RELATION BETWEEN MORSE THEORY OF CLOSED GEODESICS OF S^1 AND SYMPLECTIC FLOER THEORY OF T^*S^1

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ABSTRACT. We construct Bott-type Floer homology groups for the symplectic manifold (T^*S^1, Ω_{can}) and Bott-type Morse homology groups for the energy functional on the loop space of S^1. Both objects turn out to be isomorpic to the singular homology of the loop space of S^1. So far our objects depend on all choices involved, but the above isomorphism suggests further investigation to show independence of these choices as well as a generalization to any closed riemannian manifold.

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1. INTRODUCTION

We are going to use the set of critical points of the perturbed symplectic action functional

$$\begin{aligned} \mathcal{A}_{H_0} \;:\; \Lambda T^*S^1 &\;\to\; \mathbb{R} \\ x &\;\mapsto\; \int_{\mathbb{R}/\mathbb{Z}} x^*\Theta - \int_0^1 H_0(x(t))\, dt\ , \end{aligned} \tag{1}$$

to construct an algebraic chain complex whose homology represents the singular homology of the loop space of S^1 (and therefore also of T^*S^1). In (1)

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Θ is the Liouville form on T^*S^1, the hamiltonian $H_0 : T^*S^1 \to \mathbb{R}$ is given by $(q,p) \mapsto \frac{1}{2}g^*(p,p)$, g is the round metric of $S^1 \hookrightarrow \mathbb{C}$, g^* its dual metric. In our construction of the chain complex we follow the ideas in [RT] and [AB]. As H_0 is time independent, the critical points of $\mathcal{A}_{H_0}$ are never isolated, but come in 1-dimensional families diffeomorphic to $\mathbb{R}/\mathbb{Z}$ (as we have the action $\mathbb{R}/\mathbb{Z} \times Crit\, \mathcal{A}_{H_0} \to Crit\, \mathcal{A}_{H_0}$, $(\tau, x(\,\cdot\,)) \mapsto x(\tau + \cdot))$. Therefore usual Floer theory does not apply. We have to take into account the singular [RT] or de Rham [AB] (co)homology of $Crit\, \mathcal{A}_{H_0}$ (the set of critical points of $\mathcal{A}_{H_0}$).

In general the L^2-gradient "flow" lines of $\mathcal{A}_{H_0}$ between different connected components A, B of $Crit\, \mathcal{A}_{H_0}$ will enter the construction, but here they cannot exist as A and B lie in different connected components of ΛT^*S^1. It turns out

$$HF_k^{Bott}(T^*S^1, H_0, g; \mathbb{Z}_2) \quad = \quad \begin{cases} (\mathbb{Z}_2)^{\mathbb{Z}} & , \ k = 0, 1 \ , \\ 0 & , \ k \in \mathbb{Z} \setminus \{0,1\} \ . \end{cases}$$

k corresponds to a generalized Conley-Zehnder index, which we have to assign to any element of $Crit\, \mathcal{A}_{H_0}$.

Note that perturbing H_0 by a time dependent function (potential) on S^1, we could get $Crit\, \mathcal{A}_{H_0}$ to consist of isolated critical points and apply usual Floer theory (this is the approach we follow in [W97] in the general case).

It is known that the hamiltonian flow of H_0 gives rise to geodesics of (S^1, g) when projected to the base S^1. Moreover, $Crit\, \mathcal{A}_{H_0}$ corresponds exactly to the set of closed geodesics of (S^1, g) (which in turn may be interpreted as the elements of $Crit\, E_0$). We are going to construct an algebraic chain complex analogous as before, where we replace $\mathcal{A}_{H_0}$ by the energy functional

$$\begin{aligned} E_0 \ : \ \Lambda S^1 \quad &\to \quad \mathbb{R} \\ \gamma \quad &\mapsto \quad \tfrac{1}{2} \int_0^1 g(\dot{\gamma}(t), \dot{\gamma}(t))\, dt \ . \end{aligned} \tag{2}$$

The generalized Conley-Zehnder index is replaced by the Morse index. The resulting Bott-type Morse homology groups $HM_k^{Bott}(S^1, E_0, g; \mathbb{Z}_2)$ turn out to be isomorphic to the Floer ones.

Moreover we compute $H_*^{sing}(\Lambda S^1; \mathbb{Z}_2)$ by the procedure suggested in [Kl]; they are also isomorphic to the former ones.

The above mentioned perturbation of H_0 by the time dependent potential V would have as a consequence that the critical points of E_V couldn't be interpreted as closed geodesics any more. However, the corresponding homology groups should be the same. It would be interesting to do the perturbation in our case explicitly. One expects that the connected components of $Crit\, \mathcal{A}_{H_0}$ resp. $Crit\, E_0$, which are diffeomorphic to S^1, split in a number of isolated critical points (depending on the perturbation) of indices k and $k-1$ resp. 1 and 0, such that the corresponding local homology groups are isomorphic to $H_*^{sing}(S^1; \mathbb{Z}_2)$. This will be the subject of further investigation. In what follows we will denote E_0 by E.

2. Bott-type Morse theory for E on ΛS^1

Let S^1 be embedded in $\mathbb{C}$ as the unit circle, $i : S^1 \hookrightarrow \mathbb{C}$. By g we denote the riemannian metric on S^1 inherited from the hermitean inner product $< \cdot, \cdot >_{\mathbb{C}}$. With respect to the natural parametrization $\mathbb{R}/\mathbb{Z} \to S^1$, $u^1 \mapsto e^{2\pi i u^1}$, g is given by $g = g_{11} \cdot du^1 \otimes du^1 = 4\pi^2 du^1 \otimes du^1$ and the volume element by $vol_g = 2\pi \, du^1$. Note that any smooth metric on S^1 is of the form $g_f = f^2 \cdot g$, where $f \in C^\infty(S^1, \mathbb{R} \setminus \{0\})$.

The loop space $\Lambda S^1 = H^{1,2}(\mathbb{R}/\mathbb{Z}, S^1)$ is defined to be the completion of $C^\infty(\mathbb{R}/\mathbb{Z}, S^1)$ with respect to the norm $\| \cdot \|_{1,2}$ induced by the inner product

$$(\gamma_1, \gamma_2)_{1,2} = \int_0^1 < \gamma_1(t) \cdot \gamma_2(t) >_{\mathbb{C}} \; dt \; + \; \int_0^1 < \dot\gamma_1(t) \cdot \dot\gamma_2(t) >_{\mathbb{C}} \; dt,$$

where we parametrized the domain of γ_i by $t \in [0,1]$ and assume its range to be $\mathbb{C}$. It is known that ΛS^1 embedds into $C^0(\mathbb{R}/\mathbb{Z}, S^1)$ (Sobolev embedding theorem).

Lemma 2.1. *(cf.* [Jo] *Se. 5.4) The energy functional E is continuously differentiable.*

Lemma 2.2. *(cf.* [Jo] *Lemma 7.2.1 & Se. 5.4,* [Kl] *Thm. 1.3.11) The critical points of E in ΛS^1 are exactly the closed geodesics of (S^1, g) ,i.e. $Crit\, E = Geod(S^1, g)$.*

The Christoffel symbols of g vanish due to the constancy of $g_{11} = 4\pi^2$ in u^1. Therefore $\gamma \in C^\infty(\mathbb{R}/\mathbb{Z}, S^1)$ is a closed geodesic of (S^1, g) , if and only if

$$\tfrac{d^2}{dt^2} \gamma^1(t) = 0, \tag{3}$$

where γ^1 is the local representative of γ under the diffeomorphism $S^1 \backslash \{1\} \to [0,1]$, $e^{iu^1} \mapsto u^1/2\pi$. An element $\gamma^1 \in C^\infty(\mathbb{R}, \mathbb{R})$ is a solution of (3) if and only if

$$\gamma^1(t) = \gamma^1_{v_0,t_0}(t) = v_0 t + t_0.$$

The condition $\gamma^1_{v_0,t_0}(t+1) = \gamma^1_{v_0,t_0}(t) \pmod 1$ implies $v_0 = k \in \mathbb{Z}$, therefore

$$Crit\, E = Geod(S^1, g) = \{\gamma^1_{k,t_0}(t) = kt + t_0 \pmod 1 | k \in \mathbb{Z}, t_0 \in \mathbb{R}/\mathbb{Z}\}.$$

Now we are going to grade $Crit\, E$ by the Morse index of a closed geodesic.

Lemma 2.3. *(cf.* [Jo] *Thm. 4.1.1 & Se. 5.4) Let $\gamma \in Crit\, E$ and $X, Y \in \Gamma(\gamma^* TS^1)$ (i.e. smooth vector fields along γ), then*

$$D^2 E(\gamma) \; (X, Y) = \int_0^1 g(\dot X(t), \dot Y(t) > \; dt \; .$$

The kernel of the symmetric operator

$$\begin{aligned} L : L^2(\gamma^* TS^1) \supset H^{2,2}(\gamma^* TS^1) \;&\to\; L^2(\gamma^* TS^1) \\ X \;&\mapsto\; -\tfrac{d^2}{dt^2} X \end{aligned}$$

consists of smooth sections and equals the kernel of the symmetric bilinear form $D^2E(\gamma)$. $D^2E(\gamma)$ is negative definite on eigenspaces of L associated to negative eigenvalues.

We are going to investigate $ker\ L$: Let $\gamma \in Crit\ E$, i.e. $\gamma^1(t) = kt + t_0$, and $X \in \Gamma(\gamma^* TS^1) = C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, then the solution of

$$- \tfrac{d^2}{dt^2} X(t) = 0 \tag{4}$$

is in $C^\infty(\mathbb{R}, \mathbb{R})$ given by $X(t) = v_0 t + x_0$, $v_0, x_0 \in \mathbb{R}$. $v_0(t+1) + x_0 = X(t+1) = X(t) = v_0 t + x_0\ \forall t \in \mathbb{R}/\mathbb{Z}$ implies $v_0 = 0$, hence $X \equiv x_0, x_0 \in \mathbb{R}$, satisfies (4). Hence $Ker\ L \simeq \mathbb{R}$ and $dim\ Ker\ L = 1$.

Remark 2.4. Note that $X(t)$ and $\dot\gamma(t)$ are linearly dependent. In the general case of a Riemannian manifold (M, g) we have: γ closed geodesic $\Rightarrow \dot\gamma \in Ker\ L$. Therefore the kernel of the Hessian $D^2E(\gamma)$ is always at least 1-dimensional. (E is invariant under the action $S^1 \times \Lambda M \to \Lambda M$, $(\tau, \gamma(\cdot)) \mapsto \gamma(\tau + \cdot)$).

Now we are interested in the negative eigenvalues of L: These do not exist, because the operator $-d^2/dt^2$ is positive semidefinite on $C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$. The reason is the periodicity of the domain $\mathbb{R}/\mathbb{Z}$. To see this Fourier decompose $X(t)$ and apply $-d^2/dt^2$ to each summand. In general this follows by partial integration and the closedness of the manifold.

Definition 2.5. Let $\gamma \in Crit\, E$, then we define the **Morse index** and the **nullity** of γ

$$\begin{aligned} \mathbf{Ind_E}(\gamma) &= \#\{\text{negative eigenvalues of } L, \text{ counted with multiplicity}\}\ , \\ \mathbf{Null_E}(\gamma) &= dim\ ker\ L. \end{aligned}$$

In our case $Ind_E(\gamma) = 0$ and $Null_E(\gamma) = 1$ for all $\gamma \in Crit\ E$, i.e.

$$Crit\ E = Crit_0 E,$$

where $Crit_k E = \{\gamma \in Crit\ E | Ind_E(\gamma) = k\}$.

Lemma 2.6. *For (S^1, g) as above we have*

$$Crit_0 E = \bigsqcup_{k \in \mathbb{Z}} G_0^k\ ,\ G_0^k = \{\gamma_{k,t_0}(t) = e^{i2\pi k(t+t_0)} | t_0 \in \mathbb{R}/\mathbb{Z}\}\ ,$$

i.e. G_0^k is a submanifold of ΛS^1 diffeomorphic to S^1, called **critical submanifold**. *G_0^k is* **nondegenerate in the sense of Bott**, *i.e. $D^2E(\gamma)$ restricted to the normal bundle $N_\gamma G_0^k$ is nondegenerate for any $\gamma \in G_0^k$.*

Proof. Pick $\gamma \in G_0^k \subset Crit_0 E$, then the map $S^1 \to G_0^k$, $\tau \mapsto \gamma(\tau + \cdot)$, is a diffeomorphism (note that $G_0^k \cap G_0^l = \emptyset$, if $k \neq l$). As the kernel of $D^2E(\gamma)$ equals

$$T_\gamma G_0^k = \{X \in \Gamma(\gamma^* TS^1) | X \equiv x_0, x_0 \in \mathbb{R}\}\ ,$$

$D^2E(\gamma)$ restricted to $N_\gamma G_0^k$ is nondegenerate. $\square$

The critical submanifolds generate an algebraic chain complex as follows: The k^{th} **Bott-type Morse chain group** is defined to be the k^{th} singular chain group of $Crit_0 E$ with coefficients in $\mathbb{Z}_2$

$$\mathbf{CM_k^{Bott}(S^1, E, g; \mathbb{Z}_2)} = \begin{cases} C_k\left(\bigsqcup_{l\in\mathbb{Z}} G_0^l; \mathbb{Z}_2\right) & , k \in \mathbb{N}_0 , \\ 0 & , k \in \mathbb{Z}\setminus\mathbb{N}_0 . \end{cases}$$

If $m \neq n$, then G_0^m and G_0^n lie in different connected components of ΛS^1 (as the degree of the elements of G_0^m resp. G_0^n are m resp. n). Hence we cannot expect to have any connecting orbit (in the sense of Morse/Floer theory) between G_0^m and G_0^n. Therefore we define the **Bott-type Morse boundary operator** d_k^M just to be the singular boundary operator ∂_k^{sing} on the singular chains of our critical manifolds

$$\begin{aligned} d_k^M : CM_k^{Bott}(S^1, E, g; \mathbb{Z}_2) &\to CM_{k-1}^{Bott}(S^1, E, g; \mathbb{Z}_2) \ , \ k \in \mathbb{N}_0 \ , \\ x &\mapsto \partial_k^{sing} x \ , \end{aligned}$$

$d_k^M = 0 \ \forall k \in \mathbb{Z}\setminus\mathbb{N}_0$. Trivially $\left(CM_k^{Bott}(S^1, E, g; \mathbb{Z}_2), d_k^M\right)_{k\in\mathbb{Z}}$ is an algebraic chain complex, i.e. $d_{k-1}^M \circ d_k^M = 0$, $\forall k \in \mathbb{Z}$, so we may define the **Bott-type Morse homology groups**

$$\mathbf{HM_k^{Bott}(S^1, E, g; \mathbb{Z}_2)} = \frac{ker\ d_k^M}{im\ d_{k+1}^M}, \ k \in \mathbb{Z} \ .$$

They may be computed as follows

$$\begin{aligned} HM_k^{Bott}(S^1, E, g; \mathbb{Z}_2) &= H_k^{sing}(\textstyle\bigsqcup_{l\in\mathbb{Z}} G_0^l; \mathbb{Z}_2) \\ &= H_k^{sing}(\textstyle\bigsqcup_{l\in\mathbb{Z}} S^1; \mathbb{Z}_2) \\ &= \begin{cases} (\mathbb{Z}_2)^{\mathbb{Z}} & , \ k = 0, 1 \ , \\ 0 & , \ \text{else} \ . \end{cases} \end{aligned}$$

3. The singular homology of ΛS^1

By [Kl] Thm. 1.2.10 the inclusion $\Lambda S^1 = H^{1,2}(\mathbb{R}/\mathbb{Z}, S^1) \hookrightarrow C^0(\mathbb{R}/\mathbb{Z}, S^1)$ is a homotopy equivalence, hence

$$H_*^{sing}(\Lambda S^1; \mathbb{Z}_2) \simeq H_*^{sing}(C^0(\mathbb{R}/\mathbb{Z}, S^1); \mathbb{Z}_2) \ .$$

The set $\pi_1(S^1)$ of free homotopy classes of continuous maps equals $\mathbb{Z}$. A homotopy F_s, $s \in [0,1]$, between $f, g \in C^0(\mathbb{R}/\mathbb{Z}, S^1)$, may be considered as path in $C^0(\mathbb{R}/\mathbb{Z}, S^1)$ from f to g. Hence the homotopy classes correspond exactly to the pathwise connected components of $C^0(\mathbb{R}/\mathbb{Z}, S^1)$. On the other hand to any pathwise connected component of $C^0(\mathbb{R}/\mathbb{Z}, S^1)$ there corresponds exactly one generator of $H_0^{sing}(C^0(\mathbb{R}/\mathbb{Z}, S^1); \mathbb{Z}_2)$, thereby we have shown

$$H_0^{sing}(\Lambda S^1; \mathbb{Z}_2) \simeq H_0^{sing}(C^0(\mathbb{R}/\mathbb{Z}, S^1); \mathbb{Z}_2) = (\mathbb{Z}_2)^{\pi_1(S^1)} = (\mathbb{Z}_2)^{\mathbb{Z}} \ .$$

We are going to compute the higher homology groups via Morse theory of the energy functional E on the loop space ΛS^1. The critical submanifolds of ΛS^1 w.r.t. E are nondegenerate in the sense of Bott and they have Morse index 0. They are diffeomorphic to S^1 and given by

$$R_0^k = \{q_{k,q_0}(t) = e^{i2\pi k(t+t_0)} | t_0 \in \mathbb{R}/\mathbb{Z}\} \ , \ k \in \mathbb{Z} \ .$$

R_0^0 corresponds to the trivial (constant) geodesics, moreover

$$E|_{R_0^k} \equiv 2\pi^2 k^2 \ .$$

Pick $a \in \mathbb{R}$ and let

$$\begin{aligned} \Lambda^a S^1 &= \{c \in \Lambda S^1 \mid E(c) \leq a\} \\ \Lambda^{a-} S^1 &= \{c \in \Lambda S^1 \mid E(c) < a\} \ . \end{aligned}$$

Theorem 3.1. *([Kl], Thm. 2.4.10) Let (M,g) be a compact Riemannian manifold and assume that the set of critical points of E in $E^{-1}(a)$ is a nondegenerate critical submanifold B of ΛM. Then there exists $\epsilon > 0$, such that $\Lambda^{a+\epsilon}M$ is (equivariantly) diffeomorphic to $\Lambda^{a-\epsilon}M$ with handle bundle of type $D\mu^- \oplus D\mu^+$ attached. μ denotes the normal bundle of B and $\mu = \mu^- \oplus \mu^+$ is the decomposition in the negative and positive subbundle (w.r.t. the Hessian of E).*

In our case all critical submanifolds have Morse index 0, this implies that for $\epsilon > 0$ sufficiently small

$$\Lambda^{2\pi^2 k^2 + \epsilon} S^1 \simeq \bigsqcup_{l=-k}^{k} D\mu_l^+ \ .$$

The bundle $D\mu_l^+ \to R_0^l$ is contractible onto R_0^l, hence

$$\begin{aligned} HM_*^{sing}(\Lambda^{2\pi^2 k^2+\epsilon} S^1; \mathbb{Z}_2) &\simeq H_*^{sing}(\textstyle\bigsqcup_{l=-k}^{k} S^1; \mathbb{Z}_2) \\ &= \begin{cases} (\mathbb{Z}_2)^{2k+1} & , * = 0,1 \ , \\ 0 & , \text{else} \ . \end{cases} \end{aligned}$$

The limit $k \to \infty$ yields

$$HM_*^{sing}(\Lambda S^1; \mathbb{Z}_2) \simeq \begin{cases} (\mathbb{Z}_2)^{\mathbb{Z}} & , * = 0,1 \ , \\ 0 & , \text{else} \ . \end{cases}$$

4. Bott-type Floer homology for $(T^*S^1, \Omega = -d\Theta)$

Let (S^1, g) be as above. On T^*S^1 we introduce natural coordinates (u^1, v_1), natural in the sense that the fibre coordinate v_1 is induced by the parametrization $\psi : S^1 \to \mathbb{R}/\mathbb{Z}$, $q = e^{2\pi i u^1} \mapsto u^1$, of the base manifold. v_1 is determined by $v_1 e^1 = d\psi(q)^{*^{-1}} p$ for any element $(q,p) \in T^*S^1$ (e^1 denotes the basis of $\mathbb{R}^*$ dual to e_1). g gives rise to the Levi-Civita connection, whose Christoffel symbols with respect to the chosen parametrization vanish ($g_{11} = 4\pi^2 = const$).

On TT^*S^1 we introduce coordinates $(u^1, v_1, \xi^1, \eta_1)$ defined by

$$\begin{array}{rcl} TT\psi(q,p) \; : \; T_{q,p}T^*S^1 & \to & \mathbb{R}/\mathbb{Z} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \\ \zeta & \mapsto & (\psi(q), d\psi(q)^{*^{-1}} p, d(\psi(q), d\psi(q)^{*^{-1}} p)), \end{array} \tag{5}$$

i.e. $TT\psi(q,p)\zeta = (u^1, v_1, \xi^1, \eta_1)$. Note that ξ^1 (in general) transforms as the coordinate function of a vector under coordinate changes and has therefore an intrinsic meaning (representing an element of T_qS^1); η_1 in general does not, but here – due to $\Gamma^1_{11} = 0$ – η_1 in fact transforms as the coordinate function of a covector and may therefore be interpreted as an element of $T^*_qS^1$.

The connection map $K : TT^*S^1 \to T^*S^1$ (cf. [Kl] ,se. 1.1) is represented by

$$(u^1, v_1, \xi^1, \eta_1) \mapsto (u^1, \eta_1), \tag{6}$$

the projection $\tau^*_{S^1} : T^*S^1 \to S^1$ by

$$(u^1, v_1) \mapsto (u^1),$$

and its tangential map $T\tau^*_{S^1} : TT^*S^1 \to TS^1$ by

$$(u^1, v_1, \xi^1, \eta_1) \mapsto (u^1, \xi^1).$$

Now K and $T\tau^*_{S^1}$ induce a splitting of TT^*S^1

$$TT^*S^1 = T^hT^*S^1 \oplus T^vT^*S^1 = Ker\ K \oplus Ker\ T\tau^*_{S^1}. \tag{7}$$

We have the natural isomorphism

$$(T\tau^*_{S^1}(q,p) \oplus K(q,p)) \; : \; T^h_{(q,p)}T^*S^1 \oplus T^v_{(q,p)}T^*S^1 \xrightarrow{\simeq} T_qS^1 \oplus T^*_qS^1, \tag{8}$$

i.e. locally $((\xi^1, 0), (0, \eta_1)) \mapsto (\xi^1, \eta_1)$. Note that this isomorphism reflects the fact mentioned below equation (5) that ξ^1 resp. η_1 are coordinate functions of an element of T_qS^1 resp. $T^*_qS^1$ with respect to the natural bases $\{\partial_{u^1}\}$ resp. $\{du^1\}$.

Now we define the loop space $\Lambda T^*S^1 = H^{1,2}(\mathbb{R}/\mathbb{Z}, T^*S^1)$ to be the completion of $C^\infty(\mathbb{R}/\mathbb{Z}, T^*S^1)$ with respect to the norm $\|\cdot\|_{1,2}$ induced by the inner product

$$\begin{array}{rcl} (\gamma, \tilde{\gamma})_{1,2} & = & \displaystyle\int_0^1 < q(t), \tilde{q}(t) >_{\mathbb{C}} \, dt \; + \; \int_0^1 v_1(t) \cdot \tilde{v}_1(t) \, dt \\ & + & \displaystyle\int_0^1 < \dot{q}(t), \dot{\tilde{q}}(t) >_{\mathbb{C}} \, dt \; + \; \int_0^1 \dot{v}_1(t) \cdot \dot{\tilde{v}}_1(t) \, dt. \end{array}$$

Here we represent $\gamma : \mathbb{R}/\mathbb{Z} \to T^*S^1$ as $(q(\cdot), v_1(\cdot))$, where $q : \mathbb{R}/\mathbb{Z} \to S^1 \subset \mathbb{C}$ and $v_1(t)$ is the natural coordinate function of the fibre part of $\gamma(t)$ as above.

The **Liouville form** $\Theta : TT^*S^1 \to \mathbb{R}$ is defined by

$$\Theta(\zeta) = (\tau_{T^*S^1}\zeta)\,(T\tau^*_{S^1}\zeta),$$

i.e. in our local coordinates $(u^1, v_1, \xi^1, \eta_1)$

$$\Theta\mid_{(u^1,v_1)}(\xi^1,\eta_1) = (v_1 du^1)\,(\xi^1\partial_{u^1}) = v_1\cdot\xi^1,$$

$$\Theta\mid_{(u^1,v_1)}(\cdot) = (v_1 du^1).$$

The **canonical symplectic form** on T^*S^1 is $\Omega = -d\Theta$, i.e. in our natural coordinates

$$\Omega\mid_{(u^1,v_1)}(\cdot,\cdot) = (du^1\wedge dv_1)\,(\cdot,\cdot).$$

Therefore Ω is represented by the standard symplectic form on $\mathbb{R}^2$

$$\begin{aligned} \Omega_0\ :\mathbb{R}^2\times\mathbb{R}^2 &\to \mathbb{R} \\ (x,\tilde{x}) &\mapsto (J_0x)^T\tilde{x}, \end{aligned} \tag{9}$$

where

$$J_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

is the standard complex structure on $\mathbb{R}^2$. Note that under the natural isomorphism (8) Ω is taken to the natural symplectic form of the direct sum of a vector space and its dual space:

$$\begin{aligned} \tilde{\Omega}\ :\ (T_qS^1\oplus T_q^*S^1)\times(T_qS^1\oplus T_q^*S^1) &\to \mathbb{R} \\ \left(\begin{pmatrix}\xi\\ \eta\end{pmatrix},\begin{pmatrix}\tilde{\xi}\\ \tilde{\eta}\end{pmatrix}\right) &\mapsto \tilde{\eta}(\xi)-\eta(\tilde{\xi}). \end{aligned}$$

Now consider a time-1-periodic Hamiltonian $H\in C^\infty(\mathbb{R}/\mathbb{Z}\times T^*S^1,\mathbb{R})$ of the form

$$H(t,u^1,v_1) = \frac{1}{2}g^{11}(u^1v_1v_1 + V(t,u^1),$$

where $V\in C^\infty(\mathbb{R}/\mathbb{Z}\times S^1,\mathbb{R})$. With H we can associate the **hamiltonian vector field** X_H by requiring

$$dH(\cdot) = \Omega(X_H,\cdot). \tag{10}$$

We are interested in the critical points of the action functional $\mathcal{A}_H$ (1) because they are related to the time-1-periodic integral curves of the hamiltonian vector field X_H as follows

$$Crit\,\mathcal{A}_H = \{x\in C^\infty(\mathbb{R}/\mathbb{Z},T^*S^1)\mid \dot{x} = X_H(x)\} \stackrel{\text{def}}{=} \mathcal{P}er\ H\,.$$

Now we restrict to the case $V\equiv 0$ and denote the corresponding Hamiltonian by H_0. In this case we cannot expect $Crit\,\mathcal{A}_{H_0}$ to consist of isolated points, because we have the action

$$\begin{aligned} \mathbb{R}/\mathbb{Z}\times Crit\,\mathcal{A}_{H_0} &\to Crit\,\mathcal{A}_{H_0}, \\ (\tau,x(\cdot)) &\mapsto x(\tau+\cdot). \end{aligned}$$

The differential of $H_0(u^1,v_1) = \frac{1}{2}\left(\frac{1}{2\pi}\right)^2 v_1v_1$ is given by

$$dH_0(u^1,v_1) = \frac{1}{(2\pi)^2}v_1dv_1,$$

i.e. $dH_0(u^1,v_1)$ is represented by the row vector

$$dH_0(u^1,v_1)=\frac{1}{(2\pi)^2}(0,v_1). \tag{11}$$

The hamiltonian vector field $X_{H_0}:T^*S^1\to TT^*S^1$ is computed via its defining equation (10): The lhs has been just calculated and the rhs will be determined by using the *Ansatz*

$$X_{H_0}(u^1,v_1)=(u^1,v_1;r(u^1,v_1)\partial_{u^1}+s(u^1,v_1)\partial_{v_1})\ .$$

By (9) and (10) $\Omega_{(u^1,v_1)}(X_{H_0}(u^1,v_1),\cdot)$ is represented by

$$(J_0X_{H_0}(u^1,v_1))^T=(-s(u^1,v_1),r(u^1,v_1))\ .$$

Comparison of lhs and rhs gives

$$s(u^1,v_1)=0\ ,\ r(u^1,v_1)=\frac{v_1}{(2\pi)^2}\ ,$$

and therefore $X_{H_0}(u^1,v_1)$ is represented by

$$\frac{1}{(2\pi)^2}\begin{pmatrix}v_1\\0\end{pmatrix}\ .$$

In appendix A we present another method to compute X_{H_0}.

The time-1-periodic trajectories of the hamiltonian vector field X_{H_0} are exactly the solutions to the initial value problem

$$\begin{cases}\begin{pmatrix}\dot u^1(t)\\ \dot v_1(t)\end{pmatrix}=\frac{1}{(2\pi)^2}\begin{pmatrix}v_1(t)\\0\end{pmatrix} & ,\\ \begin{pmatrix}u^1(0)\\ v_1(0)\end{pmatrix}=\begin{pmatrix}u^{1,0}\\ v_{1,0}\end{pmatrix} & .\end{cases} \tag{12}$$

The Ansatz

$$x(t)=\begin{pmatrix}u^1(t)\\ v_1(t)\end{pmatrix}=\begin{pmatrix}\frac{v_{1,0}}{(2\pi)^2}t+u^{1,0}\\ v_{1,0}\end{pmatrix}$$

solves (12) for $x\in C^\infty(\mathbb{R},\mathbb{R}^2)$. The condition $u^1\in C^\infty(\mathbb{R}/\mathbb{Z},\mathbb{R}/\mathbb{Z})$, i.e. $u^1(t+1)=u^1(t) \pmod 1$, implies $v_{1,0}/(2\pi)^2=k\in\mathbb{Z}$, hence

$$\begin{aligned}\mathcal{C}&=\{x\in C^\infty(\mathbb{R}/\mathbb{Z},T^*S^1)|\dot x=X_{H_0}(x)\} \qquad (13)\\ &\simeq\{x_{u^{1,0},k}(t)=\begin{pmatrix}kt+u^{1,0}\\ (2\pi)^2k\end{pmatrix}\mid k\in\mathbb{Z},u^{1,0}\in\mathbb{R}/\mathbb{Z}\}\ .\end{aligned}$$

Usually in Floer theory one assigns an integer μ_{CZ}, called **Conley-Zehnder index** (cf. [CZ],[SZ]), to any element $x=x_{u^{1,0},k}\in\mathcal{C}$. Let $\phi_t:\mathbb{R}/\mathbb{Z}\times\mathbb{R}\to\mathbb{R}/\mathbb{Z}\times\mathbb{R}$ be the time-t-map associated to X_{H_0}, i.e.

$$\phi_t\begin{pmatrix}u^{1,0}\\ v_{1,0}\end{pmatrix}=x_{u^{1,0},k}(t)=\begin{pmatrix}kt+u^{1,0}\\ (2\pi)^2k\end{pmatrix}\ ,$$

$$A(t)\stackrel{\text{def}}{=}d\phi_t|_{(u^{1,0},v_{1,0})}=\begin{pmatrix}1&\frac{t}{(2\pi)^2}\\0&1\end{pmatrix}\ .$$

Now $A(0) = \mathbf{1}$, but $A(1) \notin Sp^* = \{M \in Sp(2,\mathbb{R}) | 1 \notin spec(M)\}$, hence the usual definition of the Conley-Zehnder index does not apply. On the other hand $A(t)$ is independent of $u^{1,0}$ and $v_{1,0}$, therefore – assume one had defined a generalized Conley-Zehnder index μ^g_{CZ} – this index was the same for any element of $\mathcal{C}$ (in [W96] we compute μ^g_{CZ} according to the method suggested in [RS] to generalize the Maslov index. There will also be a proof by picture. It turns out: $\mu^g_{CZ} = -\frac{1}{2}\, \forall x \in \mathcal{C}$. This observation gives a first hint that there shouldn't exist connecting orbits in the sense of Floer [Fl] between $x, y \in \mathcal{C}, x \neq y$. Here is a precise argument: Any two $x, y \in \mathcal{C}, x \neq y$, lie in two different connected components of ΛT^*S^1 (homotop the fibre part to zero and get the γ_{k,t_0} from section 2).

Grading $\mathcal{C}$ by the generalized Conley-Zehnder index we get

$$\mathcal{C} = \mathcal{C}_{-1/2} = \bigsqcup_{k\in\mathbb{Z}} C^k_{-1/2}\ ,\ S^1 \cong C^k_{-1/2} \subset \Lambda T^*S^1\ ,\ \forall k \in \mathbb{Z}\ .$$

$C^k_{-1/2}$ denotes the k^{th} connected component of generalized Conley-Zehnder index $-1/2$ of the critical manifold of the functional $\mathcal{A}_{H_0}$. Pick $x_{u^{1,0},k} \in C^k_{-1/2}$, then

$$\begin{aligned}
\mathcal{A}_{H_0}(x_{u^{1,0},k}) &= \int_{\mathbb{R}/\mathbb{Z}} {x_{u^{1,0},k}}^*\Theta - \int_0^1 H_0(x_{u^{1,0},k}(t)\, dt \\
&= \int_0^1 \Theta(\dot{x}_{u^{1,0},k}(t))\, dt - \int_0^1 \tfrac{1}{2}\tfrac{1}{(2\pi)^2} \cdot (2\pi)^4 k^2\, dt \\
&= \int_0^1 \underbrace{pr_2 \circ x_{u^{1,0},k}(t)}_{=v_{1,0}=(2\pi)^2 k} \cdot \underbrace{pr_1 \circ \dot{x}_{u^{1,0},k})}_{=\frac{v_{1,0}}{(2\pi)^2}=k}\, dt \\
&\quad - \int_0^1 2\pi^2 k^2\, dt \\
&= 2\pi^2 k^2\ ,
\end{aligned}$$

i.e. $\mathcal{A}_{H_0}|_{C^k_{-1/2}} \equiv 2\pi^2 k^2$. Because of the time independence of H_0, each $x \in \mathcal{C}$ gives rise to an orbit under the action $\mathbb{R}/\mathbb{Z} \times \mathcal{C} \to \mathcal{C}, (\tau, x(\cdot)) \mapsto x(\tau + \cdot)$, diffeomorphic to $\mathbb{R}/\mathbb{Z}$. $\mathcal{A}_{H_0}$ is invariant under this action, i.e. the orbit generated by $x \in \mathcal{C}$ is a connected component of a critical manifold of $\mathcal{A}_{H_0}$.

Remark 4.1. The Hamiltonian H_0 is of **Bott-type** in the sense of ([RT], def. 1.1), i.e. i) the fixed point set $\mathcal{F}$ of the time-1-map ϕ_1 is a submanifold of T^*S^1 and ii) if $x \in \mathcal{F}$ and $1 \in spec(d\phi_1(x))$, then the corresponding eigendirection is tangential to $\mathcal{F}$. In our case $\mathcal{F} = \{(u^{1,0}, (2\pi)^2 k) | u^{1,0} \in \mathbb{R}/\mathbb{Z}, k \in \mathbb{Z}\} = \mathbb{R}/\mathbb{Z} \times (2\pi^2)\mathbb{Z}$ and $spec(d\phi_1(u^{1,0}, v_{1,0})) = \{1\}$ with corresponding eigenvector $\xi = 1 \cdot \partial_{u^1} \in T_{(u^{1,0},v_{1,0})}\mathcal{F}$. (In general: H of Bott-type implies, that the critical sets of $\mathcal{A}_{H_0}$ are submanifolds of ΛT^*M.)

As above the critical submanifolds generate an algebraic chain complex: The k^{th} **Bott-type Floer chain group** is defined to be the k^{th} singular chain group of $\mathcal{C}_{-1/2}$ with coefficients in $\mathbb{Z}_2$

$$\mathbf{CF_k^{Bott}(T^*S^1, H_0, g; \mathbb{Z}_2)} = \begin{cases} C_k(\mathcal{C}_{-1/2} = \bigsqcup_{l\in\mathbb{Z}} C^l_{-1/2} & , k \in \mathbb{N}_0 , \\ 0 & , k \in \mathbb{Z}\setminus\mathbb{N}_0 . \end{cases}$$

The sets $C^l_{-1/2}$ are exactly the connected components of $\mathcal{C}_{-1/2}$ ($C^l_{-1/2} \subset \Lambda T^*S^1$ may be contracted to $G^l_0 \subset \Lambda S^1$: Homotop the fibre component of $x \in C^l_{-1/2}$ to 0). Therefore in defining the **boundary operator** d^F_k we don't have to take into account interaction of different connected components of $\mathcal{C}$ via connecting orbits (i.e. solutions to Floer's partial differential equation, cf. [Fl] in case of isolated critical points and [RT] in case of critical manifolds). We define d^F_k to be the singular boundary operator ∂^{sing}_k for $k \in \mathbb{N}_0$ ($d^F_k = 0$ else)

$$\begin{aligned} d^F_k \ : \ CF^{Bott}_k(T^*S^1, H_0, g; \mathbb{Z}_2) &\rightarrow CF^{Bott}_{k-1}(, T^*S^1, H_0, g; \mathbb{Z}_2) , \\ x &\mapsto \partial^{sing}_k x . \end{aligned}$$

By singular homology theory $(CF^{Bott}_k(T^*S^1, H_0, g; \mathbb{Z}_2), d^F_k)_{k\in\mathbb{Z}}$ is an algebraic chain complex. The k^{th} **Bott-type Floer homology group** is defined to be

$$\mathbf{HF_k^{Bott}(T^*S^1, H_0, g; \mathbb{Z}_2)} = \frac{ker\ d^F_k}{im\ d^F_{k+1}} , \ k \in \mathbb{Z} .$$

These groups may be computed as follows

$$\begin{aligned} HF^{Bott}_k(H_0, T^*S^1, g; \mathbb{Z}_2) &= H^{sing}_k(\mathcal{C}_{-1/2} = \bigsqcup\nolimits_{l\in\mathbb{Z}} C^l_{-1/2}; \mathbb{Z}_2) \\ &= H^{sing}_k(\bigsqcup\nolimits_{l\in\mathbb{Z}} S^l; \mathbb{Z}_2) \\ &= \begin{cases} (\mathbb{Z}_2)^{\mathbb{Z}} & , k = 0, 1 , \\ 0 & , \text{else} . \end{cases} \end{aligned}$$

Appendix A. Computation of the hamiltonian vector field X_{H_0}

We present a method to compute the hamiltonian vector field X_{H_0}, which leads us to investigate the problem from a more abstract point of view (cf. [Kl], Ch. 3). We continue using natural local coordinates as above. Because of

$$T^h_{(u^1,v_1)}T^*S^1 \cap T^v_{(u^1,v_1)}T^*S^1 = (u^1, v_1; 0, 0) ,$$

we get a metric G on TT^*S^1 by defining $T^hT^*S^1$ and $T^vT^*S^1$ being orthogonal to each other, i.e. define G by

$$G(\cdot,\cdot) = g(T\tau^*_{S^1}\cdot, T\tau^*_{S^1}\cdot) + g^*(K\cdot, K\cdot) ,$$

where K is the connection map as in (6) and g^* is the dual metric. In natural local coordinates this reads

$$G_{(u^1,v_1)}\left(\begin{pmatrix}\xi^1\\ \eta_1\end{pmatrix}, \begin{pmatrix}\tilde{\xi}^1\\ \tilde{\eta}_1\end{pmatrix}\right) = (2\pi)^2\xi^1\tilde{\xi}^1 + \tfrac{1}{(2\pi)^2}\eta_1\tilde{\eta}_1 . \qquad (14)$$

An almost complex structure $J \in End(TT^*S^1)$, i.e. $(J)^2 = -Id$, is determined by the conditions

$$\begin{array}{ll} \text{a)} & -(g \circ T\tau^*_{S^1}) \circ J = K \ , \\ \text{b)} & K \circ J = (g \circ T\tau^*_{S^1}) \ , \end{array} \tag{15}$$

where in abuse of notation

$$\begin{array}{rcl} g \, : \, TS^1 & \to & T^*S^1 \\ \xi & \mapsto & g(\xi, \cdot) \end{array}$$

is the isomorphism between vector fields and 1-forms on S^1 induced by the metric g. Using (15)a) and b) J is computed to be

$$J_{(u^1,v_1)} = \begin{pmatrix} 0 & -g^{11} \\ g_{11} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{-1}{(2\pi)^2} \\ (2\pi)^2 & 0 \end{pmatrix} \ .$$

The compatibility of G and Ω, i.e. $G(\cdot,\cdot) = \Omega(\cdot, J\cdot)$, implies $X_{H_0} = -J \circ {}^G\nabla\, H_0$. Moreover ${}^G\nabla\, H_0$ is determined by

$$dH_{0(u^1,v_1)}(\cdot) = G_{(u^1,v_1)}({}^G\nabla\, H_{0(u^1,v_1)}, \cdot) \ .$$

Setting ${}^G\nabla\, H_0(u^1,v_1) = (x^1, y_1)^t$ and using (11),(14) we get

$$\begin{aligned} \frac{v_1\eta_1}{(2\pi)^2} &= \tfrac{1}{(2\pi)^2}(0, v_1) \begin{pmatrix} \xi^1 \\ \eta_1 \end{pmatrix} = dH_{0(u^1,v_1)} \begin{pmatrix} \xi^1 \\ \eta_1 \end{pmatrix} \\ &= G_{(u^1,v_1)} \left(\begin{pmatrix} x^1 \\ y_1 \end{pmatrix}, \begin{pmatrix} \xi^1 \\ \eta_1 \end{pmatrix} \right) = (2\pi)^2 x^1 \xi^1 + \tfrac{1}{(2\pi)^2} y_1 \eta_1 \ . \end{aligned}$$

This implies $x^1 = 0$ and $y_1 = v_1$, i.e.

$${}^G\nabla\, H_{0(u^1,v_1)} = \begin{pmatrix} 0 \\ v_1 \end{pmatrix}$$

and therefore

$$X_{H_0}(u^1, v_1) = - \begin{pmatrix} 0 & \frac{-1}{(2\pi)^2} \\ (2\pi)^2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ v_1 \end{pmatrix} = \begin{pmatrix} \frac{v_1}{(2\pi)^2} \\ 0 \end{pmatrix} \ .$$

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