Complex Contact Structures and Spin^c Manifolds

Andrei Moroianu[1]

Abstract - In this paper we give an alternative (spinorial) proof of C. LeBrun's theorem stating that a complex contact manifold admitting a Kähler–Einstein metric of positive scalar curvature is necessarily the twistor space of a quaternionic Kähler manifold [7]. This theorem was independently obtained by K.-D. Kirchberg, U. Semmelmann and the author, using special eigenspinors of the Dirac operator, but only in half of the possible dimensions ([4], [8], [10]).

1 Introduction

The notion of projectable spinor were introduced in [8], in order to obtain the classification of Kähler manifolds admitting Kählerian Killing spinors. In the same time, K.-D. Kirchberg and U. Semmelmann discovered that every complex contact manifold of complex dimension $4l+3$ admitting a Kähler–Einstein metric of positive scalar curvature carries a canonical spin structure with Kählerian Killing spinors [4]. Using these results, we immediately obtained the

Theorem 1.1 (cf. [10]) Let M be a compact spin Kähler manifold of positive scalar curvature and complex dimension $4l+3$. Then the following statements are equivalent:

(i) M is the twistor space of some quaternionic Kähler manifold of positive scalar curvature;

(ii) M is Kähler-Einstein and admits a complex contact structure;

(iii) M admits Kählerian Killing spinors.

By different methods, C. LeBrun independently obtained the following

Theorem 1.2 (cf. [7]) Let Z be a Fano contact manifold. Then Z is a twistor space iff it admits a Kähler–Einstein metric.

In complex dimension $4l+3$, this is a direct corollary of Theorem 1.1. A complex contact manifold has odd complex dimension, so it remains to study the case of the complex dimension $4l+1$. The reasons for which our Theorem 1.1 fails to

[1]Supported by the SFB 288 "Differentialgeometrie und Quantenphysik" of the DFG.

hold in these dimensions are of topological nature: neither the twistor spaces, nor the complex contact manifolds of complex dimension $4l+1$ are spin (with one exception: the complex projective space). Nevertheless, this topological obstruction can be avoided in a standard way, namely by considering Spin^c structures, rather than spin structures.

The aim of this article is to construct the Spin^c analogue of Kählerian Killing spinors, and to state the corresponding modified version of Theorem 1.1 in complex dimension $4l+1$, having as corollary Theorem 1.2.

In order to keep the computations as simpler as possible, we do not construct here the whole theory of projectable spinors on Spin^c manifolds, and restrict ourselves to the situation which interests us. Generalisations of the constructions described below can be easily obtained.

The author is very indebted to Uwe Semmelmann for many helpful discussions.

2 Preliminaries

Definition 2.1 *A Spin^c structure on an oriented Riemannian manifold (M^n, g) is given by a $U(1)$ principal bundle $P_{U(1)}M$ and a Spin^c_n principal bundle $P_{Spin^c_n}M$ together with a projection $\theta : P_{Spin^c_n}M \to P_{SO(n)}M \times P_{U(1)}M$ ($P_{SO(n)}M$ means the $SO(n)$ principal bundle of oriented orthonormal frames on M), satisfying*

$$\theta(\tilde{u}\tilde{a}) = \theta(\tilde{u})\xi(\tilde{a}),$$

for every $\tilde{u} \in P_{Spin^c_n}M$ and $\tilde{a} \in Spin^c_n$, where ξ is the canonical 2–fold covering of Spin^c_n over $SO(n) \times U(1)$.

Recall that $\mathrm{Spin}^c_n = \mathrm{Spin}_n \times_{\mathbf{Z}_2} U(1)$, and that ξ is given by $\xi([u,a]) = (\phi(u), a^2)$, where $\phi : \mathrm{Spin}_n \to SO(n)$ is the canonical 2–fold covering.

If M has a Spin^c structure, we denote by ΣM the associated complex spinor bundle and by L the complex line bundle associated to $P_{U(1)}M$, which is called the auxiliary bundle. On ΣM there is a canonical hermitian product $(.,.)$, with respect to which the Clifford multiplication by vectors is anti–hermitian:

$$(X \cdot \psi, \varphi) = -(\psi, X \cdot \varphi), \quad \forall X \in TM,\ \psi, \varphi \in \Sigma M. \tag{1}$$

Every connection form A on $P_{U(1)}M$ defines, together with the Levi-Civita connection of M, a covariant derivative on ΣM denoted by ∇^A. Correspondingly, we define the Dirac operator as the composition $\gamma \circ \nabla^A$, where γ denotes the Clifford contraction. The Dirac operator can be expressed with the aid of a local orthonormal frame $\{e_1, \cdots, e_n\}$ as

$$D = \sum_{i=1}^{n} e_i \cdot \nabla^A_{e_i}.$$

Suppose now that (M^{2m}, g, J) is a Kähler manifold. We define the twisted Dirac operator $\tilde{D}$ by

$$\tilde{D} = \sum_{i=1}^{2m} J(e_i) \cdot \nabla_{e_i} = -\sum_{i=1}^{2m} e_i \cdot \nabla_{J(e_i)},$$

which satisfies

$$\tilde{D}^2 = D^2 \quad \text{and} \quad \tilde{D}D + D\tilde{D} = 0. \tag{2}$$

We also define the complex Dirac operators $D_\pm := \frac{1}{2}(D \mp i\tilde{D})$, and (2) becomes

$$D_-^2 = D_+^2 = 0 \quad \text{and} \quad D^2 = D_+D_- + D_-D_+. \tag{3}$$

Consider a local orthonormal frame $\{X_\alpha, Y_\alpha\}$ such that $Y_\alpha = J(X_\alpha)$. Then $Z_\alpha = \frac{1}{2}(X_\alpha - iY_\alpha)$ and $Z_{\bar{\alpha}} = \frac{1}{2}(X_\alpha + iY_\alpha)$ are local frames of $T^{1,0}(M)$ and $T^{0,1}(M)$, and $D_\pm$ can be expressed as

$$D_+ = 2\sum_{\alpha=1}^{m} Z_\alpha \cdot \nabla^A_{Z_{\bar{\alpha}}} \quad , \quad D_- = 2\sum_{\alpha=1}^{m} Z_{\bar{\alpha}} \cdot \nabla^A_{Z_\alpha}. \tag{4}$$

A k-form ω acts on ΣM by

$$\omega \cdot \Psi := \sum_{i_1 < \cdots < i_k} \omega(e_{i_1}, \cdots, e_{i_k}) e_{i_1} \cdot \ldots \cdot e_{i_k} \cdot \Psi \ .$$

With respect to this action, the Kähler form Ω (defined by $\Omega(X,Y) = g(X, JY)$), satisfies

$$\Omega = \frac{1}{2}\sum_{i=1}^{2m} J(e_i) \cdot e_i = -\frac{1}{2}\sum_{i=1}^{2m} e_i \cdot J(e_i). \tag{5}$$

For later use let us note that

$$\sum_{\alpha=1}^{m} Z_\alpha \cdot Z_{\bar{\alpha}} = \frac{\Omega}{2i} - \frac{m}{2} \quad , \quad \sum_{\alpha=1}^{m} Z_{\bar{\alpha}} \cdot Z_\alpha = -\frac{\Omega}{2i} - \frac{m}{2}, \tag{6}$$

where Z_α and $Z_{\bar{\alpha}}$ are local frames of $T^{1,0}(M)$ and $T^{0,1}(M)$ as before.

The action of Ω on ΣM yields an orthogonal decomposition

$$\Sigma M = \oplus_{r=0}^{m} \Sigma_r M,$$

where $\Sigma_r M$ is the eigenbundle associated to the eigenvalue $i\,\mu_r = i\,(m - 2\,r)$ of Ω. If we define $\Sigma_{-1}M = \Sigma_{m+1}M = \{0\}$, then

$$D_{\pm}\Gamma(\Sigma_r M) \subset \Gamma(\Sigma_{r\pm 1} M). \tag{7}$$

The complex volume element

$$\omega_{\mathbf{C}} := i^m e_1 \cdot \ldots \cdot e_{2m}$$

acts on ΣM by Clifford multiplication and its square is the identity. We denote by $\Sigma^{\pm} M$ the eigenbundles corresponding to the eigenvalues ± 1, and it is easy to see that $\Sigma_r M \subset \Sigma^+ M$ ($\Sigma_r M \subset \Sigma^- M$) iff r is even (respectively odd). If, with respect to the decomposition $\Sigma M = \Sigma^+ M \oplus \Sigma^- M$, a spinor ψ can be written as $\psi = \psi_+ + \psi_-$, then we define its conjugate $\bar{\psi} := \psi_+ - \psi_-$.

3 Complex contact structures

Definition 3.1 (cf. [5]) *Let M^{2m} be a complex manifold of complex dimension $m = 2k+1$. A complex contact structure is a family $\mathcal{C} = \{(U_i, \omega_i)\}$ satisfying the following conditions:*

(i) *$\{U_i\}$ is an open covering of M.*

(ii) *ω_i is a holomorphic 1-form on U_i.*

(iii) *$\omega_i \wedge (\partial\omega_i)^k \in \Gamma(\Lambda^{m,0} M)$ is different from zero at every point of U_i.*

(iv) *$\omega_i = f_{ij}\omega_j$ in $U_i \cap U_j$, where f_{ij} is a holomorphic function on $U_i \cap U_j$.*

Let $\mathcal{C} = \{(U_i, \omega_i)\}$ be a complex contact structure. Then there exists an associated holomorphic line subbundle $L_{\mathcal{C}} \subset \Lambda^{1,0}(M)$ with transition functions $\{f_{ij}^{-1}\}$ and local sections ω_i. It is easy to see that

$$D := \{Z \in T^{1,0}M \mid \omega(Z) = 0,\ \forall \omega \in L_{\mathcal{C}}\}$$

is a codimension 1 maximally non–integrable holomorphic sub–bundle of $T^{1,0}M$, and conversely, every such bundle defines a complex contact structure.

From condition (iii) immediately follows the isomorphism $L_{\mathcal{C}}^{k+1} \cong K$, where $K = \Lambda^{m,0}(M)$ denotes the canonical bundle of M.

The collection $(U_i, \omega_i \wedge (\partial\omega_i)^l)$ defines a holomorphic line bundle $L_l \subset \Lambda^{2l+1,0}M$, and from the definition of $\mathcal{C}$ we easily obtain

$$L_l \cong L_{\mathcal{C}}^{l+1}. \tag{8}$$

We now fix some $(U, \omega) \in \mathcal{C}$ and define a local section $\psi_{\mathcal{C}}$ of $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1}$ by

$$\psi_{\mathcal{C}}|_U := |\xi_\tau|^{-2}\bar{\tau} \otimes \xi_\tau, \tag{9}$$

where $\tau := \omega \wedge (\partial\omega)^l$ and ξ_τ is the ellement corresponding to τ through the isomorphism (8). The fact that $\psi_{\mathcal{C}}$ does not depend of the element $(U, \omega) \in \mathcal{C}$ shows that it actually defines a global section $\psi_{\mathcal{C}}$ of $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1}$.

We now recall ([6], Appendix D) that $\Lambda^{0,*}M$ is just the spinor bundle associated to the canonical Spin^c structure on M, whose auxiliary line bundle is K^{-1}, so that $\Lambda^{0,*}M \otimes L_{\mathcal{C}}^{l+1}$ is actually the spinor bundle associated to the Spin^c structure on M with auxiliary bundle $L = K^{-1} \otimes L_{\mathcal{C}}^{2(l+1)} \cong L_{\mathcal{C}}^{-(2l+1)} \otimes L_{\mathcal{C}}^{2(l+1)} \cong L_{\mathcal{C}}$. The section $\psi_{\mathcal{C}}$ is thus a spinor lying in $\Lambda^{0,2l+1}M \otimes L_{\mathcal{C}}^{l+1} \cong \Sigma_{2l+1}M$, so

$$\Omega \cdot \psi = -i\psi. \tag{10}$$

Proposition 3.1 *The spinor $\psi_{\mathcal{C}}$ satisfies $\nabla_Z \psi_{\mathcal{C}} = 0,\ \forall Z \in T^{1,0}M$* (in particular $D_-\psi_{\mathcal{C}} = 0$), and

$$D^2\psi_{\mathcal{C}} = D_-D_+\psi_{\mathcal{C}} = \frac{l+1}{2l+1}(\frac{R}{2}\psi_{\mathcal{C}} - i\rho \cdot \psi_{\mathcal{C}}), \tag{11}$$

where R is the scalar curvature of M.

Proof. This is actually a variant of Proposition 5 from [4], the only difference being that ξ_τ (Ψ_ω in their notations) is no more a section in $K^{1/2}$, but in $K^{(l+1)/(2l+1)}$, so the coefficients $1/2$ in formulas (8) and (9) from [4] have to be replaced by $\frac{l+1}{2l+1}$.

Q.E.D.

Using (10), (11) and the fact that $\rho = \frac{R}{8l+2}\Omega = (8l+4)\Omega$, we obtain

Corollary 3.1 *The spinor $\psi_{\mathcal{C}}$ is an eigenspinor of D^2 with respect to the eigenvalue $16l(l+1)$.*

Let us make the notations

$$\psi_- := \psi_{\mathcal{C}} \in \Gamma(\Sigma_{2l+1}M) \qquad , \qquad \psi_+ := \frac{D\psi_{\mathcal{C}}}{4l+4} \in \Gamma(\Sigma_{2l+2}M) \qquad . \tag{12}$$

By integration over M we immediately obtain from the above Corollary

$$|\psi_-|_{L^2}^2 = \frac{l+1}{l}|\psi_+|_{L^2}^2. \tag{13}$$

Proposition 3.2 *The following relations hold*

$$\nabla_Z \psi_- = 0,\ \forall Z \in T^{1,0}M. \tag{14}$$

$$\nabla_{\bar{Z}} \psi_- + \bar{Z} \cdot \psi_+ = 0,\ \forall \bar{Z} \in T^{0,1}M, \tag{15}$$

$$\nabla_{\bar{Z}} \psi_+ = 0,\ \forall \bar{Z} \in T^{0,1}M. \tag{16}$$

$$\nabla_Z \psi_+ + Z \cdot \psi_- = 0,\ \forall Z \in T^{1,0}M. \tag{17}$$

Proof. The first relation is part of Proposition 3.1. In order to prove (15), let us consider the local frames of $T^{1,0}(M)$ and $T^{0,1}(M)$ introduced in Section 2: $Z_\alpha = \frac{1}{2}(X_\alpha - iY_\alpha)$ and $Z_{\bar\alpha} = \frac{1}{2}(X_\alpha + iY_\alpha)$, where $Y_\alpha = J(X_\alpha)$, and $\{X_\alpha, Y_\alpha\}$ is a local orthonormal frame of TM. thus proving that $F = 0$ and consequently (15). In order to check the last two equations one has to make use of the operator $\tilde D$. From $D_-\psi_- = 0$ we find

$$0 = \frac{1}{4l+4} D_+^2 \psi_- = D_+ \psi_+, \tag{18}$$

so

$$\tilde D \psi_+ = -iD\psi_+. \tag{19}$$

We take a local orthonormal frame e_i and write (using (1), (5), (12) and (19))

$$\begin{aligned} 0 &\leq \sum_{j=1}^n |\nabla_{e_j}\psi_+ + \frac{1}{2}(e_j - iJ(e_j))\cdot\psi_-|^2 \\ &= |\nabla\psi_+|^2 - \Re e((D + i\tilde D)\psi_+, \psi_-) \\ &\quad -\frac{1}{4}\sum_{j=1}^n ((e_j + iJ(e_j))\cdot(e_j - iJ(e_j))\cdot\psi_-, \psi_-) \\ &= |\nabla\psi_+|^2 - 2\Re e(D\psi_+, \psi_-) + ((m - i\Omega)\cdot\psi_-, \psi_-) \\ &= |\nabla\psi_+|^2 - 8l|\psi_-|^2 + 4l|\psi_-|^2 := G \end{aligned}$$

Just as before, this shows that G is a positive function on M whose integral over M is

$$\begin{aligned} |G|^2 &= |\nabla\psi_+|^2_{L^2} - 4l|\psi_-|^2_{L^2} \\ &= (\nabla^*\nabla\psi_+, \psi_+)_{L^2} - 4l|\psi_-|^2_{L^2} \\ &= (D^2\psi_+ - \frac{R}{4}\psi_+ + \frac{i}{2}\frac{\rho}{2l+1}\cdot\psi_+, \psi_+)_{L^2} - 4l|\psi_-|^2_{L^2} \\ &= |\psi_+|^2_{L^2}\left(16l(l+1) - \frac{(8l+2)(8l+4)}{4} + \frac{i}{2}\frac{-3i(8l+4)}{2l+1} - 4(l+1)\right) = 0, \end{aligned}$$

thus proving $G = 0$. Consequently $\nabla_X\psi_+ + \frac{1}{2}(X - iJ(X))\cdot\psi_- = 0,\ \forall X \in TM$, which is equivalent to (16) and (17).

Q.E.D.

The above proposition motivates the following

Definition 3.2 *A section ψ of the spinor bundle of a given Spin^c structure on a Kähler manifold (M^{8l+2}, g, J) satisfying*

$$\nabla^A_X\psi = \frac{1}{2}X\cdot\psi + \frac{i}{2}JX\cdot\bar\psi, \quad \forall X \in TM \tag{20}$$

is called a Kählerian Killing spinor.

Defining $\psi := \psi_+ - \psi_-$ we immediately obtain the

Corollary 3.2 *Let $\mathcal{C}$ be a complex contact structure on a Kähler–Einstein manifold (M^{8l+2}, g, J). Then the Spin^c structure on M with auxiliary bundle $L_{\mathcal{C}}$ carries a Kählerian Killing spinor $\psi \in \Gamma(\Sigma_{2l+1}M \oplus \Sigma_{2l+2}M)$.*

4 Projectable spinors

In view of the above results, we will restrict our attention to Spin^c structures on Kähler-Einstein manifolds (M^n, g, J) of positive scalar curvature, whose auxiliary bundle L is a root of the canonical bundle K, i.e. $L^{\otimes r} = K$ for some $r \in \mathbf{N}^*$. The canonical connection on K, whose curvature form is just $-i\rho$ (ρ is the Ricci form), induces then a connection A on L, whose curvature form ω satisfies $\omega = -i\rho/r$. Let us denote by $\bar{M}$ the $U(1)$ principal bundle associated to L. By rescaling the metric on M if necessary, we can suppose that the scalar curvature of M is equal to $n(n+2)$, and thus $\rho = (n+2)\Omega$.

The manifold $\bar{M}$ carries a canonical 1–parameter family of Riemannian metrics g_t which make the bundle projection $\pi : \bar{M} \to M$ into a Riemannian submersion with totally geodesic fibres. These metrics are given by

$$g_t(X,Y) = g(\pi_* X, \pi_* Y) - t^2 A(X)A(Y), \ \forall X, Y \in T\bar{M},$$

and we denote by ∇^t the covariant derivative of the Levi-Civita connection of g_t and by V the unit vertical vector field on $(\bar{M}, \bar{g})$ satisfying $A(V) = i/t$. This choice of V fixes an orientation for $\bar{M}$.

Lemma 4.1 *A Spin^c structure with trivial auxiliary bundle is canonically identified with a spin structure. Moreover, if the connection A of the auxiliary bundle L is flat, then by this identification ∇^A corresponds to ∇ on the spinor bundles.*

Proof. One first remark that since the $U(1)$ bundle associated to L is trivial, we can exhibit a global section of it, that we will call σ. Denote by $P_{Spin_n}M$ the inverse image by θ of $P_{SO(n)}M \times \sigma$. It is straightforward to check that this defines a spin structure on M, and that the connection on $P_{Spin^c_n}M$ restricts to the Levi-Civita connection on $P_{Spin_n}M$ if σ can be choosen parallel, i.e. if A defines a flat connection.

Q.E.D.

Proposition 4.1 *Every Spin^c structure on M induces a canonical spin structure on $\bar{M}$.*

PROOF. By enlargement of the structure groups, the two-fold covering

$$\theta : P_{Spin^c_n} M \to P_{SO(n)} M \times P_{U(1)} M,$$

gives a two-fold covering

$$\theta : P_{Spin^c_{n+1}} M \to P_{SO(n+1)} M \times P_{U(1)} M,$$

which, by pull-back through π, gives rise to a Spin^c structure on $\bar{M}$

$$\begin{array}{ccc} P_{Spin^c_{n+1}} \bar{M} & \xrightarrow{\pi} & P_{Spin^c_{n+1}} M \\ \pi^*\theta \downarrow & & \theta \downarrow \\ P_{SO(n+1)} \bar{M} \times P_{U(1)} \bar{M} & \xrightarrow{\pi} & P_{SO(n+1)} M \times P_{U(1)} M \\ \downarrow & & \downarrow \\ \bar{M} & \xrightarrow{\pi} & M \end{array}$$

Using Lemma 4.1 we see that this construction actually yields a spin structure on $\bar{M}$. Indeed, the pull back of a G–principal bundle with respect to its own projection map is allways trivial:

$$\begin{array}{ccc} \pi^* P \simeq P \times G & \xrightarrow{\pi} & P \\ \pi^*\pi \downarrow & & \pi \downarrow \\ P & \xrightarrow{\pi} & M \end{array}$$

Nevertheless, by commodity reasons we will continue to speak about the induced Spin^c structure on $\bar{M}$.

Q.E.D.

The next step is to relate the covariant derivatives of spinors on M and $\bar{M}$. We first have to compute the flat connection on $\pi^* P$.

Lemma 4.2 *The connection form A_0 of the flat connection on $\pi^* P$ can be related with an arbitrary connection A on P by*

$$A_0((\pi^* s)_*(V)) = -A(V), \tag{21}$$

$$A_0((\pi^* s)_*(X^*)) = A(s_* X), \tag{22}$$

where V is a vertical vector field on P, X^* is the horizontal lift (with respect to A) of a vector field X on M, and s is a local section of $P \to M$.

PROOF. The identification $P \times U(1) \simeq \pi^* P$ is given by $(u, a) \mapsto (u, ua)$, $\forall (u, a) \in P \times U(1)$. For some fixed $u \in P$, take a path u_t in the fiber over $x := \pi(u)$ such that $u_0 = u$ and $\dot{u}_0 = V$. We define $a_t \in U(1)$ by $u_t = s(x)a_t$, so via the above identification we have

$$(\pi^* s)(u_t) = (u_t, s(x)) = (u_t, (a_t)^{-1}),$$

and thus

$$A_0((\pi^* s)_*(V)) = -a_0^{-1}\dot{a}_0 = -A(\dot{u}_0) = -A(V).$$

Similarly, for $x \in M$ and $X \in T_x M$, take a path x_t in M such that $x_0 = x$ and $\dot{x}_0 = X$. Let $u \in \pi^{-1}(x)$ and u_t the horizontal lift of x_t such that $u_0 = u$. We define $a_t \in U(1)$ by $s(x_t) = u_t a_t$, which by derivation gives $s_*(X) = R_{a_0}\dot{u}_0 + u_0\dot{a}_0$. Then

$$(\pi^* s)(u_t) = (u_t, s(x_t)) = (u_t, a_t),$$

and thus, using the fact that $\dot{u}_0$ is horizontal,

$$A_0((\pi^* s)_*(X^*)) = a_0^{-1}\dot{a}_0 = A(s_*(X)).$$

Q.E.D.

Recall that the complex Clifford representation Σ_n can be made into a $Cl(n+1)$–representation by defining

$$\rho(e_j) \cdot \psi = \begin{cases} e_j \cdot \psi & \text{for } j \leq n \\ i\bar{\psi} & \text{for } j = n+1 \end{cases}$$

Corresponding to this, we obtain an identification of the pull back $\pi^* \Sigma M$ with $\Sigma \bar{M}$, and with respect to this identification, if X is a vector and ψ a spinor on M, then

$$X^* \cdot \pi^* \psi = \pi^*(X \cdot \psi). \tag{23}$$

$$V \cdot \pi^* \psi = \pi^*(i\bar{\psi}). \tag{24}$$

We are now ready to compute the covariant derivative of projectable spinors on $\bar{M}$.

Proposition 4.2 *The covariant derivative ∇^t on $\Sigma \bar{M}$ induced by the Levi-Civita connection on $(\bar{M}, g_t)$ and the flat connection on $\pi^* P_{U(1)} M$ satisfy*

$$\nabla^t_{X^*}(\pi^* \psi) = \pi^*(\nabla^A_X \psi - i\frac{t(n+2)}{4r} J(X) \cdot \bar{\psi}) \qquad \forall X \in TM, \tag{25}$$

$$\nabla^t_V \pi^* \psi = -\pi^*(\frac{t(n+2)}{4r}\Omega \cdot \psi + \frac{i}{2t}\psi). \tag{26}$$

PROOF. Recall that the curvature form ω of the principal $U(1)$-bundle $\bar{M} \to M$ satisfies

$$\omega = dA = -i\frac{n+2}{r}\pi^*\Omega. \tag{27}$$

The metric g_t is given by

$$g_t(X,Y) = g(\pi_*X, \pi_*Y) - t^2 A(X)A(Y),\ \forall X, Y \in T\bar{M}. \tag{28}$$

If V denotes the unit vertical vector field, then $A(V) = i/t$, and we obtain

$$\begin{aligned} g_t(\nabla^t_{X^*}V, Y^*) &= -\frac{1}{2}g_t(V,[X^*,Y^*]) = \frac{t^2}{2}A(V)A([X^*,Y^*]) \\ &= \frac{it}{2}A([X^*,Y^*]) = -\frac{it}{2}dA(X^*,Y^*) \\ &= -\frac{t(n+2)}{2r}\Omega(X,Y) = \frac{t(n+2)}{2r}g_t(J(X)^*,Y^*), \end{aligned}$$

so finally

$$\nabla^t_{X^*}V = \frac{t(n+2)}{2r}J(X)^*. \tag{29}$$

Consider the pull-back $\pi^*\psi$ of a spinor $\psi = [\sigma, \xi]$, where $\xi : U \subset M \to \Sigma_n$ is a vector valued function, and σ is a local section of $P_{Spin^c_n}M$ whose projection onto $P_{SO(n)}M$ is a local orthonormal frame $(X_1, ..., X_n)$ and whose projection onto $P_{U(1)}M$ is a local section s. Then $\pi^*\psi$ can be expressed as $\pi^*\psi = [\pi^*\sigma, \pi^*\xi]$, and it is easy to see that the projection of $\pi^*\sigma$ onto $P_{SO(n+1)}\bar{M}$ is the local orthonormal frame $(X_1^*, ..., X_n^*, V)$ and its projection onto $P_{U(1)}\bar{M}$ is just π^*s.

Using Lemma 4.2 and (29) we obtain

$$\begin{aligned} \nabla^t_{X^*}\pi^*\psi &= [\pi^*\sigma, X^*(\pi^*\xi)] + \frac{1}{2}\sum_{j<k} g_t(\nabla^t_{X^*}X_j^*, X_k^*)X_j^* \cdot X_k^* \cdot \pi^*\psi \\ &\quad + \frac{1}{2}\sum_j g_t(\nabla^t_{X^*}X_j^*, V)X_j^* \cdot V \cdot \pi^*\psi + \frac{1}{2}A_0((\pi^*s)_*X^*)\pi^*\psi \\ &= [\pi^*\sigma, \pi^*(X(\xi))] + \frac{1}{2}\sum_{j<k} g(\nabla_X X_j, X_k)\pi^*(X_j \cdot X_k \cdot \psi) \\ &\quad - \frac{1}{2}\frac{t(n+2)}{2r}i\sum_j g(J(X), X_j)\pi^*(X_j \cdot \bar{\psi}) + \frac{1}{2}A(s_*X)\pi^*\psi \\ &= \pi^*\Big([\sigma, (X(\xi))] + \frac{1}{2}\sum_{j<k} g(\nabla_X X_j, X_k)X_j \cdot X_k \cdot \psi \\ &\quad - i\frac{t(n+2)}{4r}J(X) \cdot \bar{\psi} + \frac{1}{2}A(s_*X)\psi\Big) \\ &= \pi^*(\nabla^A_X\psi - i\frac{t(n+2)}{4r}J(X) \cdot \bar{\psi}). \end{aligned}$$

and similarly,

$$\begin{aligned}
\nabla^t_V \pi^*\psi &= [\pi^*\sigma, V(\pi^*\xi)] + \frac{1}{2}\sum_{j<k} g_t(\nabla^t_V X^*_j, X^*_k) X^*_j \cdot X^*_k \cdot \pi^*\psi \\
&\quad + \frac{1}{2}\sum_j g_t(\nabla^t_V X^*_j, V) X^*_j \cdot V \cdot \pi^*\psi + \frac{1}{2} A_0((\pi^* s)_* V)\pi^*\psi \\
&= \frac{t(n+2)}{2r}\frac{1}{2}\sum_{j<k} g(J(X_j), X_k)\pi^*(X_j \cdot X_k \cdot \psi) - \frac{1}{2}A(V)\pi^*\psi \\
&= -\frac{t(n+2)}{4r}\pi^*(\Omega\cdot\psi) - \frac{i}{2t}\pi^*\psi \\
&= -\pi^*\Big(\frac{t(n+2)}{4r}\Omega\cdot\psi + \frac{i}{2t}\psi\Big).
\end{aligned}$$

Q.E.D.

The formula (29) allows us to compute the Ricci tensor Ric^t of the Riemannian manifold $(\bar{M}, g_t)$. If we denote by $a := \frac{t(n+2)}{2r}$, then

$$\mathrm{Ric}^t(V,V) = na^2 \quad , \quad \mathrm{Ric}^t(X^*, V) = 0, \tag{30}$$

$$\mathrm{Ric}^t(X^*, Y^*) = (n + 2 - 2a^2) g(X, Y). \tag{31}$$

Let us take $t_0 = \frac{2r}{n+2}$ and denote $\bar{g} := g_{t_0}$, $\bar{\nabla} := \nabla^{t_0}$. The vertical vector field V defines then an Einstein–Sasakian structure on the manifold $(\bar{M}, \bar{g})$ (cf. [2]). We can synthetise the above results in the following

Theorem 4.1 *Let (M^n, g, J) be a Kähler-Einstein manifold with scalar curvature $R = n(n+2)$, $L := K^{\frac{1}{r}}$ a root of the canonical bundle K and $\bar{M}$ the associated $U(1)$ principal bundle with connection form A, induced by the Levi-Civita connection on K. Then the following hold:*

(i) *There is a canonical metric $\bar{g}$ on $\bar{M}$ making the bundle projection $\pi : \bar{M} \to M$ into a Riemannian submersion with totally geodesic fibres, and satisfying $\bar{\nabla}_{X^*} V = J(X)^*$.*

(ii) *With respect to the metric $\bar{g}$, V defines a regular Einstein–Sasakian structure on $\bar{M}$. The length of the fibres of the corresponding S^1–action is constant and equal to $\frac{4\pi r}{n+2}$.*

(iii) *The Spin^c structure on M defined by (L, A) induces a canonical spin structure on $\bar{M}$ and every spinor on M induces a projectable spinor $\pi^*\psi$ on $\bar{M}$, satisfying*

$$\bar{\nabla}_{X^*}(\pi^*\psi) = \pi^*(\nabla^A_X \psi - \frac{i}{2} J(X)\cdot\psi) \qquad \forall X \in TM, \tag{32}$$

$$\bar{\nabla}_V \pi^*\psi = -\frac{1}{2}\pi^*(\Omega\cdot\psi + \frac{i(n+2)}{2r}\psi). \tag{33}$$

5 Consequences

We can now state the main result of this paper:

Theorem 5.1 *Let M be a compact Kähler manifold of positive scalar curvature and complex dimension $4l+1$. Then the following statements are equivalent:*

(i) *M is the twistor space of some quaternionic Kähler manifold of positive scalar curvature;*

(ii) *M is Kähler-Einstein and admits a complex contact structure;*

(iii) *There exist a Spin^c structure on M with auxiliary bundle L and spinor bundle ΣM such that $L^{\otimes(2l+1)} \cong \Lambda^{4l+1,0}M$ and ΣM carries a Kählerian Killing spinor $\psi \in \Gamma(\Sigma_{2l+1}M \oplus \Sigma_{2l+2}M)$.*

Proof. The implications (i)$\Longrightarrow$(ii) and (ii)$\Longrightarrow$(iii) follow directly from [11] and Corollary 3.2 respectively.

Suppose now that (iii) holds. The proof of (iii)$\Longrightarrow$(i) goes roughly as in [8]. We first show that M is Kähler–Einstein. Let $\psi \in \Gamma(\Sigma_{2l+1}M \oplus \Sigma_{2l+2}M)$ be a spinor on M which satisfies (20). Taking the covariant derivative with respect to an arbitrary vector field Y we obtain

$$\nabla^A_Y \nabla^A_X \psi = \frac{1}{4}(X \cdot Y + JX \cdot JY) \cdot \psi + \frac{i}{4}(X \cdot JY - JX \cdot Y) \cdot \ bar\psi, \tag{34}$$

which easily implies

$$\mathcal{R}^A_{X,Y}\psi = \frac{1}{2}(X \cdot Y + JX \cdot JY) \cdot \psi - ig(X, JY)\bar{\psi}. \tag{35}$$

A local computation shows that the curvature operator $\mathcal{R}^A$ on the spinor bundle is given by the formula

$$\mathcal{R}^A = \mathcal{R} + \frac{i}{2}\omega, \tag{36}$$

where $i\omega := -\frac{i\rho}{2l+1}$ is the curvature form of the auxiliary bundle L, and

$$\mathcal{R}_{X,Y} = \frac{1}{2}\sum_{j<k} R(X, Y, e_j, e_k)e_j \cdot e_k \cdot \tag{37}$$

in a local orthonormal frame $\{e_1, ..., e_n\}$. Using the first Bianchi identity for the curvature tensor one obtains ([2], p.16)

$$\sum_i e_i \cdot \mathcal{R}_{e_i,X} = \frac{1}{2}\mathrm{Ric}(X), \tag{38}$$

so by (36) and (38),

$$\sum_j e_j \cdot \mathcal{R}^A_{e_j,X}\psi = \sum_j e_j \cdot (\mathcal{R}_{e_j,X}\psi + \frac{i}{2}\omega(e_j,X)\psi) = \frac{1}{2}\mathrm{Ric}(X)\cdot\psi - \frac{i}{2}X \lrcorner \omega\cdot\psi. \quad (39)$$

On the other hand, (35) yields

$$\begin{aligned}\sum_j e_j \cdot \mathcal{R}^A_{e_j,X}\psi &= (4l+2)X\cdot\psi + iJX\cdot\bar{\psi} + JX\cdot\Omega\cdot\psi \\ &= (4l+2)X\cdot\psi - 2iJX\cdot\psi,\end{aligned}$$

which, together with (39) gives

$$\Big(\frac{1}{2}\mathrm{Ric}(X) - (4l+2)X\Big)\cdot\psi = \frac{i}{2l+1}J\Big(\frac{1}{2}\mathrm{Ric}(X) - (4l+2)X\Big)\cdot\psi. \quad (40)$$

As ψ never vanishes, if the equality $A\cdot\psi = iB\cdot\psi$ holds for some real vectors A, B, then $|A| = |B|$. The above formula thus shows that $\mathrm{Ric}(X) = (8l+4)X$, $\forall X \in TM$, so M is Kähler–Einstein with scalar curvature $R = (8l+2)(8l+4)$.

The spinor $\pi^*\psi$ induces then a parallel spinor Ψ on the cone $C\bar{M}$ over $\bar{M}$, which is a Kähler manifold (cf. [1], [8], [9]). Moreover, using (**??**) we can compute the action of the Kähler form of $C\bar{M}$ on Ψ, and obtain that $\Psi \in \Sigma_{2l+3}C\bar{M}$. From C. Bär's classification [1] we know that the restricted holonomy group of $C\bar{M}$ is one of the following: $SU(4l+2)$, $Sp(2l+1)$ or 0. The fixed points of the spin representation of $SU(4l+2)$ ly in Σ_0 and Σ_{4l+2}, so as Ψ is a parallel spinor in $\Sigma_{2l+3}C\bar{M}$, the restricted holonomy group of $C\bar{M}$ cannot be equal to $SU(4l+2)$. This implies that the universal covering of $C\bar{M}$ is hyperkähler, and thus that the universal covering of $\bar{M}$ is 3–Sasakian (see [1]).

Let us denote by $\bar{M}'$ the $U(1)$–bundle associated to some maximal root of L. Using the Gysin exact sequence we deduce that $\bar{M}'$ is simply connected (see [2], p.85). Moreover, there exists a canonical covering projection $\bar{M}' \to \bar{M}$, thus proving that $\bar{M}'$ is the universal covering of $\bar{M}$. Consequently, $(\bar{M}', \bar{g}')$ is a 3–Sasakian manifold, where $\bar{g}'$ is the metric induced from $\bar{g}$ via the covering projection. On the other hand, the unit vertical vector field V' on $\bar{M}'$ defines a Sasakian structure, since this is true for its projection V on $\bar{M}$. It is well known that any Sasakian structure on a 3–Sasakian manifold P^{4k-1} of non-constant sectional curvature belongs to the 2–sphere of Sasakian structures. Indeed, the cone CP over P has restricted holonomy $Sp(k)$, and since the centraliser of $Sp(k)$ in $U(2k)$ is just $Sp(1)$, every Kähler structure on CP must belong to the 2–sphere of Kähler structures of CP, which is equivalent to our statement.

Now, $\bar{M}'$ is regular in the direction of V', so an old result of Tanno implies that it is actually a regular 3–Sasakian manifold (cf. [12]). It is then well known that the quotient of $\bar{M}'$ by the corresponding $SO(3)$ action is a

quaternionic Kähler manifold of positive scalar curvature, say N, and that the twistor space over N is biholomorphic to the quotient of $\bar{M}'$ by each of the S^1 actions given by the Sasakian vector fields, so in particular to M, which is the quotient of $\bar{M}'$ by the S^1 action generated by V'.

Q.E.D.

result of LeBrun mentioned in Section 1:

References

[1] C. BÄR, *Real Killing spinors and holonomy*, Commun. Math. Phys. **154** (1993), 509-521.

[2] H. BAUM, Th. FRIEDRICH, R. GRUNEWALD, I. KATH, *Twistor and Killing Spinors on Riemannian Manifolds,* Teubner-Verlag, Stuttgart-Leipzig, 1991.

[3] O. HIJAZI, *Eigenvalues of the Dirac operator on Compact Kähler Manifolds*, Commun. Math. Phys. **160** (1994), 563-579.

[4] K.-D. KIRCHBERG, U. SEMMELMANN, *Complex Contact Structures and the first Eigenvalue of the Dirac Operator on Kähler Manifolds*, GAFA **5** No.3 (1995), 604-618.

[5] S. KOBAYASHI, *Remarks on Complex Contact Manifolds*, Proc. Amer. Math. Soc. **10** (1959), 164-167.

[6] B. LAWSON, M.-L. MICHELSON, *Spin Geometry,* Princeton University Press, 1989.

[7] C. LEBRUN, *Fano Manifols, Contact Structures, and Quaternionic Geometry*, Int. J. Math. **6** (1995), 419-437.

[8] A. MOROIANU, *La première valeur propre de l'opérateur de Dirac sur les variétés kählériennes compactes*, Commun. Math. Phys. **169** (1995), 373-384.

[9] A. MOROIANU, *Parallel and Killing Spinors on Spin^c manifolds*, SFB 288 Preprint No. **241** (1996).

[10] A. MOROIANU, U. SEMMELMANN, *Kählerian Killing Spinors, Complex Contact Structures and Twistor Spaces*, C.R. Acad. Sci. Paris **t. 323** Série I (1996), 57-61.

[11] S. SALAMON, *Quaternionic Kähler Manifolds*, Invent. Math. **67** (1982), 143-171.

[12] S. Tanno, *Killing vectors on contact Riemannian manifolds and fiberings related to the Hopf fibrations*, Tohoku Math. J. **23** (1971), 314-333.

Andrei Moroianu
Institut für Reine Mathematik
Humboldt-Universität zu Berlin
Ziegelstr. 13a
D–10099 Berlin