On the Infinitesimal Isometries of Manifolds with Killing Spinors

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Abstract - We study the Lie algebra of infinitesimal isometries of 7–dimensional simply connected manifolds with Killing spinors. We obtain some splitting theorems for the action of this algebra on the space of Killing spinors, and as a corollary we prove that there are no infinitesimal isometry of constant length on a 7–dimensional 3–Sasakian manifold (not isometric to a space form) except the linear combinations of the Sasakian vector fields.

1 Introduction

Let M^n be a spin manifold and ΣM the fibre bundle of real spinors. A real Killing spinor on M^n is a section ψ of ΣM satisfying

$$\nabla_X \psi = \lambda X \cdot \psi, \quad \forall\, X \in TM,$$

for some real number $\lambda \neq 0$. The existence of such a spinor implies that M is an Einstein manifold with Einstein constant $4\lambda^2(n-1)$. By rescaling the metric we can always suppose that $\lambda = \pm\frac{1}{2}$, and we denote by $N_\pm$ the dimension of the (real) vector space of Killing spinors with Killing constant $\pm\frac{1}{2}$.

Definition 1.1 *A compact spin manifold is called of type N if it is simply connected and $N_+ + N_- = N$.*

We are interested here in odd–dimensional manifolds M^{2k+1}, for which we have the following picture (cf. [1]):

type of M :	M :
$N = 0$	generical
$N = 1$	proper weak G_2–structure (k=3)
$N = 2$	Einstein–Sasakian structure
$N = \frac{k+3}{3}$	3–Sasakian structure (k odd)
$N = 2^k$	S^{2k+1}

This restriction to the odd–dimensional case represents actually a very small loss of generality since the only even–dimensional manifolds of nonzero type are the spheres and the 6–dimensional nearly Kähler manifolds ([1]).

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The Lie algebra of infinitesimal isometries of 7–dimensional manifolds of type 1 is studied in [3]; one obtains several results about the zero sets of Killing vector fields, as well as a characterisation of these vector fields in terms of the weak G_2–structure.

The aim of this paper is to give several new results about the Lie algebra of infinitesimal isometries of manifolds of type 2 and 3, especially in dimension 7. Let (M^{2k+1}, g, ξ) be a manifold of type 2 (i.e. Einstein–Sasakian and not 3–Sasakian). The first result that we obtain is that each infinitesimal isometry of M is an infinitesimal automorphism of the Sasakian structure of M (Theorem 2.1). We next describe the action of the Killing vector fields on the set [2] of Killing spinors on M, which enables us to prove a splitting theorem for the Lie algebra of the infinitesimal isometries of spin Einstein–Sasakian manifolds (Theorem 2.2). In particular, we prove that the Lie algebra of infinitesimal automorphisms of a weak G_2–structure on a 7–dimensional manifold of type 2 is a subalgebra of codimension 1 in the Lie algebra of infinitesimal isometries.

The situation is more complicated for manifolds of type 3 (3–Sasakian manifolds). Here the subalgebra I spanned by the Sasakian Killing vector fields is no more abelian, so a result like Theorem 2.1 cannot hold. A closer analisys of the action of the Killing vector fields on the space of Killing spinors, allows us nevertheless to state a splitting theorem in dimension 7, and to deduce that I is in fact an ideal of the Lie algebra of infinitesimal isometries for every 7–dimensional simply connected 3–Sasakian manifold not isometric to S^7 (Theorem 3.1). As a corollary, we prove that every infinitesimal isometry of constant length on a 7–dimensional manifold of type 3 is a linear combination (with constant coefficients) of the Sasakian vector fields (Theorem 4.1 and Corollary 4.2).

2 Manifolds of type 2

A fundamental tool in all which follows is the Lie derivative of spinors with respect to Killing vector fields. We refer the reader to [7] for the definition and the basic properties of this operation. Here is a simple but useful result concerning it.

Lemma 2.1 *The Lie derivative with respect to Killing vector fields preserves the space of Killing spinors.*

Proof. Recall that the Lie derivation with respect to Killing vector fields satisfies the Leibnitz rule with respect to the Clifford product (i.e. $\mathcal{L}_X(Y \cdot \psi) = \mathcal{L}_X Y \cdot \psi + Y \cdot \mathcal{L}_X \psi$).

[2] This set is a vector space iff all Killing spinors have the same Killing constant, which holds exactly when k is odd (cf. [2])

Suppose that ψ satisfies

$$\nabla_Y \psi = aY \cdot \psi,\ \forall Y, \tag{1}$$

and let X be a Killing vector field. As $\mathcal{L}_X$ preserves the covariant derivative, by taking the Lie derivative with respect to X of the above formula we obtain

$$\nabla_{\mathcal{L}_X Y} \psi + \nabla_Y (\mathcal{L}_X \psi) = a\mathcal{L}_X Y \cdot \psi + aY \cdot \mathcal{L}_X \psi,\ \forall Y, \tag{2}$$

so, replacing Y by $\mathcal{L}_X Y$ in (1) and substracting from (2) gives

$$\nabla_Y (\mathcal{L}_X \psi) = aY \cdot \mathcal{L}_X \psi,\ \forall Y.$$

Q.E.D.

Definition 2.1 *A vector field ξ on a Riemannian manifold (M, g) is called a Sasakian structure if the following conditions are satisfied:*

1. *ξ is a Killing vector field of unit length;*
2. *The tensors $\varphi := -\nabla\xi$ and $\eta := g(\xi, .)$ are related by*

$$\varphi^2 = -Id + \eta \otimes \xi;$$

3. *$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)\, X$, for all vectors X, Y.*

Definition 2.2 *A triple (ξ_1, ξ_2, ξ_3) of Sasakian structures is called a 3-Sasakian structure on M if the following conditions are satisfied:*

1. *The frame (ξ_1, ξ_2, ξ_3) is orthonormal;*
2. *For each permutation (i, j, k) of signature δ, the tensors $\varphi_i := -\nabla\xi_i$ and $\eta_i := g(\xi_i, .)$ are related by $\varphi_i\,\varphi_j = (-1)^\delta\,\varphi_k + \eta_j \otimes \xi_i$.*

Let (M^{2k+1}, ξ, φ) be a spin Einstein–Sasakian manifold of type 2. If k is even, then $N_+ = N_- = 1$ (cf. [2]), so the action (by Lie derivative) of the infinitesimal isometries on Killing spinors is trivial. As we shall see below, this easily implies that every infinitesimal isometry of M is an infinitesimal automorphism of the Sasakian structure. From now on, we will thus suppose that k is odd.

The Killing spinors on M span a 2-dimensional sub-bundle $\Phi(\xi)$ of ΣM which can be defined by (cf. [2])

$$\Phi(\xi) = \{\psi \in \Gamma(\Sigma M) \mid (2\varphi(X) + \xi \cdot X - X \cdot \xi) \cdot \psi = 0,\ \forall X \in TM\}. \tag{3}$$

Using (3) we easily obtain the following

Lemma 2.2 *If ψ is a Killing spinor on M, then the Clifford product $\xi \cdot \psi$ defines a Killing spinor, too, and ψ and $\xi \cdot \psi$ form an orthonormal basis of $\Phi(\xi)$ everywhere on M.*

We are now ready for our first result.

Theorem 2.1 *Let X be a Killing vector field on a spin Einstein–Sasakian manifold (M^{2k+1}, g, ξ), and suppose that M is not 3–Sasakian. Then X is an infinitesimal automorphism of the Sasakian structure of M, that is, $[X, \xi] = 0$.*

Proof. Consider a Killing spinor ψ on M. By Lemma 2.2, $\xi \cdot \psi$ is a Killing spinor, too, and as M is not 3–Sasakian, every Killing spinor on M is a linear combination of ψ and $\xi \cdot \psi$. Recall that we work with real spinors, which means that we have an euclidean scalar product on ΣM, with respect to which the Clifford multiplication by a vector is anti-symmetric. In particular we have $(X \cdot \phi, \phi) = 0$ for all X and ϕ. Moreover, for every spinor ϕ of constant length (so, in particular, for every Killing spinor), we have $0 = X(\phi, \phi) = 2(L_X\phi, \phi)$, since the Lie derivation with respect to Killing vector fields preserves the euclidean product on ΣM. By Lemma 2.1 we then deduce that $\mathcal{L}_X\psi$ and $\mathcal{L}_X(\xi \cdot \psi)$ are proportional with $\xi \cdot \psi$ and ψ respectively. Let $a \in \mathbf{R}$ such that $\mathcal{L}_X\psi = a\xi \cdot \psi$. We then have

$$\mathcal{L}_X(\xi \cdot \psi) = \mathcal{L}_X\xi \cdot \psi + \xi \cdot \mathcal{L}_X\psi = \mathcal{L}_X\xi \cdot \psi - a\psi,$$

and as the left side of this equation is proportional to ψ, the same must be true for $\mathcal{L}_X\xi \cdot \psi$, which is actually perpendicular to ψ. Thus, $\mathcal{L}_X\xi \cdot \psi = 0$, which implies $\mathcal{L}_X\xi = 0$.

Q.E.D.

We now restrict our attention to the 7–dimensional case, and consider a spin manifold M^7 of type 2 (recall that this means that M is Einstein–Sasakian, but not 3–Sasakian). The existence of a Killing spinor ψ on a 7–dimensional manifold M is equivalent to that of a weak G_2–structure ω_ψ ([3], [4]), via the following relation (cf. [2])

$$\omega_\psi(X, Y, \cdot) \cdot \psi = (X \cdot Y + g(X, Y)) \cdot \psi \quad , \quad \forall X, Y \in TM. \tag{4}$$

By choosing $Y = \xi$ in (4) and using (3) we obtain:

$$d\xi = 2\xi \lrcorner\, \omega_\psi, \tag{5}$$

Corresponding to the 2-dimensional vector space $Kill(\Sigma M)$ of Killing spinors, we have a 2-dimensional vector space $G_2(M)$ of weak G_2–structures on M, and of course, this vector space is preserved by the action of the infinitesimal isometries, i.e. we have a real representation l of $i(M)$ on $G_2(M)$. Our aim is to describe this representation. By Theorem 2.1 we have an exact sequence of Lie algebras

$$0 \to \mathbf{R}\xi \to i(M) \to i(M)/\mathbf{R}\xi \to 0. \tag{6}$$

Theorem 2.2 *(splitting theorem for 7–dimensional spin Einstein–Sasakian manifolds)*
(i) The restriction of l to $\mathbf{R}\xi$ is isomorphic to the (irreducible) representation of $u(1)$ on $\mathbf{C}$.
(ii) There exists a splitting of the above exact sequence in $i(M) = \mathbf{R}\xi \oplus i_0(M)$, such that the restriction of l to $i_0(M)$ is trivial.

Proof. (i) We already know that $\mathcal{L}_\xi \psi = a\xi \cdot \psi$ for some $a \in R$, and from this it follows that $\mathcal{L}_\xi(\xi \cdot \psi) = -a\psi$. All we have to show is that a is nonzero. Suppose that $\mathcal{L}_X \psi = 0$. Then X is an infinitesimal automorphism of the weak G_2–structure ω_ψ defined by ψ, so by Theorem 6.1 in [3] we obtain:

$$\pi_7(d\xi) = -\frac{2}{3}(\xi \lrcorner \omega_\psi). \tag{7}$$

On the other hand, (5) shows that $d\xi = 2\xi \lrcorner \omega_\psi \in \Lambda^2_7 M$, so $\pi_7(d\xi) = 2\xi \lrcorner \omega_\psi$. This, together with (7) would then imply $\xi \lrcorner \omega_\psi = 0$, so $\xi = 0$, a contradiction which proves that $a \neq 0$.

(ii) Let us fix a Killing spinor ψ as before. Consider the linear function $L : i(M) \to \mathbf{R}$ given by

$$L(X) = (\mathcal{L}_X \psi, \xi \cdot \psi).$$

By the preceding discution, L can be also given by

$$\mathcal{L}_X \psi = L(X)\xi \cdot \psi.$$

This formula shows that L is in fact a Lie algebra homomorphism, since

$$\begin{aligned} \mathcal{L}_{[X,Y]}\psi &= [\mathcal{L}_X, \mathcal{L}_Y]\psi = \mathcal{L}_X(L(Y)\xi \cdot \psi) - \mathcal{L}_Y(L(X)\xi \cdot \psi) \\ &= -L(X)L(Y)\psi + L(X)L(Y)\psi = 0. \end{aligned}$$

The kernel $i_0(M)$ of L is thus an ideal of $i(M)$ of codimension 1. By (i) we see that $i_0(M) \cap \mathbf{R}\xi = 0$, so $i(M) = \mathbf{R}\xi \oplus i_0(M)$. The last statement of the theorem is now trivial, by the very definition of $i_0(M)$.

Q.E.D.

3 Manifolds of type 3

We now consider a 7–dimensional manifold M of type 3, i.e. a simply connected 3–Sasakian manifold not isometric with the euclidean 7–sphere. Denote by ξ_i and $\varphi_i := -\nabla_{\xi_i}$, $i \in \{1, 2, 3\}$, the 3-Sasakian structure. It is easily seen that the structure group of the frame of oriented orthonormal bundles on M restricts to $SU(2)$, which can be viewed (being simply connected) as a subgroup of Spin(7), so M is automatically spin. M is also Einstein, by a theorem of Kashiwada ([6]). As before, each Sasakian structure ξ_i defines a 2–dimensional sub-bundle of ΣM, $H_i := \Phi(\xi_i)$, trivialisated by Killing spinors, and conversely, each H_i determines ξ_i. If for some $x \in M$ the fibres $H_i(x)$ and $H_j(x)$ are equal, then $H_i = H_j$ (a Killing spinor is determinated by its value at any point), so $\xi_i = \xi_j$, thus implying $i = j$. Consequently, for $i \neq j$ we have $dim(H_i(x) \cap H_j(x)) \leq 1$, so $dim(H_i(x) \cup H_j(x)) \geq 3$. But the space of Killing spinors has dimension 3, so $dim(H_i(x) \cup H_j(x)) = 3$, thus showing that $dim(H_i(x) \cap H_j(x)) = 1$. We choose Killing spinors of unit norm ψ_1, ψ_2, ψ_3 spanning $H_2 \cap H_3$, $H_3 \cap H_1$ and $H_1 \cap H_2$ respectively. We first show that these three spinors are orthogonal to each other in every point of M. Let us fix some $x \in M$ and denote by $a_{ij} := (\psi_i, \psi_j)$ in x. By Lemma 2.2, ψ_1 and $\xi_2 \cdot \psi_1$ span H_2, so we can write

$$\psi_3 = a_{13}\psi_1 + a\xi_2 \cdot \psi_1 \tag{8}$$

for some real number a. Similarly we can find $b \in \mathbf{R}$ such that

$$\psi_1 = a_{12}\psi_2 + b\xi_3 \cdot \psi_2. \tag{9}$$

Using (8) and (9) we obtain

$$\psi_3 = a_{13}a_{12}\psi_2 + a_{13}b\xi_3 \cdot \psi_2 + aa_{12}\xi_2 \cdot \psi_2 + ab\xi_2 \cdot \xi_3 \cdot \psi_2, \tag{10}$$

and taking the scalar product with ψ_2 in (10) gives $a_{23} = a_{13}a_{12}$. Similarly we obtain $a_{13} = a_{23}a_{12}$, and hence $a_{23}(1 - a_{12}^2) = 0$. But $a_{12}^2 < 1$, since otherwise $\psi_1 = \pm\psi_2$, so $a_{23} = 0$, and by symmetry, $a_{12} = a_{13} = 0$.

We thus have shown that ψ_i are everywhere orthogonal to each other, which means that H_i is the orthogonal space to ψ_i in $H = \cup H_i$. Using this and Lemma 2.2 we see that $\xi_2 \cdot \psi_1$ lies in $H_1 \cap H_2$, so we can suppose without loss of generality that $\xi_2 \cdot \psi_1 = -\psi_3$, and similarly $\xi_3 \cdot \psi_1 = -\psi_2$. We also have $\xi_1 \cdot \psi_2 = \varepsilon\psi_3$, where $\varepsilon = \pm 1$. By replacing ψ_1 with $\varepsilon\psi_1$ we finally obtain the homogeneous formulas

$$\xi_i \cdot \psi_j = \delta\varepsilon\psi_k, \tag{11}$$

where δ is the signature of the permutation $\{i, j, k\}$.

Let us now compute the Lie derivative of the spinors ψ_i with respect to ξ_j. As $\mathcal{L}_{\xi_1}$ preserves H_1, $\mathcal{L}_{\xi_1}\psi_3$ lies in H_1. Moreover it is orthogonal to ψ_3, so there is some $a \in \mathbf{R}$ such that $\mathcal{L}_{\xi_1}\psi_3 = a\psi_2$. Taking the Lie derivative with respect to ξ_1 in $\xi_2 \cdot \psi_3 = \varepsilon\psi_1$ and using the definition of the 3–Sasakian structure we obtain $\varepsilon\mathcal{L}_{\xi_1}\psi_1 = \xi_3 \cdot \psi_3 + a\xi_2 \cdot \psi_2$. This is orthogonal to all of the ψ_i, as easily follows from (11). On the other hand, $\mathcal{L}_{\xi_1}\psi_1 \in H$ by Lemma 2.1, so $\mathcal{L}_{\xi_1}\psi_1 = 0$. This implies that $\xi_3 \cdot \psi_3 + a\xi_2 \cdot \psi_2 = 0$, so $a = -1$ (by (11)) and hence

$$\mathcal{L}_{\xi_i}\psi_j = \delta\psi_k, \tag{12}$$

where δ denotes the signature of the permutation $\{i, j, k\}$, with the convention $\delta = 0$ if $\{i, j, k\} \neq \{1, 2, 3\}$.

We are now ready for the main result of this section. As in the Einstein–Sasakian case, there is a natural bijection between the 3–dimensional vector space H of Killing spinors and the set of weak G_2–structures $G_2(M)$, and we consider the real representation l of $i(M)$ on $G_2(M)$ given by the Lie derivative via the above bijection. Let I denote the subalgebra of $i(M)$ spanned by ξ_1, ξ_2 and ξ_3.

Theorem 3.1 *(splitting theorem for 7–dimensional 3–Sasakian manifolds)*
(i) The restriction of l to I is isomorphic to the (irreducible) representation of $su(2) = so(3)$ on $\mathbf{R}^3$.
(ii) There exists a Lie algebra homomorphism $L : i(M) \to I$, whose restriction to I is the identity.
(iii) The restriction of l to $i_0(M) := ker(L)$ is trivial.
(iv) The induced exact sequence of Lie algebras

$$0 \to i_0(M) \to i(M) \to I \to 0 \tag{13}$$

is splitting. Equivalently, I is an ideal of $i(M)$ and $i(M) \simeq I \oplus i_0(M)$.

Proof. (i) Follows directly from (12).

(ii) Given $X \in i(M)$, we denote by $b_{ij}(X) = (\mathcal{L}_X\psi_i, \psi_j)$, and define $L(X) = \frac{1}{2}(b_{23}(X)\xi_1 + b_{31}(X)\xi_2 + b_{12}(X)\xi_3)$. Let us check that L is a Lie algebra homomorphism. Take $X, Y \in i(M)$. We will just show that the ξ_1–component of $L([X, Y])$ and $[L(X), L(Y)]$ are the same, which is clearly sufficient, by symmetry. From the definition of a 3–Sasakian structure we obtain directly

$$g([L(X), L(Y)], \xi_1) = \frac{1}{2}(b_{31}(X)b_{12}(Y) - 2b_{31}(Y)b_{12}(X)). \tag{14}$$

Using $\mathcal{L}_{[X,Y]} = [\mathcal{L}_X, \mathcal{L}_Y]$ we find

$$\begin{aligned} b_{23}([X, Y]) &= (\mathcal{L}_X\mathcal{L}_Y\psi_2, \psi_3) - (\mathcal{L}_Y\mathcal{L}_X\psi_2, \psi_3) \\ &= (\mathcal{L}_X(b_{21}(Y)\psi_1 + b_{23}(Y)\psi_3), \psi_3) \\ &\quad -(\mathcal{L}_Y(b_{21}(X)\psi_1 + b_{23}(X)\psi_3), \psi_3) \\ &= b_{21}(Y)b_{13}(X) - b_{21}(X)b_{13}(Y), \end{aligned}$$

so finally

$$g(L([X, Y]), \xi_1) = \frac{1}{2}b_{23}([X, Y]) = \frac{1}{2}(b_{31}(X)b_{12}(Y) - 2b_{31}(Y)b_{12}(X)).$$

The last assertion follows from (12).

(iii) If $X \in i_0(M)$, then $b_{ij} = 0$ so $\mathcal{L}_X\psi_i$ is orthogonal to H. As $\mathcal{L}_X\psi_i \in H$ (by Lemma 2.1), we obtain $\mathcal{L}_X\psi_i = 0$.

(iv) We have to show that I is an ideal of $i(M)$ or, equivalently, that I and $i_0(M)$ are commuting subalgebras of $i(M)$. Let $X \in i_0(M)$. Taking the Lie derivative of (11) with respect to X and using (iii), we see that $\mathcal{L}_X\xi_i \cdot \psi_j = 0$, so $\mathcal{L}_X\xi_i = 0$.
Q.E.D.

4 Infinitesimal isometries of unit length on 7–dimensional manifolds of type 3

As a first application of the above result we have the following

Corollary 4.1 *On a 7–dimensional manifold M of type 3, every Killing vector field X of unit length is either a Sasakian structure or an infinitesimal automorphism of the 3–Sasakian structure.*

Proof. We decompose X with respect to $i(M) = I \oplus i_0(M)$ as $X = A + Y$. From (iv) of the preceding theorem we know that Y commutes with the unit Killing vector fields ξ_i spanning I, i.e. it is an infinitesimal automorphism of the 3–Sasakian structure. If a_i denotes the ξ_i–component of A, we have to show that either $Y = 0$ or $a_i = 0$. Consider the functions $f_i := g(X, \xi_i)$. As $[Y, \xi_1] = 0$, we can compute $[X, \xi_1] = a_2\xi_3 - a_3\xi_2$, so

$$\begin{aligned} 0 &= -\frac{1}{2}\mathcal{L}_{\xi_1} g(X, X) = -g(\mathcal{L}_{\xi_1}X, X) = g([X, \xi_1], X) \\ &= a_2 f_3 - a_3 f_2, \end{aligned}$$

and similarly

$$a_2 f_1 = a_1 f_2 \text{ and } a_3 f_1 = a_1 f_3. \tag{15}$$

On the other hand, using the definition of the 3–Sasakian structure one easily obtains

$$\xi_i(f_i) = 0 \tag{16}$$

and

$$\xi_i(f_j) = 2\delta(f_k - a_k), \tag{17}$$

where δ denotes as usual the signature of the permutation $\{i, j, k\}$.

Suppose now that at least one of the a_i's doesn't vanish. Without loss of generality, we can suppose $a_1 \neq 0$. Differentiating in (15) with respect to ξ_1 and using (16) and (17) we obtain $f_2 = a_2$ and $f_3 = a_3$. Using $[Y, \xi_i] = 0$ we find

$0 = df_i = 2\varphi_i(Y)$ ($i = 2$ and $i = 3$), so $Y \in \mathbf{R}\xi_2 \cap \mathbf{R}\xi_3 = \{0\}$. It is now clear that X belongs to the 2–sphere of Sasakian structures of M.

Q.E.D.

Actually a closer analysis of this situation allows us to show that the second case never occurs in the above corollary. Indeed, we have the

Theorem 4.1 *Consider an infinitesimal isometry of unit length X on a simply connected 3–Sasakian manifold (M^7, g, ξ_i). If X is an infinitesimal automorphism of the 3–Sasakian structure, then M is isometric to the sphere S^7.*

Proof. It is obvious that $\{\xi_1, \xi_2, \xi_3, X, \varphi_1(X), \varphi_2(X), \varphi_3(X)\}$ is a frame in each point of M^7 where X doesn't belong to the distribution spanned by the ξ_i. The set N of such points is obviously open. Suppose that N is not dense in M. Then there exists an open subset U of M such that X is a vertical Killing vector field over U (in the sense of Ishihara and Konishi [5]). From Lemma 7 of loc. cit. we obtain that X is a linear combination with constant coefficients of ξ_i over U, and hence over M since two infinitesimal isometries wich coincide on an open set must be equal. This contradicts the fact that X is an infinitesimal automorphism of the 3–Sasakian structure, so we conclude that N is everywhere dense in M.

The next step is to prove that in any point of N, the sectional curvature of M is constant and equal to 1. In order to prove this, it suffices to show that

$$g(R(Y_1, Y_2)Y_3, Y_4) = g(Y_2, Y_3)g(Y_1, Y_4) - g(Y_1, Y_3)g(Y_2, Y_4), \tag{18}$$

whenever each Y_i belong to the frame $\{\xi_1, \xi_2, \xi_3, X, \varphi_1(X), \varphi_2(X), \varphi_3(X)\}$. First of all, when $Y_i = \xi_j$ for some i, j, then we are done by Lemma 4 at the page 78 in [2]. Actually, this simple lemma just says that the sectional curvature of every 2-plane containing one of the Sasakian directions is equal to 1.

It is thus enough to check (18) in the case when $Y_i \in \{X, \varphi_1(X), \varphi_2(X), \varphi_3(X)\}$. A priori we would have $4^4 = 256$ possibilities, but thanks to the symmetries of the curvature tensor and of the 3–Sasakian structure, the reader can easily convince himself that it is enough to prove (18) in the following cases:

a) $Y_1 = \varphi_1(X),\ Y_2 = Y_3 = X$

b) $Y_1 = X,\ Y_2 = Y_3 = \varphi_1(X)$

c) $Y_1 = \varphi_2(X),\ Y_2 = Y_3 = \varphi_1(X)$

d) $Y_1 = \varphi_3(X),\ Y_2 = \varphi_1(X),\ Y_3 = \varphi_2(X)$

e) $Y_1 = \varphi_1(X),\ Y_2 = \varphi_2(X),\ Y_3 = X$

In order to compute the curvature tensor we need the

Lemma 4.1 *The following relations hold*

$$\nabla_X X = 0 \ ; \ [X, \xi_i] = 0 \quad \forall i \in \{1, 2, 3\} \tag{19}$$

$$[X, \varphi_i(X)] = 0 \quad \forall i \in \{1,2,3\} \tag{20}$$

$$df_i = 2\varphi_i(X) \quad \forall i \in \{1,2,3\} \ (f_i = g(X, \xi_i)) \tag{21}$$

$$g(X, \varphi_i(X)) = 0 \quad ; \quad g(\varphi_i(X), \varphi_j(X)) = \delta_{ij} - f_i f_j \tag{22}$$

$$\nabla_X(\varphi_i(X)) = \xi_i - f_i X \quad \forall i \in \{1,2,3\} \tag{23}$$

$$\nabla_{\varphi_i(X)}(\varphi_i(X)) = -2 f_i \varphi_i(X) \quad \forall i \in \{1,2,3\} \tag{24}$$

$$\nabla_{\varphi_i(X)}(\varphi_j(X)) = -f_i \varphi_j(X) - f_j \varphi_i(X) - \delta \xi_k, \tag{25}$$

where in the last equation, δ denotes as usual the signature of the permutation $\{i,j,k\}$.

Proof of the Lemma. (19) is just the hypothesis on X; (20) follows from the fact that $\mathcal{L}_X$ preseves ξ_i and ∇, so it preserves φ_i; (21) and (22) are trivial, and (23), (24), (25) follow directly from (20) and the definition of a 3–Sasakian structure.

We will now check the relation (18) in each of the cases a) – e) using the definition of the 3–Sasakian structure and the above lemma.

a) $$\begin{aligned} R(\varphi_1(X), X)X &= -\nabla_X \nabla_{\varphi_1(X)} X = -\nabla_X(\xi_1 - f_1 X) \\ &= \varphi_1(X) = g(X,X)\varphi_1(X) - g(X, \varphi_1(X))X; \end{aligned}$$

b) $$\begin{aligned} R(X, \varphi_1(X))\varphi_1(X) &= \nabla_X(-2f_1\varphi_1(X)) - \nabla_{\varphi_1(X)}(\xi_1 - f_1 X) \\ &= -2f_1(\xi_1 - f_1 X) + \varphi_1^2(X) \\ &\quad + f_1(\xi_1 - f_1 X) + 2X|\varphi_1(X)|^2 \\ &= -f_1(\xi_1 - f_1 X) - X + \xi_1 f_1 + 2X|\varphi_1(X)|^2 \\ &= X|\varphi_1(X)|^2 \\ &= g(\varphi_1(X), \varphi_1(X))X - g(X, \varphi_1(X))\varphi_1(X); \end{aligned}$$

c) $$\begin{aligned} R(\varphi_2(X), \varphi_1(X))\varphi_1(X) &= \nabla_{\varphi_2(X)}(-2f_1\varphi_1(X)) \\ &\quad -\nabla_{\varphi_1(X)}(-f_2\varphi_1(X) - f_1\varphi_2(X) + \xi_3) \\ &\quad -2\nabla_{\xi_3}(\varphi_1(X)) \\ &= -2f_1(-f_2\varphi_1(X) - f_1\varphi_2(X) + \xi_3) + 4f_1 f_2 \varphi_1(X) \\ &\quad -2f_1 f_2 \varphi_1(X) - 2f_1 f_2 \varphi_1(X) + 2(1 - f_1^2)\varphi_2(X) \\ &\quad + f_1(-f_2\varphi_1(X) - f_1\varphi_2(X) - \xi_3) + \varphi_3(\varphi_1(X)) \\ &\quad +2\varphi_1(\varphi_3(X)) - 2f_3\xi_1 + 2f_1\xi_3 \end{aligned}$$

$$
\begin{aligned}
&= f_1 f_2 \varphi_1(X) + (2f_1^2 + 2 - 2f_1^2 - f_1^2)\varphi_2(X) \\
&\quad -2f_1\xi_3 - f_1\xi_3 + \varphi_2(X) + f_1\xi_3 - 2\varphi_2(X) \\
&\quad +2f_3\xi_1 - 2f_3\xi_1 + 2f_1\xi_3 \\
&= f_1 f_2 \varphi_1(X) + (1 - f_1^2)\varphi_2(X) \\
&= g(\varphi_1(X), \varphi_1(X))\varphi_2(X) \\
&\quad -g(\varphi_1(X), \varphi_2(X))\varphi_1(X);
\end{aligned}
$$

$$
\begin{aligned}
\text{d) } R(\varphi_3(X), \varphi_1(X))\varphi_2(X) &= \nabla_{\varphi_3(X)}(-f_1\varphi_2(X) - f_2\varphi_1(X) - \xi_3) \\
&\quad -\nabla_{\varphi_1(X)}(-f_2\varphi_3(X) - f_3\varphi_2(X) + \xi_1) \\
&\quad +2\nabla_{\xi_2}(\varphi_2(X)) \\
&= 2f_1 f_3 \varphi_2(X) + 2f_2 f_3 \varphi_1(X) \\
&\quad -f_1(-f_2\varphi_3(X) - f_3\varphi_2(X) + \xi_1) \\
&\quad -f_2(-f_1\varphi_3(X) - f_3\varphi_1(X) - \xi_2) - X + \xi_3 f_3 \\
&\quad -2f_1 f_2 \varphi_3(X) - 2f_1 f_3 \varphi_2(X) \\
&\quad +f_2(-f_1\varphi_3(X) - f_3\varphi_1(X) + \xi_2) \\
&\quad +f_3(-f_1\varphi_2(X) - f_2\varphi_1(X) - \xi_3) - X + \xi_1 f_1 \\
&\quad +2f_2\xi_2 - 2f_2\xi_2 + 2X - 2f_2\xi_2 \\
&= f_2 f_3 \varphi_1(X) - f_1 f_2 \varphi_3(X) \\
&= g(\varphi_1(X), \varphi_2(X))\varphi_3(X) \\
&\quad -g(\varphi_2(X), \varphi_3(X))\varphi_1(X);
\end{aligned}
$$

$$
\begin{aligned}
\text{e) } R(\varphi_1(X), \varphi_2(X))X &= \nabla_{\varphi_1(X)}(\xi_2 - X f_2) - \nabla_{\varphi_2(X)}(\xi_1 - X f_1) + 2\nabla_{\xi_3} X \\
&= \varphi_3(X) - f_1\xi_2 - f_2(\xi_1 - X f_1) + 2f_1 f_2 X \\
&\quad +\varphi_3(X) + f_2\xi_1 + f_1(\xi_2 - X f_2) - 2f_1 f_2 X - 2\varphi_3(X) \\
&= 0 = g(X, \varphi_2(X))\varphi_1(X) - g(X, \varphi_1(X))\varphi_2(X).
\end{aligned}
$$

In order to complete the proof of the theorem, we remark that, N being dense in M, the sectional curvature of M has to be constant on the whole M, so by simply connectedness, M is isometric to S^7.

Q.E.D.

Corollary 4.2 *On a 7–dimensional manifold M of type 3, the only Killing vector fields of constant length are the trivial ones, i.e. the linear combinations with constant coefficients of the Sasakian vector fields.*

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