_u B_{1\ldots n,\beta}(\underline{x}; u)\; g^{1\ldots n\beta}(\underline{x}; u). \tag{5.1}$$

For fixed positions of the particles I_+ and anti-particles I_-, if the anti-particles are sitting on the left of all particles, the function $g^{1\ldots n\beta}$ is given by

$$g^{I_-I_+\beta}(\underline{x}, u) = \prod_{i\in I_+} \psi(x_i - u)\, {f^{(1)}}^{I_-\beta} \otimes \Omega^{I_+} \tag{5.2}$$

with $\psi(x) = \frac{\Gamma(1-\frac{1}{N}+\frac{x}{2})}{\Gamma(1+\frac{x}{2})}$ (see eq. (3.17)) and by Lemma 3.6

$${f^{(1)}}^{I_-I_1} = |\, \bar{N} \ldots \bar{N} 2 \,\rangle\ , \quad \Omega^{I_+} = |\, 1 \ldots 1 \,\rangle. \tag{5.3}$$

i.e. $f^{(1)}$ is the highest weight vector in $V^{(1)}_{ref}$. Note that the function ψ appears only with respect to the parameter x_i which correspond to the particles. The action of the B-operators in eq. (5.1) can easily be obtained from the definition of the R-matrices (2.3) and (2.4). In particular we consider

Example 5.3 As a simple case of Example 5.2 we take $n_- = 1$ and $n_+ = 2$

$$f^{\bar{1}23}(x,y,z) = \sum_u \psi(y-u)\,\psi(z-u)\Big\{c(y-u)\,b(z-u)\,|\,\bar{N}21\,\rangle + c(z-u)\,|\,\bar{N}12\,\rangle\Big\}.$$

The sum over u can be performed and gives the same result as in Example 5.3 of [I].

Example 5.4 For $N = 2$ let $n_- = n_+$ be $= 1$. In addition to the trivial case of Example 5.1 with no B-operator there is only the possibility of Examples 5.2 and 5.3 with one B-operator. By Section 4 $f^{\bar{1}2}(x,y)$ is an $U(2)$-singlet vector with the weights $w = (0,0)$. For $N = 2$ we must take into account the annihilation-creation contribution, for the action of the B-operator on the reference state as well as for $f^{(1)}$ due to Lemma 3.6:

$$f^{\bar{1}2}(x,y) = \sum_u \bar{\psi}_1(x-u)\,\psi(y-u)\Big\{d(x-u)\,b(y-u)\,|\,\bar{1}1\,\rangle + c(y-u)\,|\,\bar{2}2\,\rangle\Big\}$$

with $\psi(x) = \bar{\psi}_1(x) = \frac{\Gamma(\frac{1}{2}+\frac{x}{2})}{\Gamma(1+\frac{x}{2})}$. Similar as in Example 5.3 the sum over $u = \tilde{u} - 2l,\ (l \in \mathbf{Z})$ can be performed

$$f^{\bar{1}2}(x,y) = \frac{\cos\frac{\pi}{2}(x-y)}{\cos\frac{\pi}{2}(x-\tilde{u})\cos\frac{\pi}{2}(y-\tilde{u})}\,\frac{1}{x-y+1}\Big\{\,|\,\bar{1}1\,\rangle + \,|\,\bar{2}2\,\rangle\Big\}.$$

As a generalization of this formula for arbitrary N we consider

Example 5.5 Let us take for $n_- = n_+ = 1$ the case where there is exactly one B-operator in each level of the nested Bethe Ansatz, which means that the k_{max} of Section 3 is equal to $N-1$. By Section 4 this means the weights are $w = (0,\ldots,0)$, i.e. $f^{12}(x,y)$ is an $U(N)$-singlet. The first level Ansatz reads

$$f^{\bar{1}2}(x,y) = \sum_u B_{\bar{1}2,\beta}(x,y;u)\; g^{\bar{1}2\beta}(x,y;u) \tag{5.4}$$

where $u = \tilde{u} + 2l$, $l \in \mathbf{Z}$ and

$$g^{\bar{1}20}(x,y,u) = \psi(y-u)\,f^{(1)\,\bar{1}0}(x,u)\,\Omega^2. \tag{5.5}$$

The higher level Ansätze are of the same form. One solution is

$$f^{\bar{\alpha}\beta}(x,y) = \frac{1}{x-y+1}\,\delta_{\alpha\beta}. \tag{5.6}$$

Proof: The solution (5.6) is a up to a constant the particular case $k = 0$ of a general formula valid for all levels. This general formula (again up to unimportant constants)

$$f^{(k)\bar{\alpha}\beta}(x,y) = \bar{\psi}_k(x-y)\,\delta_{\alpha\beta}\ , \quad (k < \alpha,\beta \leq N) \tag{5.7}$$

where as an extension of eq. (3.25)

$$\bar{\psi}_k(x) = \frac{\Gamma\left(\frac{1}{2}+\frac{x}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{k}{N}+\frac{x}{2}\right)} \tag{5.8}$$

will be proved inductively. For $k = N-1$ as the start of the iteration formula (5.7) follows from Lemma 3.6. For the other values of k it follows recursively from eqs. (3.20) and (3.21):

$$f^{(k-1)\bar{1}2}(x,y) = \sum_u B^{(k-1)}_{\bar{1}2\beta}(x,y;u)\, g^{(k-1)\bar{1}2\beta}(x,y;u) \tag{5.9}$$

$$\sum_u B^{(k-1)}_{\bar{1}2\beta}(x,y;u)\,|\,\bar{\alpha}k\,\rangle = d(x-u)\,b(y-u)\,|\,\bar{k}k\,\rangle + c(y-u)\,|\,\bar{\alpha}\beta\,\rangle \tag{5.10}$$

$$g^{(k-1)\bar{\alpha}\beta\gamma}(x,y;u) = \psi(y-u)\,f^{(k)\bar{\alpha}\gamma}(x,u)\,\delta_{\beta 2}. \tag{5.11}$$

We calculate the right hand side of eq. (5.9) inserting eqs. (5.10) and (5.11) with $f^{(k)}$ given by (5.7) and with $u = x+2l$

$$\begin{aligned}
&\sum_l \psi(x-u)\,\bar{\psi}_k(y-x)\left\{d(x-u)\,b(y-u)\,(N-k)\,|\,\bar{k}k\,\rangle + c(y-u)\sum_{\alpha=k+1}^{N}|\,\bar{\alpha}\alpha\,\rangle\right\} \\
&= \frac{1}{N}\sum_l\left\{\frac{\Gamma\left(-\frac{1}{N}+l\right)}{\Gamma(l)}\frac{\Gamma\left(-\frac{1}{2}+\frac{x-y}{2}+l\right)}{\Gamma\left(\frac{3}{2}-\frac{k}{N}+\frac{x-y}{2}+l\right)}(N-k)\,|\,\bar{k}k\,\rangle\right. \\
&\qquad \left. + \frac{\Gamma\left(-\frac{1}{N}+l\right)}{\Gamma(1+l)}\frac{\Gamma\left(\frac{1}{2}+\frac{x-y}{2}+l\right)}{\Gamma\left(\frac{3}{2}-\frac{k}{N}+\frac{x-y}{2}+l\right)}\sum_{\alpha=k+1}^{N}|\,\bar{\alpha}\alpha\,\rangle\right\} \\
&= const.\,\bar{\psi}_{k-1}(x-y)\sum_{\alpha=k}^{N}|\,\bar{\alpha}\alpha\,\rangle = const.\,f^{(k-1)\bar{1}2}(x,y)
\end{aligned} \tag{5.12}$$

The sums over l have been performed using the Gauss formula

$$\sum_l \frac{\Gamma(a+l)\Gamma(b+l)}{\Gamma(l)\Gamma(c+l)} = \frac{\Gamma(c-a-b-1)\Gamma(1+a)\Gamma(1+b)}{\Gamma(c-a)\Gamma(c-b)}$$

This concludes the proof of formula 5.7.

Example 5.6 The last example will be used in [2] to calculate the exact three particle form factor of the fundamental field in the chiral Gross-Neveu model. We take $n_ =$

1, $n_+ = 2$ and again exactly one B-operator in each level of the nested Bethe Ansatz. By Section 4 this means that $f^{\bar{1}23}(x,y,z)$ is an $U(N)$-vector with weights

$$w = (1, 0, \ldots, 0).$$

The first level Ansatz reads

$$f^{\bar{1}23}(x,y,z) = \sum_u B_{\bar{1}23,\beta}(x,y,z;u)\, g^{\bar{1}23\beta}(x,y,z;u) \tag{5.13}$$

The higher level ($k > 1$) Bethe Ansatz coincides with that of Example 5.5. A solution of the difference equations for this case is given by

$$\begin{aligned} f^{\bar{1}23}(x,y,z) \;=\; & \sum_u \left\{ \frac{\Gamma\left(-\frac{1}{N}+\frac{x-u}{2}\right)}{\Gamma\left(\frac{x-u}{2}\right)} \frac{\Gamma\left(-\frac{1}{N}+\frac{y-u}{2}\right)}{\Gamma\left(\frac{y-u}{2}\right)} \frac{\Gamma\left(-\frac{1}{2}+\frac{z-u}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{N}+\frac{z-u}{2}\right)} (N-1)\,|\,\bar{1}11\,\rangle \right. \\ & - \frac{\Gamma\left(-\frac{1}{N}+\frac{x-u}{2}\right)}{\Gamma\left(\frac{x-u}{2}\right)} \frac{\Gamma\left(-\frac{1}{N}+\frac{y-u}{2}\right)}{\Gamma\left(1+\frac{y-u}{2}\right)} \frac{\Gamma\left(\frac{1}{2}+\frac{z-u}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{N}+\frac{z-u}{2}\right)} \sum_{\alpha=2}^{N} |\,\bar{\alpha}\alpha 1\,\rangle \\ & \left. - \frac{\Gamma\left(-\frac{1}{N}+\frac{x-u}{2}\right)}{\Gamma\left(1+\frac{x-u}{2}\right)} \frac{\Gamma\left(1-\frac{1}{N}+\frac{y-u}{2}\right)}{\Gamma\left(1+\frac{y-u}{2}\right)} \frac{\Gamma\left(\frac{1}{2}+\frac{z-u}{2}\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{N}+\frac{z-u}{2}\right)} \sum_{\alpha=2}^{N} |\,\bar{\alpha}1\alpha\,\rangle \right\} \end{aligned} \tag{5.14}$$

where $u = \tilde{u} + 2l$, $l \in \mathbf{Z}$.

Proof: Analogously to eqs. (5.10) and (5.11) we have for ($k = 1$)

$$\begin{aligned} B_{\bar{1}23,\beta}(x,y,z;u)\,|\,\bar{\alpha}11\,\rangle \;=\; & b(x-u)\,d(y-u)\,b(z-u)\,\delta_{\alpha\beta}\,|\,\bar{1}11\,\rangle \\ & + c(y-u)\,b(z-u)\,|\,\bar{\alpha}\beta 1\,\rangle \\ & + c(z-u)\,|\,\bar{\alpha}1\beta\,\rangle \end{aligned}$$

$$g^{\bar{\alpha}\beta}(x,y,z;u) = \psi(y-u)\psi(z-u)\, f^{(1)\,\bar{\alpha}\beta}(x,u)$$

where $f^{(1)\,\bar{\alpha}\beta}(x,u)$ is the same function as in Example 5.5 and given by eq. (5.7) for $k = 1$. Inserting this into eq. (5.13) we get in analogy to eq. (5.12) the result (5.14).

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