Elliptic Calogero Moser quantum problem and deformations of algebraic surfaces

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Abstract

We deform the complex Bloch variety of a special case of the trigonometric Calogero-Moser operator (Sutherland operator for three particles) into the Bloch variety of the corresponding elliptic operator. This yields an explicit description of the complex Bloch spectral variety for this elliptic Calogero-Moser operator, which is shown to be a nonsingular projective surface. We conclude that this operator is algebraically integrable and finite-gap on a generic energy level.

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1 Introduction

One of the most remarkable one-dimensional Schrödinger operators is the classical *Lamé operator*

$$L = -\frac{d^2}{dx^2} + m(m+1)\wp(x).$$

Here $\wp(x)$ is the Weierstraß elliptic function and m is an integer. The corresponding *Lamé equation*

$$\left(-\frac{d^2}{dx^2} + m(m+1)\wp(x)\right)\psi(x) = \lambda\psi(x)$$

has been an object of the classical investigation of Lamé, Halphen, Hermite and Stieltjes (see e.g. [W-W]). In particular Hermite has found some formulas for the solutions ψ of Lamé equation in terms of classical elliptic functions. In the simplest case $m = 1$, i. e. for the equation

$$\left(-\frac{d^2}{dx^2} + 2\wp(x)\right)\psi(x) = \lambda\psi(x)$$

the solution ψ has the form

$$\psi(x,z) = \exp\left(\zeta(z)x\right)\frac{\sigma(z-x)}{\sigma(x)\sigma(z)},$$

where ζ and σ are classical elliptic functions, z is a spectral parameter such that $\lambda = -\wp(z)$. This function has the following properties

(i) for $i = 1, 2$

$$\psi(x + 2\omega_i, z) = \exp\left(2\omega_i\zeta(z) - z\eta_i\right)\psi(x,z),$$

where $2\omega_1, 2\omega_2$ are the periods of $\wp(x)$ and $\eta_i = \zeta(\omega_i)$ (see [W-W]).

(ii) for $i = 1, 2$

$$\psi(x, z + 2\omega_i) = \psi(x,z).$$

The first property means that ψ is the *complex Bloch eigenfunction* with respect to both periods $2\omega_1$ and $2\omega_2$. The second property means that the spectral parameter z takes its values in the same elliptic curve as the potential $\wp(x)$:

$$\mu^2 = 4\lambda^3 - g_2\lambda - g_3,$$

which is the *complex Bloch spectral curve*[1] of the Lamé operator for $m = 1$. This coincidence is a special property of the case $m = 1$. Already for $m = 2$ the spectral curve is a hyperelliptic curve defined by the equation

$$\mu^2 = \lambda^5 - 21/4g_2\lambda^3 - 27/4g_2\lambda^2 + 27/4g_2^2\lambda + 81/4g_2g_3.$$

[1]The real Bloch spectral curve is determined by the condition that the multiplier $\xi_1 = \exp\left(2\omega_1\zeta(z) - z\eta_1\right)$, corresponding to the real period ω_1, has the modulus 1, which imposes some reality conditions on μ and z. We will not discuss such conditions in our paper. Notice that in contrast to the classical case when a half-period shift makes the Lamé equation both real and nonsingular for its two-dimensional analogue 1 it is impossible: the operator 1 has no nonsingular real form.

For a general periodic potential $q(x)$ the variety of Bloch eigenfunctions of the Schrödinger operator $L = -d^2/dx^2 + q(x)$ is not algebraic. This is an exceptional property of the so-called 'finite-gap' operators. The general theory of such operators has been developed about twenty years ago (see [D-K-N-2] and references therein). In particular, it was shown that all hyperelliptic curves can be taken as the spectral curve for some 'finite-gap' potential $q(x)$ and for a given curve Γ of genus g all such potentials form a torus isomorphic to the Jacobi variety $J(\Gamma)$. The condition on the hyperelliptic curve to be a spectral curve for a periodic Schrödinger operator is known to be transcendental. It can be formulated in terms of conformal mappings as it has been shown by V. A. Marchenko. An alternative approach has been recently proposed by P. Grinevich and one of the authors [G-S]. The idea is to write down a system of ODE's describing the deformations of the hyperelliptic curves preserving the periods of the corresponding finite-gap Schrödinger operators or, equivalently, the periods of certain Abelian differentials on this curves (see [G-S]) for the details).

In the present paper we apply the same idea to describe explicitly the complex Bloch variety of a two-dimensional analogue of the Lamé operator corresponding to a special case of the quantum elliptic Calogero-Moser system

$$L = -\left(\frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2} + \frac{\partial^2}{\partial q_3^2}\right) + 4\left(\wp(q_1 - q_2) + \wp(q_2 - q_3) + \wp(q_3 - q_1)\right).$$

This is an operator on the three dimensional space, but one can reduce it to the plane $q_1 + q_2 + q_3 = 0$. In fact, the operator L commutes with the vector-field $\partial/\partial q_1 + \partial/\partial q_2 + \partial/\partial q_3$ and we may restrict it to the space of functions, which are constant along this vector-field. In the coordinates $x_1 = q_1 - q_2, x_2 = q_2 - q_3$ the reduced operator has the form

$$L = -2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1 \partial x_2}\right) + 4\left(\wp(x_1) + \wp(x_2) + \wp(-x_1 - x_2)\right) \tag{1}$$

and can be considered as a two-dimensional analogue of Lamé operator. Notice that Lamé operator is the reduced Calogero-Moser operator for two particles. The main result of the present paper is the following explicit description of the Bloch variety of the operator (1).

Theorem 1 *Let's consider the algebraic surface Y defined by the equations*

$$\begin{aligned} \nu^3 + a(\lambda,\mu)\nu + b(\lambda,\mu) &= 0 \\ \kappa^2 + \nu\kappa + (\nu^2 + a(\lambda,\mu)) &= 0, \end{aligned} \tag{2}$$

where

$$\begin{aligned} a(\lambda,\mu) &= 6\lambda\mu^2 - 3\nu_0^2(\lambda), \\ b(\lambda,\mu) &= -\mu^4 + (10\lambda^3 - 18g_2\lambda + 108g_3)\mu^2 + 2\nu_0^3(\lambda), \\ \nu_0(\lambda) &= \lambda^2 - 3g_2. \end{aligned}$$

There exists a function $\psi(x_1, x_2, P)$ on $\mathbb{C}^2 \times Y$, which is meromorphic on $\mathbb{C}^2\times$ the affine part of Y, has a essential singularity at infinity, and is a complex Bloch eigenfunction of the operator L (1) with respect to all periods of the potential:

$$L\psi(x_1, x_2, P) = -8\lambda(P)\psi(x_1, x_2, P), \; P \in Y.$$

We should mention that in the recent papers [F-V-1, F-V-2] G. Felder and A. Varchenko have found the description of the Bloch variety for elliptic Calogero-Moser operator in general case, when

$$L = -\Delta_N + \sum_{i<j}^{N} 2m(m+1)\wp(x_i - x_j),\ m \in \mathbb{Z}.$$

The Bethe-ansatz equations, which are algebraic on the product of the elliptic curves, define the 'Hermite-Bethe-variety'. Although this is probably the most appropriate description of the general case, it is not effective enough to investigate the property of the corresponding algebraic variety. Our formulas (2) give such a possibility in the first nontrivial case $m = 1,\ N = 3$. In this particular case the results of Felder and Varchenko were independently found by V. Inozemtsev [In].

Our idea was to consider the operator (1) as a deformation of the trigonometric operator

$$L = -2\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1 \partial x_2}\right) + 4\left(\frac{1}{\sin^2(x_1)} + \frac{1}{\sin^2(x_2)} + \frac{1}{\sin^2(-x_1 - x_2)}\right), \tag{3}$$

which is a reduction of the Sutherland operator for three particles

$$L = -\left(\frac{\partial^2}{\partial q_1^2} + \frac{\partial^2}{\partial q_2^2} + \frac{\partial^2}{\partial q_3^2}\right) + 4\left(\frac{1}{\sin^2(q_1 - q_2)} + \frac{1}{\sin(q_2 - q_3)} + \frac{1}{\sin(q_3 - q_1)}\right).$$

Combining the description of the Bloch variety of (3) found by O. Chalykh and one of the authors in [C-V] with the one-dimensional calculations in [G-S] we succeed in proving the explicit description (2) of the 'isoperiodic deformation' of the spectral surface of the operator (3) to be the Bloch variety of the elliptic Calogero-Moser-operator. As a corollary of our results this operator is shown to be algebraically integrable in the sense of [Kr, C-S-V, Sch-2], which has been conjectured in [C-S-V].

In the last section we analyze the geometry of the complex Fermi-curves λ = constant and show that the Schrödinger operator is 'finite-gap' for generic energy levels in the sense of Dubrovin, Krichever and Novikov (see [D-K-N-1, N-V-1, N-V-1]). Hence the Bloch-eigenfunction ψ may be written in terms of Prym's theta-functions of the corresponding algebraic curves.

2 The spectral variety of the Calogero-Moser operator in the trigonometric case

The Bloch varieties of the operator (3) and its generalizations related to an arbitrary root system have been described in [C-V, C-S-V]. They are shown to be singular affine varieties. In our case the root system is of type A_2 and the underlying space is the Cartan subalgebra of $su(3,\mathbb{C})$. More precisely, the period lattice Γ is equal to the weight lattice of $su(3,\mathbb{C})$, and the momentum lattice $\Gamma^\sharp$ is equal to the root lattice of $su(3,\mathbb{C})$. Let us parameterize the space by the vectors $x = x_1\lambda_1 + x_2\lambda_2$, where λ_1 and λ_2 are the two fundamental weights, and the momentum space by the vectors $k = k_1\alpha_1 + k_2\alpha_2$, where α_1 and α_2 are the simple roots. Thus the period lattice is isomorphic to the subset $\Gamma \simeq \mathbb{Z}^2 \subset \mathbb{R}^2$ of the parameter space of the Cartan subalgebra and the momentum lattice is isomorphic to the subset $\Gamma^\sharp \simeq \mathbb{Z}^2 \subset \mathbb{C}^2$ of the parameter space of the momentum space. The trigonometric Calogero-Moser operator with respect to these coordinates is equal to

$$-2\frac{\partial^2}{\partial x_1^2} - 2\frac{\partial^2}{\partial x_2^2} + 2\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{4\pi^2}{\sin^2(\pi x_1)} + \frac{4\pi^2}{\sin^2(\pi x_2)} + \frac{4\pi^2}{\sin^2(-\pi(x_1+x_2))}.$$

The normalization of the Bloch surface may be described by all polynomials $p(k_1, k_2)$ in two variables k_1 and k_2. The structure ring $\mathcal{R}$ contains all polynomials, which obey the condition, that for all roots α and all $k \in \mathbb{C}^2$, which are orthogonal to α, the value $p(k + \alpha/2)$ is equal to $p(k - \alpha/2)$. More explicitly, this yields in terms of our coordinates (k_1, k_2) the conditions:

(i) $p(k + 1/2, 2k) = p(k - 1/2, 2k)$ for all $k \in \mathbb{C}$.

(ii) $p(2k, k + 1/2) = p(2k, k - 1/2)$ for all $k \in \mathbb{C}$.

(iii) $p(k + 1/2, -k + 1/2) = p(k - 1/2, -k - 1/2)$ for all $k \in \mathbb{C}$.

The elements of this ring may be considered as the quantum integrals of the Schrödinger operator. The standard integrals [O-P] correspond to the polynomials, which are invariant under the action of the Weyl group, and therefore obviously satisfy these conditions. They form a subring, which due to Chevalley Theorem is freely generated by 2 (=rank of $su(3, \mathbb{C})$) elements. In our case the Weyl group is the permutation group of three elements and the action on the spectral variety is given by the permutation of the three variables $k_1, k_2, -k_1 + k_2$. The invariant functions are generated by the elementary symmetric functions. It is convenient for us to use the following generators:

$$\lambda = k_1(-k_2) + k_1(-k_1 + k_2) + (-k_2)(-k_1 + k_2) = k_1 k_2 - k_1^2 - k_2^2;$$

$$\mu = 3\sqrt{3}k_1(-k_2)(-k_1 + k_2) = 3\sqrt{3}k_1 k_2(k_1 - k_2).$$

The eigenvalue of the Schrödinger operator is equal to $-8\pi^2\lambda$. But the ring $\mathcal{R}$ is bigger than all invariant functions. Let us give three examples:

$$\tilde{\nu}_1 = ((k_1 + k_2)^2 - 1)((2k_1 - k_2)^2 - 1);$$

$$\tilde{\nu}_2 = ((k_1 + k_2)^2 - 1)((k_1 - 2k_2)^2 - 1);$$

$$\tilde{\nu}_3 = ((2k_1 - k_2)^2 - 1)((k_1 - 2k_2)^2 - 1).$$

It is easy to see, that these three polynomials are transformed into each other under the action of the permutation group. In particular are the elementary symmetric functions of these three polynomials $\tilde{\nu}_1 + \tilde{\nu}_2 + \tilde{\nu}_3, \tilde{\nu}_1\tilde{\nu}_2 + \tilde{\nu}_1\tilde{\nu}_3 + \tilde{\nu}_2\tilde{\nu}_3$ and $\tilde{\nu}_1\tilde{\nu}_2\tilde{\nu}_3$ invariant under the action of the permutation group and therefore polynomials in λ and μ. From a more geometric point of view this means that our singular complex surface is a 6-fold covering space over $\mathbb{C} \times \mathbb{C}$, and the covering map is given by (λ, μ). In fact, from this point of view $\tilde{\nu}_1, \tilde{\nu}_2$ and $\tilde{\nu}_3$ are the values of one function $\tilde{\nu}$ on the sheets of this covering map, respectively. This function $\tilde{\nu}$ may be defined as the solution of the equation:

$$\tilde{\nu}^3 - \tilde{\nu}^2(3 + 12\lambda + 9\lambda^2) + \tilde{\nu}(6\lambda + 3)(4\lambda^3 + \mu^2 + 9\lambda^2 + 6\lambda + 1) - (4\lambda^3 + \mu^2 + 9\lambda^2 + 6\lambda + 1)^2 = 0.$$

This polynomial of third degree with respect to $\tilde{\nu}$ can be used to describe the 6-fold covering mentioned above. In fact, all sheets are determined by a numeration of all roots of this polynomial, and the permutation group acts by permutation of the corresponding roots.

Now we want to determine, whether the functions $\lambda, \mu, \tilde{\nu}_1, \tilde{\nu}_2$ generate the ring $\mathcal{R}$. In order to do this let us calculate the singularities of the ring generated by these functions. So let us determine the zero set of the discriminant of the polynomial of third degree with respect to $\tilde{\nu}$.

$$(\tilde{\nu}_1 - \tilde{\nu}_2)(\tilde{\nu}_1 - \tilde{\nu}_3) = ((k_1 + k_2)^2 - 1)((2k_1 - k_2)^2 - 1)(k_1 + k_2)(2k_1 - k_2)3k_2(3k_1 - 3k_2).$$

The first two factors describe the singularities of the ring $\mathcal{R}$, the third and the fourth factor are part of the branching divisor of the covering map $(k_1, k_2) \mapsto (\lambda, \mu)$, only the last two factors describe singularities, which are not part of the ring $\mathcal{R}$. In fact, this calculation shows that the discriminant of our polynomial differs by the factor $k_1^2 k_2^2 (k_1 - k_2)^2$ from the singularities of the ring $\mathcal{R}$ times the branching divisor of the covering map $(k_1, k_2) \mapsto (\lambda, \mu)$, which is given by $(k_1 + k_2)^2 (2k_1 - k_2)^2 (k_1 - 2k_2)^2$. This shows that we need more generators, in order to describe the ring $\mathcal{R}$. In fact, the functions $((k_1 + k_2)^2 - 1)((2k_1 - k_2)^2 - 1)((k_1 - 2k_2)^2 - 1)k_1$ and $((k_1 + k_2)^2 - 1)((2k_1 - k_2)^2 - 1)((k_1 - 2k_2)^2 - 1)k_2$ belong to the ring $\mathcal{R}$, but cannot be written as a polynomial of of $\lambda, \mu, \tilde{\nu}_1, \tilde{\nu}_2$ (cf.[C-S-V]). It can be shown that all of them together generate the ring $\mathcal{R}$.

For reasons of simplicity let us use in the sequel another function ν instead of $\tilde{\nu}$:

$$\nu = \tilde{\nu} - 1 - 4\lambda - 3\lambda^2.$$

This function ν is the solution of the following equation

$$R(\lambda, \mu, \nu) = \nu^3 + (6\lambda\mu^2 + 3\mu^2 - 3\lambda^2(\lambda+1)^2)\nu - \mu^4 + (10\lambda^3 + 15\lambda^2 + 6\lambda + 1)\mu^2 + 2\lambda^3(\lambda+1)^3 = 0.$$

In the next chapters we will deform this singular Bloch surface into another Bloch surface. The main tool of this deformation are some holomorphic 2-forms. Let us first give the definition of these 2-forms: A form $\omega = f d\lambda \wedge d\mu$ is called admissible, if

admissible (i) ω transforms under the action of the permutation group like an element of an irreducible two-dimensional representation, and

admissible (ii) for all elements $p \in \mathcal{R}$ the sum over all sheets of the covering map induced by (λ, μ) of fp is a holomorphic function of λ and μ.

All of them are of the form $\omega =$

$$\text{some polynomial}(\lambda, \mu, \nu)\frac{\mu}{R_\nu(\lambda, \mu, \nu)} d\lambda \wedge d\mu + \text{some polynomial}(\lambda, \mu, \nu)\frac{\nu - \nu_0}{R_\nu(\lambda, \mu, \nu)} d\lambda \wedge d\mu.$$

Here ν_0 is equal to the value of ν at the singularity, which does not belong to the ring $\mathcal{R}$: $\nu_0 = \lambda(\lambda + 1)$.

3 The deformation into the elliptic case

To each element of the period lattice there corresponds one Floquet-multiplier. All of them are entire functions on our singular complex surface and have essential singularities, when the energy λ goes to infinity. All of them are products of some powers of the two Floquet-multipliers corresponding to k_1 and $-k_2$:

$$\xi_1 = \exp(2\pi\sqrt{-1}k_1) \text{ and } \xi_2 = \exp(-2\pi\sqrt{-1}k_2),$$

because they build a representation of the root lattice, which is isomorphic to $\mathbb{Z}^2$. The functions $k_1, -k_2, -k_1 + k_2$ are the roots of the polynomial

$$k^3 + \lambda k - \mu/(3\sqrt{3}) = 0.$$

Hence the covering map of the normalization of our singular surface to $\mathbb{C}^2$, which is defined by (λ, μ) may be described as follows: First we use the three fold covering, which is given by one solution of the foregoing polynomial equation. Afterwards we use the two fold covering over this three fold covering, which is given by another solution of this polynomial equation. Then we obtain a 6 fold covering, and each sheet corresponds to a numeration of the three different solutions of the foregoing polynomial equation. The Weyl group acts by permutation of the solutions, which results in a permutation of the sheets. Now it is obvious that we can use the three Floquet-multipliers $\xi_1, \xi_2, \xi_1^{-1}\xi_2^{-1}$ to get a similar description of our singular surface. Hence the spectral surface may be described by some equation

$$\xi^3 + \xi^2 \text{ times some entire function}(\lambda, \mu) +$$

$$+\xi \text{ times some entire function } (\lambda, \mu) \; - 1 = 0.$$

This description of the spectral surface of course contains much more singularities, then our singular surface, but all singularities of our spectral surface are part of this description. Now let us consider some deformation of the trigonometric potential such that

deformation condition (i) all deformed Schrödinger operators are still invariant under the Weyl group.

deformation condition (ii) the commuting differential operator corresponding to the eigenvalue μ can be deformed simultaneously.

The deformation of the trigonometric Calogero-Moser operator into the elliptic one fulfills both conditions (see [O-P, O-O-S]).

Now we claim, that in each such situation the expression

$$\frac{d\lambda}{dt} d\mu \wedge d\ln\xi - \frac{d\mu}{dt} d\lambda \wedge d\ln\xi + \frac{d\ln\xi}{dt} d\lambda \wedge d\mu$$

is an admissible 2-form of our singular spectral surface. Here t is some deformation parameter. In fact, the deformation conditions (i) and (ii) imply the condition admissible (i) of the 2-form, and the proof of condition admissible (ii) is analogous to the corresponding statement for one-dimensional Schrödinger operators [G-S, Sch-1]. In order to identify locally the different spectral surfaces we may choose two of the three functions λ, μ, ξ not to depend on the deformation parameter t. In this case the expression for the 2-form simplifies. We shall use another property of these 2-forms. In fact, in [K-T] it was proven that the infinite energy part of the spectral surfaces is given by the unions of the spectral curves of all averaged one-dimensional potentials. Hence these 2-forms have to describe in the infinite energy limit the deformation of the spectral curves of the one-dimensional Schrödinger operators. More precisely, let $\bar{\lambda}$ be the energy function of an averaged one-dimensional Schrödinger operator, which is defined on some neighborhood of the corresponding spectral curve. Then the expression

$$\frac{d\bar{\lambda}}{dt} d\ln\xi - \frac{d\ln\xi}{dt} d\bar{\lambda}$$

has to be a holomorphic 1-form on this neighborhood.

The two-dimensional trigonometric potential is a sum of three one-dimensional potentials. With the results of [K-T, Sch-2] we conclude that the singular spectral surface extend to an algebraic singular surface after adding the three singular spectral curves of the one-dimensional potentials. Let us call the energy of the one-dimensional Schrödinger operator with averaged potential $\bar{\lambda}$ and the quasi momentum in direction of some of the basic vectors of the momentum lattice $1/v$. Due to [K-T] the three cases corresponds to $1/v = k_1, -k_2$ and $-k_1 + k_2$, respectively. In the trigonometric case the energy $\bar{\lambda}$ is equal to $-(k_1/2 - k_2)^2, -(k_1 - k_2/2)^2$ and $(k_1/2 + k_2/2)^2$, respectively. Due to this normalization of the energy $\bar{\lambda}$, the singular spectral curves of the one-dimensional trigonometric potentials have a double points at $\bar{\lambda} = -1/4$. Then for very small v the functions λ and μ may be written as:

$$\lambda = -\frac{3}{4v^2} + \bar{\lambda} \text{ and } \mu = -\frac{3\sqrt{3}}{v}\left(\frac{1}{4v^2} + \bar{\lambda}\right).$$

In the limit $v \to 0$ only three 2-forms of the admissible 2-forms we found in the last section have the correct behavior:

$$\frac{\mu}{R_\nu(\lambda,\mu,\nu)} d\lambda \wedge d\mu; \frac{\nu - \nu_0}{R_\nu(\lambda,\mu,\nu)} d\lambda \wedge d\mu; \frac{\lambda\mu}{R_\nu(\lambda,\mu,\nu)} d\lambda \wedge d\mu.$$

The first and the second form correspond in the limit $v \to 0$ to no deformation of the singular curve (we will see that they do not correspond to a deformation) and the third corresponds to the deformation of the one-dimensional trigonometric Schrödinger operator into the elliptic one.

Now we will deform the trigonometric Calogero-Moser operator into the elliptic one. We want to avoid to use the transcendental functions, which always occur if we use the Floquet-multipliers. But we can use the logarithmic derivatives of the Floquet-multipliers, which are meromorphic functions. In general we assume that λ and μ do not depend on the deformation parameter t. Hence all functions locally are considered as functions depending only on the independent variables λ, μ and t. Then we have in the trigonometric case

$$\frac{d\ln\xi}{2\pi\sqrt{-1}} = -9\frac{\mu(\nu + 3\lambda^2 + 4\lambda + 1)}{R_\nu(\lambda,\mu,\nu)} d\lambda - 3\frac{(\lambda+1)(\nu - \nu_0) - \mu^2}{R_\nu(\lambda,\mu,\nu)} d\mu.$$

With the help of these formulas we may obtain some differential equations, which describe the deformation of the trigonometric case into the elliptic one. In fact, we make some general ansatz for some polynomial equation in the variables λ, μ, ν, such that the singularity, which in the trigonometric case is located at the curve $\mu = 0, \nu_0 = \lambda(\lambda+1)$, depends smoothly on the deformation parameter t, and make moreover some general ansatz for the logarithmic partial differential of ξ with respect to μ and λ, respectively:

$$R(\lambda,\mu,\nu,t) = \nu^3 + a(\lambda,\mu,t)\nu + b(\lambda,\mu,t);$$
$$\frac{1}{2\pi\sqrt{-1}}\frac{\partial \ln\xi}{\partial\lambda} = \frac{c(\lambda,\mu,t)\nu + d(\lambda,\mu,t)}{R_\nu(\lambda,\mu,\nu,t)};$$
$$\frac{1}{2\pi\sqrt{-1}}\frac{\partial \ln\xi}{\partial\mu} = \frac{e(\lambda,\mu,t)\nu + f(\lambda,\mu,t)}{R_\nu(\lambda,\mu,\nu,t)}.$$

Finally for the admissible 2-form, which describes the logarithmic derivative of the Floquet-multiplier we make the ansatz:

$$\frac{1}{2\pi\sqrt{-1}}\frac{\partial \ln\xi}{\partial t} = \frac{g(\lambda,\mu,t)\nu + h(\lambda,\mu,t)}{R_\nu(\lambda,\mu,\nu,t)}.$$

From this ansatz we obtain the differential equations

$$a\frac{\partial d}{\partial t} - d\frac{\partial a}{\partial t} + 3b\frac{\partial c}{\partial t} - c\frac{\partial b}{\partial t} = a\frac{\partial h}{\partial \lambda} - h\frac{\partial a}{\partial \lambda} + 3b\frac{\partial g}{\partial \lambda} - g\frac{\partial b}{\partial \lambda};$$
$$4a^2\frac{\partial c}{\partial t} - 2ac\frac{\partial a}{\partial t} - 9b\frac{\partial d}{\partial t} + 6d\frac{\partial b}{\partial t} = 4a^2\frac{\partial g}{\partial \lambda} - 2ag\frac{\partial a}{\partial \lambda} - 9b\frac{\partial h}{\partial \lambda} + 6h\frac{\partial b}{\partial \lambda};$$
$$a\frac{\partial f}{\partial t} - f\frac{\partial a}{\partial t} + 3b\frac{\partial e}{\partial t} - e\frac{\partial b}{\partial t} = a\frac{\partial h}{\partial \mu} - h\frac{\partial a}{\partial \mu} + 3b\frac{\partial g}{\partial \mu} - g\frac{\partial b}{\partial \mu};$$
$$4a^2\frac{\partial e}{\partial t} - 2ae\frac{\partial a}{\partial t} - 9b\frac{\partial f}{\partial t} + 6f\frac{\partial b}{\partial t} = 4a^2\frac{\partial g}{\partial \mu} - 2ag\frac{\partial a}{\partial \mu} - 9b\frac{\partial h}{\partial \mu} + 6h\frac{\partial b}{\partial \mu}.$$

This general ansatz can be restricted to the following form (this ansatz is of course justified also by success):

$$\begin{aligned}
a(\lambda,\mu,t) &= 6(\lambda-\lambda_0(t))\mu^2 - 3\nu_0^2(\lambda,t),\\
b(\lambda,\mu,t) &= -\mu^4 + (10(\lambda-\lambda_0(t))^3 - 18g_2(t)(\lambda-\lambda_0(t))^2 + 108g_3(t))\mu^2 + 2\nu_0^3(\lambda,t),\\
c(\lambda,\mu,t) &= -9\mu,\\
d(\lambda,\mu,t) &= -9\mu(3(\lambda-\lambda_0(t))^2 - 2\lambda_0(t)(\lambda-\lambda_0(t)) - 3g_2(t)),\\
e(\lambda,\mu,t) &= -3(\lambda-2\lambda_0(t)),\\
f(\lambda,\mu,t) &= 3(\mu^2 + (\lambda-2\lambda_0(t))\nu_0(\lambda,t)),\\
g(\lambda,\mu,t) &= 0,\\
h(\lambda,\mu,t) &= -3(\lambda-2\lambda_0(t) + (6\lambda_0^2(t) - 9g_2(t))\lambda_0'(t))\mu,
\end{aligned}$$

with

$$\nu_0(\lambda,t) = (\lambda-\lambda_0(t))^2 - 3g_2(t) \text{ and } \lambda_0'(t) = \frac{d\lambda_0(t)}{dt}.$$

This ansatz uses only three unknown parameters $g_2(t), g_3(t)$ and $\lambda_0(t)$, which in the trigonometric case take the values $g_2 = 1/12,\ g_3 = 1/216$ and $\lambda_0 = -1/2$.

If we insert these coefficients of the polynomials a, b, c, d, e, f, g and h in the differential equations given above, then we find the following ordinary differential equations:

$$\frac{dg_2(t)}{dt} = \frac{2\lambda_0(t)\lambda_0'(t)}{3} - \frac{1}{9};$$

$$\frac{dg_3(t)}{dt} = -\frac{\lambda_0(t)}{54} + \frac{\lambda_0^2(t) - 6g_2(t)}{9}\lambda_0'(t);$$
$$\frac{d\lambda_0(t)}{dt} = \frac{3g_2(t) - \lambda_0^2(t)}{6(54g_3(t) - (9g_2(t) - \lambda_0^2(t))\lambda_0(t))}.$$

In the last section we found three admissible 2-forms, which might describe some deformation of the trigonometric case. If we fix the value $\lambda = -1$ the singular spectral surface corresponding to the trigonometric case reduces to a singular curve. From our concrete realization we can see that it has a very complicated singularity. One can show that the 2-form, which describes the deformation in the trigonometric case, must have a zero at this curve. This reduces all possible deformations to the only one we gave in this section.

Let us now consider this deformation at the curves corresponding to the one-dimensional averaged potentials. So let us replace again the functions λ and μ by v and $\bar{\lambda}$. In the limit $v \to 0$ we will obtain the spectral curves of the one-dimensional averaged potentials corresponding to the three basic lattice direction of the momentum lattice. So let us calculate the three values of the function $\tilde{\nu}$ in the limit $v \to 0$:

$$\tilde{\nu}_1 = -\frac{27}{16v^4} - \frac{9\bar{\lambda} + 9\bar{\lambda}_0}{2v^2} + \frac{6\bar{\mu}}{v} + \bar{\lambda}^2 - 6\bar{\lambda}_0\bar{\lambda} - 3\bar{\lambda}_0^2 + g_2 + \mathbf{O}(v),$$
$$\tilde{\nu}_2 = -\frac{27}{16v^4} - \frac{9\bar{\lambda} + 9\bar{\lambda}_0}{2v^2} - \frac{6\bar{\mu}}{v} + \bar{\lambda}^2 - 6\bar{\lambda}_0\bar{\lambda} - 3\bar{\lambda}_0^2 + g_2 + \mathbf{O}(v),$$
$$\tilde{\nu}_3 = \frac{27}{8v^4} + \frac{9\bar{\lambda} + 9\bar{\lambda}_0}{v^2} - 2(\bar{\lambda}^2 - 6\bar{\lambda}_0\bar{\lambda} - 3\bar{\lambda}_0^2 + g_2) + \mathbf{O}(v),$$

where $\bar{\mu}$ is defined as the solution of

$$-\bar{\mu}^2 = 4(\bar{\lambda} - \bar{\lambda}_0)^3 - g_2(\bar{\lambda} - \bar{\lambda}_0) - g_3,$$

and $\bar{\lambda}_0$ is equal to $\lambda_0/3$. Hence this last equation gives the spectral curve of the averaged potentials in the basic directions of the momentum lattice. In the trigonometric case ($g_2 = 1/12,\ g_3 = 1/216,\ \bar{\lambda}_0 = 1/6$) we have the singular curve corresponding to the one-dimensional trigonometric Schrödinger operator. In [G-S] differential equations are given, which in the one-dimensional case describe the deformation of the trigonometric into the elliptic case. It is not hard to see that the differential equations from above transform exactly into these differential equations, if we use the parameters g_2, g_3 and $\bar{\lambda}_0$ for the spectral curves. This suggests the following Lemma:

Lemma 2 *Let Y_{affine} be the affine surface, which is the normalization of the affine surface, whose coordinate ring is generated by λ, μ and the three solutions ν_1, ν_2 and ν_3 of the equation*

$$R(\lambda, \mu, \nu) = \nu^3 + a(\lambda, \mu)\nu + b(\lambda, \mu) = 0,$$

where

$$\begin{aligned} a(\lambda, \mu) &= 6(\lambda - \lambda_0)\mu^2 - 3\nu_0^2(\lambda), \\ b(\lambda, \mu) &= -\mu^4 + (10(\lambda - \lambda_0)^3 - 18g_2(\lambda - \lambda_0) + 108g_3)\mu^2 + 2\nu_0^3(\lambda), \ , \\ \nu_0(\lambda) &= (\lambda - \lambda_0)^2 - 3g_2. \end{aligned}$$

with complex numbers g_2, g_3 and λ_0, so that $27g_3^2 - g_2^3 \neq 0$. Furthermore, let $2\omega_1$ be the first period of the Weierstraß function $\wp$, which satisfies the differential equation

$$(\wp'(x))^2 = 4\wp^3(x) - g_2\wp(x) - g_3\wp(x).$$

Then the equation

$$\frac{\omega_1 d\ln\xi}{\pi\sqrt{-1}} = -9\mu\frac{\nu + 3(\lambda-\lambda_0)^2 - 2\lambda_0(\lambda-\lambda_0) - 3g_2}{R_\nu(\lambda,\mu,\nu)}d\lambda - 3\frac{(\lambda-2\lambda_0)(\nu-\nu_0) - \mu^2}{R_\nu(\lambda,\mu,\nu)}d\mu$$

defines three holomorphic functions ξ_1, ξ_2 and ξ_3 on Y_{affine}, corresponding to the three functions ν_1, ν_2 and ν_3.

Proof: The transformation

$$(\lambda - \lambda_0, \mu, \nu) \mapsto \left((\lambda + 1/2)/(2\omega_1)^2, \mu/(2\omega_1)^3, \nu/(2\omega_1)^4\right)$$

induces an isomorphism between the surface Y_{affine}, which corresponds to $g_2 = 4/(3\omega_1^4), g_3 = 8/(27\omega_1^6)$ and the affine part of the spectral variety of the trigonometric Calogero-Moser operator. This shows that in this case the statement of the Lemma is true. Now we use the isospectral deformations of the corresponding one-dimensional spectral curves described by

$$-\bar{\mu}^2 = 4(\bar{\lambda} - \bar{\lambda}_0)^3 - g_2(\bar{\lambda} - \bar{\lambda}_0) - g_3, \text{ with } \bar{\lambda}_0 = \lambda_0/3.$$

For $g_2 = 4/(3\omega_1^4)$ and $g_3 = 8/(27\omega_1^6)$ this describes the singular spectral curve of the one-dimensional trigonometric Calogero-Moser operator with space period $2\omega_1$ and energy function $-4\pi^2\bar{\lambda}$. Due to [G-S] there exists solutions of the differential equations described above, which deform the one-dimensional trigonometric Calogero-Moser operator, with period $2\omega_1$ into the one-dimensional elliptic Calogero-Moser operator with period $2\omega_1$. The partial differential equations above show, that the functions ξ_1, ξ_2 and ξ_3 uniquely extends to functions on the affine surface Y_{affine} for arbitrary values $g_2(t), g_3(t)$, and $\lambda_0(t)$ of these solutions of the differential equations. The results of [G-S] implies, that the values of these solutions $g_2(t), g_3(t)$ and $\lambda_0(t)$ covers all values g_2, g_3 and λ_0, so that the corresponding Weierstraß function has the period $2\omega_1$. □

Let us now describe these spectral surfaces Y in more detail. The affine part of these surfaces corresponding to finite values of the three meromorphic functions λ, μ and ν is the complex space defined by these 6-fold covering, after removing the singularities at the curve $\mu = 0, \nu = \nu_0$. This means that the functions $(\nu - \nu_0)/\mu$ and $\mu/(\nu - \nu_0)$ should be considered as locally holomorphic, whenever they are locally bounded. And all these functions λ, μ, ν and eventually $(\nu-\nu_0)/\mu$ and $\mu/(\nu-\nu_0)$ generate locally the structure sheaf of holomorphic functions of these complex spaces $Y(t)$ for finite values of λ, μ and ν. Now we describe the structure sheaf of Y at infinity. Here we make use of the results of [K-T, Sch-2]. The logarithmic differentials of the three basic Floquet-multipliers are given by the functions ξ_1, ξ_2 and ξ_3 with essential singularities at infinity, such that $\xi_1\xi_2\xi_3 = 1$. Let $\tilde{Y}$ be the $\mathbb{Z}^2$-sheeted covering of Y, such that $k_1 = \ln\xi_1/(2\pi\sqrt{-1}), -k_2 = \ln\xi_2/(2\pi\sqrt{-1})$ and $-k_1 + k_2 = \ln\xi_3/(2\pi\sqrt{-1})$ extends to meromorphic functions of $\tilde{Y}$. Hence the finite part of $\tilde{Y}$ is an infinite-sheeted covering of $\mathbb{C}^2$, and the covering map is given by (k_1, k_2). Now let X be the blowing up of $\mathbb{P}^2$ at the points

$(2,1,0),(1,2,0)$ and $(1,-1,0)$. Then we assume that $\tilde{Y}$ is an infinite-sheeted covering over X, and the covering map is given by (k_1,k_2) with $k_1 = z_1/z_3, k_2 = z_2/z_3$ for $(z_1,z_2,z_3) \in \mathbb{P}^2$. This determines the structure sheaf of $\tilde{Y}$ at infinity. Moreover, the complex space Y is the quotient space of $\tilde{Y}/\mathbb{Z}^2$. The results of [Sch-2] show that in the trigonometric ($27g_3^2 = g_2^3 \neq 0$) case Y is again a complex space near infinity. Moreover, over each point $(z_1,z_2,0)$, with $z_1 - 2z_2 \neq 0, 2z_1 - z_2 \neq 0$ and $z_1 + z_2 \neq 0$, Y is locally biholomorphic to X, and over the three exceptional curves of X the surface Y is biholomorphic to the spectral curves of the three averaged potentials. Due to our construction of Y, the same is true for all these deformed surfaces. In the trigonometric case the spectral surface has only the singularities described in the first section of this paper. In the elliptic case these singularities are removed and the surface Y is a nonsingular compact complex manifold.

Now let E_1, E_2 and E_3 denote these three spectral curves in Y corresponding to the averaged potentials. They are elliptic curves over the exceptional curves of X, over the points $(2,1,0),(1,2,0),(1,-1,0) \in \mathbb{P}^2$. Finally let I denote the curve over $(z_1,z_2,0) \in \mathbb{P}^2$, without the curves E_1, E_2 and E_3 over the three exceptional curves of X. Then the intersection numbers of the curves I, E_1, E_2 and E_3 are equal to $I.I = -2, I.E_1 = I.E_2 = I.E_3 = 1, E_1.E_1 = E_2.E_2 = E_3.E_3 = -1, E_1.E_2 = E_1.E_3 = E_2.E_3 = 0$. They are equal to the intersection numbers of the corresponding curves in X.

4 The proof of Theorem 1

Let us first calculate the arithmetic genus of the algebraic surface defined in the last section. All holomorphic functions are constant and the dimension of $H^0(X,\mathcal{O})$ is equal to 1.

Now we determine the space of holomorphic 2-forms. In the first section we showed that

$$\frac{\mu d\lambda \wedge d\mu}{(\nu_1-\nu_2)(\nu_1-\nu_3)(\nu_2-\nu_3)}$$

is a holomorphic 2-form, which has no zeroes for finite λ, μ and ν. Hence the divisor of this 2-form has support at infinity. Now an explicit calculation shows that this divisor is equal to $K = 2E_1 + 2E_2 + 2E_3 + 3I$. Hence there exists one more linear independent holomorphic 2-form

$$\frac{\lambda\mu d\lambda \wedge d\mu}{(\nu_1-\nu_2)(\nu_1-\nu_3)(\nu_2-\nu_3)}.$$

This shows that the dimension of the space of holomorphic 2-forms is equal to 2.

Finally we claim that the space of holomorphic 1-forms has dimension 2. In fact,

$$\omega = \frac{6(\lambda-\lambda_0)\mu d\lambda + (\nu-\nu_0)d\mu}{R_\nu(\lambda,\mu,\nu)}$$

defines a holomorphic 1-form. Moreover, if ν is replaced by another root of the equation $R(\lambda,\mu,\nu) = 0$ the same expression defines again a holomorphic 1-form. This gives three holomorphic 1-forms corresponding to the three roots ν_1,ν_2,ν_3 of the equation $R(\lambda,\mu,\nu) = 0$. Since the nominator is a polynomial of degree 1 with respect to ν, the sum of the corresponding 1-forms is equal to zero. Thus we have two linear independent holomorphic 1-forms.

The action of the permutation group of three elements on the surface induces an action on the space of holomorphic forms. So let us decompose the space of holomorphic 1-forms into irreducible sub-representations. This decomposition may not contain any representation, which is equivalent to the trivial representation, since such a 1-form will induce a holomorphic 1-form of X, which does not have any 1-forms. The three holomorphic 1-forms mentioned above belong to an irreducible two-dimensional representation. If there exists a holomorphic 1-form, which transforms under the permutation group like the non-trivial one-dimensional representation, the wedge product of this form with the three 1-forms mentioned above will generate a two-dimensional irreducible sub-representation of the space of holomorphic 2-forms. As we have shown, such a sub-representation does not exist. Hence the space of holomorphic 2-forms contains only two-dimensional irreducible sub-representation. In general all holomorphic 1-forms, which belong to a two-dimensional irreducible sub-representation, may be written as

$$\omega = \frac{p(\lambda,\mu,\nu)d\lambda + q(\lambda,\mu,\nu)d\mu}{R_\nu(\lambda,\mu,\nu)}$$

with some polynomials $p(\lambda,\mu,\nu)$ and $q(\lambda,\mu,\nu)$, such that

$$\frac{p(\lambda,\mu,\nu)R_\mu(\lambda,\mu,\nu) - q(\lambda,\mu,\nu)R_\lambda(\lambda,\mu,\nu)}{R_\nu(\lambda,\mu,\nu)}$$

is again equal to some polynomial of λ,μ and ν. Moreover, the degree of the polynomials p and q have to be small enough such that ω is holomorphic at infinity. Now an explicit calculation shows that the space of holomorphic 1-forms is spanned by the three holomorphic 1-forms defined above[2]. This shows that the dimension of the space of holomorphic 1-forms is equal to 2.

Thus the arithmetic genus of the algebraic surface is equal to $(1-2+2)-1=0$.

In [G-L] (see also [E-V, §13.]) it is shown that the component of the Picard group of all topological trivial line bundles contains a Zariski-open set, whose elements $\mathcal{L}$ obey $\dim H^0(Y,\mathcal{L}) = \dim H^1(Y,\mathcal{L}) = 0$ and $\dim H^2(Y,\mathcal{L}) = 1$. Thus, due to Serre duality, the component of all line bundles, which are topological equivalent to the line bundle of holomorphic 2-forms, contains a Zariski-open set, whose elements $\mathcal{L}$ obey $\dim H^1(Y,\mathcal{L}) = \dim H^2(Y,\mathcal{L}) = 0$ and $\dim H^0(Y,\mathcal{L}) = 1$. We identify these line bundles with the corresponding divisors D. Let us only consider elements D of this Zariski-open set, such that the support of D is disjoint from I. Since $I.K = 0$ this is possible. Moreover, the degree of the restriction of the divisor D to E_1, E_2 and E_3 is generically equal to $E_1.K = E_2.K = E_3.K = 1$, respectively. Summing up we obtain a Zariski-open subset of the Picard group, whose elements may be represented by divisors D, so that

divisor (i) $\dim H^1(Y,\mathcal{O}_D) = \dim H^2(Y,\mathcal{O}_D) = 0$ and $\dim H^0(Y,\mathcal{O}_D) = 1$,

[2] Now it is not hard to determine the Albanese variety. It is the square of the elliptic curve corresponding to the one-dimensional averaged potentials. The action of the permutation group carries over to an action on the Albanese variety given by permutation of the elements $z_1, z_2, -z_1-z_2$, with $(z_1,z_2) \in (\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}))^2$. Moreover, an explicit calculation of the holomorphic forms shows that the Albanese map is a three-sheeted covering, whose branching divisor is given by the zero divisor of $\lambda - \lambda_0$, with the exception that the curve I is mapped onto one point. This description of the surface should be more convenient in order to generalize the results of this paper to the elliptic potentials related to other two-dimensional root systems.

divisor (ii) the support of D is disjoint with I, and

divisor (iii) the dimensions the cohomology groups of the restriction of $\mathcal{O}_D$ to E_1, E_2 and E_3 are equal to $\dim H^0(E_i, \mathcal{O}_D|_{E_i}) = 1$ and $\dim H^1(E_i, \mathcal{O}_D|_{E_i}) = 0$.

Now we may apply [Sch-2, Theorem 7.1], but let us indicate the argumentation given there. The three meromorphic differentials $d\ln\xi_1/(2\pi\sqrt{-1}), d\ln\xi_2/(2\pi\sqrt{-1})$ and $d\ln\xi_3/(2\pi\sqrt{-1})$ correspond to three generators of the Lie algebra of the Picard group, which have period 1. They implement the action of the group of space translations. Now we fix an element D of the Zariski-open set mentioned above. Hence there exists for all elements (x_1, x_2) of some dense open set $U \subset (\mathbb{R}/\mathbb{Z})^2$ a unique function $\psi(x_1, x_2)$ on Y, which has the following properties:

Bloch function (i) Outside the poles of the differentials $d\ln\xi_1, d\ln\xi_2$ and $d\ln\xi_3$ the function $\psi(x_1, x_2)$ is a section of $\mathcal{O}_D$.

Bloch function (ii) Nearby the poles of the differentials $d\ln\xi_1, d\ln\xi_2$ and $d\ln\xi_3$ the function

$$\psi(x_1, x_2)\exp\left(-2\pi\sqrt{-1}(k_1x_1 + k_2x_2)\right)$$

is a section of $\mathcal{O}_D$, where k_1 and k_2 are local integrals of the meromorphic differentials $d\ln\xi_1/(2\pi\sqrt{-1})$ and $d\ln\xi_2/(2\pi\sqrt{-1})$.

Bloch function (iii) On the curve I the function $\psi(x_1, x_2)\exp\left(-2\pi\sqrt{-1}(k_1x_1 + k_2x_2)\right)$ is equal to 1.

Now we claim that for fixed x_1, x_2 the function $8\pi^2\lambda\psi(x_1, x_2) - \triangle\psi(x_1, x_2)$ also satisfies properties (i) and (ii) of the Bloch function, where $\triangle$ is the Laplace operator on the Cartan-subalgebra of $su(3, \mathbb{C})$. First we apply the theory of one-dimensional finite-gap potentials (see e.g. [D-K-N-2]) and conclude from property (iii) of the divisors that the restrictions of the functions

$$\begin{aligned}
\psi_1 &= \psi(x_1, x_2)\exp\left(-2\pi\sqrt{-1}k_1(x_1 + x_2/2)\right), \\
\psi_2 &= \psi(x_1, x_2)\exp\left(-2\pi\sqrt{-1}k_2(x_2 + x_1/2)\right) \text{ and} \\
\psi_3 &= \psi(x_1, x_2)\exp\left(-2\pi\sqrt{-1}(k_1 - k_2)(x_1/2 - x_2/2)\right)
\end{aligned}$$

to E_1, E_2 and E_3 are one-dimensional Bloch-functions depending only on x_1, x_2 and $-x_1 - x_2$, respectively. The corresponding potentials are shifts of the functions

$$4\pi^2\wp(\pi x_1) - 8\pi^2\bar\lambda_0, 4\pi^2\wp(\pi x_2) - 8\pi^2\bar\lambda_0 \text{ and } 4\pi^2\wp(-\pi x_1 - \pi x_2) - 8\pi^2\bar\lambda_0,$$

respectively. Now the claim follows from the asymptotic behavior of the energy function λ. The claim and the first property of the divisors imply that there exists an unique potential $q(x_1, x_2)$, such that

$$(-\triangle + q(x_1, x_2))\psi(x_1, x_2) = -8\pi^2\lambda\psi(x_1, x_2).$$

Moreover, an easy calculation of the asymptotic behavior of the function

$$\psi(x_1, x_2)\exp\left(-2\pi\sqrt{-1}(k_1x_1k_2x_2\right)$$

shows that $q(x_1, x_2)$ is equal to the sum of the three one-dimensional potential. Finally, the restriction of the divisor K to E_1, E_2 and E_3 is equal to the divisor at infinity ($\bar\mu = \infty, \bar\lambda = \infty$) of the elliptic curves described by

$$-\bar\mu^2 = 4(\bar\lambda - \bar\lambda_0)^3 - g_2(\bar\lambda - \bar\lambda_0) - g_3, \text{ with } \bar\lambda_0 = \lambda_0/3.$$

This shows that the line bundle of holomorphic 2-forms of Y is equivalent to the eigen bundle of the Weyl group invariant Schrödinger operator

$$-2\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}-\frac{\partial^2}{\partial x_1\partial x_2}\right)+4\pi^2\left(\wp(\pi x_1)+\wp(\pi x_2)+\wp(-\pi x_1-\pi x_2)\right)-24\pi^2\bar{\lambda}_0.$$

The substitution $\pi x\mapsto x$ completes the proof of Theorem 1. □

5 Algebraic integrability of the elliptic Calogero-Moser operator

Let's recall the notion of algebraic integrability for Schrödinger operators (see [Kr, C-S-V]). A Schrödinger operator $L=-\Delta+q(x)$ on the $\mathbb{R}^n$ is called *integrable*, if there exists n commuting differential operators $L_1=L,\ldots,L_n$, with algebraically independent highest symbols $P_1(k)=k^2,\ldots,P_n(k)$. The operator L is called *algebraically integrable*, if there exists one more differential operator L_{n+1}, which commutes with the operators $L_1,\ldots,L_n$, and whose highest symbols takes different values on the solution of the algebraic system $P_1(k)=c_1,\ldots,P_n(k)=c_n$ for generic $c_1,\ldots,c_n$.

The elliptic Calogero-Moser-operator

$$L=-\partial_1^2-\partial_2^2-\partial_3^2+4\left(\wp_{12}+\wp_{23}+\wp_{13}\right),\ \text{with}\ \wp_{ij}=\wp(x_i-x_j),$$

has the following integrals:

$$\begin{aligned} L_0 &= \partial_1+\partial_2+\partial_3\\ L_1 &= L\\ L_2 &= \partial_1\partial_2\partial_3+2\wp_{12}\partial_3+2\wp_{13}\partial_2+2\wp_{23}\partial_1 \end{aligned}$$

and therefore is integrable. It has been conjectured in [C-V] that it is also algebraically integrable. Now we can prove it. In fact, all meromorphic functions of Y, which are holomorphic on the affine part of Y, acts on the Bloch-function like a differential operator, and all these differential operators commute with each other. This follows from [Sch-2, Theorem 7.3]. Hence all differential operators, which commute with the trigonometric Calogero-Moser operator (3), can be deformed into a differential operator, which commutes with the elliptic Calogero-Moser operator (1) and the highest symbol does not change under this deformation. In [C-V] it was proven that the trigonometric Calogero-Moser operator is algebraically integrable. This proves the following theorem.

Theorem 3 *The elliptic Calogero-Moser operator (1) is algebraically integrable.*

□

Some of these additional integrals have been found recently by L. A. Khodarinova [Kh] by direct calculations. The result has the form

$$\begin{aligned} L_{12} = {} & \partial_{13}^2\partial_{23}^2-8\wp_{23}\partial_{13}^2-8\wp_{13}\partial_{23}^2+4(\wp_{12}-\wp_{13}-\wp_{23})\partial_{13}\partial_{23}-\\ & -2(\wp_{12}'+\wp_{13}'+6\wp_{23}')\partial_{13}-2(-\wp_{12}'+6\wp_{13}'+\wp_{23}')\partial_{23}-\\ & -2\wp_{12}''-6\wp_{13}''-6\wp_{23}''+4(\wp_{12}^2+\wp_{13}^2+\wp_{23}^2)+8(\wp_{12}\wp_{13}+\wp_{12}\wp_{23}+7\wp_{13}\wp_{23}), \end{aligned}$$

where $\partial_{ij}=\partial_i-\partial_j$. Two other commuting operators L_{23} and L_{13} can be found by permuting the indices. One can check that the operator $L_3=L_{12}+1/2L_{23}$ has all the property needed for the algebraic integrability.

6 The fixed energy levels

In 1976 Dubrovin, Krichever and Novikov introduced the class of two-dimensional Schrödinger operators in magnetic fields, which are 'finite-gap on one energy level' [D-K-N-1]. Later S. P. Novikov and one of the authors found the conditions on the algebro-geometric data, which correspond to the pure potential case, when the magnetic filed is identically zero [N-V-1, N-V-1].

Let's describe the corresponding construction following to [N-V-1]. Let Γ be an algebraic curve of genus g with involution $\sigma : \Gamma \to \Gamma$, having precisely two fixed points P_+ and P_- with local parameters k_+^{-1} and k_-^{-1}, respectively. Let D be an effective divisor of degree g on Γ satisfying the relation

$$D + \sigma(D) \sim P_+ + P_- + K, \tag{4}$$

where K is the canonical divisor and $\sim$ means linear equivalence. Define the *Baker-Akhiezer function* $\psi(x, y, P)$ as a function on Γ depending on x, y as parameters with the following properties:

(i) On $\Gamma \setminus \{P_+, P_-\}$ the function ψ is a section of $\mathcal{O}_D$.

(ii) At the points P_+ and P_- the function ψ has essential singularities of the form

$$\psi(x, y, P) = \left(\exp(k_+ \left(x + \sqrt{-1}y\right)\right) \left(1 + \mathbf{O}(k_+^{-1})\right) \text{ near } P_+,$$

$$\psi(x, y, P) = \left(\exp(k_- \left(x - \sqrt{-1}y\right)\right) \left(1 + \mathbf{O}(k_-^{-1})\right) \text{ near } P_-.$$

Such a function is uniquely determined by these properties and satisfies some Schrödinger equation

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi + q(x, y)\psi = 0.$$

In other words, ψ is the eigenfunction for some two-dimensional Schrödinger operator with eigenvalue equal to zero (zero energy level). In [N-V-1] the explicit formulas for such a function ψ as well as for the potential $q(x, y)$ in terms of the Prym's theta-functions corresponding to the pair (Γ, σ) have been found. We check now that the elliptic Calogero-Moser operator (1) fit into this scheme for generic energy levels λ =constant. The description (2) of this curve has a singularity, when $\mu = 0$ and $\nu = \nu_0$. In order to resolve this singularity let's introduce new variables

$$\hat{\nu} = \frac{\nu - \nu_0}{\mu} \text{ and } \hat{\kappa} = \frac{\kappa - \nu_0}{\mu}.$$

Then the equations (2) takes the form

$$\begin{array}{rclcl} P(\mu, \hat{\nu}) & = & \mu\hat{\nu}^3 + 3\nu_0\hat{\nu}^2 + 6\lambda\mu\hat{\nu} + (-\mu^2 + 10\lambda^3 - 8g_2\lambda + 108g_3) & = & 0 \\ Q(\mu, \hat{\nu}, \hat{\kappa}) & = & \mu\hat{\kappa}^2 + (3\nu_0 + \hat{\nu}\mu)\hat{\kappa}^2 + (\mu\hat{\nu}^2 + 6\lambda\mu + 3\nu_0\hat{\nu}) & = & 0 \end{array} \tag{5}$$

The curve described by these equations is invariant under the involution

$$\sigma : (\mu, \hat{\nu}, \hat{\kappa}) \mapsto (-\mu, -\hat{\nu}, -\hat{\kappa}).$$

One can check that this involution has no fixed point at the finite part of Γ and only two fixed points at infinity

$$P_\pm : \mu \sim k^3, \hat{\nu} \sim k, \hat{\kappa} \sim \frac{-1 \pm \sqrt{3}}{2} k \text{ in the limit } k \to \infty.$$

The divisor $D = 3P_+ + 3P_-$ satisfies the relation (4). In order to show this let's consider the differential

$$\omega = \frac{d\hat{\nu}}{P_\mu Q_{\hat{\kappa}}} = \frac{d\hat{\nu}}{(\hat{\nu}^3 + 6\lambda\mu - 2\mu)(2\mu\hat{\kappa} + 3\nu_0 + \hat{\nu}\mu)}.$$

One can check that ω is regular and has two zeroes at $P_\pm$ of order 5, respectively. Thus D satisfies relation (4) and the following Theorem is proven:

Theorem 4 *The elliptic Calogero-Moser operator (1) is finite-gap on a generic energy level.*

□

This means that one can use the explicit formulas from [N-V-1, N-V-1] to express the eigenfunctions ψ in terms of the theta function of the three-dimensional Prym varieties of the curve (5). The results of Inozemtsev [In] and of Felder and Varchenko [F-V-1, F-V-2] show that these formulas can be essentially simplified to expressions in terms of classical elliptic functions. This should follow from the geometric properties of these Prym varieties, but we have not yet undertaken such investigations.

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